THE NONEXISTENCE OF INARIANT UNIVERSAL MEASURES ON SEMIGROUPS

V. KANNAN AND S. RADHAKRISHNESWARA RAJU

Abstract. We prove that if $S$ is an uncountable subsemigroup of a group, then every (left or right)-translation invariant $\sigma$-finite measure defined on all subsets of $S$ must be trivial. This answers a question posed by Ryll-Nardzewski and Telgarsky.

A universal measure on a set is, by definition, a (countably-additive, positive, extended real-valued) measure defined on all subsets of that set. A measure $\mu$ on $(X, \Sigma)$ is said to be semiregular, if whenever $A \in \Sigma$ and $\mu(A) > 0$, there is $B \in \Sigma$ such that $B \subset A$ and such that $0 < \mu(B) < \infty$. It is easily seen that every $\sigma$-finite measure is semiregular. We start with a Proposition that will be heavily used in our Theorem. $\aleph_1$ denotes the first uncountable cardinal number.

Proposition. Every universal semiregular measure is $\aleph_1$-additive.

Proof. Let us recall the definition of $\aleph_1$-additivity. This means that whenever $\{A_\alpha: \alpha \in J\}$ is a class of pairwise disjoint measurable sets and $|J| = \aleph_1$ and if $A = \bigcup \{A_\alpha: \alpha \in J\}$ is measurable, then it is true that the measure of $A$ is equal to the sum of the measures of $A_\alpha$'s. If $\mu$ is the measure, what we demand is

$$\mu(A) = \sum_{\alpha \in J} \mu(A_\alpha),$$

the sum on the right being defined in the most natural way, as

$$\text{Sup} \left\{ \sum_{\alpha \in F} \mu(A_\alpha): F \text{ is a finite subset of } J \right\}.$$

To prove the Proposition, let $\mu$ be a universal semiregular measure on a set $X$, let $J$ be an index set with cardinality $\aleph_1$, let $\{A_\alpha: \alpha \in J\}$ be a family of pairwise disjoint subsets of $X$ indexed by $J$ and let $A$ be their union. We have to prove that

$$\mu(A) = \sum_{\alpha \in J} \mu(A_\alpha). \quad (1)$$

Case 1. Let $\mu(A_\alpha) = 0$ for every $\alpha \in J$. Then we claim that $\mu(A) = 0$. If not, by the semiregularity of $\mu$, there is some $B \subset A$ such that $0 < \mu(B) < \infty$. Define a measure $\nu$ on the index set $J$ by the rule

$$\nu(E) = \mu\left( \bigcup_{\alpha \in E} B \cap A_\alpha \right)$$

It is easily checked that $\nu$ is also countably additive. In fact it is a universal measure on $J$ satisfying $\nu(J) = \mu(B)$ and hence $0 < \nu(J) < \infty$. Further if $\alpha \in J$ is
any element, we have

\[ \nu(\{ \alpha \}) = \mu(B \cap A_\alpha) \leq \mu(A_\alpha) = 0. \]

Since \( J \) is of cardinality \( \mathfrak{N}_1 \), this contradicts a well-known theorem of Ulam (see [O, Theorem 5.6, p. 25]). This contradiction proves that \( \mu(A) \) should be zero.

**Case 2.** Let \( \mu(A_\alpha) > 0 \) for uncountably many \( \alpha \) in \( J \). Then \( \Sigma_{\alpha \in J} \mu(A_\alpha) \) has to be \( \infty \). Further, there is a positive integer \( n \) such that \( \mu(A_\alpha) > 1/n \) for infinitely many (in fact, uncountably many) \( \alpha \) in \( J \). Since \( A \) contains all these \( A_\alpha \)'s, the countable additivity of \( \mu \) implies that \( \mu(A) \) is also \( \infty \). Thus the equality (1) is valid in this case also.

