What Is the Difference between a Parabola and a Hyperbola?

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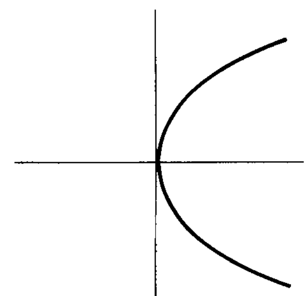
1. Parabola and Hyperbola

The *parabola* is given by the equation

$$Y^2 = X;$$

we can parametrize it by

$$X = t^2 \text{ and } Y = t.$$



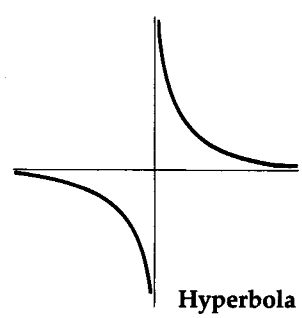
Parabola

The *hyperbola* is given by the equation

$$XY = 1;$$

we can parametrize it by

$$X = t \text{ and } Y = \frac{1}{t}.$$



Hyperbola

Thus the parabola is a *polynomial curve* in the sense that we can parametrize it by polynomial functions of the parameter t. On the other hand, for the hyperbola we need rational functions of t that are not polynomials; it can be shown that no polynomial parametrization is possible. Thus the hyperbola is not a polynomial curve, but it is a *rational curve.*

To find the reason behind this difference, let us note that the highest degree term in the equation of the parabola is Y^2, which has the only factor Y (repeated twice), whereas the highest degree term in the equation of the hyperbola is XY which has the two factors X and Y.

Shreeram Abhyankar

Shreeram Abhyankar received his Ph.D. from Harvard in 1955 under the guidance of Oscar Zariski. In his thesis, Abhyankar solved the problem of resolution of singularities of algebraic surfaces for nonzero characteristic. Ten years later, he solved the same problem for arithmetical surfaces and for three-dimensional varieties for nonzero characteristic. Since 1967, Abhyankar has been Distinguished Marshall Professor of Mathematics at Purdue University. In 1978, Abhyankar was awarded the Chauvenet Prize of the Mathematical Association of America for his article "Historical ramblings in algebraic geometry and related algebra" in the June 1976 issue of the *American Mathematical Monthly*. In that article, Abhyankar said that his father initiated him to mathematics and Sanskrit poetry by teaching him portions of Bhaskaracharya's treatise on algebra called Beejganita, which was written around A.D. 1150. In 1976, Abhyankar was also instrumental in founding the Bhaskaracharya Pratishthana, which is a research institute in Pune, India, named after this great mathematician.

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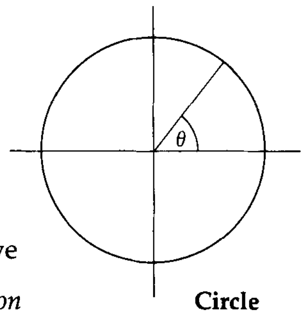
2. Circle and Ellipse

We can also note that the *circle* is given by the equation

$$X^2 + Y^2 = 1;$$

we can parametrize it by

$$X = \cos\theta \text{ and } Y = \sin\theta.$$



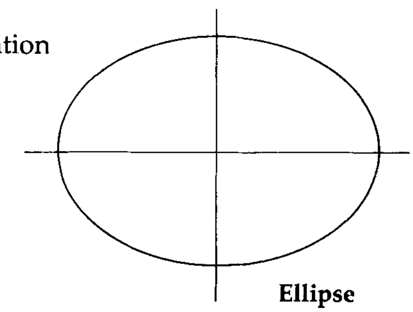
Circle

By substituting $\tan\frac{\theta}{2} = t$ we get the *rational parametrization*

$$X = \frac{1 - t^2}{1 + t^2} \text{ and } Y = \frac{2t}{1 + t^2},$$

which is not a *polynomial parametrization.* Similarly, the *ellipse* is given by the equation

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$


Ellipse

and for it we can also obtain a rational parametrization that is not a polynomial parametrization. I did not start with the circle (or ellipse) because then the highest degree terms $X^2 + Y^2$ (respectively, $(X^2/a^2) + (Y^2/b^2)$) do have two factors, but we need complex numbers to find them.

