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Classical solitons with no quantum counterparts and their supersymmetric revival

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Abstract. We demonstrate the phenomenon stated in the title, using for illustration a two-dimensional scalar-field model with a triple-well potential

$$V(\phi) = \frac{\lambda^2 \phi^2}{2m^2}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2.$$

At the classical level, this system supports static topological solitons with finite energy. Upon quantisation, however, these solitons develop infinite energy, which cannot be renormalised away. Thus this quantised model has no soliton sector, even though classical solitons exist. Finally when the model is extended supersymmetrically by adding a Majorana field, finiteness of the soliton energy is recovered.

Keywords. Solitons; 2-D scalar-field model; triple-well potential; supersymmetry.

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The procedure for quantising solitons is now well known (Jackiw 1977; Rajaraman 1982). In all the examples studied, whenever a model permits a static soliton solution at the classical level, it also yields upon quantisation a whole soliton sector of states of finite energy, of which the lowest energy state corresponds to the quantum soliton particle at rest. Numerous examples of this phenomenon have been discussed in detail in literature, especially among scalar field theories in $(1+1)$ dimensions.

If, in this paper, we discuss solitons in yet another two-dimensional toy-model, it is only because this example typifies a class of models which do not quite follow the general pattern mentioned above. While at the classical level, perfectly respectable finite-energy static solitons exist, we shall see that at the quantum level, they develop infinite-energy. This infinity is not of the ultraviolet sort, which one usually encounters in the soliton-energy calculations and which is cancelled by counter-term contributions. Rather, it is a divergence proportional to L, the infinite volume of the "box", in which the system is quantised. This divergence cannot be subtracted away, and is really there. Consequently, there is no quantum soliton sector in the model, associated with the classical soliton. However, when the model is extended supersymmetrically by adding fermions, the finiteness of the quantum soliton's energy is restored. The L-dependent divergence produced by the zero-point energy of the Boson fluctuations is cancelled by a similar divergence in the negative energy of the Fermi sea. Thus the supersymmetric model does have a soliton sector.

The model we shall use for illustration is described by the Lagrangian density

$$L = \tfrac{1}{2}[(\partial_\mu \phi)(\partial^\mu \phi) - S^2(\phi)], \tag{1}$$

with a potential

$$\tfrac{1}{2}S^2(\phi) \equiv V(\phi) = \frac{\lambda^2}{2m^2}\,\phi^2(\phi^2 - m^2/\lambda)^2, \tag{2}$$

where $\phi(x, t)$ is a real scalar field in $(1+1)$ dimensions. Later we will also consider the supersymmetric extension:

$$L = \tfrac{1}{2}[(\partial_\mu \phi)(\partial^\mu \phi) - S^2(\phi) + \bar{\psi}(i\partial\!\!\!/ - S'(\phi))\psi] \tag{3}$$

where $\psi(x, t)$ is a Majorana field.

Although we shall demonstrate the above mentioned results explicitly by computing the quantum corrections to the soliton-energy, they can be quite easily anticipated by considering the vacuum of this model (Rajaraman and Raj Lakshmi 1981). Notice that the potential $V(\phi)$ has three degenerate minima, $\phi = \pm m/\sqrt{\lambda}$ and $\phi = 0$. Two of these are related to one another by the $\phi \leftrightarrow -\phi$ symmetry of the problem, but the third $\phi = 0$ is not so related. At the classical level, the vacuum of this theory could be located either at $\phi = 0$, or at $\phi = \pm m/\sqrt{\lambda}$, the latter possibility amounting to spontaneous breaking of $\phi \leftrightarrow -\phi$ symmetry. But at the quantum level, the degeneracy between the vacua at $\phi = \pm m/\sqrt{\lambda}$ and that at $\phi = 0$ is lifted. The curvature of the potential $(\mathrm{d}^2 V/\mathrm{d}\phi^2)$ at these is different:

$$[\mathrm{d}^2 V/\mathrm{d}\phi^2]_{\phi = \pm m/\sqrt{\lambda}} = 4m^2,\ [\mathrm{d}^2 V/\mathrm{d}\phi^2]_{\phi = 0} = m^2. \tag{4}$$

