

## Homogenization of periodic optimal control problems via multi-scale convergence

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**Abstract.** The aim of this paper is to provide an alternate treatment of the homogenization of an optimal control problem in the framework of two-scale (multi-scale) convergence in the periodic case. The main advantage of this method is that we are able to show the convergence of cost functionals directly without going through the adjoint equation. We use a corrector result for the solution of the state equation to achieve this.

**Keywords.** Homogenization; optimal control; multi-scale convergence.

### 1. Introduction

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain. We denote by  $M(\alpha, \beta, \Omega)$  the set of all  $N \times N$  matrices  $A = ((a_{ij}))$  such that  $a_{ij} \in L^\infty(\Omega)$  for  $1 \leq i, j \leq N$  and

$$\alpha |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \beta |\xi|^2 \text{ a.e.}(x), \quad 0 < \alpha < \beta \quad (1.1)$$

for all  $\xi \in \mathbb{R}^N$ . (Here and in the sequel we adopt the convention of summation over repeated indices). We now describe the optimal control problem.

Let  $U_{ad} \subseteq L^2(\Omega)$  be a closed convex set. Let  $f \in L^2(\Omega)$  be a fixed function. For  $\theta \in U_{ad}$ , we define the state variable  $u = u(\theta) \in H_0^1(\Omega)$  as the (weak) solution of the following second order elliptic boundary value problem:

$$\begin{cases} -\operatorname{div}(A \nabla u) = f + \theta & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

where  $A \in M(\alpha, \beta, \Omega)$ . We then consider the cost (or objective) functional defined by

$$J(\theta) = \frac{1}{2} \int_{\Omega} B \nabla u \nabla u \, dx + \frac{N}{2} \int_{\Omega} \theta^2 \, dx, \quad (1.3)$$

where, for  $\theta \in U_{ad}$ ,  $u$  is the associated state variable (solution of (1.2)) and  $B \in M(\alpha, \beta, \Omega)$  is a symmetric matrix. The problem is then to find  $\theta^* \in U_{ad}$  (called the optimal control) such that

$$J(\theta^*) = \min_{\theta \in U_{ad}} J(\theta),$$

It is a standard result (cf. [7]) that there exists a unique optimal control  $\theta^*$ .

We are now interested in the situation where we have a family of optimal control problems of the kind described above. More precisely, let  $A_\varepsilon \in M(\alpha, \beta, \Omega)$  and  $B_\varepsilon \in M(\alpha', \beta', \Omega)$  (with  $B_\varepsilon$  symmetric) where  $\varepsilon > 0$  is a parameter which eventually tends to zero. Then we consider the problem: find  $\theta_\varepsilon^* \in U_{ad}$  such that

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta \in U_{ad}} J_\varepsilon(\theta). \quad (1.4)$$

where

$$J_\varepsilon(\theta) = \frac{1}{2} \int_{\Omega} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx + \frac{N}{2} \int_{\Omega} \theta^2 dx, \quad (1.5)$$

and  $u_\varepsilon \in H_0^1(\Omega)$  is the state variable corresponding to  $\theta \in U_{ad}$  and is the unique solution of the problem:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f + \theta & \text{in } \Omega \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

It can be shown that  $\{\theta_\varepsilon^*\}$ , where  $\theta_\varepsilon^*$  is the unique optimal control of the problem (1.4)–(1.6), is uniformly bounded in  $L^2(\Omega)$  (with respect to  $\varepsilon$ ) and so for a subsequence,  $\theta_\varepsilon^* \rightharpoonup \theta_0^*$  weakly in  $L^2(\Omega)$ . The problem is to characterize  $\theta_0^*$ . In particular, we wish to find the matrices  $A_0$  and  $B_\#$  with properties similar to those above so that  $\theta_0^*$  is the optimal control of the corresponding problem.

This problem was first studied by Kesavan and Vanninathan [6] in the case when the coefficients  $A_\varepsilon$  and  $B_\varepsilon$  are periodic (see § 3 below for a precise description of this case). Kesavan and Saint Jean Paulin [4] solved it in the case of general coefficients and in a later paper (cf. [5]) extended it to the case when the domain  $\Omega$  is replaced by a 'perforated domain  $\Omega_\varepsilon$ '.

In all the papers cited above, the energy method in homogenization theory was used. Further the adjoint state variable  $p_\varepsilon \in H_0^1(\Omega)$  was introduced via the equation

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega \\ p_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (1.7)$$

(with additional Neumann condition on the holes in case of perforated domains). The system (1.6)–(1.7) was first homogenized and from this the limit matrices  $A_0$  and  $B_\#$  were identified.

In this paper we restrict our attention to the periodic case. As technical device we use the notion of 2-scale convergence developed by Nguetseng [8] and Allaire [1]. We are then able to directly obtain the matrix  $B_\#$  without the necessity of introducing the adjoint problem.

Given the problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla w_\varepsilon) = f & \text{in } \Omega \\ w_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \quad (1.8)$$

where the  $A_\varepsilon$  are periodic, we are directly able to calculate the limit of the integral

$$\int_{\Omega} B_\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon dx.$$

Once this is done, the procedure outlined by Kesavan and Saint Jean Paulin [5] establishes the convergence of the optimal control.

Of course, the method of 2-scale convergence necessitates the assumption of some regularity of the coefficients  $A_\varepsilon$ .

We also prove, along the way, some slightly improved versions of results on 2-scale convergence compared with those of Allaire [1]. We are able to directly deal with the periodically perforated domain and we prove some corrector results for the solution of the analogue of (1.8) in that case.

Finally, we are able to easily extend our results to the multi-scale case, i.e. where there are several (well separated) scales of periodicity in the coefficients using the results of multi-scale convergence of Allaire and Briane [2].

## 2. 2-Scale convergence

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded open set. Let  $Y$  denote the unit cell  $[0, 1]^N$  in  $\mathbb{R}^N$ . In this paper we use the symbol  $\|\cdot\|_{p,\Omega}$  to denote the  $L^p$  norm of a function defined on  $\Omega$ .

For  $\varepsilon > 0$ , and  $g$  a function defined on  $\Omega \times Y$ , we define an oscillating function  $g(x, (x/\varepsilon))$  as follows. Cover  $\mathbb{R}^N$  with translates of the  $\varepsilon$ -cell,  $\varepsilon Y$ . Then any  $x \in \Omega$  falls in a translate of the cell  $\varepsilon Y$  and hence corresponds to a unique  $y$  in  $Y$ . Define  $g(x, (x/\varepsilon))$  to be the value of  $g$  at  $(x, y)$ . Here and in the sequel, we denote a function that is  $Y$ -periodic by the subscript #.

### DEFINITION 2.1

A sequence  $\{u_\varepsilon\}$  of functions in  $L^2(\Omega)$ , where  $\varepsilon$  is a parameter which tends to zero, is said to 2-scale converge to a function  $u_0 \in L^2(\Omega \times Y)$  if

$$\int_{\Omega} u_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) dx \rightarrow \int_{\Omega} \int_Y u_0(x, y) \phi(x, y) dy dx \quad \text{for all } \phi \in D(\Omega, C^\infty_\#(Y)). \quad (2.1)$$

We write  $u_\varepsilon \xrightarrow{2-s} u_0(x, y)$ . □

The relevance of this definition stems from the following result.

