

On the limit matrix obtained in the homogenization of an optimal control problem

S KESAVAN* and M RAJESH

*The Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai 600 113, India
 Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
 E-mail: kesh@imsc.ernet.in; rajesh@math.iisc.ernet.in

MS received 6 January 2001

Abstract. A new formulation for the limit matrix occurring in the cost functional of an optimal control problem on homogenization is obtained. It is used to obtain an upper bound for this matrix (in the sense of positive definite matrices).

Keywords. Homogenization; optimal control; elliptic equations.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $0 < \alpha_m < \alpha_M$. We denote by $\mathcal{M}(\alpha_m, \alpha_M, \Omega)$ the set of all $N \times N$ matrices $A = A(x)$, with coefficients in $L^\infty(\Omega)$, such that

$$\alpha_m |\xi|^2 \leq A(x)\xi \cdot \xi \leq \alpha_M |\xi|^2, \quad \text{a.e. } x, \quad \text{for all } \xi \in \mathbb{R}^N.$$

Given a family of matrices $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M, \Omega)$, let $v_\varepsilon \in H_0^1(\Omega)$ be the unique (weak) solution of the problem

$$\left. \begin{aligned} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) &= f \text{ in } \Omega \\ v_\varepsilon &= 0 \text{ on } \partial\Omega \end{aligned} \right\} \quad (1.1)$$

where $f \in L^2(\Omega)$ is given. Then $\{v_\varepsilon\}$ is bounded in $H_0^1(\Omega)$ and if $v_\varepsilon \rightharpoonup v_0$ in $H_0^1(\Omega)$, we have

$$\left. \begin{aligned} -\operatorname{div}(A_0 \nabla v_0) &= f \text{ in } \Omega \\ v_0 &= 0 \text{ on } \partial\Omega \end{aligned} \right\} \quad (1.2)$$

when A_ε H -converges to A_0 (cf. Murat [7]). We know that $\{v_\varepsilon\}$ does not converge strongly in $H_0^1(\Omega)$. Nevertheless, $\int_\Omega |\nabla v_\varepsilon|^2 dx$ is bounded and hence (at least for a subsequence) converges. We would like to know if this limit can be expressed in terms of the function v_0 . More generally, if $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$, be another family of matrices, consider the ‘energy’ defined by

$$\int_\Omega B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon dx.$$

Again, this is a bounded sequence and we would like to express its limit (when it converges) in terms of v_0 . More precisely, we would like to know if there exists a matrix $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ such that

$$\int_{\Omega} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \rightarrow \int_{\Omega} B^\# \nabla v_0 \cdot \nabla v_0 \, dx$$

and, if so, identify that matrix and estimate the constants $\tilde{\beta}_m$ and $\tilde{\beta}_M$.

When $B_\varepsilon = A_\varepsilon$, it is well-known (cf. Murat [7]) that indeed $B^\# = A_0$, the H -limit of A_ε .

It turns out that the solution to this problem is closely related to the question of homogenizing an associated optimal control problem.

Let $U_{\text{ad}} \subset L^2(\Omega)$ be a closed convex set (called the set of admissible controls) and let $f \in L^2(\Omega)$ be given. Given $\theta \in U_{\text{ad}}$, let $u_\varepsilon \in H_0^1(\Omega)$ be the unique (weak) solution of the ‘state equation’:

$$\left. \begin{aligned} -\operatorname{div} (A_\varepsilon \nabla u_\varepsilon) &= f + \theta \quad \text{in } \Omega \\ u_\varepsilon &= 0 \quad \text{on } \partial\Omega \end{aligned} \right\} \tag{1.3}$$

Then, there exists a unique ‘optimal control’ $\theta_\varepsilon^* \in U_{\text{ad}}$ such that

$$J_\varepsilon(\theta_\varepsilon^*) = \min_{\theta \in U_{\text{ad}}} J_\varepsilon(\theta), \tag{1.4}$$

where, for $\theta \in U_{\text{ad}}$, and $u_\varepsilon = u_\varepsilon(\theta)$ solution of (1.3),

$$J_\varepsilon(\theta) = \frac{1}{2} \int_{\Omega} B_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon \, dx + \frac{N}{2} \int_{\Omega} \theta^2 \, dx, \tag{1.5}$$

$N > 0$ being a fixed constant, called the ‘cost of the control’.

