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EDGE VIBRATIONS IN COMPOSITE LAMINATED SANDWICH PLATES BY USING A HIGHER ORDER DISPLACEMENT BASED THEORY

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A higher order displacement based formulation has been developed to investigate the plane strain edge vibrations or end modes in composite laminated sandwich plates. The formulation has been applied as an illustration, to the laminated sandwich plates made up of transversely isotropic laminae with the axes of symmetry lying in the plane of the lamina and core in-between. The results for isotropic and orthotropic plates are shown to be in excellent agreement with the published numerical solutions. Also, numerical results are obtained for two more examples consisting of two typical sandwich plates to obtain an insight into the physical behaviour of laminated sandwich plates. The higher order method proposed here is found to give equally accurate results by using only about half the number of degrees of freedom in comparison with the numerical techniques available in the literature.

1. INTRODUCTION

Composite materials are finding applications in a large spectrum of modern structures like spacecraft, bodies of high-speed automobiles and even in common civil engineering structures. Therefore, the prediction of exact dynamic behaviour of composite laminated structures has become essential. The complexities attendant with the dynamic analysis of laminated plates, are so much that except a few special cases, exact solutions do not exist. The anisotropic properties of the composite lamina along with the through thickness warping of cross-section make the analysis of such structures a difficult task.

Various techniques for the free vibration analysis of composite laminated plates have been reported in the literature. Cho *et al.* [1], for example, used a higher order plate theory in each individual layer of a simply supported rectangular laminated plate to determine the natural frequencies and the relative stress and deflection distribution through the thickness of the plate. The theory approximated the in-plane and normal displacements by employing third and second order functions of the thickness co-ordinate respectively. Dawe and Wang [2], on the other hand, utilized B-spline functions to define the displacement field in the analysis of composite laminated rectangular plates by Rayleigh–Ritz method. Taylor and Nayfeh [3] obtained solutions for the individual layers which relate the field variables at the upper and lower layer surfaces and used linear transformations to refer to the anisotropy of each layer to a global co-ordinate system. Wang and Lin [4] presented a finite-strip method based on higher order plate theory for determining the natural frequencies of laminated plate. This method has the advantage of dealing with only few degrees of freedom. Gorman and Ding [5] used a modified superposition-Galerkin method to obtain the solution to the problem of free vibration of clamped and simply supported laminated cross-ply rectangular plates. Filipich *et al.* [6] developed variational method named WEM for three-dimensional solids. The method was developed for solving a wide range of boundary value problems by means of extremizing a proper functional an using suitable sequence.

Liew *et al.* [7] have presented an extensive review of existing literature on the vibration analysis of thick plates. Mainly the studies based on Mindlin theory and modified Mindlin plate theories for laminated plates have been discussed and some papers employing higher order shear deformation plate theories have also been included. Liew [8] employed a global *p*-Ritz method for vibration analysis of thick rectangular laminates with various boundary conditions. Xiang *et al.* [9] investigated buckling, free vibration and vibration with initial in-plane loads for moderately thick, simply supported symmetric cross-ply rectangular laminates on Pasternak foundations. Closed-form buckling and vibration solutions have been obtained using Navier solution procedure and first order shear deformation plate theory has been incorporated in formulating the problem. Chen *et al.* [10] investigated free vibration analysis of symmetrically laminated thick rectangular plates with various combinations of free, simply supported and clamped boundary conditions. The *p*-Ritz method has been employed which uses uniquely defined polynomials for displacement and rotation functions. Also Reddy's higher order plate theory incorporating shear deformation has been followed for deriving energy integral expressions of the laminates. Liew *et al.* [11] used first order shear deformable plate theory for analyzing unsymmetric composite laminates of different boundary conditions, an arbitrary quadrilateral geometry and anisotropic material properties. Liew *et al.* [12] studied the sensitivity of the vibration responses to variations in the lamination, boundary constraints and thickness effects and also their interactions using Ritz procedure and first order shear deformable plate theory.

Plane strain edge vibrations in laminated plates have been investigated extensively by many investigators in the past. Mindlin [13], for example, provided an exact solution for edge vibrations in elastic plates. Dong and Nelson [14], on the other hand, presented a method for natural vibration analysis of laminated orthotropic plates in which a displacement field was assumed for each lamina. The displacements were characterized by a discrete number of generalized co-ordinates at the lamina-bounding planes and their mid-surfaces. Subsequently, Dong and Pauley [15] presented a finite element method for the determination of frequencies and modal patterns of vibrations and waves in an infinite anisotropic plate. Dong and Goetschel [16] used a semianalytical method using finite element interpolations over the thickness and exponential decay into plate's interior for examining the edge stress states of a laminated plate with arbitrary number of elastic, anisotropic laminae. Dong and Huang [17] investigated plane strain edge vibrations in laminated composite plates by using finite element method in which anisotropic laminate properties were considered. All these investigations were based on a parabolic variation of displacement field through thickness.

The motivation for the present work comes from the desire to achieve better results for edge vibration through the composite laminated sandwich plate using lesser number of degrees of freedom by employing a higher order displacement based formulation. The method can be applied easily to arbitrarily anisotropic laminae. However, for illustration, it is applied to the case where each lamina is transversely isotropic with the axes of symmetry lying parallel to the lamina. A higher order, cubic variation of displacement field through thickness is assumed in each lamina of the laminate. By applying the variational principle to each lamina, the stiffness and the mass matrices are calculated explicitly for a single lamina. These matrices are then assembled by enforcing compatibility of displacements and rotations at the lamina interfaces. Numerical results for both isotropic and orthotropic plates are compared with the published numerical results. The dynamic behaviour of composite laminated sandwich plates is then investigated making use of the proposed solution technique.

