A QES Band-Structure Problem in One Dimension

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Abstract

I show that the potential

$$V(x,m) = \Big[\frac{b^2}{4} - m(1-m)a(a+1)\Big]\frac{\mathrm{sn}^2(x,m)}{\mathrm{dn}^2(x,m)} - b(a+\frac{1}{2})\frac{\mathrm{cn}(x,m)}{\mathrm{dn}^2(x,m)}$$

constitutes a QES band-structure problem in one dimension. In particular, I show that for any positive integral or half-integral a, $2a+1$ band edge eigenvalues and eigenfunctions can be obtained analytically. In the limit of m going to 0 or 1, I recover the well known results for the QES double sine-Gordon or double sinh-Gordon equations respectively. As a by product, I also obtain the bound state eigenvalues and eigenfunctions of the potential

$$V(x) = \Big[\frac{\beta^2}{4} - a(a+1)\Big]\mathrm{sech}^2 x + \beta(a+\frac{1}{2})\mathrm{sech}x\tanh x$$

in case a is any positive integer or half-integer.

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In last few years, the quasi-exactly solvable (QES) problems have received wide attention in the literature [1, 2]. In these cases, the corresponding orthogonal polynomials satisfy a three-term recursion relation. Further, the Hamiltonian (or its gauged transform) can be written in terms of at most the quadratic combination of the generators of the $SL(2,R)$ group. Two of the celebrated QES potentials are the double sine-Gordon (DSG) potential

$$V(x) = \frac{b^2}{4}\sin^2 x - b(a+\frac{1}{2})\cos x \ , \tag{1}$$

and the double sinh-Gordon (DSHG) potential [3]

$$V(x) = \frac{b^2}{4}\sinh^2 x - b(a+\frac{1}{2})\cosh x \ . \tag{2}$$

In both these cases, the eigenvalues and eigenfunctions for $2a+1$ levels are analytically known in case a is any positive integer or half-integer. Further, the properties of the corresponding orthogonal polynomials have been studied in some detail [4].

The question that I would like to raise and answer in this note is the following: Instead of solving both the DSG and DSHG problems, is it not possible to solve one 'elliptic' problem of which the potentials (1) and (2) constitute special cases? In this note I show that the answer to this question is yes. In particular, I solve the Schrödinger equation for the periodic potential

$$V(x,m) = [\frac{b^2}{4} - m(1-m)a(a+1)]\frac{\mathrm{sn}^2(x,m)}{\mathrm{dn}^2(x,m)} - b(a+\frac{1}{2})\frac{\mathrm{cn}(x,m)}{\mathrm{dn}^2(x,m)} \ , \tag{3}$$

and show that it is a QES band-structure problem, i.e. $2a+1$ band edge eigenvalues and eigenfunctions can be analytically obtained in case a is any positive integer or half-integer. In particular, explicit expressions for band-edge eigenstates are given in case $a = 0, 1/2, 1, 3/2, 2$. Not surprisingly, for $m = 0, 1$ we recover the well known results for the DSG and DSHG potentials (1) and (2) respectively. Here $\mathrm{cn}(x,m), \mathrm{sn}(x,m)$ are the Jacobi elliptic functions of real elliptic modulus parameter m ($0 \le m \le 1$) with period $4K(m)$ while $\mathrm{dn}(x,m)$ has period $2K(m)$. For simplicity, now onward, we will not explicitly display the modulus parameter m as an argument of Jacobi elliptic functions [5].

I also show that in case a is any positive integer or half-integer then the (gauged) Hamiltonian can be written in terms of at most the quadratic generators of the $SL(2,R)$ group. Further, the associated orthogonal polynomials satisfy three-term recursion relation. Finally, as a by product, I also obtain the bound state eigenvalues and eigenfunctions of the well known exactly solvable potential [7]

$$V(x) = [\frac{\beta^2}{4} - a(a+1)]\mathrm{sech}^2 x - \beta(a+\frac{1}{2})\ \mathrm{sech} x \tanh x\ , \tag{4}$$

in case a is any positive integer or half-integer.

