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A CONSISTENT REFINED THEORY FOR FLEXURE OF A SYMMETRIC LAMINATE

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Introduction

Any two-dimensional plate theory is an approximation of the real three-dimensional elasticity problem. The classical laminated plate theory is based on the Kirchhoff hypothesis and ignores the effects of transverse shear deformation, normal stress, normal strain and nonlinear in-plane normal strain distribution through the plate thickness [1,2]. Two types of composite plates are generally identified in practice: (1) 'fibre reinforced laminates' in which layers of composite materials with high ratios of Young's-to-shear modulii are bonded together and (2) 'sandwiches' in which layers of isotropic materials with some layers having significantly lower elastic modulii than others, are bonded together. The effects of shear deformation are significant in these situations and thus the classical theory is inadequate. Exact elasticity solutions for flexure of some standard composite and sandwich plate problems have been obtained by Pagano [3] and Pagano and Hatfield [4]. Whitney [5] and Mau [6] have presented first-order laminate theories in which transverse shear strain is assumed constant through the thickness. This required, however, use of a transverse shear correction factor which generally varied with the lamination scheme.

Theories based on realistic displacement models which give rise to nonlinear distributions of in-plane normal strains and transverse shear strains have been developed by Murthy [7], Reddy [8], Phan and Reddy [9] and more recently by Ren and Hinton [10]. Lo, Christensen and Wu [11], Kant, Owen and Zienkiewicz [12], Kant [13] and Pandya and Kant [14] have, in addition, included the effects of transverse normal strain and stress in their theories. These

(Refs. 11-14) however, do not satisfy the conditions of zero transverse shear stresses on top and bottom bounding planes of the plate. In this paper, a novel idea of incorporating these conditions in the shear rigidity matrix instead of the displacement model is being presented for the first time. In addition this formulation gives a C^{O} finite element displacement model. Numerical results from the various theories for displacements and stress distributions through the plate thickness are compared and evaluated.

Theory

The development of the present theory starts with the assumption of the displacement field in the following form:

$$\begin{aligned} U(x,y,z) &= z\theta_x(x,y) + z^3\theta_x^*(x,y) \\ V(x,y,z) &= z\theta_y(x,y) + z^3\theta_y^*(x,y) \\ W(x,y,z) &= w(x,y) + z^2 w^*(x,y) \end{aligned} \quad \ldots (1)$$

in which, the various terms have the usual meaning except the terms θ_x^*, θ_y^* and w^* which are the corresponding higher-order terms in the Taylor's series expansion and are defined at the reference plane [12-14].

By substitution of these relations into the strain-displacement equations of the classical theory of elasticity, the following relationships are obtained:

$$\left[\begin{array}{c|c} \varepsilon_x & \gamma_{xy} \\ \varepsilon_y & \gamma_{yz} \\ \varepsilon_z & \gamma_{xz} \end{array}\right] = \left[\begin{array}{c|c} zK_x + z^3K_x^* & zK_{xy} + z^3K_{xy}^* \\ zK_y + z^3K_y^* & \phi_y + z^2\phi_y^* \\ zK_z & \phi_x + z^2\phi_x^* \end{array}\right] \quad \ldots (2a)$$

in which,

$$\left[K_x, K_y, K_{xy}\right]^t = \left[\frac{\partial\theta_x}{\partial x}, \frac{\partial\theta_y}{\partial y}, \frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right]^t$$

$$\left[K_x^*, K_y^*, K_{xy}^*, K_z\right]^t = \left[\frac{\partial\theta_x^*}{\partial x}, \frac{\partial\theta_y^*}{\partial y}, \frac{\partial\theta_x^*}{\partial y} + \frac{\partial\theta_y^*}{\partial x}, 2w^*\right]^t \quad \text{and}$$

$$\left[\phi_x, \phi_y \,\vdots\, \phi_x^*, \phi_y^*\right]^t = \left[\frac{\partial w}{\partial x} + \theta_x, \frac{\partial w}{\partial y} + \theta_y \,\vdots\, \frac{\partial w^*}{\partial x} + 3\theta_x^*, \frac{\partial w^*}{\partial y} + 3\theta_y^*\right]^t \quad \ldots (2b)$$

where, t represents transpose of an array.

