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# Is the Lowest Order Supersymmetric WKB Approximation Exact for All Shape Invariant Potentials ?

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**Abstract**

It has previously been proved that the lowest order supersymmetric WKB approximation reproduces the exact bound state spectrum of shape invariant potentials. We show that this is not true for a new, recently discovered class of shape invariant potentials and analyse the reasons underlying this breakdown of the usual proof.

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In the past few years, the supersymmetry inspired WKB approximation (SWKB) [1] has received considerable attention [2]. One reason for this was the proof [3] that the leading order SWKB quantization condition reproduces the exact bound state spectra for any shape invariant potential (SIP) [4]. Subsequently, Adhikari et al.[5] showed by explicit calculation that the higher order corrections to the energy eigenvalue spectrum vanish to $O(\hbar^6)$ for all then-known shape invariant potentials [6]. For these cases, all the higher-order corrections have since been shown to vanish [7,8]. The SWKB quantization condition has also been applied to many non-shape invariant potentials and it turns out that even though the SWKB formula does better than the usual WKB approach in most cases [5,9,10], it has never been found to be exact for these, thus suggesting that perhaps shape invariance is not only sufficient but even necessary for the lowest order SWKB formula to yield exact energy eigenvalues [10]. Barclay and Maxwell [8] proposed a very simple condition on the superpotential which ensures that the lowest order SWKB formula gives exact bound state spectra with all higher order corrections zero. By analysing this simple condition, they have suggested that there are no SIPs other than those tabulated by Dutt et al.[6] and the one example found by Levai [11]. It should be noted here that most of these potentials are also contained in the list of Infeld and Hull [12].

Recently, a large class of new SIPs has been discovered by us [13,14] disproving the conjecture that no more examples exist. These new SIPs are reflectionless and possess an infinite number of bound states. They can be looked upon as q-deformations of the symmetric Rosen-Morse potentials corresponding to one or multi- soliton solutions. Although these new potentials cannot be written in a closed form using elementary functions, the energy eigenvalue spectrum is known analytically. It is then reasonable to enquire if the lowest order SWKB formula yields the exact bound state spectra for these new SIPs or not. It is worth emphasizing that the answer to this question is not obvious. On the one hand, since the lowest order SWKB formula has been shown to be exact for all SIPs [3], it is natural to expect that even for the new SIPs, the lowest order SWKB formula must be exact. On the other hand, Barclay and Maxwell [8] have claimed that there are no other potentials for which the lowest order SWKB formula is exact and the higher order corrections are zero. Thus, it is clearly of interest to know if the lowest order SWKB formula is exact for the new SIPs or not. This is the question that we address in this letter.

In SUSY quantum mechanics, the partner potentials $V_{\pm}(x, a_0) = W^2 \pm W'(x, a_0)$ are said to be shape invariant if $(2m = 1)$

$$W^2(x, a_0) + \hbar W'(x, a_0) = W^2(x, a_1) - \hbar W'(x, a_1) + R(a_0), \tag{1}$$

where $W$ is the superpotential, $a_0$ denotes a set of parameters, $a_1 = f(a_0)$ is an arbitrary function of $a_0$ and $R(a_0)$ is independent of $x$. For all the standard SIPs [6,11] – for which the lowest order SWKB formula is exact – it turns out that the parameters $a_1$ and $a_0$ are related by translation ($a_1 = a_0 + \alpha\hbar$; $\alpha$ being a constant). On the other hand, for the new SIPs, $a_1$ and $a_0$ are related by scaling ($a_1 = qa_0$; $0 < q < 1$). Expansion of the superpotential and $R(a_0)$ in powers of $a_0$ gives

$$W(x, a_0) = \sum_{j=0}^{\infty} g_j(x) a_0^j, \quad R(a_0) = \sum_{j=0}^{\infty} R_j a_0^j, \quad a_1 = qa_0. \tag{2}$$

