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C^0 FINITE ELEMENT GEOMETRICALLY NON-LINEAR ANALYSIS OF FIBRE REINFORCED COMPOSITE AND SANDWICH LAMINATES BASED ON A HIGHER-ORDER THEORY

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Abstract—This paper presents a refined higher-order shear deformation theory for the linear and geometrically non-linear finite element analysis of fibre reinforced composite and sandwich laminates. Laminae material is assumed to be linearly elastic, homogeneous and isotropic/orthotropic. This theory accounts for parabolic distribution of the transverse shear strains through the thickness of the laminate and higher-order terms in Green's strain vector in the sense of von Karman. A simple C^0 finite element formulation is presented with a total Lagrangian approach and a nine-node Lagrangian quadrilateral element is chosen with nine degrees of freedom per node. Numerical results are presented for linear and geometric non-linear analysis of multi-layer cross-ply laminates and sandwich plates. The present theory predicts displacements and stresses more accurately than the first-order shear deformation theory. The results are compared with available closed-form and numerical solutions of both three-dimensional elasticity and plate theories.

1. INTRODUCTION

Advanced composite structures include laminated plates and shells which are composed of layers of fibre reinforced composite (FRC) material oriented in an optimal manner. A major purpose of lamination is to tailor the directional strength and stiffness of the material to match the loading conditions of the structural element. The choice of variation of the orientation for each lamina and stacking sequence gives an added degree of flexibility to the design engineer.

In the early days, the classical lamination theory [1], based on the Kirchhoff hypothesis, was adopted for the analysis of laminated composite plates. Later it was recognized that classical lamination theory ought to be modified to include certain higher-order effects like warping of the transverse cross-section. The first generalization of the classical theory, with inclusion of first-order transverse shear deformation effects was given by Reissner [2] and Mindlin [3]. The first lamination theory including bending–stretching coupling is apparently due to Reissner and Stavsky [4]. Yang *et al.* [5] presented a generalization of the Reissner–Mindlin thick plate theory for homogeneous and isotropic plates to arbitrarily laminated anisotropic plates. Whitney and Pagano [6] and Reddy and Chao [7] presented closed-form solutions to the theory when applied to certain cross-ply and angle-ply rectangular plates. Reddy [8] presented a finite-element analysis of the first-order shear deformation lamination theory.

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The origin of higher-order theories goes back to the work of Hilderbrand *et al.* [9] who made significant contributions by dispensing with all Kirchhoff's assumptions. They assumed a three-term Taylor series expansion of the displacement vector. The minimum potential energy principle along with full elasticity matrix was used in the derivation.

However, a formal plate higher-order theory, based upon the principle of stationary potential energy, resulting in eleven second-order partial differential equations to determine the eleven functions in the assumed displacement model, is credited to Lo *et al.* [10, 11]. A sub-set of the displacement model used in [10] which neglects the strain energy due to transverse normal stress–strain and higher-order in-plane modes of deformation has been adopted by Levinson [12], Murthy [13] and Reddy [14]. Later Reddy along with co-workers presented the displacement-based [15] and the mixed [16] finite element models of the theory developed earlier. The displacement formulation, however, requires C^1 continuous shape functions which are computationally inefficient and are not amenable to the popular and widely used isoparameteric formulation in present-day finite element technology.

Kant [17] derived the complete set of equations of an isotropic version of the Lo *et al.* theory and presented extensive numerical results with a numerical integration technique. Kant *et al.* [18] are the first to present a C^0 finite element formulation of this higher-order displacement model. Later, Kant along with co-workers extended this work for application to FRC and sandwich plates [19–28].

Pagano [29–31] presented three-dimensional (3D) elasticity solutions for composite laminates of finite width and infinite length [30] as well as square and rectangular composite and sandwich laminates [29, 31] subjected to sinusoidal loads with simply supported boundary conditions for cylindrical and bidirectional bending problems.

Much of the previous research in the analysis of composite plates is limited to linear problems. This may have been due to the complexity of the non-linear partial differential equations associated with the large-deflection theory of composite plates. Approximate solutions to the large-deflection theory (in the von Karman sense) of laminated composite plates were attempted by Stavsky [32, 33], Habip [34], Whitney and Leissa [35], Chia and Prabhakara [36, 37], Noor and Hartly [38], Reddy [39, 40], etc. Chia and Prabhakara [36, 37] employed the Galkerin method to reduce the governing non-linear partial differential equations to an ordinary differential system of equations. They used the perturbation technique to solve the resulting equation. In all these studies [32–37], the transverse shear effects were neglected. Although the finite element employed by Noor and Hartly [38] includes the effect of transverse shear strains, it involves 80 degrees of freedom per element. Use of such elements in the non-linear analysis of composite plates inevitably leads to large storage requirements and computational costs. The same is true even in the case of 3D FEM analysis [40].

