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Chirality of Knots 9_{42} and 10_{71} and Chern-Simons Theory.

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Abstract

Upto ten crossing number, there are two knots, 9_{42} and 10_{71} whose chirality is not detected by any of the known polynomials, namely, Jones invariants and their two variable generalisations, HOMFLY and Kauffman invariants. We show that the generalised knot invariants, obtained through $SU(2)$ Chern-Simons topological field theory, which give the known polynomials as special cases, are indeed sensitive to the chirality of these knots.

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1. Introduction

In knot theory, one associates a polynomial with each knot. These knot polynomials in most cases are capable of distinguishing isotopically different knots. But there are examples of distinct pairs of knots which have the same polynomial. An achiral knot is that which is ambient isotopic to its mirror image. Alexander polynomial $\Delta_K(q)$ [1] is the oldest known polynomial but it does not detect chirality for any knot K. Jones polynomial [2] $V_K(q)$ does distinguish most chiral knots. If the knot(K) has polynomial $V_K(q)$, then the mirror image knot (K^*) has polynomial $V_{K^*}(q) = V_K(q^{-1})$. For achiral knots we have $V_K(q) = V_K(q^{-1})$. If $V_K(q) \neq V_K(q^{-1})$, then the knot K is chiral. However the converse is not true. There are well known examples of chiral knots which have the polynomials symmetric under the exchange of $q \leftrightarrow q^{-1}$. The two variable generalisation of Jones polynomial known as HOMFLY polynomial $P_K(l, m)$[3] is a better invariant than Jones polynomial since it is able to distinguish even those chiral knots which are not distinguished from their mirror images by Jones polynomial[4]. For these polynomials the invariant for the mirror image is obtained by replacing l by l^{-1}. For example the arborscent knot whose chirality is detected by HOMFLY (i.e $P_K(l, m) \neq P_{K^*}(l, m) = P_K(l^{-1}, m)$) but not by Jones polynomial. Another two variable generalisation of Jones invariant known as Kauffman polynomial ($F_K(\alpha, q)$) [5]is shown to detect chirality of all those knots detected by HOMFLY[6]. This polynomial in fact is more powerful. For example, while the HOMFLY invariant does not detect chirality of knot 10_{48} (as listed in knot tables in Rolfsen's book[7]), but the Kauffman invariant does [6], i.e., $F_K(\alpha, q) \neq F_K(\alpha^{-1}, q)$ but $P_{10_{48}}(l, m) = P_{10_{48}}(l^{-1}, m)$.

Recently Chern-Simons field theory on arbitrary 3-manifolds has been used to study knot invariants([8]-[12]). Specifically the expectation values of Wilson loop operators, which are the observables of the theory, are topological invariants. Witten[8] has explicitly shown that Wilson loop operators associated with skein related links (V_+, V_-, V_0) in an $SU(2)$ Chern-Simons theory with doublet representation living on the knots satisfy the same skein recursion relation as the Jones polynomials. Further the recursion relation for the two-variable generalisation (HOMFLY) is obtained by considering N-dimensional representation living on the Wilson lines in $SU(N)$ Chern-Simons theory. Using $SO(N)$ Chern-Simons theory, Wu and Yamagishi[10] obtained the skein relation for Kauffman polynomial by placing the N dimensional representation of $SO(N)$ on the Wilson lines. Another approach for obtaining the link invariants involves statistical models and Yang-Baxter equations. Using this approach Akutsu and Wadati[13] obtained Jones polynomial and a class of generalisation through N state vertex models. In fact Wu and Yamgishi showed that the Kauffman polynomial for the group $SO(3)$ is same as that of Akutsu-Wadati polynomial obtained from 3 state vertex model.

Following Witten, we have developed techniques for obtaining new knot invariants by placing arbitrary representations on the Wilson lines in the $SU(2)$ or $SU(N)$ Chern-Simons theories[9]. When a 3-dimensional representation is placed on the lines of Wilson loops in an $SU(2)$ Chern-Simons theory, the polynomial coincides with Akutsu-Wadati polynomials derived from 3-state model. This is so because 3 dimensional representation of $SU(2)$ is same as the fundamental representation for $SO(3)$.