**Theorem 2.1.** [1] Every bounded sequence in  $L^2(\Omega)$  has a 2-scale convergent subsequence. □

*Remark 2.1.*

- (1) For any 2-scale convergent sequence its 2-scale limit is unique.
- (2) If  $\phi(x, y)$  is 'smooth' (for instance if it belongs to one of the spaces listed below in Remark 2.2), then  $\phi(x, (x/\varepsilon)) \xrightarrow{2-s} \phi(x, y)$ .
- (3) If  $u_\varepsilon \rightarrow u$  strongly in  $L^2(\Omega)$ , then  $u_\varepsilon \xrightarrow{2-s} u(x)$ .
- (4) If  $u_\varepsilon \xrightarrow{2-s} u_0(x, y)$ , then  $u_\varepsilon \rightarrow \int_Y u_0(x, y)$  weakly in  $L^2(\Omega)$  (take test functions depending on  $x$  alone).
- (5) As a result of the previous remark and the uniform boundedness principle, any 2-scale convergent sequence is bounded.
- (6) Supposing that  $u_\varepsilon$  admits an asymptotic expansion,

$$u_\varepsilon = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots, \quad (2.2)$$

where  $u_i(x, y)$  are assumed to be 'smooth', then  $u_\varepsilon \xrightarrow{2-s} u_0(x, y)$ . So the 2-scale limit gives the first term in the asymptotic expansion of  $u_\varepsilon$ , when the expansion is valid. □

## DEFINITION 2.2

A measurable function  $\psi: \Omega \times Y \rightarrow \mathbb{R}$  which is periodic in the variable  $y$  is said to be admissible if

$$\int_{\Omega} \psi\left(x, \frac{x}{\varepsilon}\right)^2 dx \rightarrow \int_{\Omega} \int_Y \psi(x, y)^2 dy dx. \quad (2.3)$$

More generally, let  $\{u_{\varepsilon}\}$  be a sequence which 2-scale converges to  $u_0(x, y)$ . It is said to be admissible if

$$\int_{\Omega} (u_{\varepsilon})^2 dx \rightarrow \int_{\Omega} \int_Y u_0(x, y)^2 dy dx. \quad (2.4)$$

□

**Remark 2.2** Though the most general condition under which  $\psi(x, y)$  is admissible is not known, it is known that if  $\psi$  belongs to one of the space  $L^2(\Omega, C_{\#}(Y))$ ,  $C_c(\Omega, L^{\infty}_{\#}(Y))$  or  $C(\bar{\Omega}, L^{\infty}_{\#}(Y))$  then it is admissible (cf. [1]). □

**Theorem 2.2.** Let  $u_{\varepsilon} \xrightarrow{2-s} u_0(x, y)$  and assume that  $\{u_{\varepsilon}\}$  is an admissible sequence. If  $v_{\varepsilon}$  is any sequence such that  $v_{\varepsilon} \xrightarrow{2-s} v_0(x, y)$  then,

$$u_{\varepsilon} v_{\varepsilon} \rightarrow \int_Y u_0(x, y) v_0(x, y) dy \text{ in } D'(\Omega) \quad (2.5)$$

and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_{\varepsilon} v_{\varepsilon} dx = \int_{\Omega} \int_Y u_0(x, y) v_0(x, y) dy dx. \quad (2.6)$$

Further, if  $u_0(x, (x/\varepsilon)) \xrightarrow{2-s} u_0(x, y)$  and  $u_0$  is an admissible function then,

$$\lim_{\varepsilon \rightarrow 0} \left\| u_{\varepsilon} - u_0\left(x, \frac{x}{\varepsilon}\right) \right\|_{2, \Omega} = 0. \quad (2.7)$$

□

**Remark 2.3.** (1) In fact, except for (2.6), this result is proved in [1]. The same proof can be easily adapted to give (2.6) as well. (2) The hypothesis for (2.7) is slightly more general than saying (cf. [1])  $\psi \in L^2(\Omega, C_{\#}(Y))$ . □

We now prove a result which will be used repeatedly in the sequel.

**Theorem 2.3.** Suppose that  $u_{\varepsilon} \xrightarrow{2-s} u_0(x, y)$  and  $\phi \in C(\bar{\Omega}, L^{\infty}_{\#}(Y))$  then,

$$u_{\varepsilon} \phi\left(x, \frac{x}{\varepsilon}\right) \xrightarrow{2-s} u_0(x, y) \phi_0(x, y). \quad (2.8)$$

**Proof.** Since  $\|\phi(x, (x/\varepsilon))\|_{\infty, \Omega} \leq \|\phi(x, y)\|_{\infty, \Omega \times Y}$  and  $\{u_{\varepsilon}\}$  is bounded in  $L^2(\Omega)$  we have,

$$\left\| u_{\varepsilon} \phi\left(x, \frac{x}{\varepsilon}\right) \right\|_{2, \Omega} \leq c \|u_{\varepsilon}\|_{2, \Omega} \leq c \text{ for all } \varepsilon \text{ (where } c \text{ is a generic constant).}$$

By Theorem 2.1, for every subsequence of  $\{u_{\varepsilon} \phi(x, (x/\varepsilon))\}$  there is a further subsequence (which we continue to index by  $\varepsilon$  for simplicity) and a function  $u(x, y)$  such that

$u_\varepsilon \phi(x, (x/\varepsilon))$  2-scale converges to  $u(x, y)$ . We will show that  $u(x, y) = u_0(x, y) \phi(x, y)$ . Since this limit is independent of the subsequence chosen, it shows that the entire sequence 2-scale converges to this limit.

To prove the claim made above, let  $\psi \in D(\Omega, C_\#^\infty(Y))$ . Then we have by definition,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y u \psi dy dx.$$

On the other hand,  $\phi \psi \in C(\bar{\Omega}, L_\#^\infty(Y))$  and so  $\phi(x, (x/\varepsilon)) \psi(x, (x/\varepsilon)) \xrightarrow{2-s} \phi(x, y) \psi(x, y)$  and  $\phi \psi$  is an admissible function (cf. Remark 2.2). Therefore by Theorem 2.2 we get,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} u_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) \psi\left(x, \frac{x}{\varepsilon}\right) dx = \int_{\Omega} \int_Y u_0(x, y) \phi(x, y) \psi(x, y) dx dy$$

as  $u_0$  is the 2-scale limit of  $u_\varepsilon$ . From the above and the density of  $D(\Omega, C_\#^\infty(Y))$  functions in  $L^2(\Omega \times Y)$  we conclude that  $u = u_0 \phi$ . This completes the proof.  $\square$

### COROLLARY 2.1

Let  $u_\varepsilon, u_0$  and  $\phi$  be as in Theorem 2.3. Let  $\{v_\varepsilon\}$  be an admissible sequence which 2-scale converges to  $v_0(x, y)$ . Then,

$$\int_{\Omega} u_\varepsilon \phi\left(x, \frac{x}{\varepsilon}\right) v_\varepsilon dx \rightarrow \int_{\Omega} \int_Y u_0 \phi v_0 dx dy. \quad (2.9)$$

*Proof.* By Theorem 2.3,  $u_\varepsilon \phi(x, (x/\varepsilon)) \xrightarrow{2-s} u_0(x, y) \phi(x, y)$ . Also,  $v_\varepsilon \xrightarrow{2-s} v(x, y)$  and  $\{v_\varepsilon\}$  is an admissible sequence. Therefore, (2.9) follows from Theorem 2.2.  $\square$

**Theorem 2.4.** [1] Let  $\{u_\varepsilon\}$  be a bounded sequence in  $H_0^1(\Omega)$  that weakly converges to a function  $u$  in  $H_0^1(\Omega)$ . Then there exists a function  $u_1 \in L^2(\Omega, H_\#^1(Y)/\mathbb{R})$  such that (for a subsequence)

$$\begin{aligned} u_\varepsilon &\xrightarrow{2-s} u(x) \quad \text{and,} \\ \nabla u_\varepsilon &\xrightarrow{2-s} \nabla_x u + \nabla_y u_1(x, y). \end{aligned} \quad \square$$

*Remark 2.4.* (1)  $H_\#^1(Y)$  is the space of functions in  $H^1(Y)$  which have been extended by  $Y$ -periodicity to  $\mathbb{R}^N$ . (2) In case  $u_\varepsilon$  has an asymptotic expansion then it would be of the form

$$u_\varepsilon(x) = u_0(x) + u_1\left(x, \frac{x}{\varepsilon}\right) + \dots \quad \square$$

### 3. Convergence of cost functionals

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$ . We obtain a periodically perforated domain  $\Omega_\varepsilon$  by removing from  $\Omega$  a set of periodically distributed holes,  $T_\varepsilon$ . i.e.  $\Omega_\varepsilon = \Omega \setminus \bar{T}_\varepsilon$ , where  $T_\varepsilon$  is defined as follows.