Thus the (bare) boson mass would be $2m$ if the vacuum were located around $\phi = \pm m/\sqrt{\lambda}$, while it would be *smaller* and equal to m, if the vacuum were at $\phi = 0$. Consequently, the zero point energy of the vacuum at $\phi = 0$ will be lower than those of the (now false) vacua at $\phi = \pm m/\sqrt{\lambda}$. In fact the difference of these energies, to $0(\hbar)$ is

$$\begin{aligned}&(E_{\text{vac}})_{\phi = \pm m/\sqrt{\lambda}} - (E_{\text{vac}})_{\phi = 0} \\ &= \frac{\hbar L}{2}\int \mathrm{d}k/2\pi((k^2 + 4m^2)^{1/2} - (k^2 + m^2)^{1/2}).\end{aligned} \tag{5}$$

Notice that this difference carries, apart from a logarithmic UV divergence, also a factor L which diverges since $L \to \infty$. Therefore, the states built around $\phi = \pm m/\sqrt{\lambda}$ have a higher energy density *per unit length*, and the true vacuum of the model (1) is the symmetric one at $\phi = 0$. In fact, by explicitly calculating the effective potential up to two-loop order, it can be shown that the symmetric vacuum at $\phi = 0$ survives even when the potential $V(\phi)$ is so designed that the classical minimum at $\phi = 0$ is slightly *higher* than those at $\phi = \pm m/\sqrt{\lambda}$ (Rajaraman and Raj Lakshmi 1981). This mechanism of symmetry restoration is different from the more familiar one related to the absence of Goldstone bosons in $(1+1)$ dimensions (Coleman 1973).

Equations resulting from (1) are easily integrated to obtain the following static soliton solution (Khare 1979)

$$\phi_s(x) = m/\sqrt{2\lambda} \quad (1 + \tan\mathrm{h}\, mx)^{1/2}, \tag{6a}$$

with *finite* classical energy

$$M_0 = m^3/3\lambda, \tag{6b}$$

Three more degenerate solutions are trivially obtained by exchanging $x \leftrightarrow -x$ and $\phi \leftrightarrow -\phi$. These four solutions extrapolate between $\phi = 0$ and $\phi = \pm m/\sqrt{\lambda}$. (There is no static soliton connecting $\phi = m/\sqrt{\lambda}$ to $\phi = -m/\sqrt{\lambda}$, as these are not neighbouring minima of the potential $V(\phi)$). The soliton (6a) rapidly approaches the true vacuum $\phi = 0$ at one end, but approaches the false vacuum $\phi = +m/\sqrt{\lambda}$ at the other end. Since, upon quantisation, the false vacuum has a higher energy density per unit length compared to the true vacuum, we can anticipate that the quantum soliton will also have a higher energy as compared to the vacuum by an amount that tends to infinity as $L \to \infty$. On the other hand, in the supersymmetric extension (3), all the three vacua at $\phi = 0, \pm m/\sqrt{\lambda}$ continue to be degenerate and to have zero energy, even when quantum corrections are included. This fact, related to the non-breaking of supersymmetry by radiative corrections, in such models is well-known (Murphy and O'Raifaertaigh 1983; D'Adda and Di Vecchia 1978). Correspondingly, in the supersymmetric case, we can expect the quantum soliton to have only finite energy as $L \to \infty$, since the vacua it approaches at either end are now degenerate and carry zero energy. The detailed analysis below demonstrates precisely these results.

The one-loop corrections to the soliton mass arise from two sources: (i) the energy of the quantum fluctuations and (ii) the contribution from the counter terms (see for example, Jackiw 1977; Rajaraman 1982),

$$M = \frac{\hbar}{2}\left(\sum_{\text{sol}}\omega - \sum_{\text{vac}}\tilde{\omega}\right) + [\Delta E_{\text{c.t.}}(\text{sol}) - \Delta E_{\text{c.t.}}(\text{vac})]. \tag{7}$$

The fluctuation frequencies ω, around the soliton function (6a), are governed by the eigen-value equation

$$[-\mathrm{d}^2/\mathrm{d}x^2 + V''(\phi_s)]\,\xi(x) = \omega^2 \xi(x) \tag{8}$$

$$V''(\phi_s) = \frac{m^2}{4}(15\tan\mathrm{h}^2 mx + 6\tan\mathrm{h}\, mx - 5) \tag{9}$$

