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A Class of Exact Solutions for N-anyons in a N-body potential

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Abstract

A class of exact solutions are obtained for the problem of N-anyons interacting via the N-body potential
$V(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N) = -\frac{e^2}{\sqrt{\frac{1}{N}\sum_{i<j}(\vec{x}_i - \vec{x}_j)^2}}$

Unlike the oscillator case the resulting spectrum is not linear in the anyon parameter $\alpha (0 \leq \alpha \leq 1)$. However, a la oscillator case, cross-over between the ground states is shown to occur for N-anyons $(N \geq 3)$ experiencing the above potential.

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By now it is well established [1,2] that in two spatial dimensions one can have anyonic statistics which interpolates between the Bose-Einstein and the Fermi-Dirac statistics. Such objects also arise in 2+1 dimensional field theory as classical solutions of the abelian Higgs model with a Chern-Simons term [3]. It has been suggested that anyons may provide mechanism for the fractional quantum Hall effect [4].

In the anyonic quantum mechanical systems, only the problem of two anyons in various potentials has been solved exactly and as a result only the second virial coefficient of an anyon gas has been computed exactly [2]. The exact solution of the N-body problem ($N \geq 3$) seems to be out of reach. This is rather unfortunate as the nontrivial braiding effect of anyons is expected to show up only for $N \geq 3$, since only then the 3-body anyonic interaction manifests itself. As far as I am aware off, to date, only a class of exact solutions have been obtained in case N-anyons ($N \geq 3$) experience harmonic oscillator potential [5] or are in a uniform magnetic field [6] (which actually is equivalent to the oscillator problem except for a piece coming from the angular momentum eigenvalue). In both of these problems, all the known exact solutions are such that the energy eigenvalue spectrum is linear in the anyon parameter α ($0 \leq \alpha \leq 1$ and throughout this note $\alpha = 0(1)$ will correspond to boson (fermion)).

It is clearly of interest to enquire if one can also obtain exact solutions in case N-anyons are experiencing some other potential and if in these cases also the energy varies linearly with α or not. The purpose of this letter is to present one such example. In particular, I obtain a class of exact solutions in case N-anyons are interacting via the N-body potential

$$V(\vec{x}_1, \vec{x}_2, ..., \vec{x}_N) = -\frac{e^2}{\sqrt{\frac{1}{N}\sum_{i<j}(\vec{x}_i - \vec{x}_j)^2}} \tag{1}$$

The interesting point is that unlike the oscillator case, the energy spectrum here is not linear in α. However, a la oscillator case, these exact solutions include the ground state of N-bosons but not the ground state of N-fermions ($N \geq 3$). We therefore perturbatively calculate the ground state energy of three anyons near the fermionic statistics and show that for this potential also there is a cross-over between the ground states. I show that a similar cross-over must also occur in the case of N-anyons ($N \geq 4$).

For simplicity let us first discuss the case of 3-anyons experiencing the above 3-body potential. After the separation of the center of mass (which is independent of anyons), the relative problem is best discussed in terms of the hyper-spherical coordinates ρ, θ, ϕ, ψ first proposed by Kilpatrick and Larsen [7]. In particular, one can show that the relative Hamiltonian can be written as [8]

$$H = H_0^{rad} + \frac{1}{2\mu\rho^2}(-\Lambda^2 + \alpha H_1 + \alpha^2 H_2) \tag{2}$$

where Λ, H_1 and H_2 only depend on the angular coordinates θ, ϕ, ψ while H_0^{rad} only depends on the radial variable ρ. In particular, $-\Lambda^2$ is the Laplacian on the three dimensional sphere while the anyonic pieces H_1 and H_2 are as given in [9]. It is worth emphasizing that such a separation is always possible so long as the anyons experience a potential which is a function of ρ alone. Further, such a decomposition also exists for an arbitrary number of anyons [10]. In particular, for N anyons, the relative problem is best discussed in terms of ρ and $2N - 3$ angles.