The stress-strain relationship for the L^{th} lamina of the composite laminate with reference to fibre axes (1-2-3) have the following compacted form:

$$\underset{\sim}{\sigma}_1 = \underline{C}\ \underset{\sim}{\varepsilon}_1 \quad \ldots (3a)$$

in which,

$$\underset{\sim}{\sigma}_1 = \left[\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{13}\right]^t$$

$$\underset{\sim}{\varepsilon}_1 = \left[\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{13}\right]^t \qquad \ldots (3b)$$

and $\underline{C}$ is the standard stiffness matrix [14] with reference to fibre axes (see Figure 1).

These vectors ($\underset{\sim}{\sigma}_1$, $\underset{\sim}{\varepsilon}_1$) are transformed to plate axes (x-y-z) using the usual co-ordinate transformation matrix $\underline{T}$ [14] as:

$$\underset{\sim}{\sigma}_x = \underline{T}\ \underset{\sim}{\sigma}_1 \quad \text{and} \quad \underset{\sim}{\varepsilon}_x = \underline{T}\ \underset{\sim}{\varepsilon}_1 \qquad \ldots (4a)$$

in which,

$$\underset{\sim}{\sigma}_x = \left[\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}\right]^t$$

$$\underset{\sim}{\varepsilon}_x = \left[\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}\right]^t \qquad \ldots (4b)$$

With this, the stress-strain relationship with reference to plate axes is given by,

$$\underset{\sim}{\sigma}_x = \left[\underline{T}\ \underline{C}\ \underline{T}^t\right] \underset{\sim}{\varepsilon}_x \quad \text{OR} \quad \underset{\sim}{\sigma}_x = \underline{Q}\ \underset{\sim}{\varepsilon}_x \qquad \ldots (5)$$

in which, $\underline{Q}$ is the stiffness matrix with reference to plate axes [14].

The constitutive relations involving bending moments are defined as follows:

$$\left[M_x, M_y, M_{xy}, M_z\right]^t = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \left[\sigma_x, \sigma_y, \tau_{xy}, \sigma_z\right]^t z\, dz$$

$$\left[M_x^*, M_y^*, M_{xy}^*\right]^t = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} \left[\sigma_x, \sigma_y, \tau_{xy}\right]^t z^3\, dz \qquad \ldots (6a)$$

After integration, this may be written in a compact form as,

$$\underset{\sim}{M} = \underline{D}_b\ \underset{\sim}{K} \qquad \ldots (6b)$$

in which,

$$\underset{\sim}{M} = \left[M_x, M_y, M_{xy}, M_x^*, M_y^*, M_{xy}^*, M_z\right]^t \qquad \ldots (6c)$$

$$\underset{\sim}{K} = \left[K_x, K_y, K_{xy}, K_x^*, K_y^*, K_{xy}^*, K_z\right]^t \qquad \ldots (6d)$$

If n is the number of layers in a laminate and if we set,

$$H_1 = \frac{h_L^3 - h_{L-1}^3}{3} \ ; \quad H_2 = \frac{h_L^5 - h_{L-1}^5}{5} \ ; \quad H_3 = \frac{h_L^7 - h_{L-1}^7}{7} \qquad \ldots (6e)$$

then the bending stiffness matrix can be evaluated as follows:

$$\underset{\sim}{D}_b = \sum_{L=1}^{n} \begin{bmatrix} Q_{11}H_1 & Q_{12}H_1 & Q_{14}H_1 & Q_{11}H_2 & Q_{12}H_2 & Q_{14}H_2 & Q_{13}H_1 \\ & Q_{22}H_1 & Q_{24}H_1 & Q_{12}H_2 & Q_{22}H_2 & Q_{24}H_2 & Q_{23}H_1 \\ & & Q_{44}H_1 & Q_{14}H_2 & Q_{24}H_2 & Q_{44}H_2 & Q_{34}H_1 \\ & & & Q_{11}H_3 & Q_{12}H_3 & Q_{14}H_3 & Q_{13}H_2 \\ & & & & Q_{22}H_3 & Q_{24}H_3 & Q_{23}H_2 \\ \text{Symmetric} & & & & & Q_{44}H_3 & Q_{34}H_2 \\ & & & & & & Q_{33}H_1 \end{bmatrix}^{L^{th}\ \text{layer}} \quad \ldots (6f)$$

The vanishing of the transverse shear stresses on top ($z = +h/2$) and bottom ($z = -h/2$) surfaces of the plate gives us,

$$\phi_y + \frac{h^2}{4}\phi_y^* = 0 \quad \text{and} \quad \phi_x + \frac{h^2}{4}\phi_x^* = 0 \quad \ldots (7a)$$

The constitutive relations involving shear forces are given by,

$$\underset{\sim}{Q} = \underset{\sim}{D}_s\ \underset{\sim}{\phi} \quad \ldots (7b)$$

in which,

$$[Q_x, Q_y, Q_x^*, Q_y^*]^t = \sum_{L=1}^{n} \int_{h_{L-1}}^{h_L} [\tau_{xz}, \tau_{yz}, z^2\tau_{xz}, z^2\tau_{yz}]^t\, dz \quad \ldots (7c)$$

and $\quad \underset{\sim}{\phi} = [\phi_x, \phi_y, \phi_x^*, \phi_y^*]^t \quad \ldots (7d)$

By integrating equations (7c) and introducing in it the conditions given by equations (7a), the shear rigidity matrix $\underset{\sim}{D}_s$ is obtained in the following form:

$$\underset{\sim}{D}_s = \sum_{L=1}^{n} \begin{bmatrix} Q_{66}H_4 & Q_{56}H_4 & 0 & 0 \\ & Q_{55}H_4 & 0 & 0 \\ & & Q_{66}H_5 & Q_{56}H_5 \\ \text{Symmetric} & & & Q_{55}H_5 \end{bmatrix}^{L^{th}\ \text{layer}} \quad \ldots (7e)$$

in which,

$$H_4 = \left[(h_L - h_{L-1}) - H_1 \times \frac{4}{h^2}\right]; \quad H_5 = \left[H_2 - H_1 \times \frac{h^2}{4}\right] \quad \ldots (7f)$$

The above theory is used to develop a C^o finite element model of quadrilateral elements of the Lagrangian family [12,14].

Numerical Results

A symmetrically laminated and a sandwich plate, discretised with four 9-noded quadrilateral elements are considered. The selective integration scheme namely 3 x 3 for flexure and 2 x 2 for shear contributions has been employed. The properties considered for the laminated plate are [3,4]:

$$E_1 = 25 \times 10^6,\ E_2 = E_3 = 1 \times 10^6,\ G_{12} = G_{13}\ 0.5 \times 10^6,\ G_{23} = 0.2 \times 10^6$$

$$\nu_{12} = \nu_{23} = \nu_{13} = 0.25 \quad \text{and} \quad h_i = 0.25 \times h, \quad i = 1,4$$

For sandwich plate, the material properties of face sheets are same as for laminated plate with the thickness of each face sheet as $0.1 \times h$ and $\theta = 0^o$. The properties of the middle core material are [3]:

$$E_1 = E_2 = 0.04 \times 10^6,\ E_3 = 0.5 \times 10^6,\ G_{13} = G_{23} = 0.06 \times 10^6,$$

$$G_{12} = 0.016 \times 10^6, \quad \nu_{12} = \nu_{23} = \nu_{13} = 0.25, \text{ thickness} = 0.8 \times h \text{ and } \theta = 0^o$$