**Case 3.** Let \( J_1 = \{ \alpha \in J: \mu(A_\alpha) > 0 \} \) and let \( J_1 \) be countable. Let \( B = \bigcup_{\alpha \in J_1} A_\alpha \). Then we have

\[
\mu(A) = \mu(B) + \mu(A \setminus B)
= \sum_{\alpha \in J_1} \mu(A_\alpha) + \mu(A \setminus B)
\]

by countable additivity

\[
= \sum_{\alpha \in J_1} \mu(A_\alpha) + 0
\]

by Case 1, since

\[ A \setminus B = \bigcup \{ A_\alpha: \alpha \in J \setminus J_1 \} \]

and since \( \mu(A_\alpha) = 0 \forall \alpha \in J \setminus J_1 \)

\[ = \sum_{\alpha \in J} \mu(A_\alpha) \]

since \( \mu(A_\alpha) = 0 \forall \alpha \in J \setminus J_1 \).

Thus the Proposition is proved.

**Theorem.** Let \( S \) be an uncountable semigroup embeddable in a group. Let \( \mu \) be a \( \sigma \)-finite universal right translation-invariant measure on \( S \). Then \( \mu = 0 \).

**Proof.** Let \( G \) be a group in which \( S \) is embedded as a subsemigroup. Let \( E \) be any subset of \( S \) having cardinality \( \mathfrak{N}_1 \). Let \( H \) be the subgroup of \( G \) generated by \( E \). Let \( A \) be a subset of \( G \) meeting each left coset of \( H \) in \( G \), in exactly one point. Then one easily verifies that \( Ax \) and \( Ay \) are disjoint, whenever \( x \) and \( y \) are distinct elements of \( H \). Let

\[ A_x = (Ax) \cap S \]

for every \( x \) in \( H \). Then we have

\[ S = \bigcup \{ A_x: x \in H \} \]

because we have \( G = \bigcup \{ Ax: x \in H \} \). Thus (3) represents \( S \) as the union of a class of pairwise disjoint sets, indexed by the set \( H \) having cardinality \( \mathfrak{N}_1 \). Since \( \mu \) is \( \sigma \)-finite and hence semiregular, the previous Proposition applies. Thus, we have

\[ \mu(S) = \sum_{x \in H} \mu(A_x). \]

Now consider two cases.

**Case 1.** Let \( \mu(A_x) = 0 \) for every \( x \) in \( H \). Then by (4) we have \( \mu(S) = 0 \) and thus the result is proved in this case.
Case 2. Let $\mu(A_x) > 0$ for some $x$ in $H$. Now if $y$ is any element of $E$, we have

$$A_x y = (Ax \cap S)y = (Axy) \cap Sy$$

$$\subset Axy \cap S$$ since $S$ is closed under multiplication and $y \in S$

$$= A_{xy}$$

and therefore $\mu(A_{xy}) > \mu(A_{xy}) = \mu(A_x)$ because $\mu$ is translation-invariant, $> 0$ by our assumption in this case. Thus $\{A_{xy} : y \in E\}$ is a collection of pairwise disjoint subsets of $S$ indexed by a set of cardinality $\aleph_1$, such that every member in this collection has positive measure. This contradicts the assumption that $\mu$ is $\sigma$-finite. Hence Case 2 does not arise at all.

Corollary. Let $S$ be an uncountable commutative cancellative semigroup. Then every $\sigma$-finite translation-invariant universal measure on $S$ is trivial.

Proof. Every such semigroup can be embedded in a group and therefore our Theorem applies.

Remarks. The above Corollary answers a question posed in [R-T]. The special case of the above Theorem, where $S$ itself is assumed to be a group, has been proved first in [E-M] and then by a different method in [R-T].

We conclude with the following open question.

Problem. Is every translation-invariant universal semiregular measure on a group necessarily a multiple of the counting measure?

References


University of Hyderabad, Nampally Station Road, Hyderabad 500 001, India