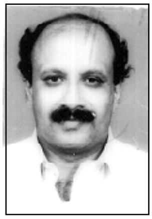
Listening to the Shape of a Drum

1. The Mathematics of Vibrating Drums

S Kesavan



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A drum vibrates at distinct frequencies. These frequencies are related to the eigenvalues of a differential operator called the Laplacian. Mathematicians are interested in knowing how much geometric information about the domain (the surface of the drum) can be retrieved from the eigenvalues.

Behind this rather obvious and apparently silly statement, for we all know that we hear *sounds* and not *shapes*, lies a lot of deep and beautiful mathematics. In 1966, Marc Kac published a paper entitled 'Can one hear the shape of a drum?' This paper, with its catchy title, spawned a lot of mathematical research and the basic problem was finally settled only in 1992.

The problem addressed by Kac is what is known as an *inverse problem*. Given a drum, the natural question to ask is, " What sounds does it make?". The inverse problem deals with the inference of the shape of the drum by just hearing the sounds it produces.

The reason for posing such a question is not just the intellectual curiosity of the mathematician's mind. Indeed, inverse problems are not only mathematically interesting but have serious applications. For instance, seismologists infer a lot about the internal structure of our planet from the 'sounds' that an earthquake produces and the way they echo, bouncing off different layers of rock. In medicine, the ultra-sound scan also deals with the inference of shapes based on sound signals received.

The ancient Greeks knew that vibrating strings (one - dimensional drums!) could produce many different musical notes depending on the number of points which are at rest

Box 1. A Bit of History

The earliest major work on vibrating strings was due to Brook Taylor in 1714. He showed that the normal modes are sinusoidal in shape and that their amplitude varies sinusoidally in time. In 1746, d'Alembert showed that the violin string has many more vibrations that are not normal modes and that are not sinusoidal. In fact, he proved that the wave can start out as almost any shape one likes. Euler was the first to write down the wave equation which he had solved by 1748. The concept of normal modes led to the notion of superposition of solutions. Thus the general solution of the wave equation could be expressed in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi ct}{L}\right)$$

and Euler gave formulae for the coefficients b_n.

But the general concept of a function was not clear and this started a controversy over the representation of an arbitrary function (d'Alembert's solution) by a series of sines (Euler's solution). This point was finally settled only about 70 years later by Dirichlet. He was the first to give precise conditions for the validity of (what we now call) Fourier series expansions. For more details on this topic, the reader is referred to the article of S Thangavelu and the book by R Bhatia (see Suggested Reading).

(these points are called *nodes*). For the fundamental frequency, only the end points are at rest. If there is one more node at the centre, the string produces a note one octave higher. The larger the number of nodes, the higher the frequency of the note will be. These higher order vibrations are called overtones.

In the same way, when a drum is struck, it vibrates at distinct frequencies via *normal modes.* The lowest or base frequency is called the fundamental tone and the higher ones are called overtones. Kac's question can now be phrased as follows: if a person who can identify all the modes of vibration hears, but cannot see, the drum, can he identify its shape just from hearing the fundamental tone and all the overtones?

It was Euler (see *Box 1* for a note on the historical development) who first wrote down the wave equation which describes the vibrations of strings (in one space dimension) and drums (in two dimensions) around the middle of the 18th Century. For a string of length L represented by the interval $[0, L]$ of the real line, the vertical displacement $u(x, t)$, where $x \in [0, L]$ and t stands for time, satisfies the differential equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where c is a constant depending on the physical characteristics of the string like the material it is made of, its tautness etc.. Likewise, in two dimensions, if a thin membrane is stretched to occupy a bounded region Ω of the plane, representing the 'drum', then the vertical displacement of the drum at a point $(x_1, x_2) \in \Omega$ and at time t is given by a function $u(x_1, x_2, t)$ satisfying the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}. \tag{2}$$

(Here, and throughout the sequel, we will normalize the constant c to be unity.) More generally, if $x = (x_1, x_2, \ldots, x_N) \in \Omega$, where Ω is a bounded domain, *i.e.* a bounded and connected open set, in the N-dimensional Euclidean space, $\mathbb{R}^N$, we say that a function $u(x, t)$ satisfies the N-dimensional wave equation if

$$\frac{\partial^2 u}{\partial t^2} = \Delta u, \tag{3}$$

where,

$$\Delta u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$$

is the Laplace operator.

To solve the wave equation, we will also need *initial* and *boundary conditions.* The initial conditions give the initial displacement $u(x,0)$ and the initial velocity $u_t(x,0)$ as known functions of $x \in \Omega$. The boundary condition prescribes the behaviour of the drum along the boundary $\partial\Omega$ for all time.