2. FORMULATION

Edge vibrations in laminated composite plate have been analyzed using higher order displacement formulation. Each lamina is assumed for simplicity to be in a state of plane strain and transversely isotropic with the symmetry axis aligned with either x- or y-axis as shown in Figure 1. The laminated composite plate consists of a number of stiff layers with layers of soft core in-between.

The local co-ordinate system (x_i, y_i, z_i) for ith layer is selected parallel to the global system $(x,\ y,\ z)$. The origin of the local system is located at the mid-plan of a lamina of thickness $2h$. By assuming the cubic variation of displacements through a lamina of a laminate, the time-dependent axial and transverse displacements of any point lying in the x–z plane can be expressed as

$$u(x, z, t) = a_0(x, t) + za_1(x, t) + z^2a_2(x, t) + z^3a_3(x, t), \quad (1)$$

$$w(x, z, t) = b_0(x, t) + zb_1(x, t) + z^2b_2(x, t) + z^3b_3(x, t). \quad (2)$$

Here, a_i, b_i, $i = 0, 1, 2, 3$, are the generalized parameters. By expressing a_i and b_i in terms of the generalized displacements and rotations at $z = \pm h$, the following equations are obtained:

$$\begin{Bmatrix} u \\ w \end{Bmatrix} = [N]\{q\}, \quad (3)$$

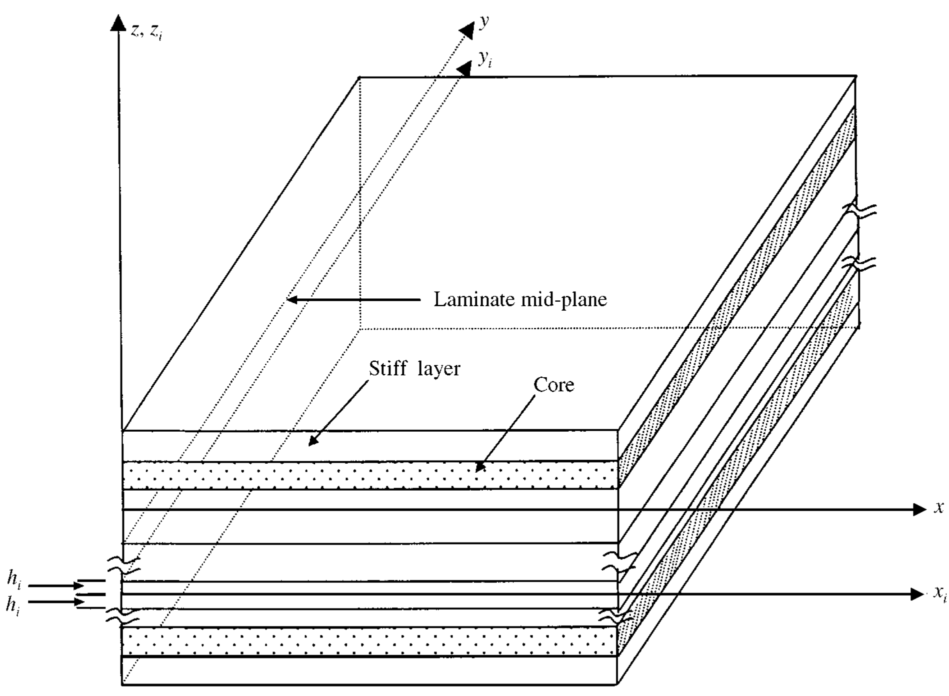

Figure 1. Laminate geometry with positive set of laminate reference axes for laminated sandwich plate.

where

$$[N] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix} \tag{4}$$

and

$$\{q\}^{\mathrm{t}} = [u_1 \quad u_2 \quad \theta_{x1} \quad \theta_{x2} \quad w_1 \quad w_2 \quad \theta_{z1} \quad \theta_{z2}]. \tag{5}$$

Here, u_i and w_i, $i = 1, 2$, are the generalized displacements along the x and z directions, respectively, at $z = (-1)^i h$. The quantities θ_{xi}, θ_{zi}, $i = 1, 2$, are rotations and are defined as $\theta_x = \partial u/\partial z$ and $\theta_z = \partial w/\partial z$.

The N_i, $i = 1, 2, 3, 4$, appearing in equation (4) are the shape functions given by

$$N_i = \tfrac{1}{4}(2 - 3\xi + \xi^3), \quad N_2 = \tfrac{1}{4}(2 + 3\xi + \xi^3),$$

$$N_3 = \frac{h}{4}(1 - \xi - \xi^2 + \xi^3) \quad \text{and} \quad N_4 = \frac{h}{4}(1 - \xi + \xi^2 + \xi^3), \tag{6}$$

where $\xi = z/h$.