Substitution into the shape invariance condition, eq. (1), and equating powers of $a_0$ permits a full determination of the quantities $g_j(x)$ for any choice of the constants $R_j$. A particularly simple choice is to take only one non-zero $R_j$. It has been shown [13,14] that this special choice gives the self-similar reflectionless potentials studied by Shabat [15] and Spiridonov [16]. In that case [13,14]

$$R_n = R_1 \delta_{n1}, \quad \beta_1 = \frac{R_1}{(1+q)}, \tag{3}$$

$$g_n(x) = \beta_n x^{2n-1} = \Big( - \frac{(1-q^n)}{(2n-1)(1+q^n)} \sum_{j=1}^{n-1} \beta_j \beta_{n-j} \Big) x^{2n-1}, \tag{4}$$

so that $W(x)$ is an odd function of $x$ and hence SUSY is unbroken. The exact bound state spectrum for $V_-(x, a_0)$ is now easily calculated and is given by [13,14]

$$E_n = R_1 a_0 \frac{(1-q^n)}{(1-q)}, \quad n = 0, 1, \ldots \tag{5}$$

For the case of unbroken SUSY, the lowest order quantization condition is

$$\int \sqrt{E - W^2}\, dx = n\pi\hbar, \quad n = 0, 1, \ldots \tag{6}$$

By construction this will always give the exact ground state energy ($E_0 = 0$), but what about the excited states ? Since $W(x)$ is not available in closed form in terms of elementary functions, the SWKB integral has to be done numerically. A method for calculating $W(x)$ has been described before [14] and careful numerical integration shows that for all $q \in (0, 1)$ the lowest order SWKB formula is not exact for the new potentials. The left hand side of equation (6) can be thought of as defining a function $n(E) \equiv (\pi\hbar)^{-1} \int \sqrt{E - W^2} dx$ which will take on the integer value $n$ for $E = E_n$, if this lowest order quantization condition is exact. In Table 1 we display the results of doing the integral using the exact results for $E = E_1$, i.e. calculating $n(E_1)$. For $q$ close to one the discrepancy is small (though still significantly greater than the

numerical uncertainties), but it diverges (roughly as $\ln q$) as $q \to 0$. At exactly $q = 0$, the Rosen-Morse potential, the condition is known to be exact, a discontinuity that is presumably a reflection of the way in which the infinitely many excited states for $q > 0$ collapse to a single one when $q = 0$.

The limit $q \to 1$ is much smoother and corresponds to the harmonic oscillator, for which the SWKB formula is also known to be exact. In fact, it appears that one can perturb about this limit for small values of $\epsilon \equiv 1 - q$. The shape invariance condition (1) can be expanded to find $W$ as a power series in $\epsilon$ and this used to evaluate the integral. One finds that

$$n(E_1) = 1 + \frac{83}{648}\epsilon^2 + O(\epsilon^3), \tag{7}$$

which accords well with the numerical results close to $q = 1$. All the above calculations are done for the simplest case where only $R_1$ is taken non-zero. It is easily checked that the lowest order SWKB formula does not give correct eigenvalues for the more general and complicated cases of new SIPs corresponding to several non-zero $R_j$ either. This is of course intuitively expected.

What is the special feature of these new potentials that interferes with the proof that condition (6) is exact for shape invariant potentials ? To understand this, let us take a fresh look at the derivation of the lowest order SWKB quantization condition, paying particular attention to the role of $\hbar$. The lowest order WKB quantization condition for the potential $V_-(x, a_0)$ is $(2m = 1)$

$$\int \sqrt{E_n - V_-(x, a_0)} \quad dx = (n + 1/2)\pi\hbar. \tag{8}$$

In terms of the superpotential $W$, this reads

$$\int \sqrt{E_n - W^2(x, a_0) + \hbar W'(x, a_0)} \quad dx = (n + 1/2)\pi\hbar. \tag{9}$$

Now comes the crucial step in the derivation. One argues that formally $W^2$ is of $O(\hbar^0)$ while $\hbar W'$ is of $O(\hbar)$ and hence one can expand the integrand on the left hand side in powers of $\hbar$ to get the lowest order SWKB quantization condition (6). This assumption is justified for all the standard SIPs [6,11] since $W^2$ is indeed of $O(\hbar^0)$ while $\hbar W'$ is indeed of $O(\hbar)$. One might object to this procedure since the resulting potential $V_-$ is then $\hbar$-dependent. However, in every one of those cases this $\hbar$-dependence can be absorbed into some dimensionful parameters in the problem. For example, consider

$$W = A \tanh x \tag{10}$$

so that

$$V_-(x) = A^2 - A(A + \hbar)\mathrm{sech}^2\mathrm{x}. \tag{11}$$

Taking $A$ such that $A(A+\hbar)$ is independent of $\hbar$ gives the desired $\hbar$-independent potential (the additive constant is irrelevant and so can contain $\hbar$). Such a move may appear to be of limited value since one cannot apply SWKB directly to a superpotential $W$ which is now $\hbar$-dependent. However, because $A$ is a free parameter, one can continue the SWKB results obtained for $A$ (and hence $W$) independent of $\hbar$ over to this superpotential and so obtain an SWKB approximation for a $\hbar$-independent potential.