Though HOMFLY and Kauffman two-variable invariants are more powerful than those of Jones, yet there are examples of isotopically distinct knots which have the same polynomial. In particular, there are two knots, namely 9_{42} and 10_{71}, upto ten crossing number whose chirality is not detected by any of the well known polynomials, namely, Jones, HOMFLY and Kauffman. Using our direct method of evaluation from $SU(2)$ Chern-Simons theory, we have explicitly derived the knot polynomial formulae for 9_{42} and 10_{71} for arbitrary representations of $SU(2)$. Using macsyma package, a general algorithm has been written to compute these formulae. We have verified that the fundamental representation gives Jones polynomials and the 3 dimensional representation gives Akutsu-Wadati /Kauffman polynomials. The 4 dimensional representation gives polynomials which are not invariant under the transformation of the variable $q \leftrightarrow q^{-1}$. Hence these polynomials distinguish the knots 9_{42} and 10_{71} from their mirror images.

In sec.2, we give a brief account of the known polynomials for the knot 9_{42} and 10_{71}. In sec.3, we recapitulate the necessary ingredients of our direct method from $SU(2)$ Chern-Simons theory[9] and do the evaluation for these specific knots in detail. In sec.4, we summarize our results.

2. Known invariants for knots 9_{42} and 10_{71}.

As stated earlier, knots 9_{42} and 10_{71} are special knots whose chirality is not detected by any of the known polynomials, Jones, HOMFLY and Kauffman. We now list these polynomials for these knots.

We begin with the knot 9_{42}. This is a non-alternating knot with writhe -1. It has signature 2 and thus is a chiral knot. This knot can be represented as the closure of a four strand braid given in terms of the generators b_1, b_2, b_3 as a word: $b_1^3\ b_3\ b_2^{-1}\ b_3\ b_1^{-2}\ b_2^{-1}$. This is drawn in fig.1a. An equivalent representation has been drawn in fig.1b. Jones polynomial for this knot can be recursively obtained using the skein recursion relation [2]: $q^{-1}V_+ \ - \ qV_- \ = \ (q^{-1/2} - \ q^{1/2})V_0$. Here V_-, V_+, V_0 denote the polynomials for the skein-triplet. In the case here, V_- corresponds to 9_{42}, V_+ represents the unknot obtained by changing the encircled undercrossing of fig.1b to overcrossing and V_0 obtained by changing the undercrossing to no crossing. The Jones polynomial for 9_{42} with the normalisation for unknot as $V_U(q) \ = \ 1$ can easily be worked out [2]:

$$V_{9_{42}}(q) = q^{-3} - q^{-2} + q^{-1} - 1 + q - q^2 + q^3 \tag{1}$$

Notice that $V_{9_{42}}(q) = V_{9_{42}}(q^{-1})$ and hence this polynomial does not detect the chirality of 9_{42}.

In a similar fashion, we can use the skein relation for the 2-variable generalization of Jones invariant referred to as HOMFLY polynomial to get[4]:

$$P_{9_{42}}(l, m) = (-2l^{-2} - 3 - 2l^2) + (l^{-2} + 4 + l^2)m^2 - m^4 \tag{2}$$

Clearly the polynomial is invariant under the transformation of $l \to l^{-1}$ which relates the knot invariant to that of its mirror image. Hence this polynomial also does not distinguish 9_{42} from its mirror image.

Kauffman has obtained another two-variable generalization of Jones invariant through a new recursion relation for unoriented knots or links. This polynomial for knot 9_{42} is [5]:

$$F_{9_{42}}(a,z) = (a+a^{-1})z^7 + (a^1+a^{-1})^2 z^6 - 5(a+a^{-1})z^5 - 5(a^1+a^{-1})^2 z^4$$
$$+ 6(a+a^{-1})z^3 + (6a^2+12+6a^{-2})z^2 - 2(a+a^{-1})z^1 - (2a^2+3+2a^{-2})z^0 \quad (3)$$

Kauffman polynomials for mirror reflected knots are obtained by $a \leftrightarrow a^{-1}$. Clearly, the invariant in eqn(3) does not change under this transformation and hence even this polynomial is not powerful enough.