The homogenization of the optimal control problem (1.3)–(1.5) was first studied in the periodic case by Kesavan and Vanninathan [5] and then in the general case under the framework of H -convergence by Kesavan and Saint Jean Paulin [3]. They also extended these results (cf. [4]) to the ‘perforated case’ where Ω is replaced by a family of ‘perforated domains’ $\Omega_\varepsilon \subset \Omega$. In all these cases, it was shown that there exists a matrix $B^\#$ such that, in the limit, there is an optimal control problem with A_0 and $B^\#$ replacing A_ε and B_ε respectively in (1.3)–(1.5).

The expression derived for $B^\#$ is complicated and the symmetry of this matrix, when all the B_ε are symmetric, requires a detailed proof (cf. [3,4]). Further, while the ellipticity could be proved, no upper bound, i.e. an estimate for $\tilde{\beta}_M$, could be derived.

In this paper, a new formula for $B^\#$ is obtained and, in the symmetric case, the symmetry can be read off directly from it. An upper bound is also derived.

The paper is organized as follows. In §2, the equivalence of the two problems stated above is studied and the existence and uniqueness of the matrix $B^\#$ is established based on the results of Kesavan and Saint Jean Paulin [3]. In §3, the new formula for $B^\#$ is derived and its properties are studied. In §4, the corresponding results for the perforated case are stated.

2. Two equivalent problems

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M, \Omega)$ and $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$ be given. We now consider two statements.

(S1) There exists a matrix $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ such that, given a strongly convergent sequence $\{g_\varepsilon\}$ in $H^{-1}(\Omega)$ and the corresponding sequence $\{v_\varepsilon\}$ of solutions in $H_0^1(\Omega)$ of the problem

$$\left. \begin{aligned} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) &= g_\varepsilon \text{ in } \Omega \\ v_\varepsilon &= 0 \text{ on } \partial\Omega \end{aligned} \right\}, \quad (2.1)$$

then, for a subsequence, $v_\varepsilon \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$ and

$$\int_\Omega B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx \rightarrow \int_\Omega B^\# \nabla v_0 \cdot \nabla v_0 \, dx, \quad (2.2)$$

$$B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup B^\# \nabla v_0 \cdot \nabla v_0 \text{ in } \mathcal{D}'(\Omega). \quad \blacksquare \quad (2.3)$$

Remark 2.1. If $g_\varepsilon \rightarrow g$ in $H^{-1}(\Omega)$, then (cf. Murat [7])

$$\left. \begin{aligned} -\operatorname{div}(A_0 \nabla v_0) &= g \text{ in } \Omega \\ v_0 &= 0 \text{ on } \partial\Omega \end{aligned} \right\}, \quad (2.4)$$

where A_0 is the H -limit of A_ε . Also

$$A_\varepsilon \nabla v_\varepsilon \rightharpoonup A_0 \nabla v_0 \quad (2.5)$$

weakly in $L^2(\Omega)^N$. \blacksquare

In order to make the second statement, we need to introduce the ‘adjoint state’ function. Let g_ε and v_ε be as above. Then we denote by $p_\varepsilon \in H_0^1(\Omega)$, the adjoint state, which is the solution of

$$\left. \begin{aligned} \operatorname{div}({}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &= 0 \text{ in } \Omega \\ p_\varepsilon &= 0 \text{ on } \partial\Omega \end{aligned} \right\}, \quad (2.6)$$

where, we have denoted the transpose of A_ε by ${}^t A_\varepsilon$.

Remark 2.2. From the hypotheses, it is evident that $\{p_\varepsilon\}$ is bounded in $H_0^1(\Omega)$. \blacksquare

Remark 2.3. The system consisting of (2.1) and (2.6) is of the type used by Lions [6] to construct the optimality system to solve the optimal control problem (1.3)–(1.5), using a duality argument. The system consisting of (2.1) and (2.6) was used by Kesavan and Saint Jean Paulin [3] to homogenize the optimal control problem (1.3)–(1.5). \blacksquare

(S2) There exists a matrix $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ such that, given g_ε strongly convergent in $H^{-1}(\Omega)$ and v_ε solution of (2.1) and p_ε solution of (2.6), then, for a subsequence, $v_\varepsilon \rightharpoonup v_0$, $p_\varepsilon \rightharpoonup p_0$ weakly in $H_0^1(\Omega)$ and $z_\varepsilon = {}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon \rightharpoonup z$ weakly in $L^2(\Omega)^N$, where

$$z = {}^t A_0 \nabla p_0 - B^\# \nabla v_0 \quad (2.7)$$

and A_0 is the H -limit of A_ε . \blacksquare

Remark 2.4. The pair (v_0, p_0) will satisfy the homogenized system:

$$\left. \begin{aligned} -\operatorname{div}(A_0 \nabla v_0) &= g \text{ in } \Omega \\ \operatorname{div}({}^t A_0 \nabla p_0 - B^\# \nabla v_0) &= 0 \text{ in } \Omega \\ v_0 = p_0 &= 0 \text{ on } \partial\Omega \end{aligned} \right\}. \quad \blacksquare \quad (2.8)$$

We now prove the equivalence of these two statements.

