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Analytical solution to the dynamic analysis of laminated beams using higher order refined theory

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Analytical solution to the natural frequency analysis of composite and sandwich beams based on a higher order refined theory is presented. This theory incorporates cubic axial, transverse shear and quadratic transverse normal strain components in the basic formulation — thus modelling the warping of cross section accurately and eliminating the need for a shear correction coefficient. Also, it considers each layer of the lamina to be orthotropic and in a two dimensional state of plane stress. The equations of equilibrium are derived using Hamilton's principle. Numerical experiments are carried out and from the results of thick and thin sections, conclusions are drawn. © 1998 Elsevier Science Ltd. All rights reserved

INTRODUCTION

The emergence of composites has had an impact in almost all industries. Bio-medical engineering field uses composites for artificial limb implants, nuclear industry for radiation monitoring and protection equipment, transportation industry for hi-tech cycles and cars, space industry for Boron–Epoxy based shuttles and sports industry for racquets, vaulting pole, etc. The application of composites has increased owing to the many advantages they offer: high strength/stiffness for lower weight, superior fatigue response characteristics, facility to vary fibre orientation, material and stacking pattern. At the same time, the fabricated material poses new problems, such as failure due to delamination and pronounced transverse shear effects due to the high ratio of in-plane modulus to transverse shear modulus. Such difficulties can be analysed to predict the behaviour of composite laminates accurately, only by theories which consider shear deformation and other secondary effects in their formulation.

The widely used classical beam theory of Euler–Bernoulli is non-shear deformable and hence is useful only for thin sections. Exact solutions for this theory for static [1], free vibration and forced vibration problems [2] are well known.

The first attempt to incorporate shear deformation was made by Timoshenko [3,4] in which a constant transverse shear strain was assumed across the cross section, resulting in the requirement for a problem dependent shear correction factor. Shear correction factors have been evaluated for various cross sections in the past [5–7].

Many works incorporating transverse shear deformation have been reported ever since. Mindlin & Goodman [8] proposed an analytical solution to the free vibration problem of Timoshenko beams with time dependent boundary conditions and it was extended later to forced vibrations of Timoshenko beams by Herrmann [9]. The vibration problem of sandwich beams with flexible cores, where shear deformation effects of the core are predominant was analysed by Mead & Sivakumuran [10] using the Stodola method. A refined first order theory was reported by Cowper [11] and Teoh & Huang [12] proposed an analytical solution to the vibration problem of fibre reinforced beams which considers both shear deformation and rotatory inertia and derives the equation of motion based on Timoshenko theory using Hamilton's principle. Myklestad's method was employed by Abarcar & Cunniff [13] to analyse the vibration problem of a cantilever by taking shear deformation and rotatory inertia into account. Chandrashekhara & Bangera [14] estimated natural frequencies of composite beam with a mass at the free end using Hamilton's principle and incorporating rotatory inertia and shear deformation effects. Quite a number of anisoparametric [15–19] and isoparametric [20] Timoshenko finite elements are also available.

A second order beam theory by Stephen & Levinson [21] considers shear curvature, transverse direct stresses and rotatory inertia and formulates differential equations of motion, similar in form to Timoshenko equations, with two coefficients — the first dependent on cross-sectional warping and the second on transverse direct stresses.

A third order theory proposed by Phan & Reddy [22] assumes in-plane displacements u, v as cubic functions of thickness coordinate of the composite plate resulting in parabolic transverse shear strain distribution and strain free conditions on the upper and lower surfaces of the plate without the need to apply a shear correction coefficient. Equations of motion are derived using Hamilton's principle. The same theory was later applied to beam formulations by Heyliger & Reddy [23]. As an extension to this formulation, Soldatos & Elishakoff [24] proposed a theory for orthotropic straight beams considering both transverse shear and transverse normal strains. While the parabolic distribution of transverse shear strains through thickness ensured zero shear conditions at the top and bottom of the beam, Hamilton's principle yielded the governing equation of equilibrium and associated boundary conditions.

A fourth order theory by Levinson [25] allows the cross section to rotate relative to the neutral surface as well as warp into a non-planar surface by taking the axial displacement as the sum of linear variation, across the thickness, of cross-sectional rotation and cubic variation of a warping function. This theory, formulated for beams with narrow rectangular cross sections is self-contained and does not require a shear coefficient derived from factors extrinsic to the theory. An improvement over this theory was reported by Rychter [26] by incorporating two-dimensional theory of elasticity to model 2D displacement pattern.