Recently, Kant and Mallikarjuna [41] presented dynamic large-deflection response of laminated composite plates using a refined theory and C^0 finite elements.

The aim of this paper is to provide a geometrically non-linear (GNL) higher-order formulation under the assumption of large displacement but small rotations and strains. In the present investigation square/rectangular FRC and sandwich laminates are considered. Numerical results for linear as well as non-linear analysis are presented for several cross-ply and sandwich plates. Comparisons are made with available results and conclusions on the use of the proposed refined theory are drawn.

2. NON-LINEAR HIGHER-ORDER THEORY OF ANISOTROPIC LAMINATES

We consider a composite laminate consisting of thin homogeneous and orthotropic/isotropic layers oriented arbitrarily and having a total thickness of h ($h_1, h_2, h_3 \ldots$, etc. are thicknesses of individual layers such that $h = h_1 + h_2 + \cdots$) (see Fig. 1).

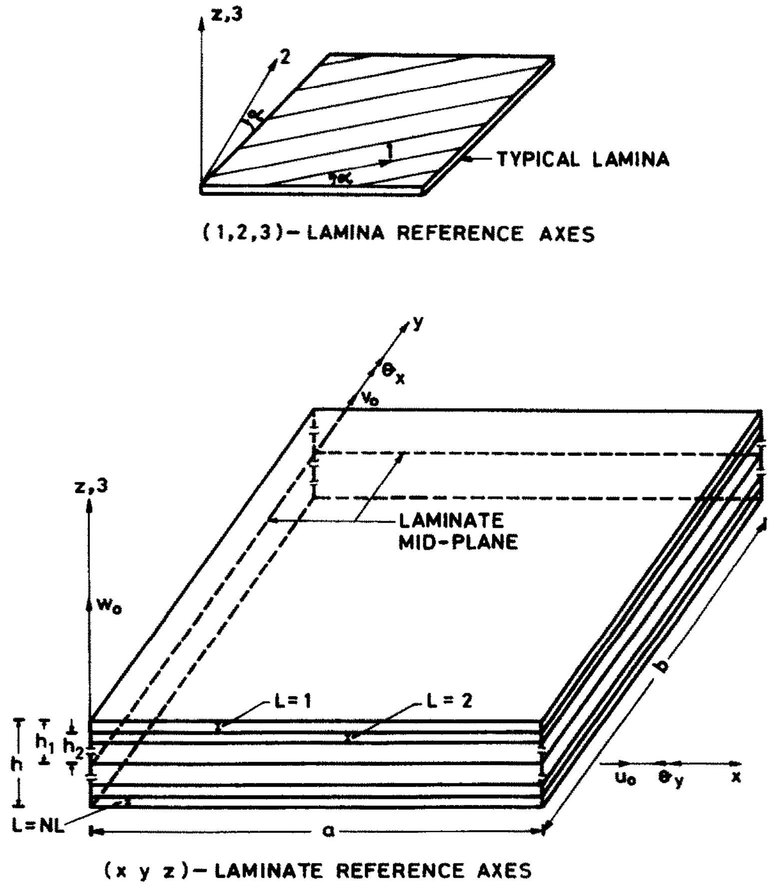

Fig. 1. Laminate geometry with positive reference set of lamina/laminate reference axes displacement components and fibre orientation.

The x–y plane coincides with the middle plane of the laminate with the z-axis oriented in the thickness direction such that x, y and z form a right-handed screw coordinate system. In the present theory, the displacement components of a generic point in the laminate are assumed to be of the form (see Kant and Pandya [21] and Reddy [14])

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\theta_x(x,y) \\ &\quad + z^2 u_0^*(x,y) + z^3\theta_x^*(x,y) \\ v(x,y,z) &= v_0(x,y) + z\theta_y(x,y) \\ &\quad + z^2 v_0^*(x,y) + z^3\theta_y^*(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

in which u_0, v_0 and w_0 are mid-plane displacements of a generic point having displacements u, v and w in x, y and z directions, respectively. The parameters u_0^* and v_0^* are higher-order terms corresponding to u and v, respectively. The parameters θ_x and θ_y are rotations of the transverse normal cross-section in the xz and yz planes, respectively. The parameters θ_x^* and θ_y^* are the corresponding higher-order terms in Taylor's series expansion. A total Lagrangian approach is adopted and stress and strain descriptions used are those due to Piola–Kirchhoff [42] and Green [43], respectively. For GNL analysis alone, undergoing large displacements with small rotations and strains is considered here, the stress–strain relationship is in terms of total values.