We start from the Schrödinger equation ($\hbar = 2m = 1$)

$$\frac{d^2\psi(x)}{dx^2} + [E - V(x)]\psi(x) = 0\ , \tag{5}$$

where $V(x)$ is as given by eq. (3). Note that the potential (3) is of period $4K(m)$, where $K(m)$ denotes the complete elliptic integral of the first kind. In fact, the potential (3) is also invariant under $x \to x + 2K(m)$ provided we also let $b \to -b$. Further, in the limit m going to 0 or 1, the potential (3) reduces to the DSG or DSHG potentials (1) and (2) respectively. In this note, we are interested in obtaining the band edge eigenvalues and eigenfunctions which if arranged in order of increasing energy $E_0 < E_1 \le E_2 < E_3 \le E_4$...are of period $4K, 8K, 8K, 4K, 4K$,... with the corresponding number of wave function nodes in the interval $4K$ being 0,1,1,2,2,... .

We substitute

$$\psi = \exp\left[-\frac{b}{2\sqrt{m(1-m)}}tan^{-1}(\sqrt{\frac{m}{1-m}}\mathrm{cn}\,x)\right]y \ , \tag{6}$$

in the Schrödinger equation (5) yielding

$$y''(x) - b\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x}y'(x) + [E + ab\frac{\mathrm{cn}\,x}{\mathrm{dn}^2\,x} + m(1-m)a(a+1)\frac{\mathrm{sn}^2\,x}{\mathrm{dn}^2\,x}]y(x) = 0 \ . \tag{7}$$

On further substituting

$$y(x) = (\mathrm{dn}\,x)^{-a}u(x) \ , \tag{8}$$

it is easily shown that $u(x)$ satisfies the equation

$$\begin{aligned} u''(x) \quad &+ \quad [2am\frac{\mathrm{sn}\,x\mathrm{cn}\,x}{\mathrm{dn}\,x} - b\frac{\mathrm{sn}\,x}{\mathrm{dn}\,x}]u'(x) \\ &+ \quad [E + am + ab\mathrm{cn}\,x + ma(a-1)\mathrm{sn}^2\,x] \ u(x) = 0 \ . \end{aligned} \tag{9}$$

We now want to show that irrespective of the value of b, eq. (9) is a QES case. In particular, we wish to show that if a is a positive integer (half-integer) than eq. (9) admits $2a+1$ algebraic solutions of period $4K$ ($8K$). For the special case of $b = 0$ this is of course well known since in that case the potential (3) essentially reduces to the Lamé potential.

Let us start from eq. (9) and substitute

$$\mathrm{sn}\,x = \sin\theta \ , \ u(x) \equiv z(\theta) \ . \tag{10}$$

Then $z(\theta)$ satisfies

$$\begin{aligned} &(1 - m\sin^2\theta)z''(\theta) + [(2a-1)m\cos\theta\sin\theta - b\sin\theta]z'(\theta) \\ &\quad + [E + am + ab\cos\theta + ma(a-1)\sin^2\theta]z(\theta) = 0 \ . \end{aligned} \tag{11}$$

On further substituting $\cos\theta = t$, one finds that $z(t)$ satisfies

$$\begin{aligned} &[mt^4 + (1-2m)t^2 - (1-m)]z''(t) \\ &+ [2m(1-a)t^3 + bt^2 + (2ma - 2m + 1)t - b]z'(t) \\ &+ [ma(a-1)t^2 - abt - E - ma^2]z(t) = 0 \ . \end{aligned} \tag{12}$$