3. Conics

In the above paragraph we have given the equations of parabola, hyperbola, circle, and ellipse in their *standard form.* Given the general equation of a conic

$$aX^2 + 2hXY + bY^2 + 2fX + 2gY + c = 0,$$

by a linear change of coordinates, we can bring it to one of the above four standard forms, and then we can tell whether the conic is a parabola, hyperbola, ellipse, or circle. Now, the nature of the factors of the highest degree terms remains unchanged when we make such a change of coordinates. Therefore we can tell what kind of a conic we have, simply by factoring the highest degree terms. Namely, if the highest degree terms $aX^2 + 2hXY + bY^2$ have only one real factor, then the conic is a parabola; if they have two real factors, then it is a hyperbola; if they have two complex factors, then it is an ellipse; and, finally, if these two complex factors are the special factors $X \pm iY$, then it is a circle. Here we are assuming that the conic in question does not degenerate into one or two lines.

4. Projective Plane

The geometric significance of the highest degree terms is that they dominate when X and Y are large. In other words, they give the behavior at infinity. To make this more vivid we shall introduce certain fictitious points, which are called the "points at infinity" on the given curve and which correspond to factors of the highest degree terms in the equation of the curve. These fictitious points may be considered as "points" in the "projective plane." The concept of the projective plane may be described in the following two ways.

A point in the *affine* (X,Y)*-plane,* i.e., in the ordinary (X,Y)-plane, is given by a pair (α,β) where α is the X-coordinate and β is the Y-coordinate. The idea of points at infinity can be made clear by introducing *homogeneous coordinates.* In this setup, the old point (α,β) is represented by all triples $(k\alpha,k\beta,k)$ with $k \neq 0$, and we call any such triple $(k\alpha,k\beta,k)$ homogeneous (X,Y,Z)-coordinates of the point (α,β). This creates room for "points" whose homogeneous Z-coordinate is zero; we call these the *points at infinity,* and we call their totality the *line at infinity.* This amounts to enlarging the affine (X,Y)-plane to the *projective* (X,Y,Z)*-plane* by adjoining the line at infinity.

More directly, the projective (X,Y,Z)-plane is obtained by considering all triples (α,β,γ), and identifying proportional triples; in other words, (α,β,γ) and (α',β',γ') represent the same point if and only if $(\alpha',\beta',\gamma') = (k\alpha,k\beta,k\gamma)$ for some $k \neq 0$; here we exclude the zero triple $(0,0,0)$ from consideration. The line at infinity is now given by $Z = 0$. To a point (α,β,γ) with $\gamma \neq 0$, i.e., to a point not on the line at infinity, there corresponds the point $(\alpha/\gamma,\beta/\gamma)$ in the affine plane. In this correspondence, as γ tends to zero, α/γ or β/γ tends to infinity; this explains why points whose homogeneous Z-coordinate is zero are called points at infinity.

To find the points at infinity on the given conic, we replace (X,Y) by $(X/Z, Y/Z)$ and multiply throughout by Z^2 to get the homogeneous equation

$$aX^2 + 2hXY + bY^2 + 2fXZ + 2gYZ + cZ^2 = 0$$

of the projective conic. On the one hand, the points of the original affine conic correspond to those points of the projective conic for which $Z \neq 0$. On the other hand, we put $Z = 0$ in the homogeneous equation and for the remaining expression we write

$$aX^2 + 2hXY + bY^2 = (pX - qY)(p^*X - q^*Y)$$

to get $(q,p,0)$ and $(q^*,p^*,0)$ as the points at infinity of the conic that correspond to the factors $(pX - qY)$ and $(p^*X - q^*Y)$ of the highest degree terms $aX^2 + 2hXY + bY^2$.

In the language of points at infinity, we may rephrase the above observation by saying that if the given conic has only one real point at infinity, then it is a parabola; if it has two real points at infinity, then it is a hyperbola; if it has two complex points at infinity, then it is an ellipse; and, finally, if these two complex points are the special points $(1,i,0)$ and $(1,-i,0)$, then it is a circle. At any rate, all the conics are rational curves, and among them the parabola is the only polynomial curve.

5. Polynomial Curves

The above information about parametrization suggests the following result.

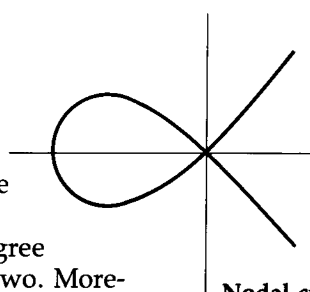
THEOREM. *A rational curve is a polynomial curve if and only if it has only one place at infinity.*

Here *place* is a refinement of the idea of a point. At a point there can be more than one place. To have *only one place at infinity* means to have only one point at infinity and to have only one place at that point. So what are the places at a point? To explain this, and having reviewed conics, let us briefly review cubics.