In the case of 3-anyons experiencing the 3-body potential (1), the relative radial Hamiltonian H_0^{rad} becomes

$$H_0^{rad} = \frac{-1}{2\mu}(\frac{\partial^2}{\partial\rho^2} + \frac{3}{\rho}\frac{\partial}{\partial\rho}) - \frac{e^2}{\rho} \tag{3}$$

Let us first obtain the exact eigenvalues and eigenfunctions of three bosons and fermions in the potential (1). They are obtained by noticing that in that case α can be taken to be zero and further, the eigenvalue of $-\Lambda^2$ is k(k+2) with k = 0,1,2,.... The resulting radial equation as obtained from eqs. (2) and (3) is nothing but the Schrödinger equation for the Coulomb potential in 4-dimensions. In this way we find that the energy eigenvalues of 3-bosons or 3-fermions in the 3-body potential (1) are

$$E_{n',k} = -\frac{\mu e^4}{2[n' + k + \frac{3}{2}]^2} \tag{4}$$

while the corresponding fermionic (bosonic) eigenfunctions, $\psi^{(\mp)}_{n',k,\nu,\lambda}$ are given by

$$\psi^{(\mp)}_{n',k,\nu,\lambda} = F^k_{n'}(\rho) Y^{(\mp)}_{k,\nu,\lambda}(\theta, \phi, \psi) \tag{5}$$

Here the normalized angular eigenfunctions $Y^{(\mp)}_{k,\nu,\lambda}(\theta, \phi, \psi)$ are identical to those in the harmonic case and have been explicitly written down in [9] while the normalized (with measure $\rho^3 d\rho$) radial eigenfunctions $F^k_{n'}(\rho)$ are given by

$$F^k_{n'}(\rho) = N_{n',2k+2} exp(-y/2) y^k L^{2k+2}_{n'}(y) \tag{6}$$

where L^{α}_N is a Laguerre polynomial, $y = 2\sqrt{2\mu \mid E \mid}\rho$ and $N_{n',2k+2}$ is the normalization constant. It is worth pointing out that for the 3-boson ground state, $n' = k = 0$ and hence $\in^B_g \equiv 2E^B_g/\mu e^4 = -4/9$ while for 3-fermion ground state $n' = 0, k = 2$ and hence $\in^F_g = -4/49$. It is also worth pointing out that the wave function (5) is also an eigenfunction of the angular momentum operator with eigenvalue λ.

Proceeding in the same way, the eigenvalues of N bosons or N fermions in the N-body potential (1) can be immediately written down. This is because in that case $-\Lambda^2$ is the Laplacian on the (2N-3)-dimensional sphere whose eigenvalues are k(k+2N-4) with k =0, 1,2,..., and whose eigenfunctions are generalized spherical harmonics. The resulting radial Schrödinger equation then takes the form

$$\Big[\frac{\partial^2}{\partial\rho^2} + \frac{2N-3}{\rho}\frac{\partial}{\partial\rho} + \frac{2\mu e^2}{\rho} - \frac{k(k+2N-4)}{\rho^2}\Big]F(\rho) = -2\mu E F(\rho) \tag{7}$$

This equation is easily solved and the resulting energy eigenvalues for N bosons or N fermions experiencing the N-body potential (1) are given by $(k, n' = 0, 1, 2, ...)$

$$\in_{n',k} = -\frac{1}{[n' + k + N - \frac{3}{2}]^2} \tag{8}$$

while the corresponding unnormalized radial eigenfunctions $F(\rho)$ are given by

$$F^k_{n'}(\rho) = exp(-y/2) y^k L^{2N+2k-4}_{n'}(y) \tag{9}$$

where as before $y = 2\sqrt{2\mu \mid E \mid}\rho$.