Notice that the potential $V''(\phi_s)$ in this Schrödinger-like equation (8) approaches different asymptotic values, namely m^2 and $4m^2$ as $x \to -\infty$ and $+\infty$ respectively. Besides one solution with discrete eigenvalue $\omega = 0$, which anyway does not contribute anything to (7), this second order differential equation admits of two sets of independent solutions with eigen-values in the continuum $\omega^2 \geqslant m^2$. The asymptotic behaviour of these solutions can be described as follows:

$$\begin{aligned} &\text{(i)} \quad \xi^{(1)}(x \to -\infty) = \exp(ikx) + A\exp(-ikx), \\ &\qquad \xi^{(1)}(x \to +\infty) = B\exp(ik'x) \end{aligned} \tag{10}$$

$$\begin{aligned} &\text{(ii)} \quad \xi^{(2)}(x \to -\infty) = B'\exp(-ikx), \\ &\qquad \xi^{(2)}(x \to +\infty) = \exp(-ik'x) + A'\exp(ik'x) \end{aligned} \tag{11}$$

where A, B, A' and B', are functions only of k, and $(k')^2 + 4m^2 = k^2 + m^2$.

Formally, solutions of the form (10–11) hold not only for $\omega^2 \geqslant 4m^2$, but also for $m^2 \leqslant \omega^2 < 4m^2$, with the understanding that for $m^2 \leqslant \omega^2 < 4m^2$, $k' = i\gamma = +i(4m^2 - \omega^2)^{1/2}$ is pure imaginary and for $\omega^2 \geqslant 4m^2$, $k' = (\omega^2 - 4m^2)^{1/2}$ is real. On the other hand $k = +(\omega^2 - m^2)^{1/2}$ is real for the whole range $\omega^2 \geqslant m^2$. The various coefficients A, B, A', B' are related by using the fact that Wronskians at $x = +\infty$ and $x = -\infty$ for the sets of solutions $(\xi^{(1)*}, \xi^{(1)})$, $(\xi^{(2)*}, \xi^{(2)})$, $(\xi^{(1)*}, \xi^{(2)})$ and $(\xi^{(1)}, \xi^{(2)})$ are equal:

(i) For $m^2 \leqslant \omega^2 < 4m^2$:

$$1-|A|^2 = 0,\ ikB' = -\gamma B, \text{ and } ikA^*B' = -\gamma B^* \tag{12}$$

(ii) For $\omega^2 \geqslant 4m^2$:

$$k(1-|A|^2) = k'|B|^2,\ k'(1-|A'|^2) = k|B'|^2$$
$$kB' = k'B,\ kA^*B' = -k'A'B^* \tag{13}$$

To evaluate the contribution of these continuum solutions to the soliton mass, we need to know the density of the continuum eigen-values. We obtain this density of states by putting the system in a "box" of length L (with $L \to \infty$), and imposing the usual second order boundary conditions on any solution $\xi(x)$:

$$\xi(-L/2) = \xi(+L/2) \text{ and } \mathrm{d}\xi/\mathrm{d}x(-L/2) = \mathrm{d}\xi/\mathrm{d}x(+L/2). \tag{14}$$

Neither $\xi^{(1)}(x)$ nor $\xi^{(2)}(x)$ has enough freedom to satisfy both conditions in (14). But a suitable linear combination $\xi(x) = \xi^{(1)}(x) + \alpha\xi^{(2)}(x)$ can satisfy (14) for some values of α and k, given by

$$\exp(-ikL/2) + (A+\alpha B')\exp(ikL/2) = (B+\alpha A')\exp(ik'L/2) + \alpha\exp(-ik'L/2)$$
$$k[\exp(-ikL/2) - (A+\alpha B')\exp(ikL/2)] = k'[(B+\alpha A')\exp(ik'L/2) - \alpha\exp(-ik'L/2)]. \tag{15}$$

The coefficient α may be eliminated from these to obtain the following constraint which discretizes the values of k:

$$1 = (AA'-BB')\exp(i(k+k')L) + 4k'B/(k+k')\exp(i(k+k')L/2)$$
$$+ (k-k')/(k+k')(A\exp(ikL) - A'\exp(ik'L)), \tag{16}$$

where use has been made of relation $kB' = k'B$.

