DO QUASI-EXACTLY SOLVABLE SYSTEMS ALWAYS CORRESPOND TO ORTHOGONAL POLYNOMIALS?

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Abstract

We consider two quasi-exactly solvable problems in one dimension for which the Schrödinger equation can be converted to Heun's equation. We show that in neither case the Bender-Dunne polynomials form an orthogonal set. Using the anti-isopectral transformation we also discover a new quasi-exactly solvable problem and show that even in this case the polynomials do not form an orthogonal set.

Some time ago, in a remarkable paper Bender and Dunne [1] showed that the eigenfunctions of the Schrödinger equation for a quasi-exactly solvable (QES) problem is the generating function for a set of orthogonal polynomials $\{P_n(E)\}$ in the energy variable E. It was further shown that these polynomials satisfy the three-term recursion relation

$$P_n(E) = EP_{n-1}(E) + C_n P_{n-2}(E) \tag{1}$$

where C_n is E independent quantity. Using the well known theorem [2,3], " the *necessary and sufficient* condition for a family of polynomials $\{P_n\}$ (with degree $P_n = n$) to form an orthogonal polynomial system is that $\{P_n\}$ satisfy a three-term recursion relation of the form

$$P_n(E) = (A_n E + B_n)\, P_{n-1}(E) + C_n P_{n-2}(E) \qquad n \geq 1 \tag{2}$$

where the coefficients A_n, B_n and C_n are independent of E, $A_n \neq 0, C_1 = 0, C_n \neq 0$ for $n \geq 1$" , it then followed that $\{P_n(E)\}$ for this problem forms an orthogonal set of polynomials with respect to some weight function, $w(E)$. Recently several authors have studied the Bender-Dunne polynomials in detail [3–6]. In fact it has been claimed that the Bender-Dunne construction is quite universal and valid for any quasi-exactly solvable model in both one as well as multi-dimensions.

The purpose of this note is to critically examine this assertion. In particular, we discuss two QES systems, one of which is in one dimension while the other one is an effective one dimensional system. We show in both the cases that though these problems are QES problems, the corresponding three-term recursion relation is not of the type as given in Eq. (2), and hence the corresponding polynomials do not form an orthogonal set. Using the anti-isospectral transformation $x \rightarrow ix$ introduced recently by Krajewska *et al.* [5], we discover a new QES problem and show that even in this case the polynomials do not form an orthogonal set.

Consider the potential,

$$V(x) = \frac{\mu^2 \left[8 \sinh^4 \frac{\mu x}{2} - 4(\frac{5}{\epsilon^2} - 1) \sinh^2 \frac{\mu x}{2} + 2\left(\frac{1}{\epsilon^4} - \frac{1}{\epsilon^2} - 2\right)\right]}{8\left[1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}\right]^2} \tag{3}$$

which arises in the context of the stability analysis around the kink solution for ϕ^6-type field theory in 1+1 dimensions [7]. It has already been shown that it is an example of QES system in which both the ground and the second excited states are exactly known in the case $\epsilon^2 = \frac{1}{2}$. Further it has also been shown that contrary to the usual belief [8], this QES problem can not be generated by the quadratic elements of an enveloping $Sl(2)$ algebra [9]. A related fact is that the Schrödinger equation for this case, after some transformations, can be written in the form of Heun's equation which has four regular singular points [7]. We shall now show that in this case one can obtain a three-term recursion relation for the polynomials $\{P_n(E)\}$ which is different from the one given in Eq. (2) and hence these polynomials do *not* form an orthogonal set even though it is a QES system.

Consider the Schrödinger equation ($\hbar = 2m = 1$)

$$\left[-\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x) \tag{4}$$

On making use of the transformation

$$y = \frac{\sinh^2 \frac{\mu x}{2}}{\left(1 + \frac{1}{\epsilon^2} + \sinh^2 \frac{\mu x}{2}\right)}, \quad \psi = (1-y)^s f, \quad s = \left(1 - \frac{E}{\mu^2}\right)^{\frac{1}{2}} \tag{5}$$

the Schrödinger equation reduces to the Heun's equation,

$$f''(y) + \left[\frac{1}{2y} + \frac{1+2s}{y-1} + \frac{1}{2(y+\epsilon^2)}\right] f'(y) + \frac{(\alpha\beta y - q)}{y(y-1)(y+\epsilon^2)} f(y) = 0 \tag{6}$$