Theorem 2.1. *If $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ is such that (S2) is true, then it also verifies (S1). The converse is true if B_ε and $B^\#$ are symmetric.*

Proof. Let (S2) hold for $B^\#$. Now, by virtue of (2.1), (2.4), (2.6) and (2.8), we have, for the relevant subsequence,

$$\begin{aligned} \int_{\Omega} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx &= \int_{\Omega} {}^t A_\varepsilon \nabla p_\varepsilon \cdot \nabla v_\varepsilon \, dx \\ &= \int_{\Omega} A_\varepsilon \nabla v_\varepsilon \cdot \nabla p_\varepsilon \, dx \\ &= \langle g_\varepsilon, p_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &\rightarrow \langle g, p_0 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ &= \int_{\Omega} A_0 \nabla v_0 \cdot \nabla p_0 \, dx \\ &= \int_{\Omega} B^\# \nabla v_0 \cdot \nabla v_0 \, dx. \end{aligned}$$

This proves (2.2). Now,

$$B_\varepsilon \nabla v_\varepsilon \cdot v_\varepsilon = -({}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) \cdot \nabla v_\varepsilon + (A_\varepsilon \nabla v_\varepsilon) \cdot \nabla p_\varepsilon.$$

Now, by virtue of (2.1) and (2.6), the divergences of the expressions within parantheses in each of the two terms in the right-hand side are strongly convergent in $H^{-1}(\Omega)$. Also, ∇v_ε and ∇p_ε converge weakly in $L^2(\Omega)^N$. Thus, by the div-curl lemma of compensated compactness theory (cf. Murat [7], Murat and Tartar [8] or Tartar [10]), we conclude that, in view of (2.5) and (2.7),

$$\begin{aligned} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon &\rightharpoonup -({}^t A_0 \nabla p_0 - B^\# \nabla v_0) \cdot \nabla v_0 + A_0 \nabla v_0 \cdot \nabla p_0 \\ &= B^\# \nabla v_0 \cdot \nabla v_0 \end{aligned}$$

in $\mathcal{D}'(\Omega)$. This proves (2.3).

Conversely, let B_ε and $B^\#$ be symmetric and assume that (S1) holds. Let $\omega \Subset \Omega$ be a (relatively compact) open subset and let $\eta \in \mathcal{D}(\Omega)$ be such that $\eta \equiv 1$ in a neighborhood of ω . Define $\eta_\varepsilon^k \in H_0^1(\Omega)$, $1 \leq k \leq N$, to be the unique solution of the problem

$$\left. \begin{aligned} -\operatorname{div}(A_\varepsilon \nabla \eta_\varepsilon^k) &= -\operatorname{div}(A_0 \nabla(\eta x_k)) \quad \text{in } \Omega \\ \eta_\varepsilon^k &= 0 \quad \text{on } \partial\Omega \end{aligned} \right\}. \quad (2.9)$$

Then, by H -convergence, $\eta_\varepsilon^k \rightharpoonup \eta x_k$ weakly in $H_0^1(\Omega)$ and $A_\varepsilon \nabla \eta_\varepsilon^k \rightharpoonup A_0 \nabla(\eta x_k)$ weakly in $L^2(\Omega)^N$. By superposition of the solutions of (2.1) and (2.9), we get

$$-\operatorname{div}(A_\varepsilon \nabla(v_\varepsilon \pm \eta_\varepsilon^k)) = g_\varepsilon \pm (-\operatorname{div}(A_0 \nabla(\eta x_k))) \quad \text{in } \Omega.$$