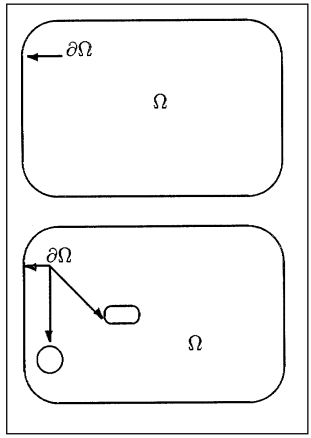

Figure 1. Bounded domains

(All the domains that we consider shall be assumed to be bounded. By the boundary, $\partial\Omega$, we mean the topological boundary. For simplicity, we assume that $\partial\Omega$ has finitely many connected components, each of which is a piecewise smooth and closed hypersurface in $I\!R^N$. Thus, if $N=2$, $\partial\Omega$ will be the disjoint union of finitely many piecewise smooth simple closed curves. See *figure 1*).

Thus, if we imagine that the drum skin is attached to the boundary, then it cannot move along the boundary and so, for all t, we have

$$u = 0 \text{ on } \partial\Omega. \tag{4}$$

In case of the string, this translates as

$$u(0,t) = u(L,t) = 0 \tag{5}$$

for all $t > 0$.
One of the ways of solving the wave equation is to use the method of separation of variables. We look for solutions of the form

$$u(x,t) = \psi(t)w(x). \tag{6}$$

This, on substitution in (3), yields

$$\frac{\psi''(t)}{\psi(t)} = \frac{\Delta w}{w} = -\lambda(\text{say}). \tag{7}$$

Thus, we look for functions $w(x)$, not identically zero, such that

$$\left.\begin{array}{rcll} \Delta w + \lambda w & = & 0 & \text{in } \Omega \\ w & = & 0 & \text{on } \partial\Omega. \end{array}\right\} \tag{8}$$

Elementary considerations show that this can be possible only if $\lambda > 0$. For, if $\lambda \leq 0$, and if w were a solution to (8), we multiply the differential equation by w and integrate by parts using Green's theorem to get

$$-\int_\Omega |\nabla w|^2 dx + \lambda \int_\Omega w^2 dx = 0.$$

(There is no term involving the integral on the boundary since $w = 0$ on $\partial\Omega$.) Since both terms are negative, we deduce that $w \equiv 0$.

Thus, we can now write the general solution for ψ. We have

$$\psi(t) = A\cos(\sqrt{\lambda}t) + B\sin(\sqrt{\lambda}t).$$

The constants A and B can be determined using the initial conditions. For instance, if $u(x, 0) = 0$, then $A = 0$. In this case, we have, upto a multiplicative constant,

$$u(x,t) = \sin(\sqrt{\lambda}t)w(x) \tag{9}$$

where w is a solution of (8). Solutions of this kind are called *normal modes.*

The Case of a String

In the one-dimensional case, the equation (8) reads as

$$\left.\begin{array}{rcll} w''(x) + \lambda w(x) & = & 0 & \text{for } 0 < x < L \\ w(0) = w(L) & = & 0. & \end{array}\right\}$$

Again, the general solution is given by

$$w(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$$

and, taking into account the boundary conditions, we deduce that

$$A = 0 \text{ and } \lambda = n^2\pi^2/L^2.$$

Thus we have non-trivial solutions only when $\lambda = \lambda_n = n^2\pi^2/L^2$ and the corresponding solution (upto a multiplicative constant) is given by

$$w_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

The numbers λ_n are called *eigenvalues* and the functions w_n are called *eigenfunctions* of the differential operator $\frac{d^2}{dx^2}$. The numbers $\sqrt{\lambda_n}$ are the *frequencies*; $\sqrt{\lambda_1}$ is the fundamental frequency and the rest are the overtones. The general solution of the wave equation (1) can be written down by superposition of the normal modes (see Box 1).

Higher Dimensions

In dimensions $N \geq 2$, we can again show (though we need more sophisticated mathematical techniques to do this) that there exists a sequence of numbers

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n \leq \cdots \tag{10}$$

which tends to infinity as $n \to \infty$, and corresponding functions $w_n(x)$ such that the pairs $(\lambda_n, w_n(x))$ are nontrivial solutions to (8) and that non-trivial solutions occur only at those numbers which are, once again, called eigenvalues and are the squares of the frequencies. In the one-dimensional case, all the frequencies were distinct. This need not be the case in higher dimensions. However, the fundamental frequency will still be distinct from the rest as shown by the strict inequality following λ_1 in (10). The higher eigenvalues, nevertheless, will only repeat themselves at most finite numbers of times. The set of eigenfunctions corresponding to an eigenvalue is a finite dimensional vector space spanned by the corresponding w_n's. Again, while the eigenfunctions in the one-dimensional case were all sinusoidal, those in higher dimensions are more complicated in shape.

If $\Omega = (0,1) \times (0,1)$, the open unit square, then the eigenfunctions are combinations of sinusoidal ripples in two perpendicular directions. Corresponding to the eigenvalue $\lambda =$

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$(m^2+n^2)\pi^2$, we have the eigenfunctions

$$\left.\begin{array}{rcl} w_{n,m}(x_1,x_2) & = & \sin(n\pi x_1)\sin(m\pi x_2) \\ w_{m,n}(x_1,x_2) & = & \sin(m\pi x_1)\sin(n\pi x_2). \end{array}\right\}$$

Thus, the fundamental frequency is $\sqrt{\lambda_1}=\sqrt{2}\pi$ while the next higher overtones are given by $\sqrt{\lambda_2}=\sqrt{\lambda_3}=\sqrt{5}\pi$.

