Black Hole Entropy from a Highly Excited Elementary String [*]

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Suggested correspondence between a black hole and a highly excited elementary string is explored. Black hole entropy is calculated by computing the density of states for an open excited string. We identify the square root of oscillator number of the excited string with Rindler energy of black hole to obtain an entropy formula which, not only agrees at the leading order with the Bekenstein-Hawking entropy, but also reproduces the logarithmic correction obtained for black hole entropy in the quantum geometry framework. This provides an additional supporting evidence for correspondence between black holes and strings.

It is now more than a decade since 't Hooft suggested a complementarity between black holes and strings [1]. Black hole horizon is governed by some conformal operator algebra on a two-dimensional surface. So is a string. These two may be equally fundamental pictures, related by a complementarity. As emphasized by 't Hooft, it may be possible to provide a black hole interpretation of strings. Also conversely, black holes will have a string representation. This idea is substantially further developed by Susskind's suggestion that the spectrum of a Schwarzschild black hole is described by the states of a highly excited (uncharged) string at Hagedorn temperature [2]. This allows a determination of the black hole entropy by counting the number of states of an excited string. Thus we have a statistical interpretation of the black hole entropy. More evidence of this correspondence has been subsequently presented in refs. [3–6] and many others, where a variety of cases of black holes, non-rotating and rotating, uncharged and charged, in different dimensions are studied from this perspective. In particular, Bekenstein-Hawking entropy of a Schwarzschild black hole in four and also arbitrary dimensions is reproduced from the density of states of an excited string.

The correspondence principle for the two spectra, of a black hole and a highly excited elementary string, may be understood as follows. As the string coupling g_{st} ($g_{st}^2 = (\ell_P/\ell_S)^{d-2}$ in d dimensions, ℓ_P is the Planck length and ℓ_S is the string length scale) increases, the Compton wave length of a high mass and low angular momentum string state shrinks to less than its Schwarzschild radius to become a black hole. On the other hand, as the coupling is reduced, the black hole eventually becomes smaller than the string size. The metric near the horizon then loses its meaning and instead of being a black hole such a system is better described as a string state. But at some intermediate size, when the black hole size and string size are equal, either description is admissible. This would imply a one-to-one correspondence between the spectra of black holes and strings [2].

At first this correspondence between the two sets of states appears to be beset with a difficulty. As functions of mass, there is an apparent striking difference between the black hole density of states and the density of string states. The latter in any dimensions grows exponentially in the first power of mass M of the excited string [7]. This implies string entropy as linearly proportional to mass M, $S_{st} \sim \sqrt{\alpha'}\, M \sim \sqrt{N}$, where α' is the inverse of the string tension and N is oscillator occupation number of the excited string state. In contrast, the density of states of a black hole grows exponentially with the second power of mass in four dimensions (and in general as $M^{(d-2)/(d-3)}$ in d dimensions). The corresponding entropy then is proportional to square of the mass in four dimensions (or to $M^{(d-2)/(d-3)}$ in arbitrary d dimensions). As suggested by Susskind [2], this apparent discrepancy can be cured by a proper identification that takes in to account a large mass-renormalization, a gravitational redshift. To do this use is made of the fact that the near horizon geometry of a Schwarzschild black hole is a Rindler space with a dimensionless time τ_R and a dimensionless energy E_R. The Rindler mass and ADM mass of a black hole are related by a huge redshift between the stretched horizon and asymptotic infinity. In d dimensions ($d \geq 4$) the dimensionless Rindler energy associated with a Schwarzschild black hole is given by [2,3]

$$E_R \;=\; \left(\frac{2}{d-2}\right) M^{(d-2)/(d-3)} \left(\frac{16\pi G}{(d-2)\; A_{d-2}}\right)^{1/(d-3)} \;=\; \left(\frac{2}{d-2}\right) M\; r_{BH}, \tag{1}$$

where A_{d-2} is the area of a unit sphere of $d-2$ dimensions, G is the Newton's constant and r_{BH} is the Schwarzschild radius associated with mass M. In terms of Rindler energy, the horizon 'area' A_H and Bekenstein-Hawking entropy S_{BH} of a Schwarzschild black hole in any arbitrary dimensions is given by

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$$A_H \;=\; 8\pi G E_R, \qquad\qquad S_{BH} \;=\; 2\pi E_R \;. \tag{2}$$

