Off-diagonal long-range order in a one-dimensional many-body problem

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Abstract

We prove the existence of an off-diagonal long-range order in a one-dimensional many-body problem.

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It is well known that in a many body system of bosons or fermions it is possible to have an off-diagonal long-range order (ODLRO) of the reduced density matrices in coordinate space representation [1]. The onset of ODLRO points at quantum phases and quantum phase transitions in a many-body system. For a many-body bosonic system, the well-known example is the Bose-Einstein condensation (BEC). It is reasonable to assume that superfluid He II and superconductors are quantum phases characterized by the existence of such an order. A generalized criterion developed by Penrose and Onsager states that a system exhibits ODLRO if the largest eigenvalue of the one-particle reduced density matrix is extensive [2]. Besides these well-known examples of three-dimensional systems, it was relatively recently shown that there exists a novel type of ODLRO in the fractional quantum Hall effect ground state [3]. Related to this, presence of ODLRO in the ground state of a Calogero-type model in two dimensions has also been demonstrated [4].

Recently, two of us (JK) [5] have presented a many-body system in one dimension with upto next-to-nearest neighbour interaction and obtained its exact ground state. For the version of the model where N particles reside on a circle, they calculated the one and two point correlation functions by using this ground state and showed that there is no long range order. Further, by applying the Penrose-Onsager criterion it was claimed that there is no ODLRO as well. Unfortunately, the Hamiltonian employed there to describe the system is not a symmetrised one and since the Penrose-Onsager criterion does not apply for the case of distinguishable particles, the claim regarding the absence of ODLRO is incorrect. In this Letter, we present the symmetrised version of the many-body system and prove that indeed there exists ODLRO. To the best of our knowledge, this is the first example of a one-dimensional system to possess ODLRO and hence, quantum phases.

Consider N particles on a circle, described by the symmetrised Hamiltonian,

$$\begin{aligned} H &= -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{P\in S_N} \\ &\quad \Theta(x_{P(1)} - x_{P(2)})...\Theta(x_{P(N-1)} - x_{P(N)})W(x_{P(1)},...,x_{P(N)})\,, \end{aligned} \tag{1}$$

where Θ is the step function and $W(x_1, ..., x_N)$ is the N-body potential

$$\begin{aligned} W(x_1,\cdots,x_N) &= g\frac{\pi^2}{L^2}\sum_{i=1}^{N}\frac{1}{\sin^2[\frac{\pi}{L}(x_i - x_{i+1})]} \\ &\quad - G\frac{\pi^2}{L^2}\sum_{i=1}^{N}\cot\left[\frac{\pi}{L}(x_{i-1} - x_i)\right]\cot\left[\frac{\pi}{L}(x_i - x_{i+1})\right] \end{aligned} \tag{2}$$

In (1), the sum is over all permutations on N symbols in the symmetric group, S_N. The exact ground state of the un-symmetrised version of (1) was obtained in [5]. Relying on the ground state found there, we introduce the symmetrized (un-normalized) wave function :

$$\psi_N(x_1, ..., x_N) = \phi_N(x_{P(1)}, ..., x_{P(N)}) \,, \tag{3}$$

where P is the permutation in S_N such that $1 > x_{P(1)} > x_{P(2)} > ... > x_{P(N)} > 0$, and ϕ_N is the (un-symmetrized) exact ground state wave function of the un-normalized Hamiltonian which is given by

$$\phi_N(x_1, ..., x_N) = \prod_{n=1}^{N} \mid \sin \pi(x_n - x_{n+1}) \mid^{\beta} ; \quad (x_{N+1} = x_1) \,, \tag{4}$$

provided $g = \beta(\beta - 1)$, $G = \beta^2$ (we have set the scale factor L equal to 1). Primitively, the function (3) is defined on the hypercube $[0, 1]^N$.