The deflection and stresses presented are non-dimensionalised using the following multiplying factors:

$$m_1 = \frac{100\, h^3 E_2}{p_o a^4} \quad ; \quad m_2 = \frac{h^2}{p_o a^2} \quad ; \quad m_3 = \frac{h}{p_o a}$$

The stresses presented using finite element technique are at the nearest gauss points. The transverse shear stresses presented using present/Mindlin theory are obtained from equilibrium equations. Results of the analysis are presented in Table 1 and Figure 2.

Conclusions

A novel approach to satisfy the zero transverse shear stress conditions on top and bottom faces of the plate is presented. This theory includes the effect of transverse normal stress, does not require arbitrary shear correction factors for transverse shear stiffnesses and results in parabolic variation of transverse shear strains/stresses through the plate thickness. The numerical results obtained using finite element technique proves the validity of the present theory for flexure of laminated as well as sandwich plates.

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TABLE 1

Deflection and Stresses in a Square Simply Supported ($w = \theta_t = w^* = \theta_t^* = 0$) Four Layer Cross-ply and Sandwich Plate ($a/h = 10$) under Sinusoidal Transverse Load (p_o)

Source	$w \times m_1$ $(\frac{a}{2}, \frac{a}{2}, 0)$	$\sigma_{x1} \times m_2$ $(\frac{a}{2}, \frac{a}{2}, \frac{h}{2})$	$\sigma_{x2} \times m_2$ $(\frac{a}{2}, \frac{a}{2}, \frac{4h}{10})$	$\sigma_y \times m_2$ $(\frac{a}{2}, \frac{a}{2}, \frac{h}{2})^*$	$\tau_{xy} \times m_2$ $(0, 0, \frac{h}{2})$	$\tau_{xz} \times m_3$ $(0, \frac{a}{2}, 0)$	$\tau_{yz} \times m_3$ $(\frac{a}{2}, 0, 0)$
			Symmetrically laminated plate (0°/90°/90°/0°)				
Present consistent higher-order theory	0.7070	0.5358	-	0.3893	0.02605	0.2732	0.1696
Higher-order theory [14]	0.7185	0.5676	-	0.3948	0.02728	0.2702	0.1715
Mindlin theory	0.6613	0.5063	-	0.3653	0.02415	0.2819	0.1600
Higher-order shear deformation theory [9]	0.7294	0.5226	-	0.3667	0.02510	0.2574	0.1534
3-D Elasticity [4]	0.7370	0.5590	-	0.4010	0.02750	0.3010	0.1960
Classical plate theory [1,4]	0.4312	0.5390	-	0.2690	0.0213	0.3390	0.1380
			Sandwich plate				
Present consistent higher-order theory	2.0179	1.1140	0.7461	0.1077	0.06640	0.2690	0.04320
Higher-order theory	2.0816	1.1680	0.6900	0.1111	0.06890	0.2676	0.04440
Mindlin theory	1.5571	1.0620	0.8595	0.08057	0.05530	0.2779	0.03640
3-D Elasticity [3]	-	1.1520	0.6290	0.1099	0.07170	0.3000	0.05270
Classical plate theory [1,3]	-	1.0970	0.8780	0.0543	0.04330	0.3240	0.02950