That is, black hole density of states grows exponentially with Rindler energy in any dimensions. The *string⇔black-hole* correspondence is then obtained when Rindler energy E_R is identified with the square root of the oscillator number $\sqrt{N}$ of the highly excited string in *any dimension*. In particular, in four dimensions, $N \sim E_R^2 \sim G^2 M^4$. This identification, then gives the same leading order expression for the density of black hole states and that for the string states allowing equality of Bekenstein-Hawking entropy of a black hole with that of an excited string in the leading order. Further support for this correspondence is the consequent identification of Hawking temperature of the black hole with red-shifted Hagedorn temperature associated with the excited string. Equivalently, the black hole size r_{BH} will get identified with the string length scale l_S.

If the correspondence *black-hole⇔string*, with the identification of Rindler energy (E_R) with the square root of the oscillator occupation number $(\sqrt{N})$ of highly excited string, is strictly true, all other features of black holes must get reflected in the string description too. In this context, $ln(area)$ corrections to the Bekenstein-Hawking entropy, first discovered in the quantum geometry framework, is of interest. In this framework, boundary degrees of freedom of a black hole in four dimensional gravity are described by an $SU(2)$ Chern-Simons theory on the horizon, with coupling k proportional to the horizon area. The dimensionality of boundary Hilbert space can thus be readily computed by counting the conformal blocks of $SU(2)_k$ conformal field theory on a two-sphere (spatial slice of the horizon) with a number of punctures carrying $SU(2)$ spin representations on them [8]. Then for large horizon area A_H, the black hole entropy S_{bh} has been found to be [9,10]

$$S_{bh} \;=\; \frac{A_H}{4G} \;-\; \frac{3}{2}\, ln\left(\frac{A_H}{4G}\right) \;+\; \cdots\cdots \tag{3}$$

There have also been other subsequent derivations of this $ln(area)$ correction with the same coefficient $-3/2$ [11–14]. It appears to be universal in the sense that it obtains for a variety of black holes and also in different dimensions.

Thus, if *black-hole⇔string* correspondence is indeed true, the same correction as in eqn.(3) should be reflected in the entropy of a highly excited string also. Over years, the counting of string states has been done in many places. In the following, we shall recalculate *the level density of a highly excited open string beyond the leading order carefully taking into account contribution of the zero modes.* It will be demonstrated that the string entropy does indeed receive a correction $-3/2 \, ln\sqrt{N}$ beyond the leading value proportional to $\sqrt{N}$. This may then be taken as an additional evidence in support of the *black-hole⇔string* correspondence.

An open string moving in a d dimensional space-time is described by d two-dimensional fields $X^\mu(\sigma,\tau)$, $\mu = 0,1,2,....(d-1)$ with $0 \le \sigma \le \pi$, satisfying the string equation $\partial^2 X^\mu(\sigma,\tau) - \partial^2 X^\mu(\sigma,\tau) = 0$ subject to boundary conditions reflecting no flow of momentum from the string ends: $\partial_\sigma X^\mu(\sigma,\tau) = 0$ at $\sigma = 0$ and $\sigma = \pi$. The string equation is satisfied by

$$X^\mu \;=\; x^\mu \;+\; \ell_S^2\, p^\mu\, \tau \;+\; i\ell_S \sum_{n\ne 0} \frac{1}{n}\, \alpha_n^\mu\, e^{-in\tau}\, cosn\sigma$$

where $\ell_S = \sqrt{2\alpha'} = 1/\sqrt{\pi T}$ is the fundamental string length (our units are $\hbar = c = 1$), T is string tension. The center of mass coordinate is x^μ and string momentum is p^μ. The commutation relations satisfied by the various operators are

$$[\, x^\mu \,,\, p^\mu \,] \;=\; i\eta^{\mu\nu}, \qquad [\, \alpha_m^\mu \,,\, \alpha_n^\nu \,] \;=\; m\, \delta_{m+n,0}\, \eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ is the flat metric in d dimensional space-time. This theory has a reparametrization invariance $(\sigma,\tau) \rightarrow (\sigma',\tau')$ generated by Virasoro constraints: $(\partial_\tau X^\mu \pm \partial_\sigma X^\mu)^2 = 0$. This may be fixed by a gauge choice, in particular by light-cone gauge introducing a preferred longitudinal direction in space: $X^+ = x^+ + \ell_S^2\, p^+\, \tau$, where the light-cone coordinates are $X^\pm = (X^0 \pm X^{d-1})/\sqrt{2}$. In this gauge all the oscillators $\alpha_n^+ = 0$ $(n \ne 0)$ and also the coordinates $X^-(\sigma,\tau)$ and hence p^- and α_n^- are not independent. Only independent variables are transverse coordinates, momenta and oscillators: x^i, p^i, α_n^i $(n \ne 0)$ with $i = 1, 2, 3,....d-2$. The τ-translation generating Hamiltonian is given by