A consistent higher order theory was published by Bickford [27] based on Levinson's theory and Hamilton's principle. Another higher order theory with C^1 continuity and without transverse normal strain was proposed by Reddy [28].

In addition, exact three dimensional solutions for rectangular laminates with simply supported edges [29,30], for simply supported orthotropic rectangular plates and laminates [31] and analytical three dimensional solution for multi layered anisotropic plate [32] are also available.

A closer study of these theories reveals the following aspects. The Euler–Bernoulli theory is applicable only for sections with high aspect ratios. The first order shear deformable theory requires a problem dependent shear correction factor. Moreover, modelling of both cross sectional warping and transverse normal strain effects are not feasible with this theory. Also, as it yields results which vary considerably compared with the elasticity solution for built-up beams [33] and fails to predict correct deflections and stresses for thick and moderately thick sandwich and composite laminates [34], its application to a wider range of problems is very much restricted.

The second order theory [21] is also problem dependent as it needs two cross-sectional shape based coefficients — one based on the warping of cross-section and another on transverse direct stresses. The third order theory [23] retains the disadvantage of C^1 continuity. As the fourth order theory [25] has been formulated only for beams with narrow rectangular cross sections, its direct application to generic beam cross sections becomes infeasible. The higher order theory [28] is also C^1 continuity based.

As the available shear deformable flexural theories are plagued with cited shortcomings, the need arises for a theoretical model free from shear correction factors, capable of modelling the warping of cross sections, of predicting the performance of deeper fibre reinforced laminates with better accuracy and finally of facilitating easier formulation and coding.

The higher order theory with cubic axial strain, quadratic transverse shear and linear transverse normal strains by Kant & Gupta [33] could model the complete flexural behaviour of deeper beams, without any recourse to problem dependent factors and by employing isoparametric elements could retain the ease and elegance of formulation. Subsequently, this theory was applied to composite and sandwich deep beam vibrations [35], statics [36] and transient dynamics [37] and its performance was found to be superior in all these cases.

While earlier flexural theories are presented with analytical as well as finite element solutions, higher order theories have so far been formulated with only C^0 finite elements and the analytical solution for the same has not been proposed yet. The objective of this paper is to exactly meet that requirement.

THEORETICAL FORMULATIONS

The axial and transverse displacements in an x–z plane using Taylor's series expansion [38] can be expressed as

$$u(x,z,t) = u_o(x,t) + z\theta_x(x,t) + z^2 u^*_o(x,t) + z^3\theta^*_x(x,t) \quad (1)$$

$$w(x,z,t) = w_o(x,t) + z\theta_z(x,t) + z^2 w^*_o(x,t) + z^3\theta^*_z(x,t) \quad (2)$$

where u_o and w_o are axial and transverse displacements defined at the neutral axis of the beam, θ_x is the rotation of the cross section and u^*_o,θ^*_x, θ_z, w^*_o and θ^*_z are higher order terms arising out of Taylor's series expansion.

The strain displacement relationships are given by

$$\varepsilon_x = \varepsilon_{xo} + zK_x + z^2\varepsilon^*_{xo} + z^3 K^*_x \quad (3)$$

$$\varepsilon_z = \varepsilon_{zo} + zK_z + z^2\varepsilon^*_{zo} \quad (4)$$

$$\gamma_{xz} = \phi + z\psi + z^2\phi^* + z^3\psi^* \quad (5)$$

where

$$(\varepsilon_{xo},K_x,\varepsilon^*_{xo},K^*_x,\varepsilon_{zo},K_z,\varepsilon^*_{zo}) = (u_{o,x},\theta_{x,x},u^*_{o,x},\theta^*_{x,x},\theta_z,2w^*_o,3\theta^*_z) \quad (5a)$$

and

$$(\phi,\psi,\phi^*,\psi^*) = ((w_{o,x}+\theta_x),(2u^*_o+\theta_{z,x}),(3\theta^*_x+w^*_{o,x}),\theta^*_{z,x}) \quad (5b)$$