Now for $\beta \geq 2$ it is easily verified that ψ_N can be continued to a multi-periodic function in the whole space $\mathcal{R}^N$ (or equivalently on the torus T^N):

$$\psi_N(x_1, ..., x_i + 1, ..., x_N) = \psi_N(x_1, ..., x_i, ..., x_N) \, ; \quad (i = 1, ..., N) \,, \tag{5}$$

which belongs to C^2 (i.e. is twice continuously differentiable). Owing to this property and the results of [5], ψ_N then obeys the Schrödinger equation (with Hamiltonian (1) and with the same ground state energy $E_0 = N(\beta\pi)^2$) not only in the sector $x_1 > x_2 > ... > x_N$ but everywhere. Thus, ψ_N describes the ground state wave function of the N-boson system. Moreover, it is translation-invariant (on $\mathcal{R}^N$):

$$\psi_N(x_1 + a, x_2 + a, ..., x_N + a) = \psi_N(x_1, x_2, ..., x_N) \, ; \quad \forall \; a \; \in \; \mathcal{R} \, . \tag{6}$$

We are interested in the one-particle reduced density matrix, given by

$$\begin{aligned} \rho_N(x - x') \quad &= \quad \frac{N}{C_N} \int_0^1 dx_1 ... \int_0^1 dx_{N-1} \\ &\quad\quad \psi_N(x_1, ..., x_{N-1}, x) \psi_N(x_1, ..., x_{N-1}, x') \,, \end{aligned} \tag{7}$$

where C_N stands for the squared norm of the wave function:

$$C_N = \int_0^1 dx_1 ... \int_0^1 dx_N \mid \psi_N(x_1, ..., x_N) \mid^2 \; . \tag{8}$$

That the right hand side (RHS) of Eq. (7) defines a (periodic) function of $(x - x')$ is an easy consequence of Eqs. (5) and (6). The normalization of ρ_N is such that $\rho_N(0) = N$, the particle density. Further, the function $\rho_N(\xi)$ is manifestly of positive type on the $U(1)$ group, which implies that its Fourier coefficients,

$$\rho_N^{(n)} = \int_0^1 d\xi e^{-2i\pi n\xi} \rho_N(\xi) \, ; \quad (n = 0, \pm 1, \pm 2, ...) \,, \tag{9}$$

are non-negative (Bochner's theorem). In fact, this directly appears if one writes their explicit expression in the form (obtained by using the periodicity property):

$$\rho_N^{(n)} = \frac{N}{C_N}\int_0^1 dx_1 ... \int_0^1 dx_{N-1} \left| \int_0^1 dx e^{2i\pi n x} \psi_N(x_1, ..., x_{N-1}, x) \right|^2 . \quad (10)$$

Since the function ρ_N is not only of positive type but also **positive** (like ψ_N), Eq. (9) shows us that

$$\rho_N^{(0)} \geq \rho_N^{(n)} \; ; \quad (n = \pm 1, \pm 2, ...) \, . \qquad (11)$$

Notice that the coefficients $\rho_N^{(n)}$, which physically represent the expectation values of the number of particles having momentum $k_n = 2\pi n$ in the ground state, are nothing but the eigenvalues of the one-particle reduced density matrix (diagonal in the k_n representation). According to the Penrose-Onsager criterion [2], no condensation can occur in the system (at least for Bose particles) if the largest of these eigenvalues is not an extensive quantity in the thermodynamic limit, that is, if

$$\lim_{N\to\infty} \frac{\rho_N^{(0)}}{N} = 0 \, . \qquad (12)$$

Accordingly, we have to evaluate

$$\frac{\rho_N^{(0)}}{N} = \frac{A_N}{C_N} \, , \qquad (13)$$

where C_N is given by Eq. (8) and

$$\begin{aligned} A_N \;\; = & \int_0^1 dx_1 ... \int_0^1 dx_{N-1} \psi_N(x_1, ..., x_{N-1}, 0) \\ & \int_0^1 dx \psi_N(x_1, ..., x_{N-1}, x). \end{aligned} \qquad (14)$$

Because of the special form, (3)-(4), of the wave function, the computation of the squared norm C_N is more difficult than in the case of N free, impenetrable particles. As a consequence, the (mainly algebraic) method introduced long ago by Lenard [26] to deal with the latter case does not apply here, and we have to resort to another device. For conciseness, we introduce the notation:

$$S(x_n - x_{n+1}) = \mid \sin \pi (x_n - x_{n+1}) \mid^{\beta} , \qquad (15)$$

and define:

$$S_2(\triangle) = \int_0^{\triangle} dx S(x) S(\triangle - x) \; ; \quad (0 \leq \triangle \leq 1) \, . \qquad (16)$$