* $z = h/4$ for laminated plate

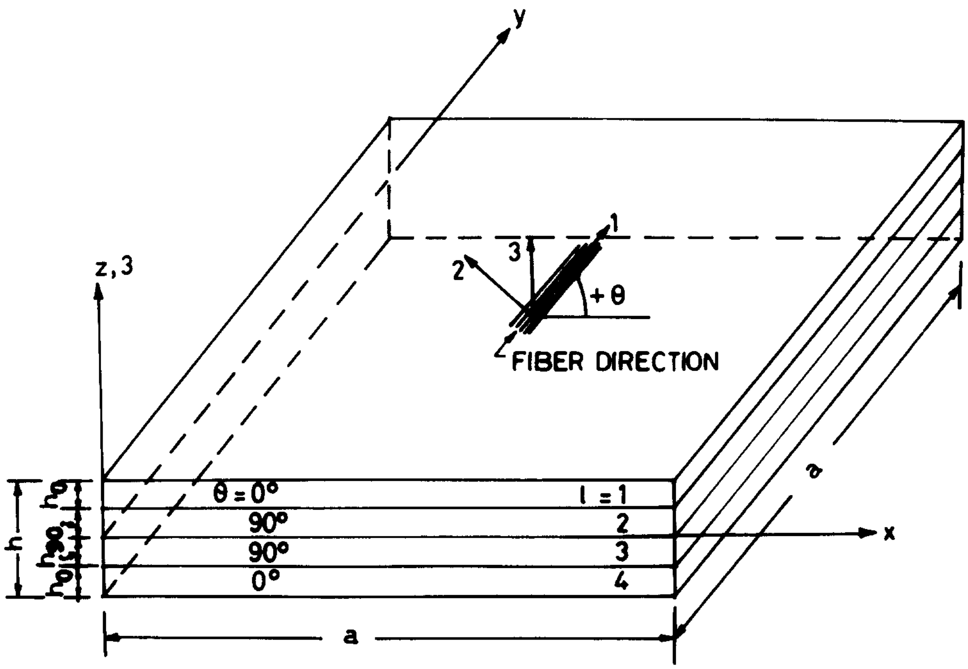

Fig. 1 - Plate and Fibre Axes

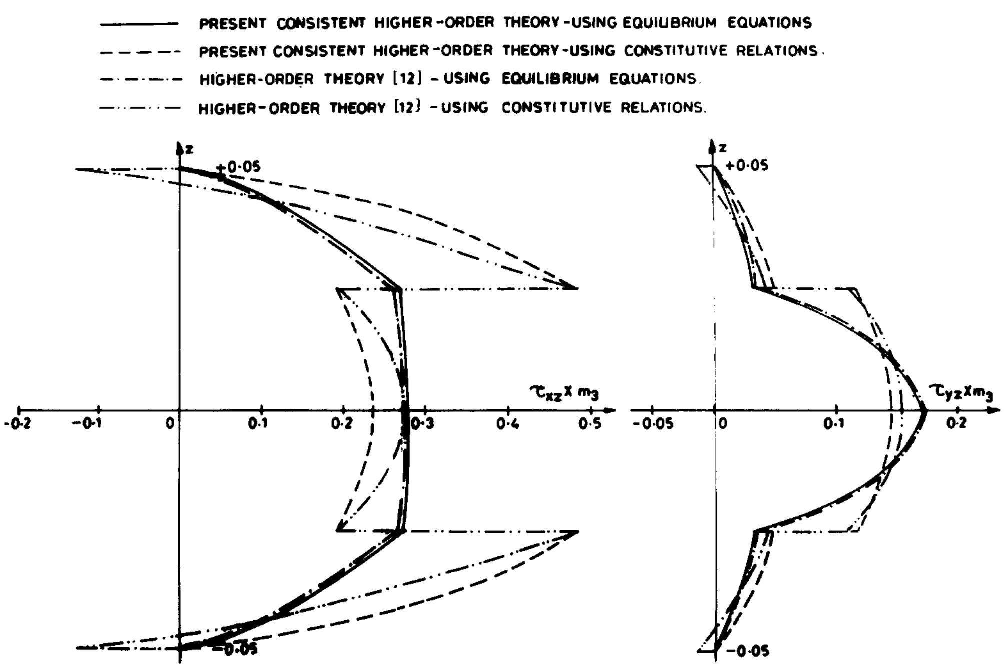

Fig. 2 - Transverse Shear Stresses