Let us now turn to the exact solutions of the N anyon problem in the presence of the N-body potential (1). On using the fact that (i) only the angular part of the Hamiltonian is affected due to the anyons (ii) the angular part is independent of the radial potential $V(\rho)$ between the anyons (iii) the radial equation for N bosons, N fermions and N anyons is same but for the coefficient of the $\frac{1}{\rho^2}$ term, one can immediately write down a class of exact solutions for N anyons experiencing the N-body potential (1). In particular, on using the exact solutions for N anyons in the oscillator potential [5] we find that the exact energy eigenvalues in our case are

$$\in_{n',\lambda}(\alpha) = -\frac{1}{[n' + \mid \lambda - \frac{N(N-1)}{2}\alpha \mid + N - \frac{3}{2}]^2} \tag{10}$$

where λ is the eigenvalue of the angular momentum operator. The corresponding eigenfunctions are

$$\psi_{n',\lambda} = exp[i\lambda \sum_{i<j} \theta_{ij}] e^{-y/2} \Pi_{i<j} \mid \vec{x}_{ij} \mid^{|\lambda-\alpha|} L^{a}_{n'}(y) \tag{11}$$

where $y = 2\sqrt{2\mu \mid E \mid}\rho$, $a = 2N - 4 + 2 \mid \lambda - \frac{N(N-1)}{2}\alpha \mid$, $\vec{x}_{ij} = \vec{x}_i - \vec{x}_j$ and $tan\theta_{ij} = \frac{(y_i - y_j)}{(x_i - x_j)}$.

It is worth pointing out that for N = 2, the expression as given by eq.(10) gives the complete spectrum [11]. For $N \geq 3$ however, it does not give the complete spectrum. For example, for N = 3, the three fermion ground state is missing from these exact solutions (the three fermion ground state energy $\in^F_g = -\frac{4}{49}$ which is not included in the expression (10)). The three boson ground state which corresponds to $n' = \lambda = \alpha = 0$ (and N = 3) has energy $\in^B_g = -4/9$ and it interpolates to the fermionic state with $\in^F = -4/81$ which is an excited state. Thus, as in the oscillator case [9], a level-crossing has to occur for the true ground state of the 3-anyon system. Infact, such a crossing must also occur for any N (≥ 3). This is because the exact N boson ground state interpolates to the fermionic state with an eigenstate of angular momentum L with eigenvalue $-N(N-1)/2$ (see eq.(10)). On the other hand, the fermionic ground state is obtained by filling the one particle levels from bottom to top. One can show that the fermionic ground state always has a total angular momentum $\mid L \mid$ less than $\frac{N(N-1)}{2}$ (for $N > 2$) [12]. We thus conclude that a la oscillator case, even in the case of the N-body potential (1), there must be a ground state cross-over at some value of α.

What is the nature of the missing states in the N-anyon spectra ? We now show that in our case, whereas for the exact solutions $(-\in)^{-1/2}$ is linear in α, for all the missing solutions $(-\in)^{-1/2}$ will have nonlinear dependence on α. Let us first recall that in the case of the oscillator potential, all those states for which energy varies linearly with α are known analytically [5]. Further, it is also known that there are several missing states whose energy varies nonlinearly with α. For N =3,4 the energy of the low lying "nonlinear states" has been estimated by using numerical and perturbative techniques [13,14]. We now show that we can borrow these oscillator results and obtain the energies of the missing states in the case of the potential (1). The point is that as argued above, only the angular part of the Hamiltonian is affected due to anyons [10] and this part is identical for both the oscillator and our potential (1) . Secondly, the only effect of the angular part is to affect the coefficient of the $1/\rho^2$ term in the radial Schrödinger equation. For example, in eq. (7), instead of $k(k+2N-4)/\rho^2$ we would have $\beta(\beta+2N-4)/\rho^2$ where β need not necessarily be an integer and would in general be a complicated function of α. As a result, the energy eigenvalues of the missing states would again be given by eq. (8) but with k replaced by β. The corresponding oscillator radial Schrödinger equation is also given by eq. (7) but with k replaced by β and $2\mu e^2/\rho$ replaced by $-\mu^2\omega^2\rho^2$. As a result, the corresponding oscillator energy eigenvalues are given by

$$\in_{n'}^{osc}(\alpha) \equiv E_{n'}^{osc}(\alpha)/\omega = (2n' + \beta + N - 1/2) \tag{12}$$