Here

$$\alpha = -\frac{5}{2} - s; \quad \beta = \frac{3}{2} - s; \quad q = (1-s^2)(1+\epsilon^2) - \frac{1}{2}s\epsilon^2 - \frac{1}{4}(1-2\epsilon^2). \tag{7}$$

On further substituting

$$t = \left(y + \frac{1}{2}\right)^{\frac{1}{2}} \tag{8}$$

it is easy to show that $f(t)$ satisfies

$$\begin{aligned}(t^2-\epsilon^2)(t^2-1-\epsilon^2)f''(t) + &\left[(t^2-1-\epsilon^2) + 2(1+2s)(t^2-\epsilon^2)\right] t f'(t) \\ &+ \left[4\alpha\beta t^2 - 4(\alpha\beta\epsilon^2 + q)\right] f(t) = 0\end{aligned} \tag{9}$$

From this equation one can easily obtain the three-term recursion relation satisfied by the polynomials associated with this QES system. In particular, on substituting

$$f(t) = \sum_{n=0}^{\infty} \frac{Q_n(s) t^n}{n!} \tag{10}$$

in Eq. (9) it is easily shown that Q_n's satisfy the three-term recursion relation

$$\begin{aligned}\epsilon^2(\epsilon^2+1)Q_{n+2}(s) - &\left[(2\epsilon^2+1)n^2 + (5\epsilon^2+2+4s\epsilon^2)n - 4s^2 + 2s\epsilon^2 + 3 - 9\epsilon^2\right] Q_n(s) \\ &+ n(n-1)\left[n^2 + n(4s-2) + 4s^2 - 4s - 15\right] Q_{n-2}(s) = 0\end{aligned} \tag{11}$$

Notice that the even and the odd Q_n's are unrelated and we can obtain two separate recursion relations for them. In particular it is easily shown that for even and odd cases, the recursion relations respectively are ($m \geq 1$)

$$\begin{aligned}\epsilon^2(\epsilon^2+1)P_m(s) - &\left[(8\epsilon^2+4)m^2 + (8s\epsilon^2 - 6\epsilon^2 - 4)m - 4s^2 - 6s\epsilon^2 + 3 - 11\epsilon^2\right] P_{m-1}(s) \\ &+ (m-1)(2m-3)\left[8m^2 + (16s-24)m + 8s^2 - 24s - 14\right] P_{m-2}(s) = 0\end{aligned} \tag{12}$$

and

$$\begin{aligned}\epsilon^2(\epsilon^2+1)P_m(s) - &\left[(8\epsilon^2+4)m^2 + (8s\epsilon^2 + 2\epsilon^2)m - 4s^2 - 2s\epsilon^2 + 2 - 12\epsilon^2\right] P_{m-1}(s) + \\ &(m-1)(2m-1)\left[8m^2 + (16s-16)m + 8s^2 - 16s - 24\right] P_{m-2}(s) = 0\end{aligned} \tag{13}$$

subject to the initial condition, $P_0(s) = 1$.

We observe that neither of these three-term recursion relations are of the form as given by Eq. (2) and hence we conclude that even though it is a QES problem, the corresponding polynomials $P_m(s)$ are not orthogonal. Note that the QES solution corresponding to the ground state is obtained from Eqs. (10) and (13) and in this case $s = 1$(*i.e.* $E = 0$) and $m = 2$(irrespective of the value of ϵ), while the solution corresponding to the second excited state is obtained from Eqs. (10) and (12) when $s = \frac{1}{2}$(*i.e.* $E = \frac{3}{4}\mu^2$) and $m = 3$ only if $\epsilon^2 = \frac{1}{2}$. Thus only for $\epsilon^2 = \frac{1}{2}$, this is a QES problem.

Why is that in this case the polynomials do not form an orthogonal set? While we do not have a definite answer, we suspect that only for those QES systems for which the symmetry group is $Sl(2)$, the corresponding polynomials form an orthonormal set. In this context recall that for the above problem, the Hamiltonian can not be written in terms of the quadratic generators of $Sl(2)$ [9]. As a support to our conjecture, we offer another QES problem where also the Schrödinger equation can be reduced to Heun's equation with four regular singular points and hence the corresponding symmetry group is not $Sl(2)$. In that case also we find the three-term recursion relation satisfied by the polynomials and show that it is not of the type as given by Eq. (2) and hence these polynomials also do not form an orthogonal set.