It is now straight forward to check that eq. (12) can be written as a quadratic combination of the operators

$$J_n^+ = t^2 \frac{d}{dt} - nt \ , \ J_n^0 = t\frac{d}{dt} - \frac{n}{2} \ , \ J_n^- = \frac{d}{dt} \ , \tag{13}$$

where $J_n's$ are the generators of the non-compact Lie group $SL(2, R)$, provided $a = n$. In particular, for $a = n$, eq. (12) can be written as

$$\left[mJ_n^+ J_n^+ + (1-2m) J_n^0 J_n^0 - (1-m) J_n^- J_n^- + nJ_n^0 + b(J_n^+ - J_n^-) + \lambda \right] z(t) = 0 \ , \tag{14}$$

where $\lambda = -(E + \frac{mn^2}{2} - \frac{n^2}{4})$. Thus, if a is a positive integer n, then the three generators $J_n^{\pm,0}$ form a representation of dimension $n+1$ of the group $SL(2, R)$.

Similarly, on substituting $z(\theta) = \sin\theta w(t = \cos\theta)$ in eq. (11), it is easily shown that the resulting equation can again be written as a quadratic combination of the generators (13) provided $a = n+1$. Thus if a is a positive integer, then the three generators $J_n^{\pm,0}$ form a representation of dimension n of the group $SL(2, R)$. In this way, we have shown that when a is a positive integer n, then one will have $n+1$ solutions of the type $F_n(t)$ and n solutions of the type $\sin\theta F_{n-1}(t)$.

Let us now turn to the case of half-integral a. On substituting

$$t = \frac{1+\cos\theta}{2} \ , \ z(\theta) = t^{1/2} w(t) \ , \tag{15}$$

in eq. (11) it is easily shown that the resulting equation can be written as a quadratic combination of the generators (13) provided $a = n + \frac{1}{2}$. A similar conclusion is also reached in case we substitute

$$t = \frac{1-\cos\theta}{2} \ , \ z(\theta) = t^{1/2} w(t) \ , \tag{16}$$

in eq. (11). Thus, in the half-integral case (i.e. $a = n + \frac{1}{2}$), we have shown that one has $n+1$ solutions of the type $\cos\frac{\theta}{2}F_n(t)$ and $n+1$ solutions of the type $\sin\frac{\theta}{2}F_n(t)$.

As an illustration, we now give explicit solutions for few values of a. In particular, we specify the eigenvalue E and eigenfunction $u(x)$ with ψ being related to u by eqs. (6) and (8).

a =0:

$$E = 0 \ , \ u(x) = constant \ . \tag{17}$$

$a = \frac{1}{2}$:

$$E = \frac{1 - 2m \mp 2b}{4} \ , \ u(x) = \sqrt{1 \pm \mathrm{cn}\, x} \ . \tag{18}$$

a=1:

$$E = 1 - 2m \ , \ u(x) = \mathrm{sn}\, x \ , \tag{19}$$

$$E = \frac{1 - 2m \pm \sqrt{1 + 4b^2}}{2} \ , \ u(x) = b - (E + m)\mathrm{cn}\, x \ . \tag{20}$$

$a = \frac{3}{2}$:

$$\begin{aligned} E &= \frac{5 - 10m - 2b}{4} \pm \sqrt{1 - m(1-m) + (1-2m)b + b^2} \ , \\ u(x) &= (\alpha + \beta \mathrm{cn}\, x)\sqrt{1 + \mathrm{cn}\, x} \ , \end{aligned} \tag{21}$$

$$\begin{aligned} E &= \frac{5 - 10m + 2b}{4} \pm \sqrt{1 - m(1-m) - (1-2m)b + b^2} \ , \\ u(x) &= (\alpha_1 + \beta_1 \mathrm{cn}\, x)\sqrt{1 - \mathrm{cn}\, x} \ . \end{aligned} \tag{22}$$

a = 2:

$$E = \frac{5-10m}{2} \pm \sqrt{9+4b^2}\ ,\ u(x) = \mathrm{sn}\,x(\alpha_2 + \beta_2 \mathrm{cn}\,x)\ , \tag{23}$$

$$\begin{aligned} x^3 &+ 2(2m-1)x^2 - (4b^2+3)x + 8(1-2m)b^2 = 0\ ,\ E = x + 1 - 2m\ , \\ u(x) &= \alpha_3 + \beta_3 \mathrm{cn}\,x + \delta_3 \mathrm{cn}^2\,x\ . \end{aligned} \tag{24}$$

In the above equations, the constants α, β etc. are easily determined.