The strain–displacement relations for a lamina can be shown to be

$$\{\varepsilon\} = [B_1]\{q\} + [B_2]\{q'\}, \tag{7}$$

where

$$\{\varepsilon\}^{\mathrm{t}} = [\varepsilon_x \ \ \varepsilon_z \ \ \gamma_{xz}] \tag{8}$$

and

$$[B_1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{N}_1 & \bar{N}_2 & \bar{N}_3 & \bar{N}_4 \\ \bar{N}_1 & \bar{N}_2 & \bar{N}_3 & \bar{N}_4 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[B_2] = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 \end{bmatrix}. \tag{9}$$

The primes denote the partial derivative with respect to x whereas the overbars denote the partial derivative with respect to z.

The stress–strain relationships of a lamina are

$$\{\sigma\} = [C]\{\varepsilon\}, \tag{10}$$

where

$$\{\sigma\}^{\mathrm{t}} = [\sigma_x \ \ \sigma_z \ \ \tau_{xz}] \tag{11}$$

and

$$[C] = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}, \tag{12}$$

with

$$C_{11} = \frac{E_x(1 - v_{yz}^2)}{(1 - v_{xz}v_{zx})(1 - v_{yz}^2) - (v_{xz}v_{zx})(1 + v_{yz})^2},$$

$$C_{12} = C_{21} = \frac{E_x v_{zx}(1 + v_{yz})}{(1 - v_{xz}v_{zx})(1 - v_{yz}^2) - (v_{xz}v_{zx})(1 + v_{yz})^2},$$

$$C_{22} = \frac{E_z(1 - v_{xz}v_{zx})}{(1 - v_{xz}v_{zx})(1 - v_{yz}^2) - (v_{xz}v_{zx})(1 + v_{yz})^2} \quad \text{and} \quad C_{66} = G_{xz}. \tag{13}$$

Here, E_x and E_z are the Young's moduli along the x and z directions, v_{xz} and v_{yz} are the in-plane and transverse-plane Poissions ratios, whereas G_{xz} is the in-plane shear modulus of elasticity.

The equation of motion for the lamina can be obtained by using the variational principle

$$\int_{t_1}^{t_2} \delta(T - U)\,\mathrm{d}t = 0, \tag{14}$$

where T and U are, respectively, the kinetic and the strain energies of a lamina. T can be computed for a unit width of a lamina from

$$T = \frac{1}{2}\int_V \rho\{\dot{u}\}^{\mathrm{t}}\{\dot{u}\}\,\mathrm{d}v. \tag{15}$$

Here, ρ is the mass density of lamina and

$$\{\dot{u}\}^{\mathrm{t}} = \left[\frac{\partial u}{\partial t}\ \frac{\partial w}{\partial t}\right]. \tag{16}$$

By substituting equation (3) into equation (15), it can be shown that

$$T = \frac{1}{2}\int [\{\dot{q}\}^{\mathrm{t}}[M]\{\dot{q}\}]\,\mathrm{d}x, \tag{17}$$

where

$$[M] = \int_{-h}^{h} ([N]^{\mathrm{t}}\rho[N])\,\mathrm{d}z \tag{18}$$

and the dot indicates derivative with respect to time "t".

The explicit form of $[M]$ has been presented in Appendix A. The internal strain energy of a lamina can be computed from

$$U = \frac{1}{2}\int_V \{\varepsilon\}^{\mathrm{t}}\{\sigma\}\,\mathrm{d}v. \tag{19}$$

The strain energy per unit width of lamina can be derived by substituting equations (10) and (7) into equation (19) as

$$U = \frac{1}{2}\int (\{q\}^{\mathrm{t}}[K_1]\{q\} + \{q\}^{\mathrm{t}}[K_2]\{q'\} + \{q'\}^{\mathrm{t}}[K_2]^{\mathrm{t}}\{q\} + \{q'\}^{\mathrm{t}}[K_3]\{q'\}\,\mathrm{d}x, \tag{20}$$

where

$$[K_1] = \int_{-h}^{h} ([B_1]^t[C][B_1])\,dz,$$

$$[K_2] = \int_{-h}^{h} ([B_1]^t[C][B_2])\,dz,$$

$$[K_3] = \int_{-h}^{h} ([B_2]^t[C][B_2])\,dz. \tag{21}$$

By substituting equations (17) and (20) into equation (14) the equations of motion for a lamina are obtained as

$$[K_1]\{q\} + [[K_2] - [K_2]^t]\{q'\} - [K_3]\{q''\} + [M]\{\ddot{q}\} = 0 \tag{22}$$

which take the form

$$[[K_1] + \lambda([\bar{K}_2] - [\bar{K}_2]^t) + \lambda^2[K_3] - \omega^2[M]]\{q_0\} = 0 \tag{23}$$

by assuming a general solution

$$\{q_0\} = \{q_{0,1}\}\sin(\lambda x)\sin(\omega t) + \{q_{0,2}\}\cos(\lambda x)\sin(\omega t). \tag{24}$$

The overbar in equation (23) indicates the modified nature of stiffness matrix K_2 after substitution of equation (24) into equation (22).

Here, $\{q_{0,1}\}$ and $\{q_{0,2}\}$ are the amplitude vectors defined as follows:

$$\{q_{0,1}\} = \{A_0\ B_0\ C_0\ D_0\ 0\ 0\ 0\ 0\}^t$$

and

$$\{q_{0,2}\} = \{0\ 0\ 0\ 0\ E_0\ F_0\ G_0\ H_0\}^t \tag{25}$$

whereas ω is the circular frequency and $\lambda = \zeta\pi/H$, with ζ being the series constant and H the total thickness of the laminated plate.