What about the new potentials ? In the simplest of these cases, the only free parameter in the problem (apart from $q$) is the combination $R_1 a_0$, on which $W$ depends as $W(x, R_1 a_0) = \sqrt{R_1 a_0} F(\sqrt{R_1 a_0} x/\hbar)$. Incorporating different dependences on $\hbar$ in $R_1 a_0$ will give different ones in $W$, $V_-$ and $E_n$, but $F$ is a sufficiently complicated function that there is no way of eliminating $\hbar$ from $W^2$. This is a direct consequence of the scaling reparameterisation $a_1 = q a_0$ not involving $\hbar$: if $W^2(x, a_0)$ were independent of $\hbar$, so would $W^2(x, a_1)$ be and in taking the lowest order of (1) one would get $W^2(x, a_0) = W^2(x, a_1)$, which corresponds to the harmonic oscillator. A reparameterisation of the form $a_1 = a_0 + \alpha\hbar$ clearly avoids this. Furthermore, with $a_1 = q a_0$, $W^2$ and $\hbar W'$ are now of a similar order in $\hbar$. The basic distinction between them involved in deriving equation (6) is thus no longer valid and we are prevented from deriving the SWKB condition for these new potentials. Similar difficulties will arise for potentials with more $R_j$ non-zero and for the other $\hbar$-independent reparameterisations considered in [14].

It might be thought that an alternative derivation of the SWKB series for these potentials could be constructed by considering $W = \sqrt{R_1 a_0} F(\sqrt{R_1 a_0} x/H)$, where $H$ is simply a free constant unrelated to $\hbar$. The series can be safely derived for all $H$ since $W$ is $\hbar$-independent and the result continued to $H = \hbar$ at the end. Unfortunately, when this is done it becomes clear that the structure of all the terms is such that the entire series is then all of $O(\hbar)$, so that one cannot usefully separate out a lowest order condition corresponding to (6) from a set of higher order corrections. It is then little surprise that an arbitrarily selected piece (6) of the full condition does not give the exact spectrum.

Since it appears to have been overlooked in the extensive literature (see e.g. [17]) on symmetric, reflectionless potentials containing infinitely many bound states, we point out that the inextricable $\hbar$-dependence of these new shape invariant potentials is actually a general consequence of them possessing these properties. Reflectionlessness implies that a potential is finitely deep, and the infinite spectrum means that the top of it is given by the limit of $E_n$ as $n \to \infty$. However, $E_n \to 0$ as $\hbar \to 0$, so the potential must also distort in the semi-classical limit; a little further reflection shows

that it always tends towards a free-particle potential. A similar conclusion can be reached for the $N$-soliton reflectionless potential (which contains $N$ bound states) by explicitly restoring $\hbar \neq 1$ in the inverse scattering formalism for these potentials [17].

We have thus seen that the SWKB quantization condition (6) is not the correct lowest order formula in the case of the new SIPs and hence it is not really surprising that eq. (6) does not give the exact eigenvalues for these potentials. In other words, it remains true that the lowest order SWKB quantization condition is exact for SIPs (if the SUSY is unbroken), but only in those cases for which the formula is applicable in the first place. It is thus still the case that the SIPs given in refs.[6,11] are the only known ones for which the lowest order SWKB formula is exact and the higher order corrections are all zero.

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| $q$ | $n(E_1)$ |
|---|---|
| 0.01 | 1.238 |
| 0.1 | 1.101 |
| 0.2 | 1.063 |
| 0.3 | 1.042 |
| 0.4 | 1.029 |
| 0.5 | 1.019 |
| 0.6 | 1.012 |
| 0.7 | 1.007 |
| 0.8 | 1.003 |
| 0.9 | 1.001 |

**Table 1:** Results of evaluating the lowest order SWKB integral using the exact value $E_1 = R_1 a_0$ for different values of $q$. An exact quantisation condition would give $n = 1$ in every case. Note that the $R_1 a_0$ dependence cancels in the integral, so these results do not depend on this free parameter.