Considering each lamina to be orthotropic and in a 2D state of plane stress, the stress–strain relationship of a lamina can be given as

$$\sigma = \underline{C}\varepsilon \quad (6)$$

where

$$\sigma = (\sigma_x \sigma_z \tau_{xz})^t \quad (6a)$$

$$\varepsilon = (\varepsilon_x \varepsilon_z \gamma_{xz})^t \quad (6b)$$

and

$$\underline{C} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & G \end{bmatrix} \quad (6c)$$

where

$$C_{11} = E_1/(1-\gamma_{12}\gamma_{21}) \quad (6d)$$

$$C_{12} = C_{21} = \gamma_{12}E_2/(1-\gamma_{12}\gamma_{21}) \quad (6e)$$

$$C_{22} = E_2/(1-\gamma_{12}\gamma_{21}) \quad (6f)$$

where E_1 and E_2 are Young's moduli along the x and z directions, γ_{12} is Poisson's ratio and G is the shear modulus.

Using Hamilton's principle, the equation of motion can be expressed as

$$\int_{t_1}^{t_2} \delta(T-U)\,dt + \int_{t_1}^{t_2} \delta W_{nc}\,dt = 0 \quad (7)$$

where T is the kinetic energy of the system, U is the internal strain energy and W_{nc} is the work done by the non-conservative forces.

The kinetic energy is given by

$$T = \frac{1}{2}\int \rho([u,_t]^2 + [w,_t]^2)\,dv \quad (8)$$

where $u,_t$ and $w,_t$ denote the first derivative of u and w with respect to time (u and w are given by Eqns (1) and (2), respectively).

On expansion, for symmetric laminates, it becomes

$$\begin{aligned} T = \frac{b}{2}\int [&(\rho_1 u_{o,t} + \rho_3 u^*_{o,t})u_{o,t} \\ &+ (\rho_3\theta_{x,t} + \rho_5\theta^*_{x,t})\theta_{x,t} + (\rho_3 u_{o,t} + \rho_5 u^*_{o,t})u^*_{o,t} \\ &+ (\rho_5\theta_{x,t} + \rho_7\theta^*_{x,t})\theta^*_{x,t} + (\rho_1 w_{o,t} + \rho_3 w^*_{o,t})w_{o,t} \\ &+ (\rho_3\theta_{z,t} + \rho_5\theta^*_{z,t})\theta_{z,t} + (\rho_3 w_{o,t} + \rho_5 w^*_{o,t})w^*_{o,t} \\ &+ (\rho_5\theta_{z,t} + \rho_7\theta^*_{z,t})\theta^*_{z,t}]\,dx \end{aligned} \quad (9)$$

where

$$\rho_n = \sum_{L=1}^{NL} \rho_L[(h^n_L - h^n_{L-1})/n],\ n = 1,3,5,7 \quad (9a)$$

The strain energy of the system can be written as

$$U=\frac{b}{2}\int \varepsilon^t\sigma\,\mathrm{d}v \tag{10}$$

where σ and ε are as given by Eqns (6a) and (6b) and it can be expanded as

$$U=\frac{b}{2}\int(\varepsilon_{xo}N_x+K_xM_x+\varepsilon^*_{xo}N^*_x+K^*_xM^*_x)\,\mathrm{d}x$$
$$+\int(\varepsilon_{zo}N_z+K_zM_z+\varepsilon^*_{zo}N^*_z)\,\mathrm{d}x$$
$$+\int(\phi Q_x+\psi S+\phi^*Q^*_x+\psi^*S^*)\,\mathrm{d}x \tag{11}$$

where the non-vanishing terms for the symmetric laminate are given as

$$N_x=\varepsilon_{xo}H_1+\varepsilon^*_{xo}H_3+\varepsilon_{zo}H^*_1+\varepsilon^*_{zo}H^*_3 \tag{11a}$$
$$M_x=K_xH_3+K^*_xH_5+K_zH^*_3 \tag{11b}$$
$$N^*_x=\varepsilon_{xo}H_3+\varepsilon^*_{xo}H_5+\varepsilon_{zo}H^*_3+\varepsilon^*_{zo}H^*_5 \tag{11c}$$
$$M^*_x=K_xH_5+K^*_xH_7+K_zH^*_5 \tag{11d}$$
$$Q_x=\phi\bar{H}_1+\phi^*\bar{H}_3 \tag{11e}$$
$$S=\psi\bar{H}_3+\psi^*\bar{H}_5 \tag{11f}$$
$$Q^*_x=\phi\bar{H}_3+\phi^*\bar{H}_5 \tag{11g}$$
$$S^*=\psi\bar{H}_5+\psi^*\bar{H}_7 \tag{11h}$$
$$N_z=\varepsilon_{xo}H^*_1+\varepsilon^*_{xo}H^*_3+\varepsilon_{zo}\hat{H}_1+\varepsilon^*_{zo}\hat{H}_3 \tag{11i}$$
$$M_z=K_xH^*_3+K^*_xH^*_5+K_z\hat{H}_3 \tag{11j}$$
$$N^*_z=\varepsilon_{xo}H^*_3+\varepsilon^*_{xo}H^*_5+\varepsilon_{zo}\hat{H}_3+\varepsilon^*_{zo}\hat{H}_5 \tag{11k}$$