Equation (23), the dispersion equation for edge vibrations in a lamina, is written in a compact form as

$$[K] - \omega^2[M] = 0, \tag{26}$$

where

$$[K] = [K_1] + \lambda([\bar{K}_2] - [\bar{K}_2]^t) + \lambda^2[K_3] \tag{27}$$

from which λ can be computed for a given ω or conversely, the frequency can be obtained for the specified λ.

Matrix $[K]$ in equation (27) has been evaluated explicitly and is presented in Appendix A. The stiffness and mass matrices thus calculated for all laminae are assembled to form $k \times k$ ($k = (NL + 1) * 4$ where NL is the number of laminae in the plate) global matrices by enforcing the compatibility of generalized displacements and rotations at the interfaces of the laminae.

3. NUMERICAL EXAMPLES

A general purpose, FORTRAN-77 computer program was developed on the basis of the theoretical formulation discussed above, for determining the natural frequencies of edge vibrations of a composite laminated sandwich plate. The natural frequencies are normalized in the program with respect to the reference frequency (ω_{ref}) which has been specified as the third lowest frequency near the cut-off for a series constant of $\xi = 0{\cdot}001$. On the other hand, the value of ω_{ref} for an isotropic plate is commonly considered as $(\pi/H)\sqrt{G/\rho}$ and thus the same value has been employed in the present work.

Plates having various properties and configurations have been solved to demonstrate the validity and applicability of the proposed method. The first two examples have been selected to validate the proposed formulation for isotropic and orthotropic plates by comparing the results with those of Dong and Nelson [14]. Further, the results obtained by the proposed method are presented for two typical composite laminated sandwich plates wherein the edge vibration phenomenon for laminated composite plates is examined with respect to that of the isotropic or orthotropic plates.

A comparison and a brief discussion of the results obtained by the proposed method and the previously published numerical results are presented next.

3.1. EXAMPLE 1

A single lamina of isotropic material with the material properties given in Table 1 was divided into 25 sub-layers. A comparison of the results obtained with the numerical solution given by Dong and Nelson [14] has been presented in Table 2. It indicates a close agreement of the results obtained by using the proposed method with those presented by Dong and Nelson [14].Various lower order theories have also been formulated by the authors by assuming various combinations of displacement fields, viz., linear, parabolic and cubic polynomials for u and w displacements. By comparing the results obtained by various models, it was observed that the proposed higher order displacement model where cubic variation of displacement field has been assumed, seems to be the most economical one, yielding accurate results with just 104 d.o.f.s for isotropic plate. On the other hand, the first order shear deformation theory failed to produce accurate results even after using as large as 804 d.o.f.s for modelling the isotropic plate. The results of various displacement models are not presented here for brevity.

3.2. EXAMPLE 2

An orthotropic plate with the lamina material properties given in Table 1 has been analyzed and the results obtained using the proposed method have been compared with those presented by Dong and Nelson [14] in Figure 2. The solid lines in Figure 2 represent the numerical solution by Dong and Nelson [14], whereas the dots represent the results obtained using the proposed method. A comparison of normalized frequencies for some typical values of series constant obtained by Dong and Nelson [14] and the proposed method has been presented in Table 3. It can be observed that the results obtained by the proposed method are in very good agreement with the published results and in this case too

TABLE 1

Material properties of various plates considered for the investigation

Data for example no.	Type of plate/(no. of sub-layers for each lamina)	Thickness of sublayer h (10^{-3} m)	Mass density ρ (kg/m^2)	E_x (N/m^2)	E_z (N/m^2)	ν_{xz}	ν_{yz}	G_{xz} (N/m^2)
1	Isotropic/(25)	1·016	$2{\cdot}7688 \times 10^{4}$	$2{\cdot}0691 \times 10^{10}$	$2{\cdot}0691 \times 10^{10}$	0·3	0·3	$7{\cdot}9581 \times 10^{9}$
2	Orthotropic/(10)	2·54	$2{\cdot}7688 \times 10^{4}$	$2{\cdot}0691 \times 10^{10}$	$2{\cdot}0691 \times 10^{10}$	0·3	0·3	$7{\cdot}9581 \times 10^{7}$
3	Sandwich	—	—	—	—	—	—	—
	Face sheet/(top/bot)	0·4572	$2{\cdot}6831 \times 10^{3}$	$6{\cdot}897 \times 10^{10}$	$6{\cdot}8970 \times 10^{10}$	0·3	0·3	$2{\cdot}6527 \times 10^{10}$
	Core/(8)	1·5875	32·8381	$2{\cdot}1519 \times 10^{8}$	$2{\cdot}1519 \times 10^{8}$	0·3	0·3	$8{\cdot}2764 \times 10^{7}$
4	Sandwich	—	—	—	—	—	—	—
	Face sheet/(top/bot)	0·508	$1{\cdot}0691 \times 10^{7}$	$6{\cdot}8970 \times 10^{10}$	$6{\cdot}8970 \times 10^{10}$	0.3	0·3	$2{\cdot}6527 \times 10^{10}$
	Core/(top/bot)	10·16	$2{\cdot}6726 \times 10^{6}$	$8{\cdot}9661 \times 10^{7}$	$8{\cdot}9661 \times 10^{7}$	0·3	0·3	3.4485×10^{7}

Table 2(A)