Hence, by (S1), for a subsequence,

$$B_\varepsilon \nabla(v_\varepsilon \pm \eta_\varepsilon^k) \cdot \nabla(v_\varepsilon \pm \eta_\varepsilon^k) \rightharpoonup B^\# \nabla(v_0 \pm \eta x_k) \cdot \nabla(v_0 \pm \eta x_k)$$

in $\mathcal{D}'(\Omega)$. Hence, using the polarization identity which requires the symmetry of B_ε and $B^\#$, we get that

$$B_\varepsilon \nabla v_\varepsilon \cdot \nabla \eta_\varepsilon^k \rightharpoonup B^\# \nabla v_0 \cdot \nabla(\eta x_k) \quad (2.10)$$

in $\mathcal{D}'(\Omega)$. We can now apply the div-curl lemma to the pair $(z_\varepsilon, \nabla \eta_\varepsilon^k)$, since $\operatorname{div} z_\varepsilon = 0$, to get

$$z_\varepsilon \cdot \nabla \eta_\varepsilon^k \rightharpoonup z \cdot \nabla(\eta x_k)$$

in $\mathcal{D}'(\Omega)$. On the other hand

$$\begin{aligned} z_\varepsilon \cdot \nabla \eta_\varepsilon^k &= A_\varepsilon \nabla \eta_\varepsilon^k \cdot \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon \cdot \nabla \eta_\varepsilon^k \\ &\rightharpoonup A_0 \nabla(\eta x_k) \cdot \nabla p_0 - B^\# \nabla v_0 \cdot \nabla(\eta x_k) \end{aligned}$$

again by applying the div-curl lemma to the first term on the right-hand side and by also using (2.10). Thus, on ω , we have

$$z \cdot e_k = {}^t A_0 \nabla p_0 \cdot e_k - B^\# \nabla v_0 \cdot e_k$$

where e_k is the standard k -th basis vector of \mathbb{R}^N . This proves (2.7) on ω and as $\omega \Subset \Omega$ was arbitrary, we have the result on Ω . ■

Remark 2.5. We can use the test functions η_ε^j to prove the uniqueness of the matrix $B^\#$, when it exists. Indeed, if we have two matrices $B_i^\#, i = 1, 2$, satisfying (S2), (or (S1), in the symmetric case), setting $v_\varepsilon = \eta_\varepsilon^j$, we then have that (cf. (2.7))

$${}^t A_0 \nabla p_0 - B_1^\# \nabla(\eta x_j) = {}^t A_0 \nabla p_0 - B_2^\# \nabla(\eta x_j).$$

Thus on ω , we have that $B_1^\# = B_2^\#$ and the result follows for all of Ω since $\omega \Subset \Omega$ is arbitrary. ■

The existence of a $B^\#$ satisfying (S2) was proved in the general case by Kesavan and Saint Jean Paulin [3]. We recall their formulation and also give another, shorter, proof of their result.

We first need to define some test functions. First of all, we recall the existence of functions $X_\varepsilon^k \in H^1(\Omega)$, for $1 \leq k \leq N$, with the following properties (cf. Murat [7]).

$$\left. \begin{aligned} X_\varepsilon^k &\rightharpoonup x_k \text{ weakly in } H^1(\Omega) \\ A_\varepsilon \nabla X_\varepsilon^k &\rightharpoonup A_0 e_k \text{ weakly in } L^2(\Omega)^N \\ \{\operatorname{div}(A_\varepsilon \nabla X_\varepsilon^k)\} &\text{ converges strongly in } H^{-1}(\Omega) \end{aligned} \right\}. \quad (2.11)$$

We now define another set of test functions $\psi_\varepsilon^k \in H_0^1(\Omega)$ for $1 \leq k \leq N$, which verify

$$\left. \begin{aligned} -\operatorname{div}({}^t A_\varepsilon \nabla \psi_\varepsilon^k + {}^t B_\varepsilon \nabla X_\varepsilon^k) &= 0 \text{ in } \Omega \\ \psi_\varepsilon^k &= 0 \text{ on } \partial\Omega \end{aligned} \right\}. \quad (2.12)$$

Then, up to a subsequence, $\{\psi_\varepsilon^k\}$ converges weakly in $H_0^1(\Omega)$ to ψ_0^k and $\{{}^t A_\varepsilon \nabla \psi_\varepsilon^k + {}^t B_\varepsilon \nabla X_\varepsilon^k\}$ converges weakly in $L^2(\Omega)^N$. Then, (cf. [3]), we define

$${}^t(B^\#)e_k = \lim_{\varepsilon \rightarrow 0} ({}^t A_\varepsilon \nabla \psi_\varepsilon^k + {}^t B_\varepsilon \nabla X_\varepsilon^k) - {}^t A_0 \nabla \psi_0^k. \quad (2.13)$$

Theorem 2.2. $B^\#$ defined by (2.13) satisfies (S2).