6. Cubics

The *nodal cubic* is given by the equation

$$Y^2 - X^2 - X^3 = 0.$$



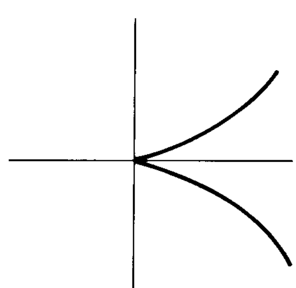
Nodal cubic

It has a *double point* at the origin because the degree of the lowest degree terms in its equation is two. Moreover, this double point at the origin is a *node*, because at the origin the curve has the two *tangent lines*

$$Y = X \text{ and } Y = -X$$

(we recall that the tangent lines at the origin are given by the factors of the lowest degree terms). Likewise, the *cuspidal cubic* is given by the equation

$$Y^2 - X^3 = 0.$$



Cuspidal cubic

It has a double point at the origin. Moreover, this double point at the origin is a *cusp*, because at the origin the curve has the only tangent line

$$Y = 0.$$

A first approximation to places is provided by the tangent lines. So the nodal cubic has two places at the origin, whereas the cuspidal cubic has only one. More precisely, the nodal cubic has two places at the origin because, although its equation cannot be factored as a polynomial, it does have two factors as a power series in X and Y; namely, by solving the equation we get

$$Y^2 - X^2 - X^3 = (Y - X(1 + X)^{1/2})(Y + X(1 + X)^{1/2}),$$

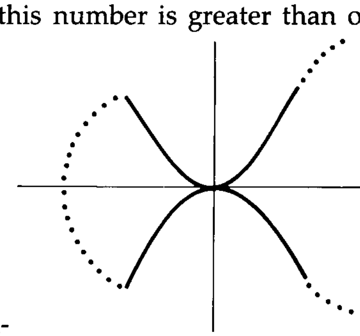
and by the binomial theorem we have

$$(1 + X)^{1/2} = 1 + (1/2)X + \dots + \frac{(1/2)[(1/2) - 1] \dots [(1/2) - j + 1)]}{j!} X^j + \dots .$$

7. Places at the Origin

Thus the number of places at the origin is defined to be equal to the number of distinct factors as power series, and in general this number is greater than or equal to the number of tangent lines. For example, the *tacnodal quintic* is given by the equation

$$Y^2 - X^4 - X^5 = 0,$$



Tacnodal quintic

which we find by multiplying the two opposite parabolas $Y \pm X^2 = 0$ and adding the extra term to make it irreducible as a polynomial. The double point at the origin is a *tacnode* because there is only one tangent line $Y = 0$ but two power series factors

$$(Y - X^2(1 + X)^{1/2})(Y + X^2(1 + X)^{1/2}).$$

So, more accurately, a cusp is a double point at which there is only one place; at a cusp it is also required that the tangent line meet the curve with *intersection multiplicity* three; i.e., when we substitute the equation of the tangent line into the equation of the curve, the resulting equation should have zero as a triple root. For example, by substituting the equation of the tangent line $Y = 0$ into the equation of the cuspidal cubic $Y^2 - X^3 = 0$, we get the equation $X^3 = 0$, which has zero as a triple root.

8. Places at Other Points

To find the number of places at any finite point, translate the coordinates to bring that point to the origin.

To find the number of places at a point at infinity, *homogenize* and *dehomogenize*. For example, by homogenizing the nodal cubic, i.e., by multiplying the various terms by suitable powers of a new variable Z so that all the terms acquire the same degree, we get

$$Y^2Z - X^2Z - X^3 = 0.$$

By putting $Z = 0$ we get $X = 0$; i.e., *the line at infinity* $Z = 0$ meets the nodal cubic only in the point P for which $X = 0$. By a suitable dehomogenization, i.e., by putting $Y = 1$, we get

$$Z - X^2Z - X^3 = 0.$$

Now, P is at the origin in the (X,Z)-plane; the left-hand side of the above equation is *analytically irreducible;* i.e., it does not factor as a power series. Thus the nodal cubic has only one place at P.