For the special case of $a = -q^{-\frac{3}{4}}$ and $z = q^{-\frac{1}{4}} + q^{\frac{1}{4}}$ this polynomial reduces to the Jones polynomial. Also Kauffman invariants, in general reduce to Akutsu-Wadati polynomial for $a = iq^2$ and $z = -i(q-q^{-1})$. In their original calculations obtained from 3-state vertex model, Akutsu, Deguchi and Wadati[14] have presented these polynomials for knots with representation in terms of closure of three strand braids. The Knot 9_{42}, however, has a representation in terms of closure of a four strand braid. Thus substituting $a = iq^2$ and $z = -i(q-q^{-1})$ in eqn(3) should yield us Akutsu-Wadati polynomial for this knot:

$$F_{9_{42}}(iq^2, -i(q-q^{-1})) = q^{-10} - q^{-9} - q^{-8} + 2q^{-7} - q^{-6} - q^{-5} + 2q^{-4} - q^{-3} + q^{-1}$$
$$- 1 + q^1 - q^3 + 2q^4 - q^5 - q^6 + 2q^7 - q^8 - q^9 + q^{10} \quad (4)$$

Now let us list the known polynomials for the other chiral knot we study here, 10_{71}. This is an alternating knot with writhe number zero and also signature zero. Its knot diagram as given in Rolfsen's book[7] is drawn in fig.2. Its Jones polynomial is [2]:

$$V_{10_{71}}(q) = -q^5 + 3q^5 - 6q^3 + 10q^2 - 12q + 13 - 12q^{-1} + 10q^{-2} - 6q^{-3} + 3q^{-4} - q^{-5} \quad (5)$$

The Kauffman two-variable polynomial for this knot can be obtained readily from Kauffman's recursion relations. Such an exercise leads us to:

$$F_{10_{71}}(a,z) = (a+a^{-1})z^9 + (3a^2+6+3a^{-2})z^8 + (4a^3+8a+8a^{-1}+4a^{-3})z^7$$

$$+(3a^4+2a^2-2+2a^{-2}+3a^{-4})z^6+(a^5-5a^3-15a-15a^{-1}-5a^{-3}+a^{-5})z^5$$
$$-(6a^4+12a^2+12+12a^{-2}+6a^{-4})z^4+(-2a^{-5}+7a^{-1}+7a^{-1}-2a^{-5})z^3$$
$$+(4a^4+10a^2+12+10a^{-2}+4a^{-4})z^2+(a^5+a^3-a-a^{-1}+a^{-3}+a^{-5})z^1$$
$$-(a^4+3a^2+3+3a^{-2}+a^{-4})z^0 \qquad (6)$$

Again, when we substitute $a=iq^2$, $z=-i(q-q^{-1})$, we obtain the Akutsu-Wadati invariant:

$$\begin{aligned} F_{10_{71}}(iq^2,-i(q-q^{-1})) &= q^{15}-3q^{14}+q^{13}+9q^{12}-17q^{11}+q^{10}+37q^9-47q^8-12q^7 \\ &+89q^6-77q^5-42q^4+140q^3-87q^2-73q+161-73q^{-1} \\ &-87q^{-2}+140q^{-3}-42q^{-4}-77q^{-5}+89q^{-6}-12q^{-7}-47q^{-8} \\ &+37q^{-9}+q^{-10}-17q^{-11}+9q^{-12}+q^{-13}-3q^{-14}+q^{-15} \end{aligned} \qquad (7)$$

Clearly, none of these invariants distinguishes 10_{71} from its mirror image.

All these invariants can also be obtained from Chern-Simons theories. Such theories, besides these yield a whole variety of new invariants which are powerful enough to detect the chirality of knots 9_{42} and 10_{71}.

3. Invariants through Chern-Simons theory.

To evaluate new Chern-Simons invariants of knots 9_{42} and 10_{71}, we shall now describe briefly the necessary aspects of the method[8, 9].

The metric independent action of Chern-Simons theory in a 3-manifold is given by

$$kS=\frac{k}{4\pi}\int tr(AdA+\frac{2}{3}A^3) \qquad (8)$$

where A is matrix valued gauge connection of the gauge group G which for our present discussion is $SU(2)$. The gauge invariant operators of this topological field theory are given in terms of Wilson loop(knot) operator:

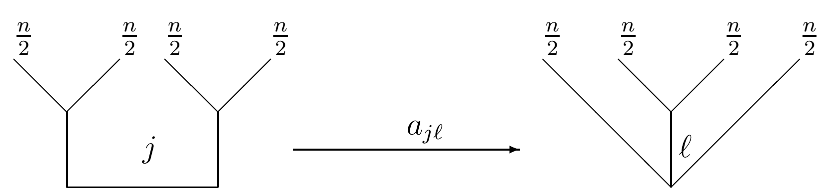
$$W_R[C]=tr_R P\exp\oint_C A$$

for an oriented knot C carrying representation R of the group G. The vacuum expectation value for this operator is given in terms of the functional integral:

$$V_R[C] = \langle W_R[C] \rangle = \frac{\int [dA] W_R[C] \exp ikS}{\int [dA] \exp ikS} \tag{9}$$

Being obtained from a topological field theory this quantity depends only on the topological properties of the knot and not on the geometric properties.