$$H \;=\; \ell_S^2\, p^+ p^- \;=\; \frac{1}{2}\, \ell_S^2\, (p^i)^2 \;+\; \mathcal{N} \;-\; a \tag{4}$$

where $a = (d-2)/24$ is the normal ordering constant and $\mathcal{N} = \sum_{i=1}^{d-2} \sum_{m=1}^{\infty} \alpha_{-m}^i\, \alpha_m^i$ is the oscillator number operator whose eigenvalues are the occupation number of the state, $\mathcal{N} \mid \psi_N >= N \mid \psi_N >$.

It is convenient to introduce the standard oscillators:

$$a_m^{i\,\dagger} = \frac{1}{\sqrt{m}}\alpha_{-m}^i, \qquad a_m^i = \frac{1}{\sqrt{m}}\alpha_m^i \qquad m > 0$$

which satisfy the standard oscillator commutation relations:

$$[\,a_m^i,\ a_n^{j\,\dagger}\,] = \delta_{mn}\,\delta^{ij}.$$

The occupation number operator in terms of number operators $\mathcal{N}_m$ for these standard oscillators is

$$\mathcal{N} = \sum_{m=1}^{\infty} m\,\mathcal{N}_m \equiv \sum_{m=1}^{\infty}\sum_{i=1}^{d-2} m\,a_m^{i\,\dagger}a_m^i, \tag{5}$$

where $\mathcal{N}_m$ has the standard oscillator eigenvalues 0, 1, 2, 3......

The mass of an excited string of level N is given by $\ell_S^2 M_N^2 = 2(N - a)$. To count the quantum states of such a string, we write the partition function as a function of a complex parameter τ as (we set $\ell_S = 1$ in the following):

$$Z(\tau) = Tr\ e^{2\pi i H \tau} = Tr\ exp\left[2\pi i\tau\left(\frac{(p^i)^2}{2} + \mathcal{N} - a\right)\right]. \tag{6}$$

Here Tr represents integration over the transverse string momentum p^i and trace over the oscillator states. That is,

$$Z(\tau) = \int (d^{d-2}p^i)\,e^{\pi i\tau(p^i)^2} tr\ exp\left[2\pi i\tau\left(\mathcal{N} - a\right)\right] = \left(\frac{1}{-i\tau}\right)^{(d-2)/2} e^{-2\pi i\tau a}\ tr\ exp\left[2\pi i\tau\mathcal{N}\right], \tag{7}$$

where now tr represents trace over the oscillator states only. Notice that

$$tr\ e^{2\pi i\tau\mathcal{N}} = \sum_{m=1}^{\infty} e^{2\pi im\tau\mathcal{N}_m} = \left(\sum_{N=1}^{\infty} p(N)e^{2\pi iN\tau}\right)^{d-2}$$

where $p(N)$ is the number of partitions of N in terms of positive integers. Now there is a standard formula in number theory:

$$f^{-1}(\tau) \equiv \sum_{N=1}^{\infty} p(N)\,e^{2\pi iN\tau} = \prod_{n=1}^{\infty}\left(1 - e^{2\pi in\tau}\right)^{-1}.$$

The function $f(\tau)$ is related to Dedekind eta function as: $\eta(\tau) = exp(i\pi\tau/12)\,f(\tau)$. Thus the partition function is

$$Z(\tau) = \left(\frac{1}{-i\tau}\right)^{(d-2)/2} \frac{1}{[\eta(\tau)]^{d-2}}\ .$$

Next note Dedekind eta function has the property: $\eta(-1/\tau) = (-i\tau)^{1/2}\,\eta(\tau)$. Using this, the partition function can be written as

$$Z(\tau) = \frac{1}{[\eta(-1/\tau)]^{d-2}} = \frac{e^{2\pi ia/\tau}}{[f(-1/\tau)]^{d-2}} \tag{8}$$