Consequently, in view of the above theorem, the nodal cubic may be expected to be a polynomial curve. To get an actual polynomial parametrization, substitute $Y = tX$ in the equation $Y^2 - X^2 - X^3 = 0$ to get

$$t^2X^2 - X^2 - X^3 = 0;$$

cancel the factor X^2 to obtain $X = t^2 - 1$ and then substitute this into $Y = tX$ to get $Y = t^3 - t$. Thus

$$X = t^2 - 1 \text{ and } Y = t^3 - t$$

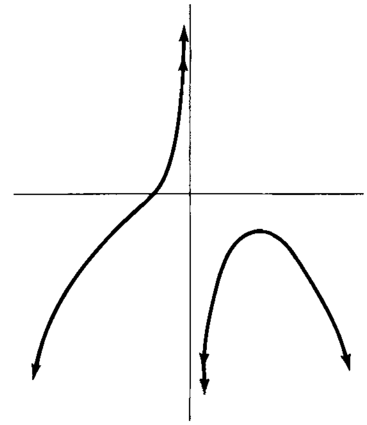
is the desired polynomial parametrization.

As a second example, recall that the nodal cubic $Y^2 - X^2 - X^3 = 0$ has two places at the origin, and the tangent line T given by $Y = X$ meets this cubic only at the origin. Therefore "by sending T to infinity" we would get a new cubic having only one point but two places at infinity; so it must be a rational curve that is not a polynomial curve. To find the equation of the new cubic, make the rotation $X' = X - Y$ and $Y' = X + Y$ to get $-X'Y' - (1/8)(X' + Y')^3 = 0$ as the equation of the nodal cubic and $X' = 0$ as the equation of T. By homogenizing and multiplying by -8 we get $8X'Y'Z' + (X' + Y')^3 = 0$ as the homogeneous equation of the nodal cubic and $X' = 0$ as the equation of T. Labeling (Y',Z',X') as (X,Y,Z), we get $8ZXY + (Z + X)^3 = 0$ as the homogeneous equation of the new cubic and T becomes the line at infinity $Z = 0$. Finally, by putting $Z = 1$, we see that the new cubic is given by the equation

$$8XY + (1 + X)^3 = 0.$$

By plotting the curve we see that one place at the point at infinity $X = Z = 0$ corresponds to the parabola-like structure indicated by the two single arrows, whereas the second place at that point corresponds to the hyperbola-like structure indicated by the two double arrows. Moreover, $Z = 0$ is the tangent to the parabola-like place, whereas $X = 0$ is the tangent to the hyperbola-like place.

So this new cubic may be called the *para-hypal cubic.* To get a rational parametrization for it, we may simply take the vertical projection. In other words, by substituting $X = t$ in the above equation, we get $Y = -(1 + t)^3/8t$. Thus



Para-hypal cubic

$$X = t \text{ and } Y = \frac{-(1 + t)^3}{8t}$$

is the desired rational parametrization; it cannot be a polynomial parametrization.

9. Desire for a Criterion

In view of the above theorem, it would be nice to have an algorithmic criterion for a given curve to have only one place at infinity or at a given point. Recently in [7] I have worked out such a criterion. See [2] to [6] for general information and [7] for details of proof; here I shall explain the matter descriptively. As a first step let us recall some basic facts about resultants.

10. Vanishing Subjects

In the above discussion I have often said "reviewing this" and "recalling that." Unfortunately, reviewing and recalling may not apply to the younger generation. Until about 30 years ago, people learned in high school and college the two subjects called "theory of equations" and "analytic geometry." Then these two subjects gradually vanished from the syllabus. "Analytic geometry" first became a chapter, then a paragraph, and finally only a footnote in books on calculus.

"Theory of equations" and "analytic geometry" were synthesized into a subject called "algebraic geometry." Better still, they were collectively called "algebraic geometry." Then "algebraic geometry" became more and more abstract until it was difficult to comprehend. Thus classical algebraic geometry was forgotten by the student of mathematics.

Engineers are now resurrecting classical algebraic geometry, which has applications in computer-aided design, geometric modeling, and robotics. Engineers have healthy attitudes; they want to solve equations concretely and algorithmically, an attitude not far from that of classical, or high-school, algebra. So let us join hands with engineers.