Proof. We extract a subsequence such that all the bounded sequences that occur below are convergent in the relevant weak topologies. Let $g_\varepsilon \rightarrow g$ strongly in $H^{-1}(\Omega)$ and $(v_\varepsilon, p_\varepsilon) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the solution of (2.1) and (2.6). Let $v_\varepsilon \rightharpoonup v_0$ and $p_\varepsilon \rightharpoonup p_0$ weakly in $H_0^1(\Omega)$ and let $z_\varepsilon \rightharpoonup z$ weakly in $L^2(\Omega)^N$. Now

$$\begin{aligned} z_\varepsilon \cdot \nabla X_\varepsilon^k &= \nabla p_\varepsilon \cdot A_\varepsilon \nabla X_\varepsilon^k - {}^t B_\varepsilon \nabla X_\varepsilon^k \cdot \nabla v_\varepsilon \\ &= (A_\varepsilon \nabla X_\varepsilon^k) \cdot \nabla p_\varepsilon - ({}^t A_\varepsilon \nabla \psi_\varepsilon^k + {}^t B_\varepsilon \nabla X_\varepsilon^k) \cdot \nabla v_\varepsilon \\ &\quad + (A_\varepsilon \nabla v_\varepsilon) \cdot \nabla \psi_\varepsilon^k \end{aligned}$$

We can pass to the limit, using the div-curl lemma, in each term to get

$$z \cdot e_k = A_0 e_k \cdot \nabla p_0 - \lim_{\varepsilon \rightarrow 0} ({}^t A_\varepsilon \nabla \psi_\varepsilon^k + {}^t B_\varepsilon \nabla X_\varepsilon^k) \cdot \nabla v_0 + A_0 \nabla v_0 \cdot \nabla \psi_0^k$$

using (2.13) from which (2.7) follows. \blacksquare

Remark 2.6. In the statements (S1) and (S2), we have required that the relevant convergences occur for a subsequence. It must be noted that the subsequence is independent of the strongly convergent sequence g_ε . Indeed, it depends only on the convergences implied in (2.11) and (2.13) (cf. Rajesh [9]). \blacksquare

3. Properties of $B^\#$

We are now interested in properties like the symmetry and ellipticity of the matrix $B^\#$ defined in the previous section. Kesavan and Saint Jean Paulin [3] proved that it is symmetric when all the B_ε are symmetric and that $\tilde{\beta}_m = \beta_m$. However, the problem of estimating $\tilde{\beta}_M$ was left open. In this section, a new formula for $B^\#$ will be given from which the symmetry can be just read off and which will also enable us to estimate $\tilde{\beta}_M$.

First of all, we recall the corrector matrices occurring in the study of H -convergence of A_ε , as introduced by Murat [7]. If X_ε^k are the test functions introduced in the previous section (cf. (2.11)), the corrector matrices are defined by

$$M_\varepsilon e_k = \nabla X_\varepsilon^k, \quad 1 \leq k \leq N. \tag{3.1}$$

Then, the following properties hold (cf. Murat [7] or Murat and Tartar [8]):

$$\left. \begin{aligned} M_\varepsilon &\rightharpoonup I \text{ weakly in } L^2(\Omega)^{N^2} \\ A_\varepsilon M_\varepsilon &\rightharpoonup A_0 \text{ weakly in } L^2(\Omega)^{N^2} \\ {}^t M_\varepsilon A_\varepsilon M_\varepsilon &\rightharpoonup A_0 \text{ in } \mathcal{D}'(\Omega)^{N^2} \\ \{\text{div}(A_\varepsilon M_\varepsilon)\} &\text{ converges strongly in } H^{-1}(\Omega)^N \end{aligned} \right\}. \tag{3.2}$$

We now prove the main result of this section.

Theorem 3.1. $B^\#$ defined by (2.13) is the limit, in the sense of distributions, of ${}^t M_\varepsilon B_\varepsilon M_\varepsilon$.

Proof. It is enough to show that, for $1 \leq i, j \leq N$,

$${}^t M_\varepsilon B_\varepsilon M_\varepsilon e_k \cdot e_j \rightharpoonup B^\# e_k \cdot e_j$$

in $\mathcal{D}'(\Omega)$. Now

$$\begin{aligned} {}^t M_\varepsilon B_\varepsilon M_\varepsilon e_k \cdot e_j &= {}^t B_\varepsilon M_\varepsilon e_j \cdot M_\varepsilon e_k \\ &= ({}^t A_\varepsilon \nabla \psi_\varepsilon^j + {}^t B_\varepsilon M_\varepsilon e_j) \cdot M_\varepsilon e_k - {}^t A_\varepsilon \nabla \psi_\varepsilon^j \cdot M_\varepsilon e_k \\ &= ({}^t A_\varepsilon \nabla \psi_\varepsilon^j + {}^t B_\varepsilon \nabla X_\varepsilon^j) \cdot \nabla X_\varepsilon^k - A_\varepsilon \nabla X_\varepsilon^k \cdot \nabla \psi_\varepsilon^j. \end{aligned}$$