The knot invariants in eqn.(9) can be calculated by using a close relationship between Chern-Simons theories on a three-manifold with boundary and the corresponding Wess-Zumino conformal field theory on that boundary[8]-[11]. The Chern-Simons functional integral over a manifold is represented as a vector in the Hilbert space of the conformal blocks associated with the boundary. For example, for a three-ball containing two Wilson lines as shown in fig.3a, the functional integral is a vector $|\psi_0\rangle$ in the Hilbert space $\mathcal{H}$ associated with boundary, four-punctured S^2. Functional integral over the same ball but with opposite orientation of its boundary (fig.3b) is represented by the state $\langle\psi_0|$ in the dual Hilbert space $\bar{\mathcal{H}}$. The dimension of the space is given by the number of 4-point conformal blocks on S^2. For our purpose where spin $\frac{n}{2}$ ($n+1$ dimensional representation) live on the punctures, the dimensionality of the Hilbert space is $min(n, k-n)+1$. These states can be expanded in terms of a complete set of basis. Two convenient choices of these bases are those in which the braid matrix for side two strands or central two strands is diagonal, $|\phi_l^{side}\rangle$ or $|\phi_l^{cent}\rangle, l = 0, 1, \cdots, min(n, k-n)$. These basis vectors correspond to two equivalent sets of conformal blocks for four-point correlators in $SU(2)_k$ Wess-Zumino model. These bases are related to each other by orthogonal duality matrices as depicted below. The duality matrices $a_{j\ell}$ are given in terms of quantum Racah coefficients as [9]:



where

$$a_{j\ell} = (-)^{\ell+j-n}\sqrt{[2j+1][2\ell+1]}\begin{pmatrix} n/2 & n/2 & j \\ n/2 & n/2 & \ell \end{pmatrix}$$

and the quantum Racah coefficients are given by

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{pmatrix} &= \Delta(j_1,j_2,j_{12})\Delta(j_3,j_4,j_{12})\Delta(j_1,j_4,j_{23})\Delta(j_3,j_2,j_{23}) \\ & \sum_{m\geq 0}(-)^m[m+1]!\Big\{[m-j_1-j_2-j_{12}]! \\ & [m-j_3-j_4-j_{12}]![m-j_1-j_4-j_{23}]! \\ & [m-j_3-j_2-j_{23}]![j_1+j_2+j_3+j_4-m]! \\ & [j_1+j_3+j_{12}+j_{23}-m]![j_2+j_4+j_{12}+j_{23}-m]!\Big\}^{-1}\end{aligned}$$

and

$$\Delta(a,b,c) = \sqrt{\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!}}$$

Here $[a]! = [a][a-1][a-2]...[2][1]$. The $SU(2)$ spins are related as $\vec{j}_1 + \vec{j}_2 + \vec{j}_3 = \vec{j}_4$, $\vec{j}_1 + \vec{j}_2 = \vec{j}_{12}$, $\vec{j}_2 + \vec{j}_3 = \vec{j}_{23}$ subject to the fusion rules of $SU(2)_k$ conformal field theory. Here the number in square bracket is the q-number defined as $[n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$. The parameter q is related to the coupling constant k of the Chern-Simons theory as : $q = \exp(\frac{2\pi i}{k+2})$.

The state $|\psi_0\rangle$ and its dual $\langle\psi_0|$ corresponding to the fig.3a and fig.3b respectively can be written in the eigen basis discussed above as follows:

$$|\psi_0\rangle = \sum_{l=0}\sqrt{[2l+1]}|\phi_l^{side}\rangle = [n+1]|\phi_0^{cent}\rangle \tag{10}$$

$$\langle\psi_0| = \sum_{l=0}\sqrt{[2l+1]}\langle\phi_l^{side}| = [n+1]\langle\phi_0^{cent}| \tag{11}$$

Now, glueing these two figures along their oppositely oriented boundaries gives two disjoint unknots and the link invariant given in eqn.(9) is represented by the inner product $\langle\psi_0|\psi_0\rangle$. This gives the unknot polynomial for the Wilson lines carrying spin $\frac{n}{2}$ representation to be: $V_n[U] = [n+1]$.