11. Victim

Vis-a-vis the "theory of equations," one principal victim of this vanishing act was the resultant. At any rate, the *Y-resultant* $\mathrm{Res}_Y(F,G)$ of two polynomials

$$F = a_0 Y^N + a_1 Y^{N-1} + \ldots + a_N \quad \text{and} \quad G = b_0 Y^M + b_1 Y^{M-1} + \ldots + b_M$$

is the determinant of the $N + M$ by $N + M$ matrix

$$\begin{matrix} a_0, a_1, \ldots\ldots\ldots, a_N, 0, \ldots\ldots\ldots\ldots 0 \\ 0, a_0, \ldots\ldots\ldots\ldots, a_N, 0, \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_0, b_1, \ldots\ldots\ldots, b_M, 0, \ldots\ldots\ldots\ldots 0 \\ 0, b_0, \ldots\ldots\ldots\ldots, b_M, 0, \ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{matrix}$$

with M rows of the a's followed by N rows of the b's. This concept was introduced by Sylvester in his 1840 paper [10]. It can be shown that if $a_0 \neq 0 \neq b_0$ and

$$F = a_0 \prod_{j=1}^{N} (Y - \alpha_j) \text{ and } G = b_0 \prod_{k=1}^{M} (Y - \beta_k),$$

then

$$\mathrm{Res}_Y(F,G) = a_0^M \prod_j G(\alpha_j) = (-1)^{NM} b_0^N \prod_k F(\beta_k) = a_0^M b_0^N \prod_{j,k} (\alpha_j - \beta_k).$$

In particular, F and G have a common root if and only if $\mathrm{Res}_Y(F,G) = 0$.

12. Approximate Roots

Henceforth let us consider an algebraic plane curve C defined by the equation

$$F(X,Y) = 0,$$

where $F(X,Y)$ is a monic polynomial in Y with coefficients that are polynomials in X, i.e.,

$$F = F(X,Y) = Y^N + a_1(X)Y^{N-1} + \ldots + a_N(X),$$

where $a_1(X), \ldots, a_N(X)$ are polynomials in X. We want to describe a criterion for C to have only one place at infinity. As a step toward this, given any positive integer D such that N is divisible by D, we would like to find the Dth root of F. We may not always be able to do this, because we wish to stay within polynomials. So we do the best we can. Namely, we try to find

$$G = G(X,Y) = Y^{N/D} + b_1(X)Y^{(N/D)-1} + \ldots + b_{N/D}(X),$$

where $b_1(X), \ldots, b_{N/D}(X)$ are polynomials in X, such that G^D is as close to F as possible. More precisely, we try to minimize the Y-degree of $F - G^D$. It turns out that if we require

$$\deg_Y(F - G^D) < N - (N/D),$$

then G exists in a unique manner; we call this G the *approximate* Dth *root of* F and we denote it by app(D,F). In a moment, by generalizing the usual decimal expansion, we shall give an algorithm for finding app(D,F). So let us revert from high-school algebra to grade-school arithmetic and discuss decimal expansion.

13. Decimal Expansion

We use decimal expansion to represent integers without thinking. For example, in decimal expansion

$$423 = (4 \text{ times } 100) + (2 \text{ times } 10) + 3.$$

We can also use binary expansion, or expansion to the base 12, and so on. Quite generally, given any integer $P > 1$, every nonnegative integer A has a unique *P-adic expansion*, i.e., A can uniquely be expressed as

$$A = \Sigma A_j P^j \text{ with nonnegative integers } A_j < P,$$

where the summation is over a finite set of nonnegative integers j. We can also change bases continuously. Namely, given any finite sequence $n = (n_1, n_2, \ldots, n_{h+1})$ of positive integers such that $n_1 = 1$ and n_{j+1} is divisible by n_j for $1 \leq j \leq h$, every nonnegative integer A has a unique *n-adic expansion;* i.e., A can uniquely be expressed as

$$A = \sum_{j=1}^{h+1} e_j n_j,$$

where $e = (e_1, \ldots, e_{h+1})$ is a sequence of nonnegative integers such that $e_j < n_{j+1}/n_j$ for $1 \leq j \leq h$.

In analogy with P-adic expansions of integers, given any

$$G = G(X,Y) = Y^M + b_1(X)Y^{M-1} + \ldots + b_M(X),$$

where $b_1(X), \ldots, b_M(X)$ are polynomials in X, every polynomial $H = H(X,Y)$ in X and Y has a unique *G-adic expansion*

$$H = \Sigma H_j G^j,$$

where the summation is over a finite set of nonnegative integers j and where H_j is a polynomial in X and Y whose Y-degree is less than M. In particular, if N/M equals a positive integer D, then as G-adic expansion of F we have

$$F = G^D + B_1G^{D-1} + \ldots + B_D,$$

where $B_1, \ldots, B_D$ are polynomials in X and Y whose Y-degree is less than N/D. Now clearly,

$$\deg_Y(F - G^D) < N - (N/D) \text{ if and only if } B_1 = 0.$$

In general, in analogy with Shreedharacharya's method of solving quadratic equations by completing the square, for which reference may be made to [8] (and assuming that in our situation $1/D$ makes sense), we may "complete the Dth power" by putting $G' = G + (B_1/D)$ and by considering the G'-adic expansion

$$F = G'^D + B'_1G'^{D-1} + \ldots + B'_D,$$

where $B'_1, \ldots, B'_D$ are polynomials in X and Y whose Y-degree is less than N/D. We can easily see that if $B_1 \neq 0$ then $\deg_Y B'_1 < \deg_Y B_1$. It follows that by starting with any G and repeating this procedure D times we get the approximate Dth root of F.