We can pass to the limit in each of the two terms on the right-hand side using the div-curl lemma to get

$${}^t M_\varepsilon B_\varepsilon M_\varepsilon e_k \cdot e_j \rightharpoonup {}^t B^\# e_j \cdot e_k = B^\# e_k \cdot e_j. \quad \blacksquare$$

COROLLARY 3.1

If the B_ε are symmetric, then so is $B^\#$. \blacksquare

Theorem 3.2. $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$, where $\tilde{\beta}_m = \beta_m$ and

$$\tilde{\beta}_M = \beta_M \left(\frac{\alpha_M}{\alpha_m} \right)^2.$$

Proof. That $\tilde{\beta}_m = \beta_m$ has already been proved in [3]. Let $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$ and let $\xi \in \mathbb{R}^N$. Then

$$\begin{aligned} \int_\Omega B_\varepsilon M_\varepsilon \xi \cdot M_\varepsilon \xi \varphi \, dx &\leq B_M \int_\Omega |M_\varepsilon \xi|^2 \varphi \, dx \\ &\leq \frac{\beta_M}{\alpha_m} \int_\Omega A_\varepsilon M_\varepsilon \xi \cdot M_\varepsilon \xi \varphi \, dx \\ &= \frac{\beta_M}{\alpha_m} \int_G {}^t M_\varepsilon A_\varepsilon M_\varepsilon \xi \cdot \xi \varphi \, dx. \end{aligned}$$

Passing to the limit, using Theorem 3.1 and (3.2), we get

$$\int_\Omega B^\# \xi \cdot \xi \varphi \, dx \leq \frac{\beta_M}{\alpha_m} \int_\Omega A_0 \xi \cdot \xi \varphi \, dx \leq \frac{\beta_M}{\alpha_m} \frac{\alpha_M^2}{\alpha_m} |\xi|^2 \int_\Omega \varphi \, dx,$$

since we know that $A_0 \in \mathcal{M}(\tilde{\alpha}_m, \tilde{\alpha}_M, \Omega)$, where $\tilde{\alpha}_m = \alpha_m$ and $\tilde{\alpha}_M = \alpha_M^2/\alpha_m$ (cf. Murat [7]). Since φ was arbitrary, this proves the theorem. \blacksquare

Remark 3.1. Consider the one-dimensional case. Let $0 < \alpha_M \leq a_\varepsilon(x) \leq \alpha_M$ and $0 < \beta_M \leq b_\varepsilon(x) \leq \beta_M$. Let

$$\begin{aligned} -\frac{d}{dx} (a_\varepsilon \frac{du_\varepsilon}{dx}) &= f \quad \text{in } (0, 1) \\ u_\varepsilon(0) &= u_\varepsilon(1) = 0. \end{aligned}$$

Then, it has been shown by Kesavan and Saint Jean Paulin [3] that

$$\int_0^1 b_\varepsilon \frac{du_\varepsilon}{dx} \frac{du_\varepsilon}{dx} \, dx \rightarrow \int_0^1 b^\# \frac{du_0}{dx} \frac{du_0}{dx} \, dx$$

with

$$b^\# = \frac{a_0^2}{g_0},$$

where

$$\frac{1}{a_\varepsilon} \rightharpoonup \frac{1}{a_0} \quad \text{and} \quad \frac{1}{g_\varepsilon} \equiv \frac{b_\varepsilon}{a_\varepsilon^2} \rightharpoonup \frac{1}{g_0} \quad \text{in } L^\infty(0, 1) \text{ weak }^*.$$

This yields precisely the bound obtained above for $b^\#$. ■

4. The perforated case

We now briefly describe the problem in the perforated case and state the results without proofs, since those of the corresponding results in the previous case carry over *mutatis mutandis*.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and for $\varepsilon > 0$, let $S_\varepsilon \subset \Omega$ be a closed set (the set of perforations). We call $\Omega_\varepsilon = \Omega \setminus S_\varepsilon$ the perforated domain. Following Briane, Damlamian and Donato [1], we say that a family $\{S_\varepsilon\}$ of holes is admissible if the following conditions are fulfilled.