and

$$H_n=\sum_{L=1}^{NL}(C_{11})_L[(h^n_L-h^n_{L-1})/n],\ n=1,3,5,7 \tag{11l}$$

$$H^*_n=\sum_{L=1}^{NL}(C_{12})_L[(h^n_L-h^n_{L-1})/n],\ n=1,3,5 \tag{11m}$$

$$\hat{H}_n=\sum_{L=1}^{NL}(C_{22})_L[(h^n_L-h^n_{L-1})/n],\ n=1,3,5, \tag{11n}$$

$$\bar{H}_n=\sum_{L=1}^{NL}G_L[(h^n_L-h^n_{L-1})/n],\ n=1,3,5,7 \tag{11o}$$

The external work done by the edge stresses is given as

$$W_{ex}=b\int(\bar{\sigma}_xu+\bar{\tau}_{xz}w)\,\mathrm{d}z \tag{12}$$

where the bars indicate the edge quantities. This work done, on edge $x=\text{constant}$, is expressed after suitable substitutions as

$$W_{ex}=b[\bar{N}_xu_o+\bar{M}_x\theta_x+\bar{N}^*_xu^*_o+\bar{M}^*_x\theta^*_x+\bar{Q}_xw_o$$
$$+\bar{S}\theta_z+\bar{Q}^*_xw^*_o+\bar{S}^*\theta^*_z] \tag{13}$$

where the stress resultants with bar are exactly same as those of Eqn (11) but are generated by edge strains.

By taking the variation of kinetic energy, strain energy and the external work done and by invoking Eqn (7), the following equilibrium equations with consistent boundary conditions are obtained:

$$\rho_1u_{o,,t}+\rho_3u^*_{o,,t}$$
$$=H_1u_{o,,x}+H_3u^*_{o,,x}+H^*_1\theta_{z,x}+3H^*_3\theta^*_{z,x} \tag{14}$$

$$\rho_3\theta_{x,,t}+\rho_5\theta^*_{x,,t}$$
$$=(H_3\theta_{x,,t}-\bar{H}_1\theta_x)+(H_5\theta^*_{x,,x}-3\bar{H}_3\theta^*_x)$$
$$-\bar{H}_1w_{o,x}+(2H^*_3-\bar{H}_3)w^*_{o,x} \tag{15}$$

$$\rho_3u_{o,,t}+\rho_5u^*_{o,,t}$$
$$=H_3u_{o,,x}+(H_5u^*_{o,,x}-4\bar{H}_3u^*_o)$$
$$+(H^*_3-2\bar{H}_3)\theta_{z,x}+(3H^*_5-2\bar{H}_5)\theta^*_{z,x} \tag{16}$$

$$\rho_5\theta_{x,,t}+\rho_7\theta^*_{x,,t}$$
$$=(H_5\theta_{x,,x}-3\bar{H}_3\theta_x)+(H_7\theta^*_{x,,x}-9\bar{H}_5\theta^*_x)$$
$$-3\bar{H}_3w_{o,x}+(2H^*_5-3\bar{H}_5)w^*_{o,x} \tag{17}$$

$$\rho_1w_{o,,t}+\rho_3w^*_{o,,t}$$
$$=\bar{H}_1\theta_{x,x}+3\bar{H}_3\theta^*_{x,x}+\bar{H}_1w_{o,,x}+\bar{H}_3w^*_{o,,x} \tag{18}$$

$$\rho_3\theta_{z,,t}+\rho_5\theta^*_{z,,t}$$
$$=-H^*_1u_{o,x}-(H^*_3-2\bar{H}_3)u^*_{o,x}$$
$$-(\hat{H}_1\theta_z-\bar{H}_3\theta_{z,,x})-(3\hat{H}_3\theta^*_z-\bar{H}_5\theta^*_{z,,x}) \tag{19}$$