As expected, for $m = 0, 1$, these eigenvalues and eigenfunctions agree with the well known results for the DSG and DSHG potentials (1) and (2) respectively.

For the special case of $m = \frac{1}{2}$, the cubic eq. (24) is easily solved yielding $E = 0, \pm\sqrt{4b^2+3}$. It is amusing to notice that for $m = \frac{1}{2}$, the eigenvalues given in eqs. (17) to (24) are symmetric about $E = 0$. We do not know if there is any deeper reason for it.

As is well known, whenever one obtains QES solutions, the associated orthogonal polynomials satisfy a three-term recursion relation and this is also true in the present case. In this context we recall the detailed work of Finkel et al. [6]. In their language, our periodic problem as given by eq. (12) corresponds to the case (7) in their notation (see their eqs. (8) and (20)) and hence on running through the steps given there it follows that in our case the orthogonal polynomials satisfy a three-term recursion relation.

Before ending this note we show that as a by product, we also obtain the band edge eigenvalues and eigenfunctions of the periodic potential

$$V(x,m) = [\frac{\beta^2}{4} - ma(a+1)]\mathrm{cn}^2(x,m) + \beta(a+\frac{1}{2})\mathrm{sn}(x,m)\mathrm{dn}(x,m)\ , \tag{25}$$

in case a is either an integer or a half-integer. Since in the limit $m \to 1$, this potential goes over to the exactly solvable potential (4), hence we also obtain the bound state eigenvalues and eigenfunctions of the potential (4) in that case. The proof is rather simple. Since under $x \to x + K(m)$

$$\mathrm{sn}\, x \to \frac{\mathrm{cn}\, x}{\mathrm{dn}\, x}\ ,\ \mathrm{cn}\, x \to -\sqrt{1-m}\frac{\mathrm{sn}\, x}{\mathrm{dn}\, x}\ ,\ \mathrm{dn}\, x \to \frac{\sqrt{1-m}}{\mathrm{dn}\, x}\ , \tag{26}$$

hence, under this transformation the potential (3) goes over to the potential (25) with $\beta = -\frac{b}{\sqrt{1-m}}$. Thus the band-edge eigenvalues and eigenfunctions of the potential (25) are immediately obtained from those of the potential (3) by making the substitution

$$b \to -\sqrt{1-m}\,\beta\ ,\ \mathrm{sn}\, x \to -\frac{\mathrm{cn}\, x}{\mathrm{dn}\, x}\ ,\ \mathrm{cn}\, x \to \sqrt{1-m}\frac{\mathrm{sn}\, x}{\mathrm{dn}\, x}\ ,\ \mathrm{dn}\, x \to \frac{\sqrt{1-m}}{\mathrm{dn}\, x}\ . \tag{27}$$

It is amusing to note that while the eigenvalues so obtained are in general β dependent, as $m \to 1$, this β dependence completely disappears from the eigenvalues (note, one is replacing b by $\sqrt{1-m}\,\beta$), as it should since it is well known [7] that the eigenvalues of the potential (4) are β independent. Note, however, that the corresponding eigenfunctions are still β-dependent.

Summarizing, it is nice to know that the QES solutions of the 'elliptic' potential (3) automatically give us the well known QES solutions of the DSG and DSHG potentials (1) and (2) respectively. Further, at the same time, it also gives us the bound state eigenvalues and eigenfunctions of the well known shape invariant potential (4) [7] in case a is positive integer or half-integer. It will be interesting to explore if other QES problems can also be dealt with in this unified manner.

References

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