Another building block for our calculation is the functional integral over an S^3 from which two three-balls have been carved out. The two boundaries so formed are connected by four Wilson lines as shown in fig.4. The Chern-Simons functional integral over this three-manifold operates like an identity and is given in terms of the basis vectors $|\phi_i^{(1)}\rangle$ and $|\phi_i^{(2)}\rangle$ refering to the two boundaries as[9]:

$$\nu_2 = \sum_i |\phi_i^{(1)}\rangle|\phi_i^{(2)}\rangle \tag{12}$$

Here the summation runs from 0 to minimum of n and $k-n$ for spin $\frac{n}{2}$ representation living on the Wilson lines. In eqn.(12) the basis vectors $|\phi_i^{(1)}\rangle$ and $|\phi_i^{(2)}\rangle$ can both refer to either the side strands or the central strands.

Yet another useful functional integral is over an S^3 with three three-balls removed from it. The consequent three boundaries (each an S^2) are connected by Wilson lines as shown in fig.5. The Chern-Simons functional integral over this manifold is given by[9]:

$$\nu_3 = \sum_{i,j,l,m=0} \frac{1}{\sqrt{[2m+1]}} a_{im}a_{jm}a_{lm}|\phi_i^{(1)cent}\rangle|\phi_j^{(2)cent}\rangle|\phi_l^{(3)cent}\rangle \tag{13}$$

$$= \sum_{m=0} \frac{1}{\sqrt{[2m+1]}} |\phi_m^{(1)side}\rangle|\phi_m^{(2)side}\rangle|\phi_m^{(3)side}\rangle \tag{14}$$

where superscripts $1,2,3$ refers to the conformal block bases on the three S^2 boundaries $1,2,3$.

In the functional integral corresponding to fig.3, fig.4 and fig.5 we can introduce any number of braids in various strands through the braiding matrix. The braiding matrix that introduces half-twist in the side two strands is diagonal in the basis $|\phi_l^{side}\rangle$. On the other hand braid matrix that twists the central two strands is diagonal in the basis $|\phi_l^{cent}\rangle$. The eigenvalues for these braiding matrices depend on the relative orientation of the strands they twist. These eigenvalues are obtained from conformal field theory and are given for right handed half-twists in parallely and antiparallely oriented strands respectively by [9]:

$$\lambda_l^{(+)} = (-1)^{n-l} q^{(n(n+2)-l(l+1))/2}$$

$$\lambda_l^{(-)} = (-1)^l q^{l(l+1)/2}, \quad l \; = \; 0, 1, \cdots, min(n, k-n)$$

The properties of Chern-Simons functional integrals listed above can now be directly used to compute the invariants for knot 9_{42} and 10_{71}. We begin with knot 9_{42}. Split the knot as represented in fig.1b at marked points 1 to 4 by vertical planes. This breaks the manifold into five pieces as shown in fig.6a-6e. The functional integral over each one of them can now be readily computed. For example, the functional integral over the manifold with one boundary in fig.6a, can be obtained by half-twisting the side two strands of fig.3a three times to yield:

$$\nu_1(P_1) = \sum_{l_1=0} \sqrt{[2l+1]}(-1)^{3(n-l_1)} q^{-3/2[n(n+2)-l_1(l_1+1)]} |\phi_{l_1}^{(1)}\rangle \tag{15}$$

Similarly, the functional integral over the manifold in fig.6e, again with one boundary is:

$$\nu_1(P_4) = \sum_{l_5=0} (-1)^{n-l_5} q^{-1/2[n(n+2)-l_5(l_5+1)]} |\phi_{l_5}^{(1)}\rangle \tag{16}$$

The rest of the three manifolds have two boundaries. These can be evaluated from fig.4 and fig.5 and using the duality relationship between the two sets of bases [9]. The state corresponding to fig.6b is

$$\nu_2(P_1; P_2) = \sum_{i_1, j_1, l_2, r=0} \frac{a_{l_1 r} a_{j_1 r} a_{l_2 r} \sqrt{[2l_2+1]}}{\sqrt{[2r+1]}} q^{n(n+2)-l_2(l_2+1)} |\phi_{i_1}^{(1)}\rangle |\phi_{j_1}^{(2)}\rangle \tag{17}$$