Let  $T$  be an open subset of the unit cell  $Y = [0, 1]^N$  with Lipschitz boundary. Set,

$$T_\varepsilon = \bigcup_{k \in \mathbb{Z}^N} \varepsilon(k + T).$$

We denote the 'material part' of the unit cell by  $Y^*$ , i.e.  $Y^* = Y \setminus \bar{T}$ .

The boundary of  $\Omega_\varepsilon$  has two parts-one comprises the union of boundaries of holes strictly contained in  $\Omega$ , denoted by  $\partial_{\text{int}}\Omega_\varepsilon$ .

$$\partial_{\text{int}}\Omega_\varepsilon = \bigcup_{\mathbf{k} \in \mathbb{Z}^N} \{\partial_\varepsilon(\mathbf{k} + T) : \varepsilon(\mathbf{k} + \bar{T}) \subset \Omega\}.$$

The second part is the exterior boundary,

$$\partial_{\text{ext}}\Omega_\varepsilon = \partial\Omega_\varepsilon \setminus \partial_{\text{int}}\Omega_\varepsilon.$$

We make the following assumptions:

$$\Omega_\varepsilon \text{ is a connected set.} \quad (3.1)$$

Let  $A$  be the  $N \times N$  matrix  $A = ((a_{ij}(x, y)))$  in  $M(\alpha, \beta, \Omega \times Y)$  such that

$$a_{ij} \in C(\bar{\Omega}, L^\infty_\#(Y)). \quad (3.2)$$

Consider a sequence  $\{f_\varepsilon\}$  in  $L^2(\Omega_\varepsilon)$  and a function  $f \in L^2(\Omega)$  such that,

$$\tilde{f}_\varepsilon \rightarrow \lambda f \text{ weakly in } L^2(\Omega), \quad (3.3)$$

where  $\lambda = |Y^*|$ , the volume of the material part of the unit cell and the  $\sim$  denotes the extension by zero outside  $\Omega_\varepsilon$ .

We consider the following problem posed in  $\Omega_\varepsilon$  with a Neumann condition on interior holes:

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f_\varepsilon & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 & \text{on } \partial_{\text{int}}\Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial_{\text{ext}}\Omega_\varepsilon. \end{cases} \quad (3.4)$$

Introduce the space,

$$V_\varepsilon = \{u \in H_0^1(\Omega_\varepsilon) : u = 0 \text{ on } \partial_{\text{ext}}\Omega_\varepsilon\}$$

and the bilinear form,  $a_\varepsilon : V_\varepsilon \times V_\varepsilon \rightarrow \mathbb{R}$

$$a_\varepsilon(u, v) = \int_{\Omega} A_\varepsilon \nabla u \nabla v dx,$$

where  $A_\varepsilon$  is the matrix  $A(x, (x/\varepsilon))$ .

Then (3.4) admits the weak formulation,

$$\begin{cases} \text{Find } u_\varepsilon \in V_\varepsilon \text{ such that} \\ a_\varepsilon(u_\varepsilon, v) = (f_\varepsilon, v) \text{ for all } v \in V_\varepsilon. \end{cases} \quad (3.5)$$

It is known that  $a_\varepsilon$  is coercive and hence (3.5) admits a unique solution  $u_\varepsilon$  in  $V_\varepsilon$ . It is also known that  $\{\tilde{u}_\varepsilon\}$  is a bounded sequence in  $L^2(\Omega)$  (cf. [1]). Hence assume for a subsequence that

$$\tilde{u}_\varepsilon \rightarrow \lambda u \text{ weakly in } L^2(\Omega) \text{ for a function } u \in L^2(\Omega). \quad (3.6)$$

**Theorem 3.1.** *Let  $u_\varepsilon$  be the solution of (3.5). Then for a subsequence*

$$\begin{cases} \tilde{u}_\varepsilon \xrightarrow{2-s} \chi(y)u(x) \\ \tilde{\nabla} u_\varepsilon \xrightarrow{2-s} \chi(y)(\nabla_x u(x) + \nabla_y u_1(x, y)) \end{cases} \quad (3.7)$$

where  $u \in H_0^1(\Omega)$ ,  $u_1 \in L^2(\Omega, C_\#^1(Y)/\mathbb{R})$  and  $\chi$  is the characteristic function of the set  $Y^*$ .

In fact, if we choose  $u_1$  such that  $\int_{Y^*} u_1(x, y) dy = 0$  then,  $(u, u_1)$  is the unique solution in  $H_0^1(\Omega) \times L^2(\Omega, C_\#^1(Y)/\mathbb{R})$  of the 2-scale homogenized problem

$$\begin{cases} -\operatorname{div}_y(A(x, y)(\nabla_x u + \nabla_y u_1(x, y))) = 0 & \text{in } \Omega \times Y^* \\ A(x, y)(\nabla_x u + \nabla_y u_1(x, y)) \cdot n_y = 0 & \text{on } \partial Y^* \setminus \partial Y \\ -\operatorname{div}_x(\int_Y \chi(y) A(x, y)(\nabla_x u + \nabla_y u_1(x, y)) dy) = \lambda f & \text{in } \Omega. \end{cases} \quad (3.8) \quad \square$$

**Remark 3.1.** (1) This theorem is proved in [1] for  $f_\varepsilon \equiv f$  in  $\Omega_\varepsilon$ . The same proof can be adapted to prove Theorem 3.1 where the right hand side in equation (3.5) is a sequence  $\{f_\varepsilon\}$  such that  $\tilde{f}_\varepsilon \rightarrow \lambda f$ . (2) The extra regularity of  $u_1$  comes from the smoothness of the coefficients  $a_{ij}(x, y)$ . (3) The equations (3.8) may be decoupled by setting

$$u_1(x, y) = \frac{\partial u}{\partial x_i} X^i(x, y),$$

where,  $X^i(x, y)$  is the solution of the problem:

$$\begin{cases} -\operatorname{div}_y(A(x, y)(e^i + \nabla X^i(x, y))) = 0 & \text{in } Y^* \text{ a.e. } x \\ \int_{Y^*} X^i(x, y) dy = 0 & \text{a.e. } x \\ y \mapsto X^i(x, y) \text{ is } Y\text{-periodic} \end{cases} \quad (3.9)$$

for  $i = 1, 2, \dots, N$  and  $u$  is the solution of

$$\begin{cases} -\operatorname{div}_x(A_0(x) \nabla u) = \lambda f & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases} \quad (3.10)$$

where the matrix  $A_0$  given by,

$$(A_0(x))_{ij} = \int_Y \chi(y) \left( a_{ij}(x, y) + a_{ik}(x, y) \frac{\partial X^i}{\partial y_k}(x, y) \right) dy,$$

is the  $H_0$ -limit of the sequence  $\{A_\varepsilon\}$ .  $\square$

We now prove a corrector result. First we need the following preliminary result.

**Lemma 3.1.** Let  $u_\varepsilon$  be the solution of (3.5) and  $(u, u_1)$  be as in Theorem 3.1. Then  $\{\chi_\varepsilon(\partial u / \partial x_i + \partial u_1 / \partial y_i(x, (x/\varepsilon)))\}$  is an admissible sequence for  $i = 1, 2, \dots, N$ , where  $\chi_\varepsilon$  is the characteristic function of the set  $\Omega_\varepsilon$ .