It will be helpful to consider the domains, $m^2 \leqslant \omega^2 < 4m^2$ and $\omega^2 \geqslant 4m^2$ separately.

(A) $m^2 \leqslant \omega^2 < 4m^2$

Here $ik' = -\gamma$ is real and $\exp(-\gamma L)$ can be dropped in (16) for $L \to \infty$, to yield

$$\exp(-ikL) = ((k-i\gamma)/(k+i\gamma))A = A\exp(i\theta) \tag{17}$$

Since from (12), $1-|A|^2 = 0$, A is a pure phase. From this we obtain

$$-k_nL + 2n\pi = \theta - i\ln A,\ n = 0, 1, 2, \ldots \tag{18}$$

and therefore, the density of these boson fluctuations is given by

$$\mathrm{d}n_B/\mathrm{d}k = L/2\pi + 1/2\pi\,(\mathrm{d}\theta/\mathrm{d}k) - i/2\pi\,(\mathrm{d}/\mathrm{d}k)\ln A \tag{19}$$

and contribution to the one-loop soliton mass is

$$\frac{\hbar}{2}\sum_{\omega^2<4m^2}\omega = \frac{\hbar}{2}\int_0^{\sqrt{3}m}\frac{\mathrm{d}k}{2\pi}(k^2+m^2)^{1/2}(L + \mathrm{d}\theta/\mathrm{d}k - i(\mathrm{d}/\mathrm{d}k)\ln A) \tag{20}$$

(B) $\omega^2 \geqslant 4m^2$

Here k' is real. Using the set of relations (13), one can easily show that $AA'-BB'$ is a pure phase. In fact writing $B = |B|\exp(i\phi)$,

$$A' = \frac{-A^*B}{B^*} = -A^*\exp(2i\phi) \tag{21a}$$

and

$$AA' - BB' = -\exp(2i\phi) \tag{21b}$$

Using these facts, (16) can be rewritten as

$$1 = -\exp(2i\phi')[1 - \exp(-i\phi')|X|] \tag{22}$$

where $\phi' = \phi + (k+k')L/2$ and $|X|$ can be read directly from (16). Equation (22) is satisfied by

$$|X| = 2\cos\phi' \tag{23}$$

which admits two solutions

$$\phi' = +\cos^{-1}\frac{|X|}{2} + 2n_1\pi, \; \phi' = -\cos^{-1}\frac{|X|}{2} + 2n_2\pi,$$
$$n_1, n_2 = 0, 1, 2, \ldots . \tag{24}$$

The corresponding densities of these fluctuations as $L \to \infty$ can be written as

$$\frac{dn_{1,2}}{dk} = \frac{1}{2\pi}\left[\left(1+\frac{k}{k'}\right)\frac{L}{2} + \frac{d\phi}{dk} \mp \frac{d}{dk}\cos^{-1}\frac{|X|}{2}\right] \tag{25a}$$

Altogether,

$$dn_B/dk = dn_1/dk + dn_2/dk \tag{25b}$$

Therefore, the contribution to the soliton mass from these fluctuations is

$$\frac{\hbar}{2}\sum_{\omega^2 \geqslant 4m^2}\omega = \frac{\hbar}{2}\int_{\sqrt{3}m}^{\infty}\frac{dk}{2\pi}(k^2+m^2)^{1/2}\left[\left(1+\frac{k}{k'}\right)L + 2\frac{d\phi}{dk}\right] \tag{26}$$

Thus adding (20) and (26), we have the total contribution to the soliton's fluctuation energy as

$$\begin{aligned}\frac{\hbar}{2}\sum_{\text{sol}}\omega &= \frac{\hbar L}{2}\int_0^{\infty} dk(dn_B/dk)(k^2+m^2)^{1/2} \\ &= \frac{\hbar L}{2}\int_0^{\infty} dk/2\pi\,(k^2+m^2)^{1/2} + \frac{\hbar L}{2}\int_0^{\infty} dk'/2\pi\,(k'^2+4m^2)^{1/2} \\ &\quad + \frac{\hbar}{2}\int_0^{\sqrt{3}m} dk/2\pi\,(k^2+m^2)^{1/2}(d\theta/dk - i(d/dk)\ln A) \\ &\quad + \frac{\hbar}{2}\int_{\sqrt{3}m}^{\infty} dk/2\pi\,(k^2+m^2)^{1/2}(-i(d/dk)\ln(B/B^*))\end{aligned} \tag{27}$$