Both isotropic and anisotropic situations can be accommodated with arbitrary thicknesses for different layers. By invoking von Karman large deflection assumptions, we have the following Green–Lagrangian strain displacement relations

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \gamma_{zx} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \end{aligned} \tag{2}$$

To develop the finite element equations, we consider the virtual work equations for a continuum written in total Lagrangian coordinate system under the assumption of small strain and conservative loading [43] as

$$\int_v \delta\boldsymbol{\epsilon}^t\boldsymbol{\sigma}\,\mathrm{d}v = \int_v \rho\,\delta\mathbf{u}^t\mathbf{q}\,\mathrm{d}v + \int_A \delta\mathbf{u}^t\mathbf{P}\,\mathrm{d}A, \tag{3}$$

where v is the undeformed volume, $\boldsymbol{\sigma}$ is the Piola–Kirchhoff stress vector, $\delta\epsilon$ is the virtual Green's strain vector due to virtual displacement $\delta\mathbf{u}$, ρ is mass density, $\mathbf{q}$ is the body force per unit mass and $\mathbf{P}$ is surface traction acting over the undeformed area A.

The laminate constitutive relations, details of which are given elsewhere [19–28] are

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{bmatrix} \begin{bmatrix} \mathbf{D}_m & \mathbf{D}_c & \mathbf{0} \\ \mathbf{D}_c^t & \mathbf{D}_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{D}_s \end{bmatrix} \begin{Bmatrix} \epsilon_P + \epsilon_{NL} \\ \epsilon_b + 0 \\ \epsilon_s + 0 \end{Bmatrix} \tag{4}$$

or symbolically

$$\boldsymbol{\sigma} = \mathbf{D}(\epsilon_L + \epsilon_{NL}) = \mathbf{D}\epsilon. \tag{5}$$

In eqns (4) and (5) $\mathbf{N}$, $\mathbf{M}$ and $\mathbf{Q}$ are vectors of in-plane stress resultants, stress moments and transverse shear forces, respectively. The coefficients D_{mij}, D_{cij}, D_{bij} and D_{sij} are respectively the in-plane, bending-in plane coupling, bending and transverse shear stiffness coefficients [19–28] and ϵ_L and ϵ_{NL} are the generalized linear and non-linear strains, respectively.

3. C^0 FINITE ELEMENT FORMULATION

The finite element used here is a nine-node isoparameteric quadrilateral element. The laminate displacement field in the element can be expressed in terms of nodal variables, such that

$$\mathbf{d}(\xi,\eta) = \sum_{i=1}^{N} N_i(\xi,\eta)\,\mathbf{d}_i, \tag{6}$$

where NN represents number of nodes in the element, $N_i(\xi,\eta)$ defines interpolation function associated with node i in terms of normalized coordinates ξ, η and $\mathbf{d}_i$ is generalized displacement vector of the mid-plane, such that

$$\mathbf{d}^t = (u_0, v_0, w_0, \theta_x, \theta_y, u_0^*, v_0^*, \theta_x^*, \theta_y^*).$$

The generalized vectors of Green strain and its variation ϵ and $\delta\epsilon$, respectively, are written in terms of nodal displacements $\mathbf{d}$, displacement gradient $\boldsymbol{\theta}_{NL}$ and cartesian derivatives of shape functions [42]

$$\begin{aligned} \epsilon &= (\mathbf{B}_0 + \tfrac{1}{2}\mathbf{B}_{NL})\,\mathbf{d}_i \\ \delta\epsilon &= (\mathbf{B}_0 + \mathbf{B}_{NL})\,\mathbf{d}_i \\ \delta\boldsymbol{\sigma} &= \mathbf{DB}\,\delta\mathbf{d}, \end{aligned} \tag{7}$$

where $\mathbf{B}_0$ is the linear strain displacement matrix [21] and $\mathbf{B}_{NL}$ non-linear strain displacement matrix which is linearly dependent upon the nodal displacement $\mathbf{d}$ [42].

Simplifying eqn (3)

$$\delta \mathbf{d}^t \int_v \mathbf{N}^t \rho \mathbf{q} \, \mathrm{d}v + \delta \mathbf{d}^t \int_A \mathbf{N}^t \mathbf{P}_0 \, \mathrm{d}A = \delta \mathbf{d}^t \int \mathbf{B}^t \boldsymbol{\sigma} \, \mathrm{d}v. \tag{8}$$

From this the fundamental equilibrium equation can be established as

$$\int_v \mathbf{N}^t \rho \mathbf{q} \, \mathrm{d}v + \int_A \mathbf{N}^t \mathbf{P}_0 \, \mathrm{d}A = \int_v \mathbf{B}^t \boldsymbol{\sigma} \, \mathrm{d}v$$

$$\boldsymbol{\psi}(d) = \mathbf{P} - \mathbf{R} = 0,$$

where

$$\mathbf{P} = \int B^t \sigma \, \mathrm{d} \quad \text{and} \quad \mathbf{R} = \int_v \mathbf{N}^t \rho \mathbf{q} \, \mathrm{d}v + \int_A \mathbf{N}^t \mathbf{P}_0 \, \mathrm{d}A. \tag{9}$$