H1 If χ_ε is the characteristic function of S_ε , then every weak- * limit of $\{\chi_\varepsilon\}$ in $L^\infty(\Omega)$ is positive a.e. ■

H2 Let

$$V_\varepsilon = \{u \in H^1(\Omega_\varepsilon) \mid u = 0 \text{ on } \partial\Omega\}.$$

Then, there exists an extension operator $P_\varepsilon: V_\varepsilon \rightarrow H_0^1(\Omega)$ and a constant $C_0 > 0$, independent of ε such that, for any $u \in V_\varepsilon$,

$$P_\varepsilon u|_{\Omega_\varepsilon} = u \quad \text{and} \quad \|\nabla P_\varepsilon u\|_{L^2(\Omega)^N} \leq C_0 \|\nabla u\|_{L^2(\Omega_\varepsilon)^N}. \quad \blacksquare \quad (4.1)$$

Analogous to the theory of H -convergence, we have a theory of H_0 -convergence (cf. [1]).

We now assume that we have two families of matrices $A_\varepsilon \in \mathcal{M}(\alpha_m, \alpha_M, \Omega)$ and $B_\varepsilon \in \mathcal{M}(\beta_m, \beta_M, \Omega)$. We denote by $P_\varepsilon^*: H^{-1}(\Omega) \rightarrow V_\varepsilon^*$, the adjoint of P_ε . For functions in $L^2(\Omega_\varepsilon)$, we also have the trivial extension operator Q_ε which extends the functions by zero across the holes to give a function in $L^2(\Omega)$. We denote the unit outward normal (with respect to Ω_ε) on ∂S_ε be n_ε .

(S3) There exists a matrix $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ such that, given a strongly convergent sequence $\{g_\varepsilon\}$ in $H^{-1}(\Omega)$ and the corresponding sequence of solutions $v_\varepsilon \in V_\varepsilon$ of

$$\left. \begin{aligned} -\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) &= P_\varepsilon^* g_\varepsilon && \text{in } \Omega_\varepsilon \\ A_\varepsilon \nabla v_\varepsilon \cdot n_\varepsilon &= 0 && \text{on } \partial S_\varepsilon \\ v_\varepsilon &= 0 && \text{on } \partial\Omega \end{aligned} \right\}, \quad (4.2)$$

then, for a subsequence, $P_\varepsilon v_\varepsilon \rightharpoonup v_0$ weakly in $H_0^1(\Omega)$ and

$$\left. \begin{aligned} \int_{\Omega_\varepsilon} B_\varepsilon \nabla v_\varepsilon \cdot \nabla v_\varepsilon \, dx &\rightharpoonup \int_{\Omega} B^\# \nabla v_0 \cdot \nabla v_0 \, dx \\ \chi_\varepsilon B_\varepsilon \nabla(P_\varepsilon v_\varepsilon) \cdot \nabla(P_\varepsilon v_\varepsilon) &\rightharpoonup B^\# \nabla v_0 \cdot \nabla v_0 \text{ in } \mathcal{D}'(\Omega) \end{aligned} \right\}. \quad \blacksquare \quad (4.3)$$

(S4) There exists a matrix $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ such that given a strongly convergent sequence $\{g_\varepsilon\}$ in $H^{-1}(\Omega)$ and the sequence $\{v_\varepsilon\}$ of solutions of (4.2) and the sequence $\{p_\varepsilon\}$ in V_ε of solutions of

$$\left. \begin{aligned} \operatorname{div}({}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) &= 0 \text{ in } \Omega_\varepsilon \\ ({}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) \cdot n_\varepsilon &= 0 \text{ on } \partial S_\varepsilon \\ p_\varepsilon &= 0 \text{ on } \partial \Omega \end{aligned} \right\}, \quad (4.4)$$

then, for a subsequence, $P_\varepsilon v_\varepsilon \rightharpoonup v_0$, $P_\varepsilon p_\varepsilon \rightharpoonup p_0$ weakly in $H_0^1(\Omega)$, $z_\varepsilon = Q_\varepsilon({}^t A_\varepsilon \nabla p_\varepsilon - B_\varepsilon \nabla v_\varepsilon) \rightharpoonup z$ weakly in $L^2(\Omega)^N$ where

$$z = {}^t A_0 \nabla p_0 - B^\# \nabla v_0, \quad (4.5)$$

A_0 being the H_0 -limit of $\{A_\varepsilon\}$. ■

Theorem 4.1. *If $B^\# \in \mathcal{M}(\tilde{\beta}_m, \tilde{\beta}_M, \Omega)$ is such that (S4) is true, then so is (S3). The converse holds when the B_ε and $B^\#$ are symmetric.* ■