Notice that in (27), the first two terms are proportional to L, while the last two are L-independent. Equation (27) gives the first term in (7). The remaining terms are easy to obtain. We have, for the true vacuum which is at $\phi = 0$,

$$-\frac{\hbar}{2}\sum_{\text{vac}}\tilde{\omega} = -\hbar L\int_0^{\infty} dk/2\pi\,(k^2+m^2)^{1/2} \tag{28}$$

The one-loop renormalisation counter term, associated with the Lagrangian (1) has the form

$$L_{\text{c.t.}} = \frac{\hbar C}{2}\left((S')^2 + SS''\right) \tag{29}$$

with

$$C = \frac{1}{2\pi}\int_0^\infty \mathrm{d}k/(k^2+m^2)^{1/2} \tag{30}$$

The contribution of (29) to the energy (7) is

$$\begin{aligned}\Delta E_{\text{c.t.}}(\text{sol}) - \Delta E_{\text{c.t.}}(\text{vac}) &= -\int_{-L/2}^{L/2} \mathrm{d}x[L_{\text{c.t.}}(\phi_s(x)) - L_{\text{c.t.}}(\phi=0)] \\ &= \left(-\frac{3}{4}m^2L + \frac{15}{4}m\right)\hbar C\end{aligned} \tag{31}$$

Adding (27), (28) and (31), we obtain the one-loop radiative corrections to the soliton mass as

$$M = \frac{\hbar L}{2}\int_0^\infty \frac{\mathrm{d}k}{2\pi}\left[(k^2+4m^2)^{1/2} - (k^2+m^2)^{1/2} - \frac{3}{2}\frac{m^2}{(k^2+m^2)^{1/2}}\right. \\ + (L \text{ independent terms}) \tag{32}$$

The UV divergences in the first term in (32) go away due to the counter term (29). But the finite piece is proportional to L and will diverge as $L\to\infty$. The remaining L-independent terms in (29) cannot remove this divergence. In other words, the quantum soliton does not exist, even though classically it had finite energy.

Supersymmetric case

Now, let us turn to the other part of our claim, namely that, in the supersymmetric version as described by the Lagrangian given in (3), the quantum soliton is revived. As is well-known (Murphy and O'Raifaerfaigh 1983; D'Adda and Di Vecchia 1978), the vacuum energy (including counter terms) vanishes exactly for the supersymmetric case. Therefore the one-loop soliton mass requires no vacuum subtractions. We have,

$$M = M_0 + \left(\frac{\hbar}{2}\sum_{\text{sol}}\omega_B - \frac{\hbar}{2}\sum_{\text{sol}}\omega_F\right) + \Delta M^{\text{susy}}_{\text{c.t.}}(\text{sol}) \tag{33}$$

where ω_B and ω_F are the frequencies of the Bose and Fermi fluctuations about the solution. M can be calculated in the same manner as has been done for the soliton of double-wall potential case earlier (Schonfeld 1979; Kaul and Rajaraman 1983). The Bose fluctuations energy, governed by (8) has already been obtained. The Fermi fluctuations are given by

$$[i\partial\!\!\!/ - S'(\phi_s)]\psi = 0.$$

Writing

$$\psi(x,t) = u(x)\exp(-i\omega_F t) + u^*(x)\exp(i\omega_F t), \tag{34}$$

$$u(x) = \frac{1}{\sqrt{2}}\begin{bmatrix} u_+(x) \\ u_-(x) \end{bmatrix},$$

we obtain the linear coupled eigen-value equations for the Fermi-fluctuations:

$$[\mathrm{d}/\mathrm{d}x + S'(\phi_s)]u_-(x) = i\omega_F u_+(x) \tag{35a}$$

$$[\mathrm{d}/\mathrm{d}x - S'(\phi_s)]u_+(x) = i\omega_F u_-(x) \tag{35b}$$

which leads to

$$[-\mathrm{d}^2/\mathrm{d}x^2 + (S'^2 + S''S)_{\phi_s}]u_+(x) = \omega_F^2 u_+(x) \tag{36a}$$

$$[-\mathrm{d}^2/\mathrm{d}x^2 + (S'^2 - S''S)_{\phi_s}]u_-(x) = \omega_F^2 u_-(x) \tag{36b}$$