Recently Bhaduri *et al.* [10] have considered a two body problem characterized by

$$H = -\frac{1}{2}(\vec{\nabla}_1^2 + \vec{\nabla}_2^2) + \frac{1}{2}\left[r_1^2 + r_2^2\right] + \frac{g_1}{2}\left(\frac{r_1^2 + r_2^2}{X^2}\right) \tag{14}$$

where $X = x_1 y_2 - x_2 y_1$. They were able to solve the problem in hyperspherical coordinates, (R, θ, ϕ, ψ). In particular they showed that the angular equation is

$$\left(\Lambda^2 + \frac{4g_1}{\sin^2(2\theta)}\right)\Phi = \beta(\beta+2)\Phi \quad \beta \geq 1 \tag{15}$$

while the radial equation is of the type of harmonic oscillator in 4-dimensions.

$$\frac{d^2F}{dR^2} + \frac{3}{R}\frac{dF}{dR} + \left(2E - R^2 - \frac{\beta(\beta+2)}{R^2}\right)F = 0 \tag{16}$$

Here Λ^2 is the Laplacian on the sphere S^3. On making the ansatz,

$$\Phi(\theta, \phi, \psi) = P(x) e^{iq\phi}\ e^{il\psi} \tag{17}$$

where q and l are integers and

$$P(x) = |x|^a (1-x)^b (1+x)^c \Theta^{a,b,c}(x) \tag{18}$$

it was shown that $\Theta(x)$ satisfies Heun's equation,

$$\begin{aligned}(1-x^2)\frac{d^2\Theta}{dx^2} &+ 2\left[\frac{a}{x} - (b-c) - (a+b+c+1)x\right]\frac{d\Theta}{dx} \\ &+ \left[\frac{(\beta+1)^2}{4} - (a+b+c+\frac{1}{2})^2 + \frac{2a(c-b)}{x}\right]\Theta(x) = 0\end{aligned} \tag{19}$$

Here $b = \frac{|l+q|}{4}$, $c = \frac{|l-q|}{4}$, $a(a-1) = g_1$. In order to obtain the recursion relation satisfied by the polynomials associated with this QES problem we substitute

$$\Theta(x) = \sum_{n=0}^{\infty} \frac{P_n(\beta) x^n}{n!} \tag{20}$$

in Eq. (19). It is easy to show that $P_n(\beta)$ satisfies the three-term recursion relation

$$\begin{aligned}(n+2a-1)P_n - 2(b-c)(n-1+a)P_{n-1} \\ +(n-1)\left[\frac{(\beta+1)^2}{4} - \left(a+b+c-\frac{3}{2}\right)^2\right] P_{n-2} = 0\end{aligned} \tag{21}$$

when $n \geq 1$ and $P_0 = 1$. We again observe that the recursion relation is not of the type given by Eq. (2) and hence it follows that the polynomials $P_n(\beta)$ do not form an orthogonal set. The QES solutions in this case have already been discussed in Ref. [10].

Summarizing, we have seen that there are QES problems in one dimension and effective one dimension where the symmetry group is not $Sl(2)$, the Schrödinger equation get converted to Heun's equation, and the corresponding polynomials do not form an orthogonal set. It will be interesting to find the symmetry group in these cases.

Before finishing this note we would like to point out a nontrivial application of the anti-isospectral transformation induced recently by Krajewska *et al.* [5]. In particular, applying it we are able to discover a new QES problem in quantum mechanics. Consider the potential given in Eq. 3 and apply the transformation $x \rightarrow i\theta$. We then obtain a new QES problem which has not been discussed in the literature so far. In particular the ground and second excited state energies of the new (periodic) potential

$$V(\theta) = -\frac{\mu^2 \left[8 \sin^4 \frac{\mu\theta}{2} + 4(\frac{5}{\epsilon^2} - 1) \sin^2 \frac{\mu\theta}{2} + 2\left(\frac{1}{\epsilon^4} - \frac{1}{\epsilon^2} - 2\right)\right]}{8\left[1 + \frac{1}{\epsilon^2} - \sin^2 \frac{\mu\theta}{2}\right]^2} \tag{22}$$

are $E_0 = -\frac{3}{4}\mu^2, E_2 = 0$ in case $\epsilon^2 = \frac{1}{2}$. The corresponding eigenfunctions can easily be written down. Needless to say in this case also one can obtain a three-term recursion relation and show that it is not of the type as given by Eq. (2). Thus this provides a third QES problem where the Bender-Dunne polynomials do not form an orthogonal set. The details will be given elsewhere [11] where we will also discuss other applications of the anti-isospectral transformation.

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