Our starting point will be the following representations of C_N and A_N:

$$C_N = (N-1)!\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ix}\tilde{F}(x)^N \,, \tag{17}$$

$$A_N = (N-1)!\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ix}\tilde{F}(x)^{N-3}\Big[\tilde{F}(x)\tilde{G}(x)+(N-2)\tilde{H}(x)^2\Big]\,, \tag{18}$$

where

$$\begin{aligned}
\tilde{F}(x) &= \int_0^1 d\triangle e^{i\triangle x} S(\triangle)^2\,, \\
\tilde{G}(x) &= \int_0^1 d\triangle e^{i\triangle x} S_2(\triangle)^2\,, \\
\tilde{H}(x) &= \int_0^1 d\triangle e^{i\triangle x} S(\triangle)S_2(\triangle)\,.
\end{aligned} \tag{19}$$

The representations (17)-(19) follow from the convolution structure of the expressions (8) and (14) of C_N and A_N, when written in terms of appropriate variables. Their proof will be given elsewhere [6]. Our aim is to extract from them the large N behaviour of C_N and A_N. Their form is especially suited for this purpose, because the integrands in Eqs. (17) and (18) are entire functions, as polynomial combinations of Fourier transforms of functions with compact support (Eq. (19)). Indeed, we are then allowed to, first, shift the integrand and then apply the residue theorem to meromorphic pieces of the integrands. However, it turns out that the calculations needed for arbitrary (integer) values of β are quite cumbersome. So, in order to keep the argument clear enough, we shall content ourselves to present below these calculations in the simplest case, namely $\beta = 1$ (recall that, strictly speaking, this value is not allowed), being understood that similar results are obtained for all integers $\beta \geq 2$. To be sure, after the illustration of the calculation for the case of $\beta = 1$, we give the final results for arbitrary integer value of β.

For $\beta = 1$, $S(\triangle) = \sin\pi\triangle$, and Eq. (19) gives, after reductions:

$$\begin{aligned}
\tilde{F}(x) &= \frac{2\pi^2}{i}\frac{1-e^{ix}}{x(x^2-4\pi^2)}\,, \\
\tilde{G}(x) &= \frac{4\pi^4}{i}\frac{5x^2-4\pi^2}{x^3(x^2-4\pi^2)^3} + e^{ix}R^{(-1)}(x)\,, \\
\tilde{H}(x) &= -\frac{4\pi^3}{i}\frac{1}{x(x^2-4\pi^2)^2} + e^{ix}R^{(-2)}(x)\,,
\end{aligned} \tag{20}$$

where $R^{(n)}(x)$ is a generic notation for rational functions behaving like x^n when $x \to \infty$, and the precise form of which will be eventually of no importance. This produces, for the functions to be integrated in Eqs. (17) and

(18):

$$\tilde{F}(x)^N = \left(\frac{2\pi^2}{i}\right)^N \left[\frac{1}{[x(x^2-4\pi^2)]^N} + \sum_{n=1}^{N} e^{inx} R_n^{(-3N)}(x)\right], \qquad (21)$$

$$\tilde{F}(x)^{N-3}[\tilde{F}(x)\tilde{G}(x) + (N-2)\tilde{H}(x)^2]$$
$$= i\left(\frac{2\pi^2}{i}\right)^N \left\{\frac{(2N+1)x^2-4\pi^2}{[x(x^2-4\pi^2)]^{N+1}} + \sum_{n=1}^{N+1} e^{inx} R_n^{(-3N-1)}(x)\right\}. \qquad (22)$$

Let us stress again that these functions, when analytically continued, are holomorphic in the whole complex plane (the poles appearing in the first term are exactly cancelled by the remaining ones).