*Proof.* We have, by Theorem 2.3,

$$\chi_\varepsilon \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right) \xrightarrow{2-s} \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right) \chi(y).$$

Since  $\partial u_1 / \partial y_i(x, (x/\varepsilon))$  vanishes on the holes, we can write

$$\begin{aligned} \int_\Omega \chi_\varepsilon \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx &= \int_\Omega \left( \chi_\varepsilon \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx \\ &= \int_\Omega \chi_\varepsilon \left( \frac{\partial u}{\partial x_i} \right)^2 dx + 2 \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) dx \\ &\quad + \int_\Omega \left( \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx. \end{aligned}$$

Now,

$$\int_{\Omega} \chi_{\varepsilon} \left( \frac{\partial u}{\partial x_i} \right)^2 dx \rightarrow \int_{\Omega} \int_Y \chi(y) \left( \frac{\partial u}{\partial x_i} \right)^2 dy dx$$

since  $\chi_{\varepsilon} \rightarrow S_Y \chi(y) dy$  in  $L^{\infty}(\Omega)$  weak \* and  $(\partial u / \partial x_i)^2 \in L^1(Y)$ . Next,

$$\begin{aligned} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) dx &\rightarrow \int_{\Omega} \left( \int_Y \frac{\partial u_1}{\partial y_i}(x, y) dy \right) \frac{\partial u}{\partial x_i} dx \\ &= \int_{\Omega} \int_Y \chi(y) \frac{\partial u}{\partial x_i} \frac{\partial u_1}{\partial y_i}(x, y) dy dx \end{aligned}$$

since  $\partial u_1 / \partial y_i(x, (x/\varepsilon)) \rightarrow \int_Y \partial u_1 / \partial y_i(x, y) dy$  weakly in  $L^2(\Omega)$  and  $\partial u_1 / \partial y_i(x, y) = \chi(y) \partial u_1 / \partial y_i(x, y)$ . Finally,

$$\begin{aligned} \int_{\Omega} \frac{\partial u_1^2}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) dx &\rightarrow \int_{\Omega} \int_Y \left( \frac{\partial u_1}{\partial y_i}(x, y) \right)^2 dy dx \text{ (since } u_1 \text{ is smooth.)} \\ &= \int_{\Omega} \int_Y \chi(y) \left( \frac{\partial u_1}{\partial y_i}(x, y) \right)^2 dy dx. \end{aligned}$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \chi_{\varepsilon} \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx = \int_{\Omega} \int_Y \left\{ \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right) \right\}^2 dy dx$$

which proves the lemma.  $\square$

We are now in a position to prove the following corrector result.

**Theorem 3.2.** *Let  $u_{\varepsilon}$  be the solution of (3.5). Then*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_i} - \frac{\partial u}{\partial x_i} \chi_{\varepsilon} - \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right\|_{2, \Omega} = 0 \text{ for } i = 1, 2, \dots, N \quad (3.11)$$

where  $(u, u_1)$  are as in Theorem 3.1.

*Proof.* Let

$$r_{\varepsilon}^i = \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_i} - \frac{\partial u}{\partial x_i} \chi_{\varepsilon} - \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon}.$$

Then,

$$\begin{aligned} \alpha \sum_{i=1}^N \left\| r_{\varepsilon}^i \right\|_{2, \Omega}^2 &\leq \\ &\int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_j} - \frac{\partial u}{\partial x_j} \chi_{\varepsilon} - \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \left( \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_i} - \frac{\partial u}{\partial x_i} \chi_{\varepsilon} - \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) dx. \end{aligned}$$

Therefore, using equation (3.4), this can be written as

$$\alpha \sum_{i=1}^N \left\| r_{\varepsilon}^i \right\|_{2, \Omega}^2 \leq \int_{\Omega} f_{\varepsilon} u_{\varepsilon} dx - \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\partial \widetilde{u}_{\varepsilon}}{\partial x_j} \left( \frac{\partial u}{\partial x_i} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) dx$$



$$\begin{aligned}
& - \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial u}{\partial x_j} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \frac{\widetilde{\partial u_{\varepsilon}}}{\partial x_i} dx + \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \\
& \quad \times \left( \frac{\partial u}{\partial x_j} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \left( \frac{\partial u}{\partial x_i} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) dx.
\end{aligned} \tag{3.12}$$

We make the following observations,

$$\begin{aligned}
& \frac{\widetilde{\partial u_{\varepsilon}}}{\partial x_i} \xrightarrow{2-s} \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right) \\
& a_{ij}(x, y) \in C(\bar{\Omega}, L^{\infty}_{\#}(Y)), \\
& \left( \frac{\partial u}{\partial x_i} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \xrightarrow{2-s} \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right).
\end{aligned}$$

Further by lemma 3.1,  $\{(\partial u / \partial x_i) \chi_{\varepsilon} + (\partial u_1 / \partial y_i)(x, (x/\varepsilon) \chi_{\varepsilon})\}$  is an admissible sequence.

So by Corollary 2.1 and from the observations made above it follows that

$$\begin{aligned}
& \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\widetilde{\partial u_{\varepsilon}}}{\partial x_i} \left( \frac{\partial u}{\partial x_j} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) dx \rightarrow \\
& \int_{\Omega} \int_Y \chi(y) A(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx, \\
& \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial u}{\partial x_j} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \frac{\widetilde{\partial u_{\varepsilon}}}{\partial x_i} dx \rightarrow \\
& \int_{\Omega} \int_Y \chi(y) A(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} a_{ij} \left( x, \frac{x}{\varepsilon} \right) \left( \frac{\partial u}{\partial x_j} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_j} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) \left( \frac{\partial u}{\partial x_i} \chi_{\varepsilon} + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_{\varepsilon} \right) dx \rightarrow \\
& \int_{\Omega} \int_Y \chi(y) A(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx.
\end{aligned}$$

Summing over the limits of the second, third and fourth terms on right hand side of equation (3.12) we get

$$- \int_{\Omega} \int_Y \chi(y) A(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx$$

which, by (3.8), is  $\int_{\Omega} \lambda f u dx$ . Now notice that

$$\|u_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})} \leq c \text{ (independent of } \varepsilon)$$

$$u_{\varepsilon} = 0 \text{ on } \partial_{\text{ext}} \Omega_{\varepsilon}$$

$$\tilde{u}_{\varepsilon} \rightharpoonup \lambda u \text{ and,}$$

$$\tilde{f}_{\varepsilon} \rightharpoonup \lambda f.$$

So by a strong compactness result of Allaire and Nandakumar [3],

$$\int_{\Omega_{\varepsilon}} f_{\varepsilon} u_{\varepsilon} dx \rightarrow \int_{\Omega} \lambda f u dx.$$

Therefore, it follows from (3.12) that  $\|r_\varepsilon^i\|_{2,\Omega} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , which completes the proof.  $\square$

*Remark 3.4.* If  $f_\varepsilon$  is the restriction of a given function  $f$  in  $L^2(\Omega)$  to  $\Omega_\varepsilon$  for all  $\varepsilon$ , then

$$\int_{\Omega_\varepsilon} f_\varepsilon u_\varepsilon dx = \int_{\Omega} f \tilde{u}_\varepsilon dx \rightarrow \int_{\Omega} f \lambda u dx,$$

since  $\tilde{u}_\varepsilon \rightharpoonup \lambda u$  weakly in  $L^2(\Omega)$ . In this case we do not use the compactness result of Allaire and Nandakumar.  $\square$

### COROLLARY 3.1

$\widetilde{\partial u_\varepsilon / \partial x_i}$  is an admissible sequence for all  $i = 1, 2, \dots, N$ . i.e.,

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \right)^2 dx = \int_{\Omega} \int_Y \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} \right)^2 dy dx.$$

*Proof.* We have

$$\int_{\Omega} \left( \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \right)^2 dx = \int_{\Omega} \left( r_\varepsilon^i + \frac{\partial u}{\partial x_i} \chi_\varepsilon + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_\varepsilon \right)^2 dx.$$

So,

$$\begin{aligned} \int_{\Omega} \left( \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \right)^2 dx &= \int_{\Omega} (r_\varepsilon^i)^2 dx + 2 \int_{\Omega} r_\varepsilon^i \left( \frac{\partial u}{\partial x_i} \chi_\varepsilon + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \chi_\varepsilon \right) dx \\ &\quad + \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \chi_\varepsilon + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx. \end{aligned} \quad (3.13)$$

By Theorem 3.2,  $r_\varepsilon^i \rightarrow 0$  in  $L^2(\Omega)$  strongly. Also,  $\{\chi_\varepsilon (\partial u / \partial x_i + \partial u_1 / \partial y_i (x, (x/\varepsilon)))\}$  is a bounded sequence in  $L^2(\Omega)$ . So the first two terms in the right hand side of (3.13) converge to zero as  $\varepsilon \rightarrow 0$ . Therefore,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \right)^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \chi_\varepsilon + \frac{\partial u_1}{\partial y_i} \left( x, \frac{x}{\varepsilon} \right) \right)^2 dx \\ &= \int_{\Omega} \int_Y \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i} (x, y) \right)^2 \chi(y) dy dx \quad (\text{by lemma 3.1}), \end{aligned}$$

i.e.,  $\{\widetilde{\partial u_\varepsilon / \partial x_i}\}$  is an admissible sequence.  $\square$

For  $u_\varepsilon$  as above i.e. solving (3.5) we have the following result on the convergence of cost functionals.