Kesavan and Saint Jean Paulin [4] gave a formula for a matrix $B^\#$ such that (S4) holds. If we define the corrector matrices \tilde{M}_ε by

$$\tilde{M}_\varepsilon e_k = \nabla(P_\varepsilon \tilde{X}_\varepsilon^k),$$

where the \tilde{X}_ε^k are test functions with properties analogous to those mentioned in (2.11) (cf. [1] or [4]), it can be shown that (cf. Rajesh [9])

$$\chi_\varepsilon {}^t \tilde{M}_\varepsilon B_\varepsilon \tilde{M}_\varepsilon \rightharpoonup B^\#$$

in the sense of distributions. Thus, if B_ε are symmetric, so is $B^\#$ and we can show that

$$\tilde{\beta}_M = \beta_M \left(\frac{\alpha_M}{\alpha_m} \right)^2.$$

Kesavan and Saint Jean Paulin [4] proved that for all $w \in H_0^1(\Omega)$

$$\int_{\Omega} B^\# \nabla w \cdot \nabla w \, dx \geq \beta_m C_0^{-2} \|w\|_{H_0^1(\Omega)}^2.$$

This, in fact, implies that $\tilde{\beta}_m = \beta_m C_0^{-2}$. We give a proof of this below. It is adapted from a similar proof by Casado-Díaz [2], but with different test functions.

Lemma 4.1. *Let $A = A(x)$ be a symmetric $N \times N$ matrix with coefficients in $L^\infty(\Omega)$ such that*

$$\int_{\Omega} A \nabla w \cdot \nabla w \, dx \geq 0 \quad (4.6)$$

for all $w \in H_0^1(\Omega)$. Then

$$A(x)\xi \cdot \xi \geq 0 \quad (4.7)$$

a.e. in Ω for all $\xi \in \mathbb{R}^N$.

Proof. Let $\xi \in \mathbb{R}^N$ and let $\varphi \geq 0$ be a C_0^1 -function. Define

$$\begin{aligned} v_\varepsilon(x) &= \varepsilon \cos(\varepsilon^{-1}\xi \cdot x)\varphi(x), \\ w_\varepsilon(x) &= \varepsilon \sin(\varepsilon^{-1}\xi \cdot x)\varphi(x). \end{aligned}$$

Then $v_\varepsilon, w_\varepsilon \in H_0^1(\Omega)$. Applying (4.6) to both v_ε and w_ε and adding the resulting inequalities, we get

$$\varepsilon^2 \int_{\Omega} A \nabla \varphi \cdot \nabla \varphi \, dx + \int_{\Omega} (A \xi \cdot \xi) \varphi^2 \, dx \geq 0.$$

Passing to the limit as $\varepsilon \rightarrow 0$, using the arbitrariness of φ , we get (4.7) by standard arguments. ■

References

- [1] Briane M, Damlamian A and Donato P, H -convergence for perforated domains, in: Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Vol. XIII, Pitman Research Notes in Mathematics, 1996
- [2] Casado-Díaz J, Personal Communication
- [3] Kesavan S and Saint Jean Paulin J, Homogenization of an optimal control problem, *SIAM J. Control Optim.* **35** (1997) 1557–1573
- [4] Kesavan S and Saint Jean Paulin J, Optimal control on perforated domains, *J. Math. Anal. Appl.* **229** (1999) 563–586
- [5] Kesavan S and Vanninathan M, L'homogénéisation d'un problème de contrôle optimal, *C. R. Acad. Sci., Paris, Série A*, **285** (1977) 441–444
- [6] Lions JL, Optimal Control of Systems Governed by Partial Differential Equations (Berlin: Springer-Verlag) (1971)
- [7] Murat F, H -convergence, Mimeographed notes, Séminaire d'Analyse Fonctionnelle et Numérique, Université d'Alger, 1977/78
- [8] Murat F and Tartar L, H -convergence, in: Topics in the Mathematical Modelling of Composite Materials (eds) A Cherkhaev and R Kohn (Birkhauser) (1997) 21–43
- [9] Rajesh M, Some Problems in Homogenization, Thesis (Indian Statistical Institute, Calcutta) (2000)
- [10] Tartar L, Compensated compactness and applications to partial differential equations, in: Nonlinear Analysis and Mechanics, Heriott Watt Symposium (ed) R J Knops, Pitman Research Notes in Mathematics, **39** (1979) 136–212.