$$\rho_3w_{o,,t}+\rho_5w^*_{o,,t}$$
$$=-(2H^*_3-\bar{H}_3)\theta_{x,x}-(2H^*_5-3\bar{H}_5)\theta^*_{x,x}$$
$$+\bar{H}_3w_{o,,x}-(4\hat{H}_3w^*_o-\bar{H}_5w^*_{o,,x}) \tag{20}$$

$$\rho_5\theta_{z,,t}+\rho_7\theta^*_{z,,t}$$
$$=-3H^*_3u_{o,x}+(2\bar{H}_5-3H^*_5)u^*_{o,x}$$
$$+(\bar{H}_5\theta_{z,,x}-3\hat{H}_3\theta_z)+(\bar{H}_7\theta^*_{z,,x}-9\hat{H}_5\theta^*_z) \tag{21}$$

along with the following boundary conditions,

$$N_x=\bar{N}_x,\ M_x=\bar{M}_x,\ N^*_x=\bar{N}^*_x,\ M^*_x=\bar{M}^*_x \tag{22}$$
$$Q_x=\bar{Q}_x,\ S=\bar{S},\ Q^*_x=\bar{Q}^*_x,\ S^*=\bar{S}^* \tag{23}$$

where $u_{o,x}$, $u_{o,,x}$ and $u_{o,,t}$ refer to single and double differentiation of u_o with respect to x

and time, respectively, and similarly for all other variables.

SOLUTION TO EQUATIONS OF MOTION

The boundary conditions of a simply supported beam, which is considered in this study, can be defined as

$$u_o(0,t) = u_o(L,t) = 0 \quad (24a)$$
$$M_x(0,t) = M_x(L,t) = 0 \quad (24b)$$
$$u^*_o(0,t) = u^*_o(L,t) = 0 \quad (24c)$$
$$M^*_x(0,t) = M^*_x(L,t) = 0 \quad (24d)$$
$$w_o(0,t) = w_o(L,t) = 0 \quad (24e)$$
$$S(0,t) = S(L,t) = 0 \quad (24f)$$
$$w^*_o(0,t) = w^*_o(L,t) = 0 \quad (24g)$$
$$S^*(0,t) = S^*(L,t) = 0 \quad (24h)$$

The solution to the displacement variables satisfying these boundary conditions can be expressed in the following form:

$$u_o(x,t) = \Sigma A_n \sin(n\pi x/L) \sin(\omega_n t) \quad (25a)$$
$$\theta_x(x,t) = \Sigma B_n \cos(n\pi x/L) \sin(\omega_n t) \quad (25b)$$
$$u^*_o(x,t) = \Sigma C_n \sin(n\pi x/L) \sin(\omega_n t) \quad (25c)$$
$$\theta^*_x(x,t) = \Sigma D_n \cos(n\pi x/L) \sin(\omega_n t) \quad (25d)$$
$$w_o(x,t) = \Sigma E_n \sin(n\pi x/L) \sin(\omega_n t) \quad (25e)$$
$$\theta_z(x,t) = \Sigma F_n \cos(n\pi x/L) \sin(\omega_n t) \quad (25f)$$
$$w^*_o(x,t) = \Sigma G_n \sin(n\pi x/L) \sin(\omega_n t) \quad (25g)$$
$$\theta^*_z(x,t) = \Sigma I_n \cos(n\pi x/L) \sin(\omega_n t) \quad (25h)$$

where ω_n is the natural frequency of the system, A_n to I_n are unknown variables and n varies from 1 to infinity in all these equations.

When these solutions (Eqs. (25a)–(25h)) are substituted in the equations of equilibrium (Eqs. (14)–(21)), they assume a new form as

$$(H_1(n\pi/L)^2 - \rho_1\omega_n^2)A_n + (H_3(n\pi/L)^2 - \rho_3\omega_n^2)C_n + H^*_1(n\pi/L)F_n + 3H^*_3(n\pi/L)I_n = 0 \quad (26)$$

$$(\bar{H}_1 + H_3(n\pi/L)^2 - \rho_3\omega_n^2)B_n + (H_5(n\pi/L)^2 + 3\bar{H}_3 - \rho_5\omega_n^2)D_n + \bar{H}_1(n\pi/L)E_n + (\bar{H}_3 - 2H^*_3)(n\pi/L)G_n = 0 \quad (27)$$