Let  $B$  be the  $N \times N$  matrix  $((b_{ij}(x, y)))$ . Assume that  $b_{ij} \in C(\bar{\Omega}, L^\infty_\#(Y))$  for all  $i, j$ . Denote the matrix  $(b_{ij}(x, (x/\varepsilon)))$  by  $B_\varepsilon$ .

**Theorem 3.3.** *With the above assumptions on  $u_\varepsilon$  and  $B$  we have,*

$$\begin{aligned} \int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx &\rightarrow \\ \int_{\Omega} \int_Y \chi(y) B(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx. \end{aligned}$$

*Proof.* We may write

$$\int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx = \int_{\Omega} b_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial u_\varepsilon}{\partial x_i} dx.$$

Note the following,

$$\begin{aligned} b_{ij} &\in C(\bar{\Omega}, L^\infty_\#(Y)) \\ \frac{\partial u_\varepsilon}{\partial x_i} &\xrightarrow{2-s} \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right) \\ \left\{ \frac{\partial u_\varepsilon}{\partial x_i} \right\} &\text{ is an admissible sequence.} \end{aligned}$$

So, by Corollary 2.1,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} b_{ij} \left( x, \frac{x}{\varepsilon} \right) \frac{\partial u_\varepsilon}{\partial x_j} \frac{\partial u_\varepsilon}{\partial x_i} dx \\ &= \int_{\Omega} \int_Y b_{ij}(x, y) \left( \frac{\partial u}{\partial x_j} + \frac{\partial u_1}{\partial y_j}(x, y) \right) \chi(y) \left( \frac{\partial u}{\partial x_i} + \frac{\partial u_1}{\partial y_i}(x, y) \right) \chi(y) dy dx \\ &= \int_{\Omega} \int_Y \chi(y) B(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx. \quad \square \end{aligned}$$

*Remarks 3.5.* (1) We can write, when  $B$  is symmetric,

$$\begin{aligned} \int_{\Omega} \int_Y \chi(y) B(x, y) (\nabla_x u + \nabla_y u_1(x, y)) (\nabla_x u + \nabla_y u_1(x, y)) dy dx \\ = \int_{\Omega} B_\# \nabla u \nabla u dx \end{aligned}$$

where,

$$(B_\#)_{ij} = (B_0)_{ij} + \int_Y \chi(y) B \nabla_y (X^i - Y^i) \nabla_y (X^j - Y^j) dy.$$

Here,  $Y^i$  is the solution of the  $i$ th cell problem for  $B$  viz.,

$$\begin{aligned} -\operatorname{div}_y (B(x, y) (e^i + \nabla_y Y^i(x, y))) &= 0 \quad \text{in } Y^* \\ B(x, y) (e^i + \nabla_y Y^i(x, y)) \cdot n &= 0 \quad \text{on } \partial Y^* \setminus \partial Y \\ \int_{Y^*} Y^i(x, y) dy &= 0 \\ y \mapsto Y^i(x, y) &\text{ is } Y\text{-periodic} \end{aligned}$$

and  $B_0$  is the  $H_0$  limit of the matrices  $B_\varepsilon$  and is given by the formula

$$(B_0)_{ij} = \int_Y \chi(y) \left( b_{ij} + b_{ik} \frac{\partial Y^j}{\partial y_k} \right) dy.$$

(2) If there are no holes then  $\chi(y) \equiv 1$ . Hence we obtain the same formula for  $B_\#$  as in [6, 4].

(3) In the above mentioned papers, the existence of  $B_\#$  was proved in the context of an optimal control problem. The authors obtained the matrix  $B_\#$  by introducing the adjoint state variable and analysing the corresponding adjoint equation. While their

analysis works for general coefficients, we have obtained the cost functional in the periodic case directly, without involving the adjoint problem.

(4) Once we have the convergence of cost functionals we can complete the study of the optimal control problem as in [5].  $\square$

#### 4. Reiterated homogenization

In physical problems involving more than one microscopic scale it is useful to have the notion of multi-scale convergence. The definition and main results of multi-scale convergence as introduced by Allaire and Briane [2] are recalled. We later apply this method to obtain the limit of quadratic functionals, thereby generalizing the results obtained in § 3 to a multiply perforated domain.

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$ . We consider functions which depend on one macroscopic variable and  $n$  microscopic variables.

Let  $\{a_1(\varepsilon)\}, \{a_2(\varepsilon)\}, \dots, \{a_n(\varepsilon)\}$  be  $n$  sequences such that

$$\lim_{\varepsilon \rightarrow 0} a_i(\varepsilon) = 0 \text{ for } i = 1, 2, \dots, n \quad (4.1)$$

and such that  $\exists m > 0$  with

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{a_i(\varepsilon)} \left[ \frac{a_{i+1}(\varepsilon)}{a_i(\varepsilon)} \right] = 0 \text{ for } i = 1, 2, \dots, n-1. \quad (4.2)$$

If (4.2) is satisfied, then we say that the scales  $a_i(\varepsilon)$  are well-separated.

*Example.* Let  $0 < k_1 < k_2 < \dots < k_n$  be  $n$  numbers. Then we may take  $a_i(\varepsilon) = \varepsilon^{k_i}$ .

##### DEFINITION 4.1

For any  $Y_k$ -periodic function (for all  $k = 1, 2, \dots, n$ )  $\phi(x, y_1, \dots, y_n)$ , the oscillating function  $[\phi]_\varepsilon$  is defined by

$$[\phi]_\varepsilon(x) = \phi\left(x, \frac{x}{a_1(\varepsilon)}, \dots, \frac{x}{a_n(\varepsilon)}\right). \quad \square$$

##### DEFINITION 4.2

A sequence  $u_\varepsilon \in L^2(\Omega)$  is said to  $(n+1)$ -scale converge to a function  $u \in L^2(\Omega \times Y_1 \times \dots \times Y_n)$  if

$$\int_{\Omega} u_\varepsilon [\phi]_\varepsilon(x) dx \rightarrow \int_{\Omega} \int_{Y_1} \dots \int_{Y_n} (u\phi)(x, y_1, \dots, y_n) dy_n \dots dy_1 dx \quad (4.3)$$

for all  $\phi \in D(\Omega, C_{\#}^\infty(Y_1 \times \dots \times Y_n))$ , where  $Y_i = [0, 1]^N$  for  $i = 1, 2, \dots, n$ . We write  $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$ .  $\square$

The following theorem justifies the definition of  $(n+1)$ -scale convergence.