Again, in analogy with n-adic expansion, given any sequence $g = (g_1, \ldots, g_{h+1})$, where g_j is a monic polynomial of degree n_j in Y with coefficients that are polynomials in X, every polynomial H in X and Y has a unique *g-adic expansion*

$$H = \Sigma H_e \prod_{j=1}^{h+1} g_j^{e_j} \text{ where } H_e \text{ is a polynomial in } X$$

and where the summation is over all sequences of nonnegative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leqslant j \leqslant h$.

14. Places at Infinity

As the next step toward the criterion, we associate several sequences with F as follows. The case when Y divides F being trivial, we assume the contrary. Now let

$$d_1 = r_0 = N, \ g_1 = Y, \ r_1 = \deg_X \mathrm{Res}_Y(F,g_1)$$

and

$$\left[\begin{array}{l} d_2 = \mathrm{GCD}(r_0,r_1), \ g_2 = \mathrm{app}(d_2,F), \\ r_2 = \deg_X \mathrm{Res}_Y(F,g_2) \end{array}\right.$$

and

$$\left[\begin{array}{l} d_3 = \mathrm{GCD}(r_0,r_1,r_2), \ g_3 = \mathrm{app}(d_3,F), \\ r_3 = \deg_X \mathrm{Res}_Y(F,g_3) \end{array}\right.$$

and so on, where we agree to put

$$\deg_X \mathrm{Res}_Y(F,g_i) = -\infty \text{ if } \mathrm{Res}_Y(F,g_i) = 0$$

and

$$\left[\begin{array}{l} \mathrm{GCD}(r_0,r_1, \ldots, r_i) = \mathrm{GCD}(r_0,r_1, \ldots, r_j) \\ \text{if } r_0, r_1, \ldots, r_j \text{ are integers and } j < i \text{ and} \\ \quad r_{j+1} = r_{j+2} = \ldots = r_i = -\infty. \end{array}\right.$$

Since $d_2 \geqslant d_3 \geqslant d_4 \geqslant \ldots$ are positive integers, there exists a unique positive integer h such that $d_2 > d_3 > \ldots > d_{h+1} = d_{h+2}$. Thus we have defined the two sequences of integers $r = (r_0,r_1, \ldots, r_h)$ and $d = (d_1,d_2, \ldots, d_{h+1})$ and a third sequence $g = (g_1,g_2, \ldots, g_{h+1})$, where g_j is a monic polynomial of degree $n_j = d_1/d_j$ in Y with coefficients that are polynomials in X. Now, for the curve C defined by $F(X,Y) = 0$, we are ready to state the criterion.

CRITERION for having only one place at infinity. *C has only one place at infinity if and only if $d_{h+1} = 1$ and $r_1d_1 > r_2d_2 > \ldots > r_hd_h$ and g_{j+1} is degreewise straight relative to (r,g,g_j) for $1 \leqslant j \leqslant h$ (in the sense we shall define in a moment).*

To spell out the definition of degreewise straightness, for every polynomial H in X and Y we consider the g-adic expansion

$$H = \Sigma H_e \prod_{j=1}^{h+1} g_j^{e_j}, \text{ where } H_e \text{ is a polynomial in } X$$

and where the summation is over all sequences of nonnegative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leqslant j \leqslant h$. We define

$$\mathrm{fing}(r,g,H) = \max\left(\sum_{j=0}^{h} e_jr_j\right) \text{ with } e_0 = \deg_X H_e,$$

where the max is taken over all e for which $H_e \neq 0 = e_{h+1}$; here fing is supposed to be an abbreviation of the phrase "degreewise formal intersection multiplicity," which in turn is meant to suggest some sort of analogy with intersection multiplicity of plane curves.