4. SOLUTION TECHNIQUE

The solution algorithm is established for solving the discrete assembled non-linear equilibrium equation (9) based on the Newton–Raphson method [42] which is summarized by the recurrence relationship

$$\mathbf{d}_{i+1} - \mathbf{d}_i = \Delta \mathbf{d}_i = -\mathbf{K}_T^{-1} \boldsymbol{\psi}_i, \tag{10}$$

where

$$\mathbf{K}^T = \left[\frac{\partial \psi}{\partial \mathbf{d}}\right] = \left[\frac{\partial(\psi_1, \psi_2 \cdots, \psi_n)}{\partial(d_1, d_2 \cdots, d_n)}\right],$$

where n is total number of degrees of freedom.

Equation (10) means that the solution to the non-linear equation (9) is achieved by a series of linear solutions carried out until $\boldsymbol{\psi}$, known as the residual force vector, is sufficiently close to zero. To assist the numerical stability of this process the load $\mathbf{R}$ is applied in a series of steps from 1 to n, a typical step having a total applied load $\mathbf{R}^n$, where

$$R^n = \sum_{J=1}^{n} \Delta R^J \tag{11}$$

and equilibrium is established for each $\mathbf{R}^n$. The total displacements $\mathbf{d}^n$ are similarly defined within each load step n; a number of iterations i take place until convergence is achieved, consequently the total incremental displacement $\Delta \mathbf{d}^n$ is made up as follows:

$$\Delta \mathbf{d}^n = \sum_{j=1}^{i} \Delta d_j^n. \tag{12}$$

5. NUMERICAL RESULTS

5.1. *Preliminary remarks*

In the present study the laminate is discretized with four nine-noded Lagrangian quadrilateral isoparameteric elements in a quarter plate. Selective integration scheme based on Gauss quadrature rules, namely 3 × 3 for membrane, flexure and coupling between membrane and flexure terms, and 2 × 2 for shear terms in the energy expression, is employed in the evaluation of the element stiffness property. For the numerical computations two computer programs FOST and HOST with five and nine degrees of freedom per node, respectively, are developed. All the computations were carried out in single precision CDC Cyber 180/840 computer with 64-bit word length. All the stress values were evaluated at the Gauss points. To facilitate the comparison of present results with existing results, the following property sets are adopted, even though they are not satisfying the symmetry condition, that is, $\nu_{12}/E_1 \neq \nu_{21}/E_2$.

Material 1. The dimensionless material properties, taken from [29], are

$$\frac{E_1}{E_2} = 25, \quad \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5, \quad \frac{G_{23}}{E_2} = 0.2, \quad \nu = 0.25.$$

Material 2. The dimensionless material properties, taken from [46], are

$$\frac{E_1}{E_2} = 40, \quad \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5, \quad \frac{G_{23}}{E_2} = 0.2, \quad \nu = 0.25.$$

Material 3. The properties, taken from [29] are given below. Face sheets

$$\frac{E_1}{E_2} = 25, \quad \frac{G_{12}}{E_2} = \frac{G_{13}}{E_2} = 0.5, \quad \frac{G_{23}}{E_2} = 0.2,$$

$$E_2 = 10^6 \text{ psi}, \quad \nu = 0.25.$$

Thickness of each face sheet $= 0.10h$. Core of sandwich plate

$$E_1 = E_2 = 0.04 \times 10^6 \text{ psi},$$

$$E_3 = 0.5 \times 10^6 \text{ psi},$$

$$G_{13} = G_{23} = 0.06 \times 10^6 \text{ psi},$$

$$G_{12} = 0.016 \times 10^6 \text{ psi},$$

$$\nu = 0.25.$$

Material 4. The properties, taken from [47], are given below

$$E_1 = 1.8282 \times 10^6 \text{ psi}, \quad E_2 = 1.8315 \times 10^6 \text{ psi},$$

$$G_{12} = G_{13} = G_{23} = 3.125 \times 10^5 \text{ psi}, \quad \nu = 0.23913.$$

Table 1. Non-dimensional displacement values for cross-ply (0°/90°/0°) laminate (Material 1)

	Square laminate					Rectangular laminate ($b = 3a$)			
a/h	Exact [29]	3D-FEM [40]	FOST	HOST	HOST (4 × 4) mesh	Exact [29]	3D-FEM [40]	FOST	HOST
2	5.075	5.051	4.4621	5.2582	5.2561	8.170	8.410	6.3127	8.0271
4	1.937	1.906	1.7102	1.9035	1.9023	2.820	2.905	2.2855	2.6425
10	0.737	0.728	0.6631	0.7209	0.7204	0.919	0.915	0.8019	0.8681
20	0.513	0.506	0.4915	0.5082	0.5078	0.610	0.589	0.5783	0.5956
50	0.445	0.438	0.4332	0.4440	0.4440	0.520	0.494	0.5156	0.5180
100	0.435	0.429	0.4290	0.4346	0.4344	0.508	0.481	0.5065	0.5070