**Theorem 4.1.** (Compactness)(cf. [2]) Every bounded sequence in  $L^2(\Omega)$  has an  $(n+1)$ -scale convergent subsequence.  $\square$

- Remark 4.1.* (1) If  $u(x, y_1, \dots, y_n)$  is smooth, then  $[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$ .  
 (2) If  $u_\varepsilon \rightarrow u$  in  $L^2(\Omega)$  strongly, then  $u_\varepsilon \xrightarrow{(n+1)-s} u(x)$ .  
 (3) If  $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$ , then  $u_\varepsilon \rightharpoonup \int_{Y_1} \dots \int_{Y_n} u(x, y_1, \dots, y_n) dy_n \dots dy_1$  weakly in  $L^2(\Omega)$ .  
 (4) Thus any  $(n+1)$ -scale convergent sequence  $\{u_\varepsilon\}$  is bounded in  $L^2(\Omega)$ . This follows from the previous remark and the uniform boundedness principle.  
 (5) If  $u_\varepsilon$  has the expansion

$$u_\varepsilon(x) = [u]_\varepsilon(x) + \sum_{i=1}^n a_i(\varepsilon)[u_i]_\varepsilon(x) + \dots$$

where the functions  $u, u_1, \dots, u_n$  are assumed to be smooth, then

$$u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n).$$

Hence the  $(n+1)$ -scale limit of a sequence determines the first term in the asymptotic expansion of  $u_\varepsilon(x)$ .  $\square$

#### DEFINITION 4.3

- (a) A measurable function  $u(x, y_1, \dots, y_n)$  is said to be admissible if

$$\int_{\Omega} ([u]_\varepsilon(x))^2 dx \rightarrow \int_{\Omega} \int_{Y_1} \dots \int_{Y_n} u(x, y_1, \dots, y_n)^2 dy_n \dots dy_1 dx. \quad (4.4)$$

- (b) A sequence  $\{u_\varepsilon\}$  which  $(n+1)$ -scale converges to an  $u(x, y_1, \dots, y_n)$  is said to be admissible if

$$\int_{\Omega} (u_\varepsilon)^2 dx \rightarrow \int_{\Omega} \int_{Y_1} \dots \int_{Y_n} u(x, y_1, \dots, y_n)^2 dy_n \dots dy_1 dx. \quad (4.5)$$

$\square$

*Remark 4.2.* (1) The space  $L^\infty(\Omega, C_\#(Y_1 \times \dots \times Y_n))$  is an example of a space of admissible functions. In general, continuity in  $n$  of the variables is sufficient in addition to the appropriate measurability properties (cf. [2]).

- (2) If  $u \in L^\infty(\Omega, C_\#(Y_1 \times \dots \times Y_n))$ , then  $[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$ .  $\square$

The following theorem is in a certain sense, a theorem on strong  $(n+1)$ -scale convergence. This is made precise as follows.

**Theorem 4.2.** [2] Suppose that  $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$  and  $\{u_\varepsilon\}$  is an admissible sequence. Also suppose that  $\{v_\varepsilon\}$  is another sequence which  $(n+1)$ -scale converges to  $v(x, y_1, \dots, y_n)$ . Then,

$$u_\varepsilon v_\varepsilon \rightarrow \int_{Y_1} \dots \int_{Y_n} (uv)(x, y_1, \dots, y_n) dy_n \dots dy_1 \text{ in } D'(\Omega). \quad (4.6)$$

Also,

$$\int_{\Omega} u_\varepsilon v_\varepsilon dx \rightarrow \int_{\Omega} \int_{Y_1} \dots \int_{Y_n} (uv)(x, y_1, \dots, y_n) dy_n \dots dy_1 dx. \quad (4.7)$$

Further, if the  $(n+1)$ -scale limit  $u(x, y_1, \dots, y_n)$  is admissible and

$$[u]_\varepsilon(x) \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n),$$

then

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - [u]_\varepsilon(x)\|_{2,\Omega} = 0. \quad (4.8)$$

□

*Remark 4.3.* (a) In Theorem 4.2, the conclusion (4.7) is new though the proof of (4.7) is similar to that of (4.6).

(b) In fact, (4.6) can be improved to weak convergence in  $L^2(\Omega)$  if  $u_\varepsilon v_\varepsilon \in L^2(\Omega)$ . This happens, for instance, if  $u_\varepsilon = \chi_\varepsilon \doteq \prod_{i=1}^n \chi_i(x/a_i(\varepsilon))$  where  $\chi_i(y_i)$  are the characteristic functions of  $Y_i^* \subseteq Y_i$  ( $Y_i^*$  are certain subsets of the unit cell when we look at certain problems over a perforated domain). □

#### COROLLARY 4.1

Let  $u_\varepsilon \xrightarrow{(n+1)-s} u(x, y_1, \dots, y_n)$  and  $\phi \in L^\infty(\Omega, C_\#(Y_1 \times \dots \times Y_n))$ . Then

$$u_\varepsilon [\phi]_\varepsilon \xrightarrow{(n+1)-s} (u\phi)(x, y_1, \dots, y_n). \quad (4.9)$$

*Proof.* The proof is similar to the proof appearing in the 2-scale case (cf. Theorem 2.3). □

#### COROLLARY 4.2

Let  $\{u_\varepsilon\}$  be a sequence which  $(n+1)$ -scale converges to  $u(x, y_1, \dots, y_n)$ . Assume that  $\{u_\varepsilon\}$  is an admissible sequence. Let  $\phi \in L^\infty(\Omega, C_\#(Y_1 \times \dots \times Y_n))$ . Let  $w_\varepsilon$  be another sequence which  $(n+1)$ -scale converges to  $w(x, y_1, \dots, y_n)$ . Then,

$$\int_\Omega u_\varepsilon [\phi]_\varepsilon w_\varepsilon dx \rightarrow \int_\Omega \int_{Y_1} \dots \int_{Y_n} (u\phi w)(x, y_1, \dots, y_n) dy_n \dots dy_1 dx. \quad (4.10)$$

*Proof.* Apply Corollary 4.1 to  $\phi, w_\varepsilon$  and take  $\phi w_\varepsilon$  for  $v_\varepsilon$  in Theorem 4.2. □

**Theorem 4.3.** [2] Let  $\{u_\varepsilon\}$  be a sequence in  $H^1(\Omega)$  which converges weakly to  $u$  in  $H^1(\Omega)$ . Then for a subsequence (which we continue to index by  $\varepsilon$ ) and functions  $u_k(x, y_1, \dots, y_k) \in L^2(\Omega \times Y_1 \times \dots \times Y_{k-1}, H_\#^1(Y_k^* \setminus \mathbb{R}))$ ,  $k = 1, 2, \dots, n$  we have,

$$u_\varepsilon \xrightarrow{(n+1)-s} u(x) \text{ and } \nabla u_\varepsilon \xrightarrow{(n+1)-s} \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k. \quad (4.11)$$

□

### 5. Convergence of quadratic functionals

We now apply the results of the previous section to obtain the limit of  $J_\varepsilon(u_\varepsilon)$  where  $J_\varepsilon$  is a quadratic functional and  $u_\varepsilon$  is the solution to a Neumann problem in a multiscale perforated domain. As before we assume that the scales  $a_1(\varepsilon), a_2(\varepsilon), \dots, a_n(\varepsilon)$  are well separated. We now define a multi-scale periodically perforated domain  $\Omega_\varepsilon$  for a fixed  $\varepsilon > 0$ .

Let  $T_i$  ( $i = 1, 2, \dots, n$ ) be open subsets of  $Y = [0, 1]^N$  with smooth boundary. Let  $Y_i^* = Y_i \setminus T_i$ . Define

$$S^\varepsilon = \bigcup_{\mathbf{k} \in \mathbb{Z}^N} \bigcup_{i=1}^n a_i(\varepsilon)(\mathbf{k} + T_i)$$

and set  $T^\varepsilon = \Omega \cap S^\varepsilon$ .