For $1 \leqslant j \leqslant h$ let $u(j) = n_{j+1}/n_j$ and consider the g_j-adic expansion

$$g_{j+1} = g_j^{u(j)} + \sum_{k=1}^{u(j)} g_{jk}g_j^{u(j)-k},$$

where g_{jk} is a polynomial in X and Y whose Y-degree is less than n_j. We say that g_{j+1} is *degreewise straight relative to* (r,g,g_j) if

$$(u(j)/k)\text{fing}(r,g,g_{jk}) \leq \text{fing}(r,g,g_{ju(j)}) = u(j)[\text{fing}(r,g,g_j)]$$

for $1 \leq k \leq u(j)$; the adjective *straight* is meant to suggest that we are considering some kind of generalization of Newton Polygon (for Newton Polygon, see [9], Part II, pp. 382–397, where it is called Newton Parallelogram).

15. Places at a Given Point

To discuss places of the curve C defined by $F(X,Y) = 0$ at a given finite point, we may suppose that the point has been brought to the origin by a translation and rotation of coordinates and that neither X nor Y divides F. By the Weierstrass Preparation Theorem (see [1], p. 74), we can write

$$F(X,Y) = \delta(X,Y)F^*(X,Y),$$

where $\delta(X,Y)$ is a power series in X and Y with $\delta(0,0) \neq 0$ and F^* is a distinguished polynomial; i.e.,

$$F^* = F^*(X,Y) = Y^{N^*} + a_1^*(X)Y^{N^*-1} + \ldots + a^*_{N^*}(X)$$

and $a_1^*(X), \ldots, a^*_{N^*}(X)$ are power series in X that are zero at zero. By ord_X of a power series in X we mean the degree of the lowest degree term present in that power series. We also note that in the present situation, the approximate roots of F^* are monic polynomials in Y whose coefficients are power series in X. Now let

$$d_1 = r_0 = N^*, g_1 = Y, r_1 = \text{ord}_X\text{Res}_Y(F^*,g_1)$$

and

$$\left[\begin{array}{l} d_2 = \text{GCD}(r_0,r_1), g_2 = \text{app}(d_2,F^*), \\ r_2 = \text{ord}_X\text{Res}_Y(F^*,g_2) \end{array}\right.$$

and

$$\left[\begin{array}{l} d_3 = \text{GCD}(r_0,r_1,r_2), g_3 = \text{app}(d_3,F^*), \\ r_3 = \text{ord}_X\text{Res}_Y(F^*,g_3) \end{array}\right.$$

and so on, where we agree to put

$$\text{ord}_X\text{Res}_Y(F^*,g_i) = \infty \text{ if } \text{Res}_Y(F^*,g_i) = 0$$

and

$$\left[\begin{array}{l} \text{GCD}(r_0,r_1, \ldots, r_i) = \text{GCD}(r_0,r_1, \ldots, r_j) \\ \text{if } r_0,r_1, \ldots, r_j \text{ are integers and } j < i \text{ and} \\ r_{j+1} = r_{j+2} = \ldots = r_i = \infty. \end{array}\right.$$

Since $d_2 \geq d_3 \geq d_4 \geq \ldots$ are positive integers, there exists a unique positive integer h such that $d_2 > d_3 > \ldots > d_{h+1} = d_{h+2}$. Thus we have defined the two sequences of integers $r = (r_0,r_1, \ldots, r_h)$ and $d = (d_1,d_2, \ldots, d_{h+1})$ and a third sequence $g = (g_1, g_2, \ldots, g_{h+1})$, where g_j is a monic polynomial of degree $n_j = d_1/d_j$ in Y with coefficients that are power series in X. For the curve C defined by $F(X,Y) = 0$, we are ready to state the main result of this section.

CRITERION for having only one place at the origin. *C has only one place at the origin if and only if $d_{h+1} = 1$ and $r_1d_1 < r_2d_2 < \ldots < r_hd_h$ and g_{j+1} is straight relative to (r,g,g_j) for $1 \leq j \leq h$ (in the sense which we shall define in a moment).*

To spell out the definition of straightness, first note that in the present situation, the coefficients of a g-adic expansion are power series in X. Now for every polynomial H in Y with coefficients that are power series in X, we consider the g-adic expansion

$$H = \Sigma H_e \prod_{j=1}^{h+1} g_j^{e_j}, \text{ where } H_e \text{ is a power series in } X$$

and where the summation is over all sequences of nonnegative integers $e = (e_1, \ldots, e_{h+1})$ such that $e_j < n_{j+1}/n_j$ for $1 \leq j \leq h$. We define

$$\text{fint}(r,g,H) = \min\left(\sum_{j=0}^{h} e_jr_j\right) \text{ with } e_0 = \text{ord}_XH_e,$$

where the min is taken over all e for which $H_e \neq 0 = e_{h+1}$; here fint is supposed to be an abbreviation of the phrase "formal intersection multiplicity," which in turn is meant to suggest some sort of analogy with intersection multiplicity of plane curves.