The boundary conditions used for simply supported and clamped plates are as follows:

Simply supported (S1)

$$\text{along } x\text{-axis: } u_0 = w_0 = \theta_x = u_0^* = \theta_x^* = 0$$
$$\text{along } y\text{-axis: } v_0 = w_0 = \theta_y = v_0^* = \theta_y^* = 0. \quad (13)$$

Simply supported (S2)

$$\text{along } x\text{-axis: } u_0 = v_0 = w_0 = \theta_x = u_0^* = v_0^* = \theta_x^* = 0$$
$$\text{along } y\text{-axis: } u_0 = v_0 = w_0 = \theta_y = u_0^* = v_0^* = \theta_y^* = 0. \quad (14)$$

Clamped (C)

$$\text{along all edges: } u_0 = v_0 = w_0 = \theta_x = \theta_y = u_0^* = v_0^* = \theta_x^* = \theta_y^* = 0. \quad (15)$$

Along the centre line in case of quarter plate symmetry

$$\text{along } x\text{-axis: } v_0 = \theta_y = v_0^* = \theta_y^*$$
$$\text{along } y\text{-axis: } u_0 = \theta_x = u_0^* = \theta_x^*. \quad (16)$$

5.2. *Examples and discussion*

5.2.1. *Linear analysis.* In this section we show the validity of the present formulation by comparing linear analysis results with the corresponding 3D elasticity results [29–31]. The following cases are considered for this purpose

(a) a three layer symmetric cross-ply (0°/90°/0°) laminate and
(b) a square sandwich plate.

In both the cases, the loading considered is sinusoidal load with

$$q = q_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \quad (17)$$

and boundary conditions are simply supported (S1). The following non-dimensional factors are used for the displacements and stresses

$$\bar{w}_0 = \left(\frac{E_2 h^3}{q_0 a^4}\right) w_0, \quad (\bar{\sigma}_1 \text{ or } \bar{\tau}_{12}) = \left(\frac{h^2}{q_0 a^2}\right)(\sigma_1 \text{ or } \tau_{12}). \quad (18)$$

Table 1 shows the non-dimensional displacements for square laminate (0°/90°/0°) with unequal thick layers of $h_1 = h_3 = h/4$, $h_2 = h/2$ and rectangular laminate ($b = 3a$, 0°/90°/0°) of equal thick layers, along with the exact (3D elasticity) values given by Pagano [29–31]. The material properties used are given by material set 1. Table 2 shows the non-dimensional stresses for square cross-ply laminates (0°/90°/0°) of unequal thick layers, $h_1 = h_3 = h/4$, $h_2 = h/2$ and equal thick layers, along with exact solution (3D elasticity) values given by Pagano [29].

Table 2. Non-dimensional stress values for a square cross-ply (0°/90°/0°) laminate (material 1)

		Unequal thick laminae			Equal thick laminae			
a/h	Stress type	Exact [31]	FOST	HOST	Exact [29]	FOST	HOST	HSDT [44]
2	$\bar{\sigma}$	1.3880	—	1.0943	0.9380	0.3625	1.1083	
	$\bar{\tau}$	0.0863	—	0.0790	0.7020	0.0494	0.0808	
4	$\bar{\sigma}$	0.7200	0.4093	0.7161	0.7550	0.4405	0.7681	0.7345
	$\bar{\tau}$	0.0467	0.0311	0.0464	0.0505	0.0373	0.0502	
10	$\bar{\sigma}$	0.5590	0.5063	0.5687	0.5900	0.5212	0.5929	0.5684
	$\bar{\tau}$	0.0275	0.0242	0.0274	0.0289	0.0252	0.0282	
20	$\bar{\sigma}$	0.5430	0.5345	0.5506	0.5520	0.5392	0.5579	
	$\bar{\tau}$	0.0230	0.0222	0.0231	0.0234	0.0224	0.0232	
50	$\bar{\sigma}$	0.5390	0.5424	0.5432	0.5410	0.5430	0.5460	
	$\bar{\tau}$	0.0216	0.0216	0.0217	0.0216	0.0216	0.0217	
100	$\bar{\sigma}$	0.5390	0.5362	0.5369	0.5390	0.5364	0.5373	0.5390
	$\bar{\tau}$	0.0214	0.0218	0.0218	0.0213	0.0218	0.0218	