Even though (36a) is the same equation as the Bose fluctuation equation (8) the state-density $\mathrm{d}n_F/\mathrm{d}k$ of Fermi fluctuations is not the same as $\mathrm{d}n_B/\mathrm{d}k$. Rather, it is given by (Schonfeld 1979; Kaul and Rajaraman 1983; Imbimbo and Mukhi 1984);

$$\mathrm{d}n_F/\mathrm{d}k = \tfrac{1}{2}[\mathrm{d}n_+/\mathrm{d}k + \mathrm{d}n_-/\mathrm{d}k] \tag{37}$$

where $\mathrm{d}n_\pm/\mathrm{d}k$ are the densities of states associated with the second order equations (36a) and (36b). Of course,

$$\mathrm{d}n_B/\mathrm{d}k \equiv \mathrm{d}n_+/\mathrm{d}k \tag{38}$$

and hence,

$$\frac{\hbar}{2}\sum_{\text{sol}}\omega_B - \frac{\hbar}{2}\sum_{\text{sol}}\omega_F = \frac{\hbar}{4}\int_0^\infty \mathrm{d}k[\mathrm{d}n_+/\mathrm{d}k\,(k) - \mathrm{d}n_-/\mathrm{d}k\,(k)]\,\omega(k) \tag{39}$$

The density $\mathrm{d}n_+/\mathrm{d}k = \mathrm{d}n_B/\mathrm{d}k$ has already been evaluated above (equation (19) for $m^2 \leqslant \omega^2 < 4m^2$ and equation (25) for $\omega^2 \geqslant 4m^2$). We are left with evaluating the density $\mathrm{d}n_-/\mathrm{d}k(k)$.

The density $\mathrm{d}n_-/\mathrm{d}k$ is evaluated similarly as was done for $\mathrm{d}n_+/\mathrm{d}k = \mathrm{d}n_B/\mathrm{d}k$, but with (36a) $\equiv$ (8), replaced by (36b). The solutions of (36b) analogous to (10) and (11) may be written as

$$\begin{aligned} u_-^{(1)}(x \to -\infty) &= \exp(ikx) + C\exp(-ikx) \\ u_-^{(1)}(x \to +\infty) &= D\exp(ik'x) \end{aligned} \tag{40}$$

and

$$\begin{aligned} u_-^{(2)}(x \to -\infty) &= D'\exp(ikx) \\ u_-^{(2)}(x \to +\infty) &= \exp(-ik'x) + C'\exp(ik'x) \end{aligned} \tag{41}$$

Proceeding exactly as before, we have, analogous to (27),

$$\begin{aligned} \frac{\hbar}{2}\int &\mathrm{d}k(\mathrm{d}n_-/\mathrm{d}k)(k^2+m^2)^{1/2} \\ &= \frac{\hbar L}{2}\int_0^\infty \mathrm{d}k/2\pi\,(k^2+m^2)^{1/2} + \frac{\hbar L}{2}\int_0^\infty \mathrm{d}k'/2\pi\,(k'^2+4m^2)^{1/2} \\ &\quad + \frac{\hbar}{2}\int_0^{\sqrt{3}m} \mathrm{d}k/2\pi\,(k^2+m^2)^{1/2}(\mathrm{d}\theta/\mathrm{d}k - i(\mathrm{d}/\mathrm{d}k)\ln C) \\ &\quad + \frac{\hbar}{2}\int_{\sqrt{3}m}^\infty \mathrm{d}k/2\pi\,(k^2+m^2)^{1/2}(-i(\mathrm{d}/\mathrm{d}k)\ln(D/D^*)) \end{aligned} \tag{42}$$