Comparison of normalized frequencies of modes 1–5 *for selected values of series constants for isotropic plate in Example* 1 *using Dong and Nelson* [14] *and the present study*

Series constant ζ	Normalized frequency Ω									
	Mode 1		Mode 2		Mode 3		Mode 4		Mode 5	
	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present
0·001	1·77E-06	0·00000	0·00169	0·00169	1·00000	1·00000	1·87082	1·87082	2·00000	2·00000
0·1	0·01507	0·01507	0.16890	1·16890	1·01694	1·01694	1·84976	1·84977	2·02880	2·02880
0·5	0·28601	0·28601	0·82521	0·82521	1·33046	1·32946	1·73912	1·73911	2·33302	2·33301
1·0	0·77450	0·77450	1·41421	1·41421	2·00713	1·90725	2·00122	2·00122	2·82554	2·82553
1·5	1·27690	1·27690	1·70532	1·70532	2·43446	2·43446	2·72221	2·72220	3·36031	3·36030
2·0	1·77234	1·77234	2·03357	2·03557	2·82942	2·82842	3·40363	3·40362	3·86798	3·86798
2·5	2·26021	2·26021	2·42276	2·42275	3·17617	3·17617	3·87468	3·87468	4·43179	4·43178
3·0	2·74192	2·74192	2·84330	2·84329	3·54097	3·54096	4·24264	4·24262	4·90994	4·90992
3·5	3·21876	3·21875	3·28220	3·28219	3·93423	3·93432	4·59687	4·59686	5·29977	5·29974
4·0	3·69177	3·69176	3·73134	3·73136	4·35291	4·35290	4·96357	4·96355	5·65683	5·65683
4·5	4·14187	4·16185	4·18642	4·18641	4·79110	4·79110	5·34826	5·34827	6·01294	6·01294
5·0	4·62981	4·62981	4·64494	4·64492	5·24396	5·24395	5·75050	5·75050	6·37946	6·37945
5·5	5·09630	5·09627	5·10552	5·10549	5·70762	5·70761	6·16802	6·16802	6·75950	6·75947
6·0	5·54174	5·56171	5·56734	5·56730	6·17926	6·17925	6·59841	6·59840	7·15305	7·15305
6·5	6·02654	6·02650	6·02993	6·02987	6·65679	6·65679	7·03944	7·03942	7·55915	7·55912
7·0	6·49096	6·49087	6·49287	6·49288	7·13875	7·13874	7·48924	7·48922	7·97632	7·97628
7·5	6·95510	6·95498	6·95630	6·95618	7·62405	7·62404	7·94623	7·94623	8·40314	8·40314
8·0	7·41890	7·41893	7·41960	7·41964	8·11191	8·11190	8·40923	8·40921	8·83843	8·83841
9·0	8·34604	8·34657	8·34711	8·34682	9·09317	9·07317	9·34916	9·34915	9·72975	9·72975
10·0	9·27405	9·27406	9·27462	9·27415	10·07952	10·07952	10·30300	10·30301	10·64300	10·64294

TABLE 2(B)

Comparison of normalized frequencies of modes 6–10 *for selected values of series constants for isotropic plate in Example* 1 *using Dong and Nelson* [14] *and the present study*

Series constant ζ	Normalized frequency Ω									
	Mode 6		Mode 7		Mode 8		Mode 9		Mode 10	
	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present
0·001	3·00000	2·99999	3·74165	3·74164	4·00000	3·99999	5·00003	4·99999	5·61246	5·61246
0·1	2·99956	2·99957	3.74709	3·74708	4·00107	4·00106	5·00133	5·00130	5·61442	5·61442
0·5	2·99930	2·99928	3·86804	3·86803	4·02793	4·02793	5·03346	5·03345	5·65894	5·65891
1·0	3·07718	3·07718	4·12762	4·12762	4·17184	4·17183	5·14347	5·14343	5·77434	5·77430
1·5	3·36523	3·36523	4·34232	4·34232	4·54624	4·54625	5·35953	5·35949	5·92526	5·92520
2·0	3·96524	3·96523	4·71897	4·71895	4·93137	4·93137	5·71403	5·71399	6·11445	6·11477
2·5	4·68826	4·68826	5·24636	5·24636	5·35272	5·35272	6·19023	6·19019	6·36861	6·36853
3·0	5·39273	5·39270	5·87202	5·87198	5·89074	5·89073	6·72063	6·72060	6·73051	6·73045
3·5	5·93473	5·93473	6·44472	6·44468	6·64583	6·64581	7·19654	7·19649	7·29425	7·29416
4·0	6·35149	6·35149	6·96655	6·96647	7·39390	7·39383	7·80677	7·80669	7·87921	7·29416
4·5	6·71761	6·71755	7·40043	7·40037	7·98701	7·98691	8·47404	8·47390	8·56957	8·56950
5·0	7·07111	7·07104	7·77781	7·77778	8·44633	8·44632	9·02313	9·02312	9·37531	9·37537
5·5	7·42773	7·42772	8·13329	8·13325	8·83726	8·83701	9·49242	9·49205	10·03930	10·03889
6·0	7·79352	7·79355	8·48540	8·48525	9·19743	9·19735	9·89509	9·89525	10·53630	10·53560
6·5	8·17044	8·17039	8·84321	8·84206	9·54808	9·54809	10·26270	10·26219	10·95430	10·95242
7·0	8·55842	8·55839	9·20724	9·20718	9·89977	9·89946	10·61410	10·61407	11·32810	11·32715
7·5	8·95702	8·95694	9·58171	9·58155	10·25630	10·25622	10·96310	10·96257	11·68260	11·68187
8·0	9·36525	9·36518	9·96570	9·96554	10·62050	10·62050	11·31420	11·31367	12·03040	12·02943
9·0	10·20700	10·02695	10·76043	10·76043	11·37430	11·37398	12·03450	12·03389	12·72840	12·72788
10·0	11·07670	11·07665	11·58710	11·58691	12·16010	12·15977	12·78400	12·78337	13·44840	13·44732