Table 3. Non-dimensional displacement and stress values for sandwich plate (material 3)

a/h	Variable	Exact [29]	FOST	HOST
2	$\bar{w}$	21.6531	14.9369	21.3707
	$\bar{\sigma}$	2.6530	0.7435	2.79852
	$\bar{\tau}$	0.2338	0.3192	0.2371
4	$\bar{w}$	7.3017	4.7691	7.1502
	$\bar{\sigma}$	1.5120	0.8991	1.4989
	$\bar{\tau}$	0.1480	0.0917	0.1428
10	$\bar{w}$	2.2440	1.5612	2.0864
	$\bar{\sigma}$	1.1520	1.0619	1.1657
	$\bar{\tau}$	0.0717	0.0553	0.0692
20	$\bar{w}$	1.2022	1.0530	1.1947
	$\bar{\sigma}$	1.1100	1.0985	1.1246
	$\bar{\tau}$	0.0511	0.0467	0.0506
50	$\bar{w}$	0.9063	0.9772	0.9299
	$\bar{\sigma}$	1.0990	1.1070	1.1118
	$\bar{\tau}$	0.0446	0.0442	0.0448
100	$\bar{w}$	0.8903	0.8856	0.8915
	$\bar{\sigma}$	1.0980	1.1044	1.1058
	$\bar{\tau}$	0.0437	0.0439	0.0440

Table 3 shows the non-dimensional displacement and stresses for a three-layer sandwich plate with each facing of $h/10$ thickness and the material properties used is given by material set 3, along with exact solutions given by Pagano [29].

The following observations are made from the numerical results presented in Tables 1–3. For a thick plate with $a/h = 4$ and less, displacement and stresses, are close to 3D elasticity result [29]. The stress components evaluated are more accurate for moderately thick to thin plates ($a/h \geqslant 10$) with either of the two theories, i.e. FOST or HOST.

5.2.2. *Non-linear analysis*. We show here the validity and credibility of the present higher-order theory in non-linear analysis by comparing our present results with available closed form solutions [46, 48], finite element results [47] and our own first-order shear deformation theory (FOST) results. The following non-dimensional parameters are used for load $\bar{q}$ displacement $\bar{w}_0$ and stress $(\bar{\sigma})$ in the non-linear analysis

$$\bar{w}_0 = \frac{w_0}{h}, \quad \bar{q} = \frac{1}{E_2}\left(\frac{a}{h}\right)^4 \cdot q, \quad \bar{\sigma} = \frac{1}{E_2}\left(\frac{a}{h}\right)^2 \sigma. \tag{19}$$

Example 1. We consider first a clamped isotropic plate with material properties $E = 3 \times 10^7\ \mathrm{lb/in}^2$, $\nu = 0.316$ and $a/h = 100$. This example has become a standard one, used by many authors, e.g. Levy [49], Pica *et al.* [50], Reddy [51], etc., to check the GNL analysis of plate formulations. The analytical solution given by Levy [49], who solved von Karman's equations using a double Fourier series, is quoted as having a possible error of less than 2%. The comparative results for displacement and stresses are presented in Table 4. From these results it can be concluded that the use of present formulation in the GNL contest is acceptable.

Example 2. We consider a square clamped (C) four-ply (0°/90°/90°/0°) symmetric laminate with $a/h = 125$, material properties set four- and an eight-ply (0°/90°/0° $\cdots$ 90°) unsymmetric laminate with $a/h = 10$ and material properties set 1, subjected to an uniformly distributed load. The present HOST results are compared with higher-order shear deformation theory results given by Reddy [47]. These are

Table 4. Non-dimensional displacement and stress values for a clamped square isotropic plate under uniformly distributed load

Load $\bar{q}$	Variable	Reddy [51]	Pica *et al.* [50]	Levy [49]	FOST	HOST
17.79	$\bar{w}$	0.2455	0.2368	0.2370	0.2385	0.2385
	$\bar{\sigma}$	2.4590	2.6319	2.6000	2.6723	2.6733
38.33	$\bar{w}$	0.4784	0.3699	0.4710	0.4725	0.4725
	$\bar{\sigma}$	5.1290	5.4816	5.2000	5.5733	5.5733
63.40	$\bar{w}$	0.7045	0.6915	0.6950	0.6948	0.6948
	$\bar{\sigma}$	7.8340	8.3258	8.0000	8.4833	8.4867
95.00	$\bar{w}$	0.9147	0.9029	0.9120	0.9065	0.9065
	$\bar{\sigma}$	10.4600	11.1030	11.1000	11.3433	11.3500
134.90	$\bar{w}$	1.1189	1.1063	1.1210	1.1100	1.1100
	$\bar{\sigma}$	13.0900	13.8270	13.3000	14.1633	14.1700
184.00	$\bar{w}$	1.3189	1.3009	1.3230	1.3046	1.3046
	$\bar{\sigma}$	15.7500	16.4970	15.9000	16.9266	16.9367
245.00	$\bar{w}$	1.5155	1.4928	1.5210	1.4963	1.4963
	$\bar{\sigma}$	18.4800	19.2250	19.2000	19.7533	19.7633
318.00	$\bar{w}$	1.7020	1.6786	1.7140	1.6820	1.6820
	$\bar{\sigma}$	21.2100	21.9940	21.9000	22.6233	22.6367
402.00	$\bar{w}$	1.8760	1.8555	1.9020	1.8589	1.8590
	$\bar{\sigma}$	23.8900	24.7800	25.1000	25.5200	25.5367