However, C and D are related to A and B, due to the Dirac equation (35), which relates the $u_-^{(1,2)}$ to the $u_+^{(1,2)}$. These $u_+^{(1,2)}$ are the same set of solution as $\xi^{(1,2)}$ given in (10–11). It is easy to see that

$$C = A\exp(i\delta_1); \quad D = B\exp(i\Delta) \tag{43}$$

where

$$\Delta = \tan^{-1}(k/m) + \tan^{-1}(k'/2m)$$

and

$$\delta_1 = 2\tan^{-1}(k/m) \tag{44}$$

Notice that the undesirable L-dependent pieces in (42) are the same as in (27), and will cancel each other in (39). Using (39), (27) and (42–44) we have

$$\begin{aligned}\frac{\hbar}{2}\left(\sum_{\text{sol}}\omega_B - \sum_{\text{sol}}\omega_F\right) &= -\frac{\hbar}{4}\int_0^{\sqrt{3}m} \mathrm{d}k/2\pi\ (k^2+m^2)^{1/2}\mathrm{d}\delta_1/\mathrm{d}k \\ &\quad -\frac{\hbar}{4}\int_{\sqrt{3}m}^{\infty} \mathrm{d}k/2\pi\ (k^2+m^2)^{1/2}2(\mathrm{d}\Delta/\mathrm{d}k). \\ &= -\frac{3\hbar m}{2}\int_0^{\infty}\mathrm{d}k/2\pi\ \frac{1}{(k^2+m^2)^{1/2}}\end{aligned} \tag{45}$$

The L-dependent divergence is no longer there, but the UV divergence in (45) still needs to be removed. This is done as usual by the counter term, which for the supersymmetric case is

$$L_{\text{c.t.}} = \frac{\hbar C}{2} SS'' \tag{46}$$

instead of (29). Here again C is as given in (30). This gives the counter-term contribution to the soliton mass as

$$\Delta M_{\text{c.t.}}^{\text{susy}} = -\frac{\hbar C}{2}\int_{-\infty}^{\infty}\mathrm{d}x(SS'')_{\phi_s} = \frac{3}{2}Cm\hbar \tag{47}$$

Here again, unlike in (31) for the non-supersymmetric case, there are no L-dependent pieces.

Finally adding (45) and (47), we obtain the one-loop soliton mass for the supersymmetric system as

$$M = M_0 = m^3/3\lambda \tag{48}$$

It would appear that not only is the quantum correction to the mass free of all divergences, but is in fact zero. However, the result (48) is in terms of bare parameters m and λ. If (48) were re-expressed in terms of one-loop corrected boson mass and coupling constant, the mass would not have the classical form, but would carry finite corrections (Schonfeld 1979; Kaul and Rajaraman 1983).

Equation (45) involving the difference between the densities $\mathrm{d}n_+/\mathrm{d}k$ and $\mathrm{d}n_-/\mathrm{d}k$ could have been obtained directly by involving the Callias–Bott–Seeley trace theorem (Calias 1978; Bott and Seeley 1978, Imbimbo and Mukhi 1984; Kaul 1984). We have chosen instead to explicitly derive these densities separately, so that the L-dependence of the soliton energy and their cancellation may be made transparent.

It is evident from the principles used in our derivation, that this phenomenon—whereby a finite-energy classical soliton does *not* lead to a finite energy soliton sector upon quantising the Bose field—will happen not only for the triple-well case we have studied, but also for any potential $V(\phi)$ which has

(i) degenerate absolute minima, so that a static finite-energy classical soliton will exist in the first place; and
(ii) unequal curvatures at neighbouring absolute minima.

Indeed, among the full set of all possible potentials $V(\phi)$ having degenerate minima, this behaviour will be more the rule than the exception. The more familiar examples like the kink $[V(\phi) \sim (\phi^2 - 1)^2]$ or the sine-Gordon soliton $(V(\phi) \sim 1 - \cos \phi)$, where a finite-energy quantum soliton does emerge, are really special cases, enjoying some symmetry relating the different minima of $V(\phi)$. Of course, once the theory is supersymmetrised as per (3), quantum solitons regain finite energies in all the cases. This is yet another minor triumph for supersymmetry!

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