Define  $\Omega_\varepsilon$  to be  $\Omega \setminus \overline{T^\varepsilon}$ . We assume  $\Omega_\varepsilon$  to be connected. The boundary of  $\Omega_\varepsilon$  consists of two parts; the union of boundaries of holes entirely contained in  $\Omega$  viz.,  $\partial_{\text{int}}\Omega_\varepsilon = \bigcup_{k \in \mathbb{Z}^N} \partial B(k)$  where,

$$B(k) = \bigcup_{i=1}^n \{a_i(\varepsilon)(k + T^i)\} \text{ and } \overline{B(k)} \subset \Omega$$

and the exterior boundary,  $\partial_{\text{ext}}\Omega_\varepsilon = \partial\Omega_\varepsilon \setminus \partial_{\text{int}}\Omega_\varepsilon$ . Consider the Neumann problem,

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla u_\varepsilon \cdot n_\varepsilon = 0 & \text{on } \partial_{\text{int}}\Omega_\varepsilon \\ u = 0 & \text{on } \partial_{\text{ext}}\Omega_\varepsilon \end{cases}, \quad (5.1)$$

where  $A_\varepsilon(x) = (([a_{ij}]_\varepsilon(x)))$  and we have the following assumptions on  $A = ((a_{ij}))$ .

- (1)  $A \in M(\alpha, \beta, \Omega \times Y_1 \times \cdots \times Y_n)$ .
- (2)  $a_{ij} \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n))$ .

It is known that (5.1) has a unique solution  $u_\varepsilon$  and that the sequence  $\{\tilde{u}_\varepsilon\}$  is bounded in  $L^2(\Omega)$  independently of  $\varepsilon$ . Assume that  $\tilde{u}_\varepsilon \rightarrow \lambda u$  in  $L^2(\Omega)$  where  $\lambda = \prod_{i=1}^n |Y_i^*|$  and  $u \in L^2(\Omega)$ . Then it is known that  $u$  solves the homogenized problem given by the following theorem.

**Theorem 5.1.** [2] *Let  $u_\varepsilon$  be the solution of (5.1). Then,*

$$\begin{aligned} \tilde{u}_\varepsilon &\xrightarrow{(n+1)-s} u(x) \chi(y_1, \dots, y_n) \\ \tilde{\nabla} u_\varepsilon &\xrightarrow{(n+1)-s} \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) \chi(y_1, \dots, y_n), \end{aligned} \quad (5.2)$$

where  $\chi(y_1, \dots, y_n) \doteq \prod_{i=1}^n \chi_i(y_i)$  and  $\chi_i$  is the characteristic function of the set  $Y_i^*$ . Also,  $(u, u_1, \dots, u_n)$  is the unique solution in

$$V = H_0^1(\Omega) \times \prod_{i=1}^n (L^2(\Omega \times Y_1 \times \cdots \times Y_{i-1}, H_\#^1(Y_i^*)))$$

of the  $(n+1)$ -scale homogenized problem:

$$\left\{ \begin{aligned} &-\operatorname{div}_{y_n}(A(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k)) = 0 && \text{in } Y_n^* \\ &A(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k) \cdot n = 0 && \text{on } \partial T_n \\ &\int_{Y_n} \chi_n(y_n) u_n dy_n = 0 \\ &-\operatorname{div}_{y_j}(\int_{Y_{j+1}} \cdots \int_{Y_n} \prod_{k=j+1}^n \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k) dy_n \cdots dy_{j+1}) = 0 && \text{in } Y_j^* \\ &(\int_{Y_{j+1}} \cdots \int_{Y_n} \prod_{k=j+1}^n \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k) dy_n \cdots dy_{j+1}) \cdot n = 0 && \text{on } \partial T_j \\ &\int_{Y_j} \chi_j(y_j) u_j dy_j = 0 \\ &\text{for } j = 1, 2, \dots, n-1 \text{ and finally,} \\ &-\operatorname{div}_x \int_{Y_1} \cdots \int_{Y_n} \prod_{k=1}^n \chi_k(y_k) A(\nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k) dy_n \cdots dy_1 = \lambda f && \text{in } \Omega \\ &u = 0 && \text{on } \partial\Omega. \end{aligned} \right. \quad (5.3)$$

□

## COROLLARY 5.1

The function  $u$  is also the unique solution of the homogenized problem

$$\begin{cases} -\operatorname{div}(A^0 \nabla u) = \lambda f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (5.4)$$

where  $A^0$  is obtained from the iterative formulae,

$$\begin{cases} A^n = A \\ A^{j-i} \xi = \int_{Y_j} \chi_k(y_k) A^j (\xi + \nabla w_j^\xi) dy_j \end{cases} \quad (5.5)$$

for  $1 \leq j \leq n$  and, for  $\xi \in \mathbb{R}^N$ ,  $w_j^\xi$  is the unique solution of the cell equation:

$$\begin{cases} -\operatorname{div}_{y_j}(A^j(\xi + \nabla w_j^\xi)) = 0 & \text{in } Y_j^* \\ A^j(\xi + \nabla w_j^\xi) \cdot n = 0 & \text{on } \partial T_j \\ \int_{Y_j} \chi_j(y_j) w_j^\xi(x, y_1, \dots, y_j) dy_j = 0 \end{cases} \quad (5.6)$$

where  $w_j^\xi \in L^2(\Omega \times Y_1 \times \dots \times Y_{j-1}; H^1(Y_j^*))$ .  $\square$

We now consider cost-functionals

$$J_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} B_\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dx$$

where  $B_\varepsilon \equiv ([b_{ij}]_\varepsilon(x))$  and  $b_{ij} \in L^\infty(\Omega, C_\#(Y_1 \times \dots \times Y_n))$ . In all that follows let  $u_\varepsilon$  be the solution of (5.1) and  $(u, u_1, \dots, u_n)$  be the solution of the homogenized problem (5.3). Let us assume that  $u_k \in L^2(\Omega, C_\#^1(Y_1 \times \dots \times Y_k))$  so that  $u_k$  may be used as test functions in multi-scale convergence. Then we show that,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon) &= \int_{\Omega} \int_{Y_1} \dots \int_{Y_n} \chi(y) B \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) \\ &\quad \times \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) dy_n \dots dy_1 dx, \end{aligned} \quad (5.7)$$

where  $y \equiv (y_1, \dots, y_n)$  with  $y_i \in Y_i$ . The proof of the convergence of cost-functionals i.e. the result (5.7) goes along the same lines as in the 2-scale case. First we require the following lemma.

**Lemma 5.1.** *With  $(u, u_1, \dots, u_n)$  as above and assumed to be regular, the sequence*

$$\left\{ \chi_\varepsilon \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \left( x, \frac{x}{a_1(\varepsilon)}, \dots, \frac{x}{a_k(\varepsilon)} \right) \right) \right\}$$

*is admissible for  $i = 1, 2, \dots, N$ .*

**Proof.** First we note that,

$$\chi_\varepsilon \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \left( x, \frac{x}{a_1(\varepsilon)}, \dots, \frac{x}{a_k(\varepsilon)} \right) \right) \xrightarrow{(n+1)-s} \chi \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right)$$

and

$$\begin{aligned} \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \chi_\varepsilon + \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \chi_\varepsilon \right)^2 dx &= \int_{\Omega} \left( \frac{\partial u}{\partial x_i} \right)^2 \chi_\varepsilon + 2 \frac{\partial u}{\partial x_i} \chi_\varepsilon \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \\ &\quad + \left( \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \right)^2 dx, \end{aligned}$$



where as usual

$$\left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \doteq \frac{\partial u_k}{\partial y_{ki}} \left( x, \frac{x}{a_1(\varepsilon)}, \dots, \frac{x}{a_k(\varepsilon)} \right).$$

Now,

$$\chi_\varepsilon \rightarrow \int_{Y_1} \cdots \int_{Y_n} \chi(y_1, \dots, y_n) dy_n \cdots dy_1 \text{ in } L^\infty(\Omega) \text{ weak } *$$

and,

$$\left( \frac{\partial u}{\partial x_i} \right)^2 \in L^1(\Omega).$$

Therefore,

$$\int_\Omega \left( \frac{\partial u}{\partial x_i} \right)^2 \chi_\varepsilon dx \rightarrow \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \left( \frac{\partial u}{\partial x_i} \right)^2 \chi dy_n \cdots dy_1 dx.$$