For $1 \leq j \leq h$ let $u(j) = n_{j+1}/n_j$ and consider the g_j-adic expansion

$$g_{j+1} = g_j^{u(j)} + \sum_{k=1}^{u(j)} g_{jk}g_j^{u(j)-k},$$

where we note that in the present situation, the coefficients g_{jk} are polynomials of degree less than n_j in Y whose coefficients are power series in X. We say that g_{j+1} is *straight relative to* (r,g,g_j) if

$$(u(j)/k)\text{fint}(r,g,g_{jk}) \geq \text{fint}(r,g,g_{ju(j)}) = u(j)[\text{fint}(r,g,g_j)]$$

for $1 \leq k \leq u(j)$; again, the adjective *straight* is meant to suggest that we are considering some kind of generalization of Newton Polygon.

16. Problem

Generalize the above criterion by finding a finitistic algorithm to count the number of places at infinity or at a given point.

17. Example

To illustrate the criterion for having only one place at the origin, let us take

$$F = F(X,Y) = (Y^2 - X^3)^2 + X^pY - X^7,$$

where p is a positive integer to be chosen. Now

$$F^* = F \text{ and } d_1 = r_0 = N^* = N = 4 \text{ and } g_1 = Y$$

and hence

$$\mathrm{Res}_Y(F,g_1) = F(X,0) = X^6 - X^7 \text{ and } r_1 = \mathrm{ord}_X\mathrm{Res}_Y(F,g_1) = 6.$$

Therefore,

$$d_2 = \mathrm{GCD}(r_0,r_1) = \mathrm{GCD}(4,6) = 2$$

and hence

$$g_2 = \mathrm{app}(d_2,F) = Y^2 - X^3 = (Y - X^{3/2})(Y + X^{3/2}).$$

Consequently,

$$\begin{aligned}\mathrm{Res}_Y(F,g_2) &= F(X,X^{3/2})F(X,-X^{3/2}) \\ &= (X^{p+(3/2)} - X^7)(-X^{p+(3/2)} - X^7) \\ &= -X^{2p+3} + X^{14}\end{aligned}$$

and hence

$$r_2 = \mathrm{ord}_X\mathrm{Res}_Y(F,g_2) = \begin{cases} 14 & \text{if } p > 5 \\ 2p + 3 & \text{if } p \leqslant 5. \end{cases}$$

Therefore,

$$d_3 = \begin{cases} 2 \text{ if } p > 5 \\ 1 \text{ if } p \leqslant 5 \end{cases} \text{ and } h = \begin{cases} 1 \text{ if } p > 5 \\ 2 \text{ if } p \leqslant 5 \end{cases}$$

and

$$r_1d_1 = \begin{cases} 24 < 26 = (2p + 3)d_2 = r_2d_2 \text{ if } p = 5 \\ 24 \geqslant 22 \geqslant (2p + 3)d_2 = r_2d_2 \text{ if } p < 5. \end{cases}$$

Now, if $p = 5$, then

$$g_{11} = 0, \text{ and } g_{21} = 0$$

and

$$g_{12} = X^3 \text{ and } \mathrm{fint}(r,g,X^3) = 3r_0 = 12 = 2r_1$$

and

$$g_{22} = X^5Y - X^7 \text{ and } \mathrm{fint}(r,g,X^5Y - X^7) = 5r_0 + r_1 = 26 = 2r_2$$

and hence g_{j+1} is straight relative to (r,g,g_j) for $1 \leqslant j \leqslant 2$.

Thus we see that if $p > 5$, then $h = 1$ and $d_{h+1} = 2$, whereas if $p < 5$, then $h = 2$ and $d_{h+1} = 1$ and $r_1d_1 > r_2d_2$; finally, if $p = 5$, then $h = 2$ and $d_{h+1} = 1$ and $r_1d_1 < r_2d_2$ and g_{j+1} is straight relative to (r,g,g_j) for $1 \leqslant j \leqslant 2$. Therefore, by the criterion we conclude that C has only one place at the origin if and only if $p = 5$.

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Mathematics is the science that yields the best opportunity to observe the working of the mind . . . and has the advantage that by cultivating it, we may acquire the habit of a method of reasoning which can be applied afterwards to the study of any subject and can guide us in the pursuit of life's object.

Marie Jean Condorcet