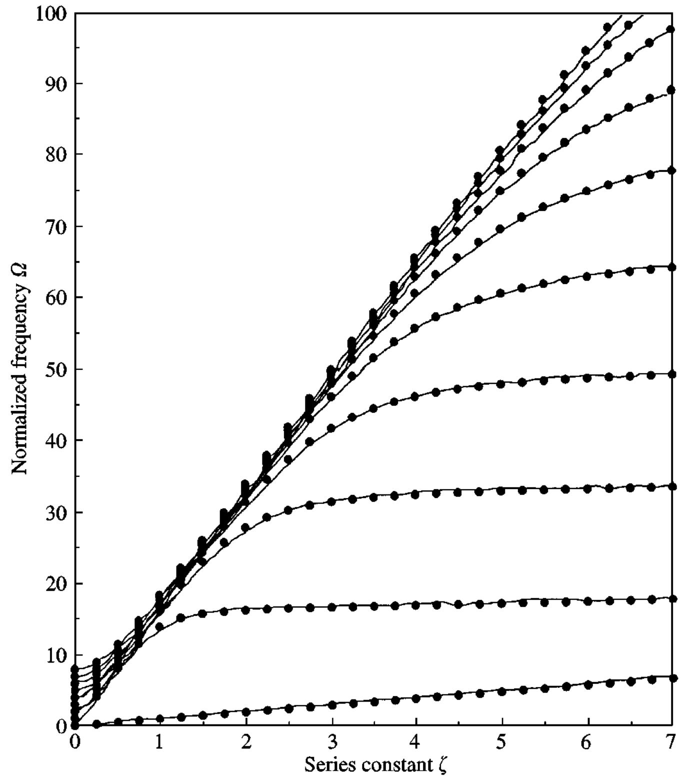

Figure 2. Comparison of dispersion curves for the orthotropic plate of Example 2.

in comparison with Dong and Nelson [14] almost half the number of d.o.f.s have yielded equally accurate results.

3.3. EXAMPLE 3

A sandwich plate composed of top and bottom face sheets having a core (eight sub-layers) in-between has been analyzed using the present formulation. The material properties have been tabulated under Table 1. The results obtained have been presented in Figure 3. It is observed by comparing Figures 2 and 3 that the frequency spectra follow more or less asymptotic path as the series constant ξ reduces for a wide range of Ω for the orthotropic plate whereas the frequency spectra seems to be well dispersed for the sandwich plate due to the inherent anisotropy. Moreover, the cut-off frequencies are higher for the sandwich plate indicating higher stiffness.

3.4. EXAMPLE 4

A sandwich plate fabricated using top, bottom and middle face sheets and two cores in-between has been considered in this example. The material properties are presented in Table 1. The frequency spectra presented in Figure 4 can be observed to have higher cut-off

TABLE 3(A)

Comparison of normalized frequencies of modes 1–5 *for selected values of series constants for orthotropic plate in Example* 2 *using Dong and Nelson* [14] *and the present study*

Series constant ζ	Normalized frequency Ω									
	Mode 1		Mode 2		Mode 3		Mode 4		Mode 5	
	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present
0·001	0·00001	2·00E-05	0·01667	0·01667	0·98612	0·98612	1·97205	1·97204	2·95807	2·95802
0·5	0·47409	0·47408	8·12201	8·12201	8·34895	8·34895	8·51548	8·51547	8·81109	8·81102
1·0	0·96767	0·96770	13·94297	13·94296	16·26651	16·26648	16·60406	16·60402	16·79374	16·79362
1·5	1·46163	1·46190	15·77040	15·77040	23·13048	23·13041	24·41266	24·41250	24·82032	24·82006
2·0	1·95587	1·95645	16·29086	16·29093	27·88608	27·88594	31·53800	31·53770	32·55984	32·55928
2·5	2·45018	2·45103	16·53830	16·53848	30·35363	30·35344	37·49561	37·49508	39·79694	39·79607
3·0	2·94443	2·94545	16·70788	16·70818	31·54318	31·54300	41·82994	41·82910	46·26626	46·26495
3·5	3·43858	3·43969	16·85273	16·85312	32·18625	32·18612	44·59117	44·59007	51·65667	51·65469
4·0	3·93260	3·93375	16·99207	16·99256	32·58602	32·58595	46·26761	46·26638	55·77596	55·77324
4·5	4·42650	4·42765	17·13420	17·13475	32·86605	32·86601	47·31711	47·31582	58·70208	58·69875
5·0	4·92028	4·92141	17·28309	17·28370	33·08218	33·08218	48·01538	48·01406	60·71343	60·70966
5·5	5·41397	5·41507	17·44075	17·44140	33·26247	33·26250	48·51039	48·50908	62·10466	62·10060
6·0	5·90757	5·90864	17·60822	17·60890	33·42224	33·42229	48·88216	48·88085	63·09438	63·09014
6·5	6·40111	6·40213	17·78599	17·78670	33·57043	33·57050	49·17592	49·17461	63·82400	63·81964
7·0	6·89458	6·89556	17·97427	17·97500	33·71256	33·71265	49·41854	49·41724	64·38161	64·37717