We consider first C_N, now given by

$$C_N = (N-1)!\left(\frac{2\pi^2}{i}\right)^N$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{-ix}\left\{\frac{1}{[x(x^2-4\pi^2)]^N} + \sum_{n=1}^{N} e^{inx} R_n^{(-3N)}(x)\right\}. \qquad (23)$$

Since the function within the curly bracket is an entire one, we can shift the integration path to $I \equiv \{z = x + ia \quad | \; x \; \varepsilon \; \mathcal{R}\}$. Let us choose $a > 0$. Then, by Cauchy theorem

$$\int_I dz e^{-iz} \sum_{n=1}^{N} e^{inz} R_n^{(-3N)}(z) = 0\,. \qquad (24)$$

Indeed, the integrand is holomorphic above I and is bounded there by const. $| \; z \; |^{-3N}$, which allows us to close the integration path at infinity in the **upper** complex plane. We end up with

$$C_N = (N-1)!\left(\frac{2\pi^2}{i}\right)^N \frac{1}{2\pi}\int_I dz \frac{e^{-iz}}{z^N(z^2-4\pi^2)^N}\,. \qquad (25)$$

Similarly, we are allowed to close the integration path at infinity in Eq. (25), but this time in the **lower** complex plane. For obtaining the large-N asymptotics, we write

$$\int_I dz \frac{e^{-iz}}{z^N(z^2-4\pi^2)^N} = \frac{1}{(N-1)!}\frac{d^{N-1}}{d\alpha^{N-1}} \,|_{\alpha=4\pi^2} \int_I dz \frac{e^{-iz}}{z^N(z^2-\alpha)}$$
$$= \frac{-2i\pi}{(N-1)!}\frac{d^{N-1}}{d\alpha^{N-1}} \,|_{\alpha=4\pi^2} \left[R_+(\alpha) + R_-(\alpha) + R_0(\alpha)\right], \qquad (26)$$

where $R_\pm(\alpha)$ and $R_0(\alpha)$ are the residues of the last integrand at $z = \pm\sqrt{\alpha}$ and $z = 0$ respectively. They are readily computed and sum up to

$$R_+(\alpha) + R_-(\alpha) + R_0(\alpha) = (-1)^{M+1}\sum_{s=0}^{\infty}\frac{(-1)^s}{(2M+2s+2)!}\alpha^s \qquad (27)$$

for $N = 2M + 1$.

Using Eqs. (25), (26) and (27) we then obtain

$$
\begin{aligned}
C_N &= \left(\frac{2\pi^2}{i}\right)^N (-1)^{M+1}(-i)\frac{d^{N-1}}{d\alpha^{N-1}}\,|_{\alpha=4\pi^2} \sum_{s=0}^{\infty} \frac{(-1)^s}{(2M+2s+2)!}\alpha^s \\
&= (2\pi^2)^N \sum_{n=0}^{\infty} \frac{(N+n-1)!}{n!(3N+2n-1)!}(-4\pi^2)^n \,. \qquad (28)
\end{aligned}
$$

The result is exactly the same for even N. It suffices now to observe that the last series alternates in sign and is decreasing to deduce

$$
C_N = (2\pi^2)^N \frac{(N-1)!}{(3N-1)!}\left[1 + O(\frac{1}{N})\right] \,. \qquad (29)
$$

Our procedure for evaluating A_N is quite similar, and we give below the final result, viz.,

$$
A_N = (2N+1)(2\pi^2)^N \frac{(N-1)!}{(3N)!}\left[1 + O(\frac{1}{N}\right] \,. \qquad (30)
$$

Finally, using Eqs. (13), (29) and (30), we obtain

$$
\frac{\rho_N^{(0)}}{N} = \frac{2}{3}\left[1 + O(\frac{1}{N})\right] \,. \qquad (31)
$$

The same procedure applies for all integer values of β, although the algebra becomes quite involved. The general result (for any integer β) is:

$$
\lim_{N\to\infty} \frac{\rho_N^{(0)}}{N} = \frac{(\beta!)^4[(3\beta+1)!]^2}{[(2\beta)!]^2[(2\beta+1)!]^3} \,. \qquad (32)
$$

Our method does not generalize in a straightforward manner to the case of non-integer values of β, but there is clearly no reason to expect a different outcome for such intermediate values.

To summarize, the Penrose-Onsager criterion (12) is **not** met for bosons in this model as the limit of the ratio in (32) is non-zero. The onset of such an order leads to a new thermodynamic phase of the system. Thus, we reach the remarkable conclusion that Bose-Einstein condensation is indeed **possible** in the bosonic version of the N-body model discussed in [5]. We recall once again that, to the best of our knowledge, this is the only example in one-dimensional statistical mechanics where there exists ODLRO and hence quantum phases.

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