$$(H_3(n\pi/L)^2 - \rho_3\omega_n^2)A_n + (H_5(n\pi/L)^2 + 4\bar{H}_3 - \rho_5\omega_n^2)C_n + (H^*_3 - 2\bar{H}_3)(n\pi/L)F_n + (3H^*_5 - 2\bar{H}_5)(n\pi/L)I_n = 0 \quad (28)$$

$$(H_5(n\pi/L)^2 + 3\bar{H}_3 - \rho_5\omega_n^2)B_n + (H_7(n\pi/L)^2 + 9\bar{H}_5 - \rho_7\omega_n^2)D_n + 3\bar{H}_3(n\pi/L)E_n + (3\bar{H}_5 - 2H^*_5)(n\pi/L)G_n = 0 \quad (29)$$

$$\bar{H}_1(n\pi/L)B_n + 3\bar{H}_3(n\pi/L)D_n + (\bar{H}_1(n\pi/L)^2 - \rho_1\omega_n^2)E_n + (\bar{H}_3(n\pi/L)^2 - \rho_3\omega_n^2)G_n = 0 \quad (30)$$

$$H^*_1(n\pi/L)A_n + (H^*_3 - 2\bar{H}_3)(n\pi/L)C_n + (\hat{H}_1 + \bar{H}_3(n\pi/L)^2 - \rho_3\omega_n^2)F_n + (3\hat{H}_3 + \bar{H}_5(n\pi/L)^2 - \rho_5\omega_n^2)I_n = 0 \quad (31)$$

$$(\bar{H}_3 - 2H^*_3)(n\pi/L)B_n + (3\bar{H}_5 - 2H^*_5)(n\pi/L)D_n + (\bar{H}_3(n\pi/L)^2 - \rho_3\omega_n^2)E_n + (4\hat{H}_3 + \bar{H}_5(n\pi/L)^2 - \rho_5\omega_n^2)G_n = 0 \quad (32)$$

$$3H^*_3(n\pi/L)A_n - (2\bar{H}_5 - 3H^*_5)(n\pi/L)C_n + (3\hat{H}_3 + \bar{H}_5(n\pi/L)^2 - \rho_5\omega_n^2)F_n + (9\hat{H}_5 + \bar{H}_7(n\pi/L)^2 - \rho_7\omega_n^2)I_n = 0 \quad (33)$$

In order to obtain an unique non-trivial solution for the unknowns A_n to I_n, the determinant of coefficients of these variables has to be set to zero. By solving this determinant, using the standard eigen value routines [39], the natural frequencies (ω^2-eigen values) and the corresponding mode shapes (eigen vectors) can be obtained.

NUMERICAL EXPERIMENTS

Numerical experiments were carried out to validate this solution procedure on a 586 platform. All the details about the material data used here are given in Table 1.

Thin beam sections

First, a few problems were chosen for which solutions are available in the literature to facilitate comparison. A sandwich beam by Ahmed [40], Khatua & Cheung [41] and a composite beam of Chandrashekhara & Bangera [14] were reanalysed using the Higher Order Beam Theory (HOBT) (Eqns (1) and (2)) and results are presented in Tables 2–4. The composite beam frequencies are non-dimensionalised using the following relationship.

$$\bar{\omega} = \omega^* L^2[\rho/(E_x^* d^2)]^{1/2} \quad (34)$$

where d is the depth of the cross section.

It can be observed that the higher order theory computes higher frequencies than those