The functional integral corresponding to fig.6c is

$$\nu_2(P_2; P_3) = \sum_{l_3=0} q^{l_3(l_3+1)} |\phi_{l_3}^{(1)}\rangle |\phi_{l_3}^{(2)}\rangle \tag{18}$$

Similarly the fig.6d corresponds to

$$\nu_2(P_3; P_4) = \sum_{i_2, j_2, l_4=0} (-1)^{l_4} q^{-l_4(l_4+1)/2} a_{l_4 i_2} a_{l_4 j_2} |\phi_{i_2}^{(1)}\rangle |\phi_{j_2}^{(2)}\rangle \tag{19}$$

Glueing these five pieces (fig.6a-6e) along the appropriate oppositely oriented boundaries gets us back to knot 9_{42} in S^3. The final result is :

$$V_n[9_{42}] = (-1)^n q^{-3/2n(n+2)} \sum_{r,l_1,l_2,j_1,j_2=0} \sqrt{[2l_1+1]}\sqrt{[2l_2+1]}\sqrt{[2j_2+1]}$$
$$a_{l_1 r} a_{l_2 r} a_{j_1 r} a_{j_1 j_2} (-1)^{l_1} q^{3/2 l_1(l_1+1)} q^{3/2 j_1(j_1+1)} q^{-l_2(l_2+1)} q^{j_2(j_2+1)} \tag{20}$$

for spin $\frac{n}{2}$ representation living on the Wilson lines. In obtaining the final result (20), the following identity involving the q-Racah coefficient has been used[9]:

$$\sum_{l_4} (-1)^{n-l_4} q^{(n(n+2)-l_4(l_4+1))/2} a_{j_1 l_4} a_{j_2 l_4} = (-1)^{j_1+j_2} q^{(j_1(j_1+1)+j_2(j_2+1))/2} a_{j_1 j_2} \tag{21}$$

The invariant (20) can be evaluated explicitly for definite values of spin $\frac{n}{2}$. This has been done on computer by using macsyma package. The results are:

a) For $n=1$, we get

$$V_1[9_{42}] = q^{7/2} + q^{-7/2}$$

This is same as eq.(1) when divided by the unknot polynomial $V_1[U] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$.

b) For $n=2$, we get

$$V_2[9_{42}] = q^{11} - q^9 + q^3 + 1 + q^{-3} - q^{-9} + q^{-11}$$

This is same as Akutsu-Wadati/Kaufmann polynomial (4) when divided by the unknot polynomial $V_2[U] = q^{-1} + 1 + q$.

c) For $n=3$, the polynomial is

$$V_3[9_{42}] = q^{45/2} - q^{41/2} - q^{39/2} + q^{35/2} + q^{23/2} + q^{21/2} - q^{19/2} - q^{17/2} + q^{13/2} - q^{9/2}$$
$$+ q^{5/2} + q^{3/2} + q^{-3/2} + q^{-5/2} - q^{-13/2} - q^{-15/2} + q^{-21/2} + 2q^{-23/2}$$
$$- q^{-27/2} + 2q^{-31/2} - 3q^{-35/2} - q^{-37/2} + q^{-39/2} + q^{-41/2} \tag{22}$$

Let us now study the knot 10_{71}. The Chern-Simons functional integral for this knot can be obtained from the functional integrals over four 3-manifolds I, II, III and IV as shown in fig.7. The three 3-manifolds I, II and III are three-balls with one boundary each (S^2). The manifold IV has three boundaries, each an S^2. Glueing these pieces together along their appropriate boundaries as shown in fig.7 yields knot 10_{71} in S^3. The rules of obtaining Chern-Simons functional integrals stated above can now readily be applied to obtain these functional integrals. The final answer for the knot invariant is:

$$V_n[10_{71}] = (-1)^n q^{\frac{n(n+2)}{2}} \sum_{i,r,s,u,m=0} \sqrt{\frac{[2r+1][2s+1][2u+1]}{[2m+1]}} a_{im} a_{ms} a_{rm} a_{iu}$$
$$(-1)^s \; q^{-i(i+1)} q^{m(m+1)} q^{-r(r+1)} q^{u(u+1)} q^{\frac{3}{2}s(s+1)} \tag{23}$$

Here the identity given in eqn.(21) has again been used.