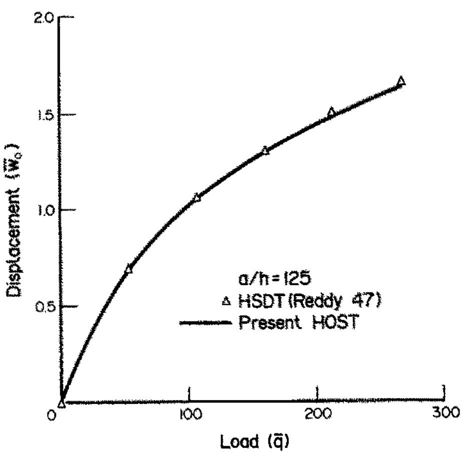

Fig. 2. Displacement vs load for a square clamped (C) symmetric cross-ply (0°/90°/90°/0°) laminate.

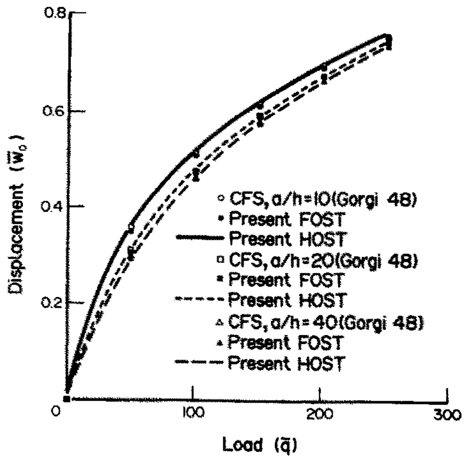

Fig. 4. Displacement vs load for a simply supported (S2) symmetric cross-ply (0°/90°/90°/0°) laminate.

plotted in Figs 2 and 3, respectively for the symmetric and unsymmetric laminates.

Example 3. We consider here a square simply supported (S2) four-ply symmetric cross-ply (0°/90°/90°/0°) laminate with material properties set 1, subjected to an uniformly distributed load. The present results for various a/h ratios are compared with analytical results given by Gorji [48] and these are shown in Fig. 4.

Example 4. A symmetric three-layer (0°/90°/0°) simply supported (S1) square cross-ply laminate with $a/h = 10$, material properties set 1 and subjected to an uniformly distributed load is considered. The present results are compared with 3D FEM results given by Reddy [40] and these are plotted in Fig. 5.

Example 5. We analyse an unsymmetric two-layer (0°/90°) simply supported (S1) square laminate with $a/h = 10$, material properties set 2 and subjected to an uniformly distributed load. The results of the present analysis are compared with double Fourier series result given by Chia [46] and these are plotted Figs 6 and 7.

It is seen from the results presented earlier that the present higher-order shear deformation theory results in non-linear analysis are in good agreement with the results of higher-order shear deformation theory given by Reddy [47], 3D FEM results given by Reddy [40], first-order shear deformation theory given by Reddy [51, 52], double Fourier series results given by Gorji [48] and Levy [49] and with perturbation solution given by Chia [46] under different support conditions, a/h ratios. This establishes the correctness and effectiveness of our formulation.

Example 6. A simply supported [S1] square sandwich laminate subjected to an uniformly distributed load with $a/h = 10$ and 100, and material properties set 3 is considered. The present results for displacement and stresses are plotted in Figs 8 and 9, respectively. In the thin case the present HOST results coincide with the FOST because of negligible shear deformation effects. But in the thick to moderate thick cases, the present HOST results differ from FOST because of transverse shear deformation effects and the way these are included in the formulation.

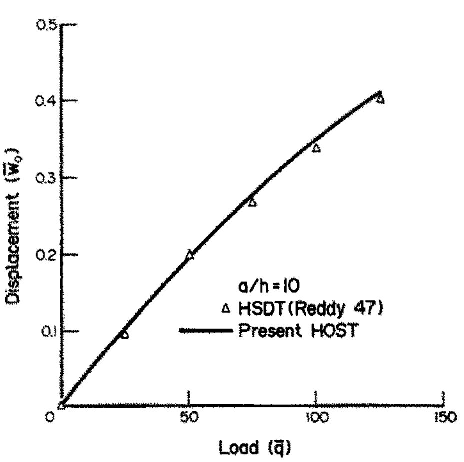

Fig. 3. Displacement vs load for a square clamped unsymmetric cross-ply (0°/90°/0°/···/90°) eight-layer laminate.