Table 1. Material properties data

No.	Description	Ref.
DATA-1	$L = 36$ in (914·4 mm) $b = 1$ in (25·4 mm) $d = 0{\cdot}536$ in (13·614 mm) *Face properties* t_f (outer/inner) = 0·018 in (0·4572 mm) $E_f = 10^7$ lb/in^2 (68·97 kN/mm^2) $\rho_f = 167{\cdot}5$ lb/ft^3 (2·6831E3 N-sec^2/m^4) *Core properties* $t_c = 0{\cdot}5$ in (12·7 mm) $G_c = 12000$ lb/in^2 (82·764 N/mm^2) $\rho_c = 2{\cdot}05$ lb/ft^3 (32·8381 N-sec^2/m^4) $\nu = 0{\cdot}3$ No. of layers of c/s = eight	[40]
DATA-2	$L = 20$ in (508 mm) $b = 1$ in (25·4 mm) $d = 0{\cdot}86$ in (21·844 mm) *Face properties* t_f (bot/mid/top) = 0·02 in (0·508 mm) $E_f = 10^7$ lb/in^2 (68·97 kN/mm^2) $\rho_f = 1{\cdot}0$ lb sec^2/in^4 (1·0691E7 N-sec^2/m^4) *Core properties* t_c (bot/top) = 0·4 in (10·16 mm) $G_c = 5000$ lb/in^2 (34·485 N/mm^2) $\rho_c = 0{\cdot}25$ lb sec^2/in^4 (2·6726E6 N-sec^2/m^4) $\nu = 0{\cdot}3$ No. of layers of c/s = five	[41]
DATA-3	$L = 15$ m, $b = 1$ m, $d = 1$ m $E_1 = 144{\cdot}8$ GPa $E_2 = 9{\cdot}65$ GPa $G_{12} = 4{\cdot}14$ GPa $\rho = 1389{\cdot}23$ N-sec^2/m^4 $\nu_{12} = 0{\cdot}3$ No. of layers of c/s = eight Material: AS4/3501-6/Graphite/Epoxy Lamination scheme: 0/90/90/0	[14]
DATA-4	$L = 762$ mm, $b = 25{\cdot}4$ mm, $d = 152{\cdot}4$ mm $E_x = 0{\cdot}525$E6 N/mm^2 $E_z = 0{\cdot}21$E5 N/mm^2 $G_{xz} = 0{\cdot}105$E5 N/mm^2 $\rho = 800$ N-sec^2/m^4 $\nu = 0{\cdot}3$ No. of layers of c/s = six Lamination scheme: 0/0/90/90/0/0	[42]
DATA-5	$d = 3$ m Rest are same as DATA-3	[14]
DATA-6	$L = 762$ mm, $b = 25{\cdot}4$ mm, $d = 152{\cdot}4$ mm *Face properties* Material: Graphite/Epoxy $t_f = 15{\cdot}24$ mm $E_{fx} = 0{\cdot}12$E6 N/mm^2 $E_{fz} = 0{\cdot}79$E4 N/mm^2 $G_{fxz} = 0{\cdot}55$E4 N/mm^2 $\rho_f = 1{\cdot}58$ kN-sec^2/m^4 $\nu = 0{\cdot}3$ *Core properties* Material: US commercial aluminium, honeycomb 0·25 in cell size, 0·007 in foil $t_c = 121{\cdot}92$ mm $G_{cxz} = 140{\cdot}7$ N/mm^2 $\rho_c = 34{\cdot}15$ N-sec^2/m^4 $t_c/t_f = 8$ No. of layers of c/s = six Lamination scheme: 0/90/core/90/0	[43] [44]

of First Order Beam Theory (FOBT), except for a few fundamental modes. With the higher order modelling having more degrees of freedom, thus making the beam more flexible, its frequencies are naturally, to be lower than those of the comparatively stiffer Timoshenko theory. This apparent conflict can be resolved by considering the following displacement model from Eqns (1) and (2):

$$u(x,z,t) = u_o(x,t) + z\theta_x(x,t) + z^2 u^*_o(x,t) \tag{35}$$

$$w(x,z,t) = w_o(x,t) \tag{36}$$

Let this be designated as HOBT4, with four degrees of freedom. Its frequencies are given in Tables 2 and 5. This particular configuration of higher order model is specifically chosen here as it would have identical degrees of freedom to FOBT, with u^*_o being ineffective for symmetric laminates. Also, the results of a five degrees of freedom model [35] HOBT5 (with u_o, w_o, θ_x, u^*_o, θ^*_x) are presented for various cases.

It can be seen that HOBT frequencies for all modes are consistently lower than those of HOBT4 and HOBT5. This clearly confirms that with the addition of more degrees of freedom the beam indeed becomes flexible and predicts lower frequencies. As both HOBT4 and FOBT possess identical degrees of freedom, the only factor that could reduce its frequencies compared with HOBT4 is the assumption of constant shear strain across the depth and the usage of shear correction factor.

Thick beam sections

The deep composite beam of DATA-4 is analysed and from the results of Table 5, the flexible nature of HOBT is confirmed with its lower frequencies for all modes compared to HOBT4 and HOBT5. In deep beams also, the frequencies of Timoshenko theory are lower compared with HOBT4, in spite of having same degrees of freedom, due to the shear correction factor. A similar pattern can be observed for the composite beam of DATA-5 for an aspect ratio of five in Table 6.