This invariant has been evaluated for specific values of the spin $\frac{n}{2}$ by macsyma package. For $n = 1$ and $n = 2$, it reduces to the Jones and Akutsu-Wadati/Kauffman polynomials (eqn.5 and 7) upto the normalisation factor $V_1[U] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$ and $V_2[U] = q + 1 + q^{-1}$ respectively. For n=3, we have the polynomial:

$$\begin{aligned} V_3[10_{71}] \;\; = \;\; & -q^{63/2} + 2q^{61/2} + q^{59/2} - 3q^{57/2} - 4q^{55/2} + 7q^{53/2} + 10q^{51/2} - 15q^{49/2} \\ & -21q^{47/2} + 22q^{45/2} + 44q^{43/2} - 25q^{41/2} - 79q^{39/2} + 17q^{37/2} + 119q^{35/2} \\ & +8q^{33/2} - 150q^{31/2} - 54q^{29/2} + 172q^{27/2} + 99q^{25/2} - 166q^{23/2} - 144q^{21/2} \\ & +137q^{19/2} + 174q^{17/2} - 95q^{15/2} - 180q^{13/2} + 46q^{11/2} + 167q^{9/2} + 7q^{7/2} \\ & -138q^{5/2} - 56q^{3/2} + 101q^{1/2} + 101q^{-1/2} - 56q^{-3/2} - 138q^{-5/2} \\ & +7q^{-7/2} + 168q^{-9/2} + 46q^{-11/2} - 182q^{-13/2} - 96q^{-15/2} + 175q^{-17/2} \\ & +138q^{-19/2} - 144q^{-21/2} - 165q^{-23/2} + 99q^{-25/2} + 171q^{-27/2} - 54q^{-29/2} \\ & -148q^{-31/2} + 8q^{-33/2} + 115q^{-35/2} + 16q^{-37/2} - 77q^{-39/2} - 23q^{-41/2} \\ & +44q^{-43/2} + 21q^{-45/2} - 21q^{-47/2} - 15q^{-49/2} + 10q^{-51/2} + 7q^{-53/2} \\ & -4q^{-55/2} - 3q^{-57/2} + q^{-59/2} + 2q^{-61/2} - q^{-63/2} \end{aligned} \tag{24}$$

Clearly, unlike Jones, HOMFLY and Kauffman/Akutsu-Wadati polynomials for knot 9_{42} and 10_{71}, the spin $\frac{3}{2}$ polynomials for these knots (eqns. 22 and 24 respectively) do indeed change under chirality tranformation $q \leftrightarrow q^{-1}$.

4. Concluding Remarks

It appears that spin $\frac{1}{2}$, 1, $\frac{3}{2}$,$\cdots$ polynomials are progressively more powerful. Upto ten crossing, there are six chiral knots (9_{42}, 10_{48}, 10_{71},10_{91}, 10_{104} and 10_{125}) which are not distinguished from their mirror images by Jones or its two-variable generalisation, HOMFLY polynomial. Kauffman /Akutsu-Wadati polynomial is more powerful. It does detect chirality of the knots 10_{48}, 10_{91}, 10_{104} and 10_{125}, but not of knots 9_{42} and 10_{71}. We have demonstrated that the new polynomials obtained from $SU(2)$ Chern-Simons theory corresponding to spin $\frac{3}{2}$ living on the knot is powerful enough to distinguish even these knots from their mirror images.

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Figure Captions:

Fig.1 Knot 9_{42} as (a) the closure of a four-strand braid and (b) an equivalent representation.

Fig.2 Knot 10_{71}

Fig.3 Diagrammatic representation of (a) state $|\psi_0\rangle$ and (b) its dual $\langle\psi_0|$ in terms of three-balls.

Fig.4 Diagrammatic representation of the functional integral ν_2 for a manifold with two boundaries.

Fig.5 Diagrammatic representation of the functional integral ν_3 for a manifold with three boundaries.

Fig.6 Knot 9_{42} obtained by glueing five building blocks, (a)–(e), with suitable entanglements.

Fig.7 Knot 10_{71} obtained by glueing four building blocks, (I)–(IV), with suitable braidings.