Next,

$$\frac{\partial u}{\partial x_i} \chi_\varepsilon \xrightarrow{(n+1)-s} \frac{\partial u}{\partial x_i} \chi(y_1, \dots, y_n)$$

and  $\{\sum_{k=1}^n [\partial u_k / \partial y_{ki}]_\varepsilon (x)\}$  is an admissible sequence since we have assumed the  $u_k$ s to be regular. Therefore by Theorem 4.2,

$$\int_\Omega \frac{\partial u}{\partial x_i} \chi_\varepsilon \left( \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \right) dx \rightarrow \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi \frac{\partial u}{\partial x_i} \left( \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right) dy_n \cdots dy_1 dx.$$

Finally,

$$\chi_\varepsilon \left( \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \right) \xrightarrow{(n+1)-s} \chi \left( \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right)$$

by Corollary 4.1. Therefore,

$$\int_\Omega \left( \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \right)^2 \chi_\varepsilon dx \rightarrow \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi \left( \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right)^2 dy_n \cdots dy_1 dx$$

by Corollary 4.2. Hence,

$$\int_\Omega \chi_\varepsilon \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \right)^2 dx \rightarrow \int_\Omega \int_{Y_1} \cdots \int_{Y_n} \chi \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right)^2 dy_n \cdots dy_1 dx$$

which proves the admissibility of the said sequence.  $\square$

Then we obtain, as in the 2-scale case, the following corrector result. The proof being similar, we omit it.

**Theorem 5.2.** *If  $u_\varepsilon$  is the solution of the Neumann problem (5.1) and we assume further that  $(u, u_1, \dots, u_n)$  are sufficiently regular then,*

$$\lim_{\varepsilon \rightarrow 0} \left\| \frac{\partial u_\varepsilon}{\partial x_i} - \frac{\partial u}{\partial x_i} \chi_\varepsilon - \sum_{k=1}^n \left[ \frac{\partial u_k}{\partial y_{ki}} \right]_\varepsilon (x) \chi_\varepsilon \right\|_{2,\Omega} = 0 \text{ for } i = 1, 2, \dots, N. \quad \square$$

We now show that  $\{\widetilde{\partial u_\varepsilon}/\partial x_i\}$  is an admissible sequence. Before that we prove the following lemma.

**Lemma 5.2.** Let  $a_\varepsilon \xrightarrow{(n+1)-s} a(x, y_1, \dots, y_n)$  and assume that  $\{a_\varepsilon\}$  is an admissible sequence. Let  $\{b_\varepsilon\}$  be a sequence of functions in  $L^2(\Omega)$  such that

$$\lim_{\varepsilon \rightarrow 0} \|a_\varepsilon - b_\varepsilon\|_{L^2(\Omega)} = 0. \quad (5.8)$$

Then  $\{b_\varepsilon\}$  is also admissible.

*Proof.* Note that we can write,  $b_\varepsilon = a_\varepsilon + t_\varepsilon$  for a sequence  $t_\varepsilon \in L^2(\Omega)$  such that  $t_\varepsilon \rightarrow 0$  strongly in  $L^2(\Omega)$ . We have that  $t_\varepsilon \xrightarrow{(n+1)-s} 0$  (since  $t_\varepsilon \rightarrow 0$  strongly in  $L^2(\Omega)$ ). Therefore,  $b_\varepsilon \xrightarrow{(n+1)-s} a(x, y_1, \dots, y_n)$  and,

$$\int_{\Omega} (b_\varepsilon)^2 dx = \int_{\Omega} (a_\varepsilon)^2 dx + 2 \int_{\Omega} a_\varepsilon t_\varepsilon dx + \int_{\Omega} (t_\varepsilon)^2 dx. \quad (5.9)$$

Hence,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (b_\varepsilon)^2 dx &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} (a_\varepsilon)^2 dx \\ &= \int_{\Omega} \int_{Y_1} \cdots \int_{Y_n} a(x, y_1, \dots, y_n)^2 dx \end{aligned}$$

(since the second and third terms in right hand side of (5.9) tend to zero by (5.8)). Hence,  $\{b_\varepsilon\}$  is an admissible sequence.  $\square$

Let  $\{u_\varepsilon\}$ ,  $u, u_1, \dots, u_n$  and the matrix  $B$  be as before. Then we have the following theorem.

**Theorem 5.3.** Assuming that  $u, u_1, \dots, u_n$  are sufficiently smooth we have,

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon) = \int_{\Omega} \int_{Y_1} \cdots \int_{Y_n} B \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) \cdot \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) dy_n \cdots dy_1 dx.$$

*Proof.* We have,

$$J_\varepsilon(u_\varepsilon) = \int_{\Omega} [b_{ij}]_\varepsilon \frac{\widetilde{\partial u_\varepsilon}}{\partial x_j} \frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} dx.$$

It follows from Lemma 5.1, Theorem 5.2 and Lemma 5.2 that  $\{\widetilde{\partial u_\varepsilon}/\partial x_i\}$  is an admissible sequence. Further

$$\frac{\widetilde{\partial u_\varepsilon}}{\partial x_i} \xrightarrow{(n+1)-s} \chi \left( \frac{\partial u}{\partial x_i} + \sum_{k=1}^n \frac{\partial u_k}{\partial y_{ki}} \right) \quad \forall i = 1, 2, \dots, N \text{ and,}$$

$$b_{ij} \in L^\infty(\Omega, C_\#(Y_1 \times \cdots \times Y_n)).$$

So, by Corollary 4.2,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} B_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx \\ &= \int_{\Omega} \int_{Y_1} \cdots \int_{Y_n} \chi B \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) \cdot \left( \nabla_x u + \sum_{k=1}^n \nabla_{y_k} u_k \right) dy_n \cdots dy_1 dx. \end{aligned}$$

□

*Remark 5.2.* By an iterative formula we can write the limit of  $J_\varepsilon(u_\varepsilon)$  as  $\int_{\Omega} B^0 \nabla u \cdot \nabla u dx$  where  $B^0$  is given by the iterative formulae,

$$B^n = B,$$

$$(B^{k-1})_{ij} = (H_0 \text{ limit of } B^k)_{ij} + \int_{Y_k} \chi_k(y_k) B^k \nabla_{y_k} (w_k^i - \eta_k^i) \nabla_{y_j} (w_k^j - \eta_k^j) dy_k$$

for all  $1 \leq i, j \leq N$  and for  $k = 1, 2, \dots, n$ ; where,  $\eta_k^i$  is the unique solution of the  $i$ th cell equation for  $B^k$  viz.,

$$\begin{aligned} -\operatorname{div}_{y_k} (B^k (e^i + \nabla_{y_k} \eta_k^i)) &= 0 \quad \text{in } Y_k^* \\ B^k (e^i + \nabla_{y_k} \eta_k^i) \cdot n &= 0 \quad \text{on } \partial T_k \\ \int_{Y_k} \chi_k(y_k) \eta_k^i dy_k &= 0. \end{aligned}$$

and  $w_k^i$  are obtained from (5.6) by taking  $\xi = e^i$ .

## References

- [1] Allaire G, Homogenization and two-scale convergence, *SIAM J. Math. Anal.* **23** (1992) 1482–1518
- [2] Allaire G and Briane M, Multiscale convergence and reiterated homogenization, *Publ. du Laboratoire d'Analyse Numérique*, R94019 (1994)
- [3] Allaire G and Murat M, Homogenization of the Neumann problem with nonisolated holes, *Asymptotic Anal.* **7** (1993) 81–95
- [4] Kesavan S and Saint Jean Paulin J, Homogenization of an optimal control problem, *SIAM J. Control. Optim.* **35** (1997) 1557–1573
- [5] Kesavan S and Saint Jean Paulin J, *Optimal control on perforated domains*, to appear (1997)
- [6] Kesavan S and Vanninathan M, L'homogénéisation d'un problème de contrôle optimal, *C.R.A.S Série A*, t. **285** (1977) 441–444
- [7] Lions J L, *Optimal Control of Systems Governed by Partial Differential Equations* (Springer-Verlag) (1971)
- [8] Nguetseng G, A general convergence result of a functional related to the theory of homogenization, *SIAM J. Math. Anal.* **20** (1989) 608–623