The HOBT4 results are presented here for only one thin and one thick section, as the observations drawn hold good for all other experiments and also are established in a more detailed finite element analysis [35] on the term-by-term contribution of the higher order model to the frequencies of beams with dif-

Table 2. Comparison of natural frequencies (in Hz) of a thin sandwich beam (DATA-1)

n	FOBT	HOBT4	HOBT5 (Ref. 35)	HOBT	Ref. [40]
1	59·508	59·715	57·041	57·040	56·028
2	225·204	227·751	218·361	218·361	–
3	466·961	477·624	460·754	459·958	457·120
4	754·554	780·178	758·692	754·554	–
5	1065·383	1112·334	1097·055	1079·866	1090·260
6	1386·399	1459·133	1457·064	1420·617	–
7	1710·916	1811·183	1849·380	1766·620	1809·800
8	2034·000	2164·507	2275·916	2115·169	–

Table 3. Comparison of natural frequencies of a slender sandwich beam (DATA-2)

n	FOBT	HOBT5 (Ref. 35)	HOBT	Ref. [41]
1	10·74	10·72	10·72	10·9117
2	29·67	31·04	31·02	32·2084
3	48·80	52·15	52·10	54·6588
4	67·49	72·93	72·75	76·7496
5	85·89	93·85	93·04	98·4750
6	104·10	114·40	113·10	119·9644
7	122·20	136·00	133·00	141·3274
8	140·20	158·90	152·70	162·8256
9	158·20	182·90	172·30	184·0243

Table 4. Comparison of non-dimensional natural frequencies of a thin composite beam (DATA-3)

n	FOBT	HOBT
1	2·512	2·516
2	8·589	8·669
3	16·045	16·320
4	23·795	24·371

Table 5. Comparison of non-dimensional natural frequencies of a deep composite beam (DATA-4)

n	FOBT	HOBT4	HOBT5 (Ref. 35)	HOBT
1	1·639	1·736	1·656	1·657
2	3·803	4·115	3·923	3·910
3	5·895	6·416	6·191	6·138
4	7·953	8·675	8·470	8·323
5	9·998	10·916	10·803	10·440
6	12·033	13·145	13·117	12·469
7	14·065	15·370	15·561	14·385
8	16·088	17·595	18·151	16·161
9	18·122	19·806	20·889	17·771

Table 6. Comparison of non-dimensional natural frequencies of a thick composite beam (DATA-5)

n	FOBT	HOBT
1	1·794	1·820
2	4·396	4·528
3	6·921	7·201
4	9·395	9·814
5	11·841	12·341
6	14·273	14·684
7	16·697	16·638
8	19·121	17·152

ferent materials and aspect ratios. The results of the sandwich beam of DATA-6, presented in Table 7, are non-dimensionalised using the following expression

$$\bar{\omega} = \omega^* L^2[\rho_f/(E_{fx}{}^* d^2)]^{1/2} \tag{37}$$

Here, the reduction of frequencies by the refined theory is quite significant — of the order of more than 50% of Timoshenko's theory frequencies — for first three modes.

CONCLUSIONS

A closed form solution of higher order refined theory is presented for the vibration analysis of composite and sandwich beams. Using Hamilton's principle, equations of dynamic equilibrium are derived and with standard

Table 7. Comparison of non-dimensional natural frequencies of a deep sandwich beam (DATA-6)

n	FOBT	HOBT5 (Ref. 35)	HOBT
1	2·229	1·293	1·293
2	5·515	2·787	2·744
3	8·699	4·315	4·073

eigenvalue routines, natural frequencies are estimated. Axial, transverse shear and normal strains are obtained using cubic, cubic and quadratic variation respectively across the thickness to enable the formulation, model the cross sectional warping exactly, without any shear correction factors.

It can be observed from the numerical experiments that the higher order model is quite flexible resulting in lower frequencies and on the other hand frequencies of first order theory are influenced by its shear correction factor. In the case of thick sandwiches, higher order model predicts very low frequencies. It is worthwhile to observe that the order of difference of predictions between the refined theory and first order theory for thick sandwiches is more than 50% while for thick composites and thin sections this difference becomes marginal.

Moreover, it emerges from this study that this solution scheme aptly fills the gap in higher order literature for an analytical solution and also is quite adequate to evaluate the fundamental flexural frequencies of sandwich and composite laminates.

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