To find the density $d(N)$ of string states with occupation number N, we write

$$Z(\tau) = \sum_{N=0}^{\infty} d(N)\,e^{2\pi i(N - a)\tau}\ . \tag{9}$$

Equating the two expressions in eqns.(8) and (9) and inverting for the level density, we have

$$d(N) = \int d\tau\ \frac{exp\left[-2\pi i\left(\,(N - a)\tau - (a/\tau)\right)\right]}{[f(-1/\tau)]^{d-2}}\ . \tag{10}$$

For large N, this can now be approximately evaluated by the saddle point method. The saddle point is at $\tau_0 = i\sqrt{a/(N - a)}$. Expanding around this point $\tau = \tau_0 + u$ and performing the Gaussian integration over u yields the level density as

$$d(N) \simeq \left(\frac{a^{1/2}}{2(N-a)^{3/2}} \right)^{1/2} exp\left(4\pi \sqrt{a(N-a)} \right),$$

where $f(-1/\tau_0) \to 1$ for large N has been used. Finally density of string states for large occupation number N can be asymptotically written as:

$$d(N) \simeq C \frac{a\, exp\left(4\pi \sqrt{aN} \right)}{(aN)^{3/4}} + \cdots \cdots , \tag{11}$$

where C is an N independent irrelevant constant and $a = (d-2)/24$. This is the same formula as that obtained by Carlip for density of states with eigenvalues of Virasoro operator L_0 as $\Delta = N - a$ in a general rational conformal field theory of central charge $c = 24a$ [11]. The asymptotic level density of string states has been calculated in many places, the earliest computation for a string in $d = 26$ dimensions was done in refs. [7] where the rapidly growing exponential dependence on $\sqrt{N}$ was correctly obtained. However, there is now an additional factor of $N^{-3/4}$. This introduces a logarithmic correction to the entropy S_{st} of a highly excited string:

$$S_{st} = ln\, d(N) \simeq 4\pi \sqrt{aN} - \frac{3}{2} ln\sqrt{aN} + lna - \cdots \cdots \tag{12}$$

Finally following Susskind we identify Rindler energy of the black hole with $\sqrt{N}$ as $E_R = 2\sqrt{aN}$, and rewrite the entropy of a highly excited string as

$$S_{st} = 2\pi E_R - \frac{3}{2} lnE_R - \cdots \cdots \tag{13}$$

Clearly this entropy has same logarithmic correction beyond the Bekenstein-Hawking area law as that obtained for the black hole entropy in quantum geometry framework [9]. This correction exits not only for excited strings describing Schwarzschild black holes in any arbitrary dimensions as above, but also all other cases discussed in refs. [3,4,6]. We emphasize that this thus provides an additional evidence in favour of the *excited-string⇔black-hole* correspondence.

In the derivation of level density above, it is important to take account of the zero modes carefully. This has been done by including the integration over p^i in the partition function in eqn.(7). If this were not included, then non-exponential part of level density formula (11) above would have changed from dimension independent factor $N^{-3/4}$ to $N^{-(d+1)/4}$ as has been found in some of the early calculations of level density (for example, see ref. [15] for the case of $d = 26$).

Our discussion here has been for generic black holes, say, Schwarzschild black holes in any dimensions. Though level density above has been calculated for an open string, the asymptotic formula (11) is valid in general in any string theory. In particular, this also obtains for the level density of BPS elementary string states of the superstring theories. For extremal black holes of these superstring theories for which no mass renormalization takes place, the correspondence between strings and black holes sharing the same macroscopic quantum numbers may be applied directly. This allows a counting of weakly coupled BPS string states which can be directly related to degeneracy of these extremal black holes reproducing the Bekeintein-Hawking entropy in the leading order [16,17]. The correction beyond Bekenstein-Hawking entropy obtained here holds for the Bogomol'nyi saturated elementary string states too [11].

Like black holes, entropy of de Sitter space can also be given a string interpretation [18]. Here also geometry near the cosmological horizon is given by a Rindler space. Thus de Sitter space may well be described by a string on the stretched horizon through the same identification of Rindler energy with square root of the oscillator number. The entropy so calculated, beyond the usual area law would also have the same logarithmic correction as discussed above.

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