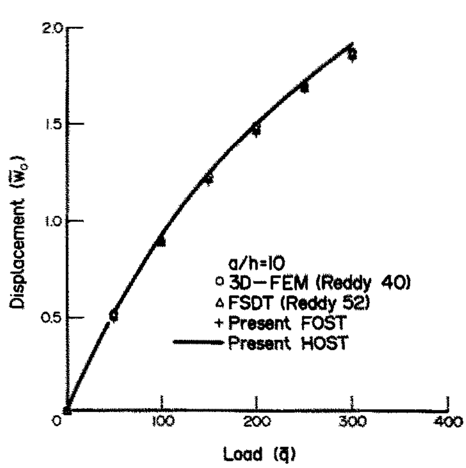

Fig. 5. Displacement vs load for a simply supported (S1) symmetric cross-ply (0°/90°/0°) laminate.

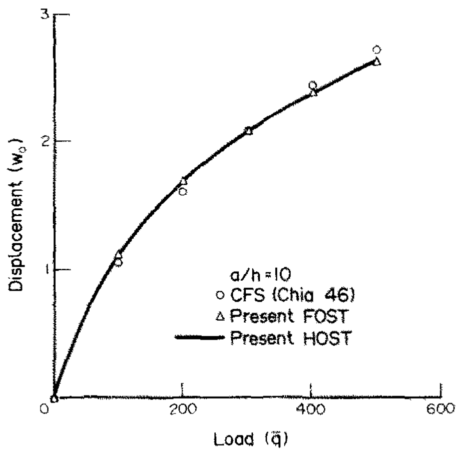

Fig. 6. Displacement vs load for a simply supported (S1) square unsymmetric cross-ply (0°/90°) laminate.

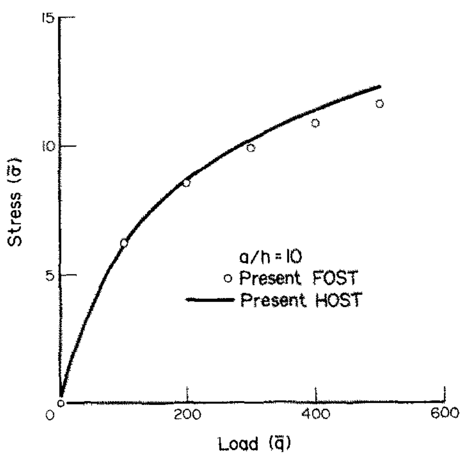

Fig. 7. Stress vs load for a simply supported (S1) square unsymmetric cross-ply (0°/90°) laminate.

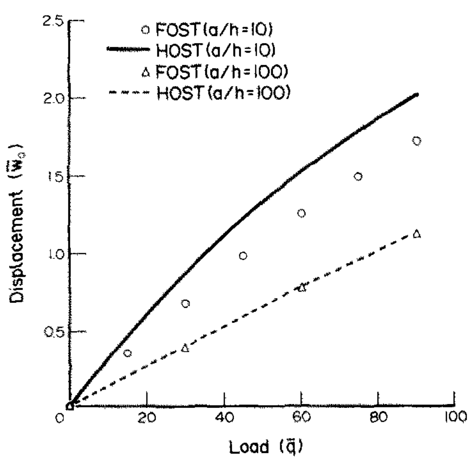

Fig. 8. Displacement vs load for a simply supported (S1) square sandwich laminate.

We have already shown earlier that the linear analysis results are close to 3D elasticity solution. Thus, we can conclude that the results of the present

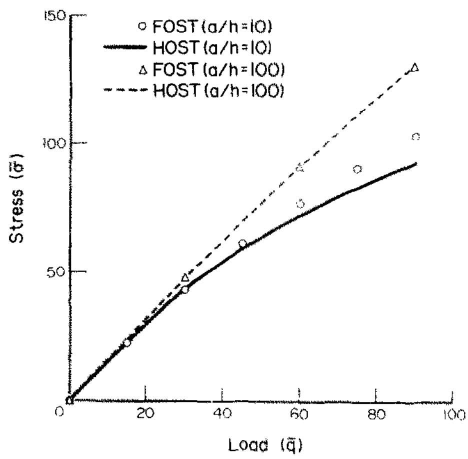

Fig. 9. Stress vs load for a simply supported (S1) square sandwich laminate.

HOST formulation will be more reliable in the non-linear regime in all situations.

6. CONCLUSIONS

A geometrically non-linear formulation for composite and sandwich laminates is proposed with the help of a simple C^0 isoparametric formulation of an assumed higher-order displacement model. The present shear deformable theory does not require the usual shear correction factors generally associated with the Mindlin–Reissner type of theories. The present higher-order theory results are found to be in excellent agreement with exact 3D elasticity solution [29] from thick to thin range in the linear analysis of composite and sandwich laminates.

In the non-linear analysis the present finite element results for displacements and stresses are in good agreement with available solutions. The difference between HOST and FOST solutions is significant in the case of thick sandwich laminates.

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