TABLE 3(B)

Comparison of normalized frequencies of modes 6–10 *for selected values of series constants for orthotropic plate in Example* 2 *using Dong and Nelson* [14] *and the present study*

Series constant ζ	Normalized frequency Ω									
	Mode 6		Mode 7		Mode 8		Mode 9		Mode 10	
	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present	Reference [14]	Present
0·001	3·94438	3·94411	4·93124	4·93052	5·91901	5·91769	6·90849	6·90630	7·90051	7·89744
0·5	9·19427	9·19403	9·66359	9·66299	10·20456	10·20334	10·81081	10·80873	11·47127	11·46823
1·0	17·01859	17·01825	17·29672	17·29595	17·61892	17·61744	17·98715	17·98468	18·39881	18·39520
1·5	25·05463	25·05413	25·27355	25·27251	25·52133	25·51940	25·79919	25·79605	26·10622	26·10171
2·0	33·01374	33·01287	33·29401	33·29261	33·52797	33·52556	33·76940	33·76550	34·03142	34·02582
2·5	40·70838	40·70700	41·19499	41·19296	41·51628	41·51327	41·77435	41·76966	42·02216	42·01523
3·0	48·00686	48·00497	48·85904	48·85639	49·37047	49·36686	49·72726	49·72208	50·01261	50·00439
3·5	54·74312	54·74055	56·19647	56·19322	57·01286	57·00884	57·54445	57·53943	57·93219	57·92430
4·0	60·71406	60·71053	63·09377	63·08974	64·37756	64·37318	65·17112	65·16668	65·72012	65·71463
4·5	65·73531	65·73060	69·41228	69·40718	71·38219	71·37730	72·55643	72·55262	73·33534	73·33302
5·0	69·72657	69·72062	75·00947	75·00299	77·92733	77·92169	79·63799	79·63470	80·73731	80·73816
5·5	72·75822	72·75114	79·77773	79·76960	83·90579	83·89902	86·34156	86·33857	87·87727	87·88109
6·0	75·00722	74·99923	83·68564	83·67574	89·22031	89·21200	92·58556	92·58247	94·69803	94·70451
6·5	76·67269	76·66401	86·79148	86·77987	93·80773	93·79745	98·28955	98·28583	101·13616	101·14491
7·0	77·92226	77·91306	89·21654	89·20339	97·65883	97·64632	103·38745	103·38246	107·12655	107·13691

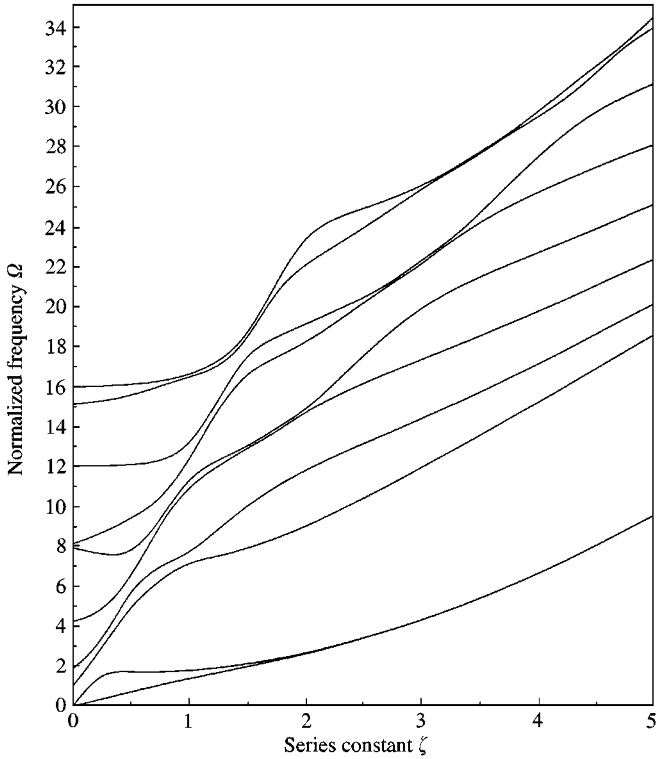

Figure 3. Dispersion curves for the laminated sandwich plate of Example 3.

frequencies of edge vibrations and still higher frequencies for larger values of series constants. More light on the edge vibration effect in composite sandwich plates can be thrown by investigating the mode shapes. However, such extensive modal analysis has not been performed in the present work.

4. CONCLUSIONS

A higher order displacement-based model has been presented to analyze edge vibrations in a laminated composite plate. The interlayer continuity of displacements and rotations has been imposed while assembling the stiffness and mass matrices of each layer. The results for isotropic as well as orthotropic plates have been shown to be in close agreement with the published numerical results. The higher order displacement-based method proposed here is observed to yield equally accurate results by using only about half the number of d.o.f.s in comparison with the numerical techniques available in the literature. Significant differences between the frequency spectra for homogeneous fibre-reinforced plate and laminated sandwich plate have been observed.