One can therefore immediately eliminate β from the two eqs. (12) and (8) (with k replaced by β) and obtain a general relation between the eigenvalues of our potential (1) and the oscillator potential given by

$$\in_{n'}(\alpha) = -\frac{1}{[\in_{n'}^{osc}(\alpha) - n' - 1/2]^2} \tag{13}$$

We can therefore immediately borrow all the known results about the missing nonlinear states in the oscillator case and obtain corresponding conclusions in our case. For example, whereas for all the analytically known states $(-\in)^{-1/2}$ changes by $\pm N(N-1)/2$ for the missing states the energy will change by $\frac{N(N-1)}{2}-2, \frac{N(N-1)}{2}-4, ... -[\frac{N(N-1)}{2}-2]$ as one will go from bosons to fermions. For example, in the 3-anyon case, $(-\in)^{-1/2}$ changes by ± 3 in the case of the exactly known solutions while it will change by ± 1 in the case of the missing states as one will go from bosons to fermions. In particular, the 3-fermion ground state at $\in = -4/49$ will interpolate to the bosonic state at $\in = -4/81$ and near the fermionic end, the energy of the corresponding anyonic state is given by [10]

$$\in = -\frac{1}{[3.5 + 1.29(1-\alpha)^2]^2} \tag{14}$$

On the other hand, the anyonic state starting from the bosonic ground state at $\in = -4/9$ is given by (see eq.(10))

$$\in = -\frac{4/9}{(1+2\alpha)^2} \tag{15}$$

The two curves cross at $\alpha = 0.71$. It is a curious numerical fact that for both the oscillator [9,13] and our N-body case, the cross-over occurs at almost the same point.

Finally it is worth pointing out that the degeneracy of the exact energy levels coming from the angular part is same for both the oscillator and our N-body potential (note the same factor $\mid \lambda - \frac{N(N-1)}{2}\alpha \mid$ occurs in both the cases). This will infact be true for any anyon potential which only depends on ρ. However, the degeneracy coming from the radial part is different in the two cases since whereas in the oscillator case one has the factor $2n' + \mid \lambda - \frac{N(N-1)}{2}\alpha \mid$, in our case the corresponding factor is $n' + \mid \lambda - \frac{N(N-1)}{2}\alpha \mid$. As a result, compared to the oscillator case, here the degeneracy is much more. In particular, for a given energy, both even and odd angular momentum states are present in general in the spectra. As a result, if one plots $(-\in)^{-1/2}$ as a function of α, then one will find that one will not only have those levels which are present in the oscillator spectrum but there will be few extra states in our case which are not there in the oscillator case. For example, in the 3-anyon case we have an extra state for which $(-\in)^{-1/2}$ changes linearly from 1.5 to 4.5 as one goes from the bosonic to the fermionic end (see for example Fig. 1 of [10] for the low lying 3-anyon spectrum in the oscillator potential).

Are there other potentials for which a class of exact N-anyon eigen states can be found? We believe that the answer is no since only the Coulomb and the oscillator problems are analytically solvable in N dimensions ($N \geq 2$). All other potentials are atbest quasi-exactly solvable and hence for a given potential, eigenstates could be analytically obtained for atbest some specific values of the angular momentum λ.

This work raises several issues like number of crossings in the ground state for N anyons [15], possible supersymmery for N anyons [16], pseudo-integrability of the N-anyon problem [17], solutions in the presence of the uniform magnetic field [18], scattering solutions etc. which need to be discussed carefully in the context of the potential (1). Some of these issues as well as details of this work will be published elsewhere [19].

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