If Ω were the unit circle in the plane (the familiar circular drum!), the eigenfunctions involve more complicated expressions called *Bessel functions*.

It must be mentioned, however, that whatever the domain, the eigenfunction corresponding to the fundamental frequency vanishes only on $\partial\Omega$. It has no zeros inside Ω.

For a brief idea about the mathematical principles from which these properties are deduced, see *Box 2*.

Kac's Question

We are now in a position to formulate mathematically the question posed by Kac. Two domains Ω_1 and Ω_2 in $\mathbb{R}^N$ are said to be *isospectral* if they have the same set of eigenvalues (*iso* means *same* in Greek; the spectrum, in this case, is the collection of eigenvalues of the Laplacian).

As we know from high school geometry, two plane domains are said to be *congruent* if we can cut out one of them and place it on the other so that the two coincide exactly. A more formal way of saying this is that we can map one domain onto the other using a combination of (i) rotations about the z-axis, (ii) reflection about some line in the $x-y$ plane passing through the origin and (iii) translations in the $x-y$ plane. It turns out that these transformations are precisely those which preserve the Euclidean distance (the mathematical jargon is *metric*) between points in the plane. Thus, congruent domains are also called *isometric*.

Thus Kac's question reads: "*Are isospectral domains in the plane isometric?*" Kac attributes this question to Bochner

Box 2. The Spectrum of the Laplacian

For those readers who have some knowledge of functional analysis, the following remarks will outline the procedure for obtaining the eigenvalues and eigenfunctions of the Laplacian.

Let Ω be a bounded domain in $I\!R^N$ and consider the real Hilbert space $L^2(\Omega)$ of square integrable real-valued functions on Ω. It can be shown that if $f \in L^2(\Omega)$, then there exists a unique solution (in a subspace of $L^2(\Omega)$ consisting of functions vanishing on the boundary and whose first order derivatives are also square integrable) of the problem:

$$\left.\begin{array}{rcll} \Delta u + f & = & 0 & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right\}$$

The mapping $f \mapsto G(f) := u$, can be shown to be a self-adjoint, compact linear operator on $L^2(\Omega)$. The problem (8) is now equivalent to solving

$$w = \lambda G(w)$$

and the spectral theory of compact, self-adjoint linear operators tells us that there is a sequence of eigenvalues $\{\lambda_n\}$ increasing to infinity (which are in fact the reciprocals of the eigenvalues of G). The associated eigenfunctions $\{w_n\}$ can be chosen to form an *orthonormal basis* for $L^2(\Omega)$, *i.e.* if $g \in L^2(\Omega)$, then

$$g = \sum_{n=1}^{\infty} (g, w_n) w_n$$

(in the sense of $L^2(\Omega)$) where, $(g,h) = \int_\Omega ghdx$ is the inner-product in $L^2(\Omega)$.

The fact that λ_1 is a simple eigenvalue and that w_1 does not vanish in the interior of the domain are consequences of what is known as the *maximum principle*, a property basic to 'second-order elliptic partial differential operators' of which the Laplacian is the most important example.

and the non-mathematical interpretation in terms of drums to L. Bers.

While Kac posed the question for plane domains, one can ask the same about domains in any N-dimensional space.

More generally, if M is a Riemannian manifold (see [2]) , it is equipped with a metric and we can also perform calculus on functions on M. We can define the Laplace operator by

$$\Delta(f) = \mathrm{div}(\mathrm{grad}(f)).$$

Once again Δ has a sequence of eigenvalues and one can ask if isospectral manifolds are isometric.

The essence of Kac's question is whether a domain in Euclidean space or a manifold is completely determined by the spectrum of the associated Laplace operator. On the face of it there is no reason to see why this should be true and even Kac felt that it was rather ambitious to think so. However, we will see in the next part that the spectrum does hold a lot of geometrical information about the domain. We will also trace the developments which ultimately led to the settling (negatively!) of the conjecture of Kac.

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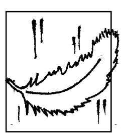
Suggested Reading

[1] R Bhatia. *Fourier Series.* TRIM series, Hindustan Book Agency.
[2] K H Paranjape. Geometry. *Resonance.* Vol. 1. No.6. June, 1996.
[3] I Sneddon. *Elements of Partial Differential Equations.* International student. Edition, McGraw-Hill Kogakusha.
[4] S Thangavelu. Fourier Series. *Resonance.* Vol.1. No.10. October, 1996.



The men of experiment are like the ant; they only collect and use. The reasoners resemble spiders, who make cobwebs out of their own substance. But the bee takes a middle course; it gathers its material from the flowers of the garden and of the field, but transforms and digests it by a power of its own.

Leonardo Da Vinci