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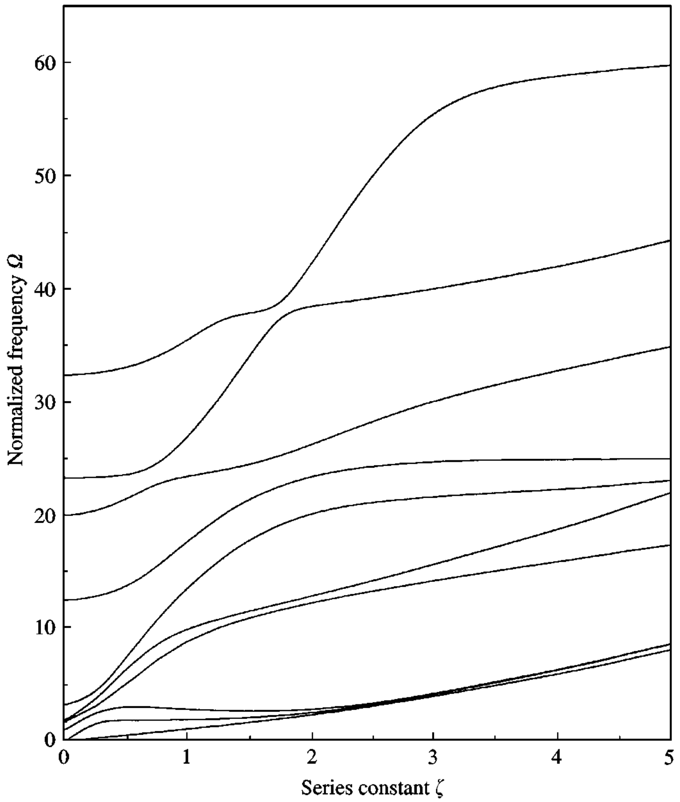

Figure 4. Dispersion curves for the laminated sandwich plate of Example 4.

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APPENDIX A

The mass matrix $[M]$ for a lamina of thickness $2h$, has been explicitly evaluated as

$$[M] = \begin{bmatrix} [M_1] & 0 \\ 0 & [M_1] \end{bmatrix} \quad \text{and} \quad [M_1] = \frac{\rho h}{105}\begin{bmatrix} 78 & 27 & 22h & -13h \\ 27 & 78 & 13h & -22h \\ 22h & 13h & 8h^3 & -6h^3 \\ -13h & -22h & -6h^3 & 8h^3 \end{bmatrix}. \tag{A1}$$

The stiffness matrix $[K]$ for a lamina is obtained to be

$$[K] = \begin{bmatrix} [K_4] & [K_5] \\ [K_6] & [K_7] \end{bmatrix}. \tag{A2}$$

Here

$$[K_4] = \begin{bmatrix} 18A + 78B & -18A + 27B & 3Ah + 22Bh & 3Ah - 13Bh \\ -18A + 27B & 6A + 78B & -3Ah + 13Bh & -3Ah - 22Bh \\ 3Ah + 22Bh & -3Ah + 13Bh & 8Ah^2 + 8Bh^3 & -2Ah^2 - 6Bh^3 \\ 3Ah + 13Bh & -3Ah - 22Bh & -2Ah^2 - 6Bh^3 & 8Ah^2 + 8Bh^3 \end{bmatrix}, \tag{A3}$$

$$[K_5] = \begin{bmatrix} -15F - 15E & 15F + 15E & 6Fh + 6Eh & -6Fh - 6Eh \\ -15F - 15E & 15F - 15E & -6Fh - 6Eh & 6Fh + 6Eh \\ -6Fh - 6Eh & 6Fh + 6Eh & 0 & -2Fh^2 - 2Eh^2 \\ 6Fh + 6Eh & -6Fh - 6Eh & 2Fh^2 + 2Eh^2 & 0 \end{bmatrix}, \tag{A4}$$

$$[K_6] = \begin{bmatrix} 15E - 15F & -15E - 15F & -6Eh - 6Fh & 6Eh + 6Fh \\ 15E + 15F & -15E + 15F & 6Eh + 6Fh & -6Eh - 6Fh \\ 6Eh + 6Fh & -6Eh - 6Fh & 0 & 2Eh^2 + 2Fh^2 \\ -6Eh - 6Fh & 6Eh + 6Fh & -2Eh^2 - 2Fh^2 & 0 \end{bmatrix}, \quad \text{(A5)}$$

$$[K_7] = \begin{bmatrix} 18P + 78Q & -18P + 27Q & 3Ph + 22Qh & 3Ph - 13Q\mathrm{h} \\ -18P + 27Q & 6P + 78Q & -3Ph + 13Qh & -3Ph - 22Qh \\ 3Ph + 22Qh & -3Ph + 13Qh & 8Ph^2 + 8Qh^3 & -2Ph^2 - 6Qh^3 \\ 3Ph - 13Qh & -3Ph - 22Qh & -2Ph^2 - 6Qh^3 & 8Ph^2 + 8Qh^3 \end{bmatrix}, \quad \text{(A6)}$$

with

$$A = \frac{C_{55}}{30h}, \qquad B = \frac{C_{11}\lambda^2 h}{105}, \qquad E = \frac{\lambda C_{55}}{30}, \qquad F = \frac{\lambda C_{13}}{30}, \qquad P = \frac{C_{33}}{30h}, \qquad Q\frac{C_{55}\lambda^2 h}{105}.$$