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LOCAL COMPACTNESS AND SIMPLE EXTENSIONS OF DISCRETE SPACES

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ABSTRACT. Hereditarily locally compact spaces are characterized as those locally compact spaces which are simple extensions of discrete spaces.

Introduction. If a simple extension of a discrete space is locally compact, then it is hereditarily so. Surprisingly the converse also holds, i.e. every hereditarily locally compact space is a simple extension of a discrete space. The aim of this note is to prove this fact.

Preliminaries. All spaces in question are supposed to be Hausdorff. A dense embedding e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is called a *simple extension* of $(X, \mathfrak{T})$, provided $e[X]$ is open in $(Y, \mathfrak{S})$ and the subspace of $(Y, \mathfrak{S})$, determined by the set $Y \setminus e[X]$, is discrete (see B. Banaschewski [1]). A simple extension e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ of $(X, \mathfrak{T})$ is called a *simple local compactification* of $(X, \mathfrak{T})$, provided $(Y, \mathfrak{S})$ is locally compact. A simple local compactification e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is called *maximal*, provided there does not exist any proper local compactification c: $(Y, \mathfrak{S}) \to (Z, \mathfrak{R})$ such that $c \circ e$: $(X, \mathfrak{T}) \to (Z, \mathfrak{R})$ is a simple extension of $(X, \mathfrak{T})$. A space $(Y, \mathfrak{S})$ is called a (maximal) simple local compactification of a space $(X, \mathfrak{T})$, provided there exists a map e: $X \to Y$ such that e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is a (maximal) simple local compactification.

For any set A and any set $\mathfrak{B}$ of infinite subsets of A such that any two members of $\mathfrak{B}$ have finite intersection, we will construct a simple local compactification $e_{(A,\mathfrak{B})}$: $(A, \mathfrak{P}A) \to (X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ as follows: (1) $\mathfrak{P}A = \{C \mid C \subset A\}$ is the discrete topology on A. (2) $X_{(A,\mathfrak{B})}$ is the disjoint union of A and $\mathfrak{B}$. (3) e: $A \to X_{(A,\mathfrak{B})}$ is the natural embedding, defined by $e(a) = a$ for each $a \in A$. (4) $\mathfrak{T}_{(A,\mathfrak{B})}$ is the set of those subsets D of $X_{(A,\mathfrak{B})}$, satisfying the condition that $B \in D \cap \mathfrak{B}$ implies that $B \setminus D$ is finite. [In B. Banaschewski's suggestive terminology [1], $e_{(A,\mathfrak{B})}$: $(A, \mathfrak{P}A) \to (X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ is the simple extension of the discrete space $(A, \mathfrak{P}A)$, determined by the family $\{\mathfrak{F}_B \mid B \in \mathfrak{B}\}$ of trace-filters $\mathfrak{F}_B = \{C \subset \mathfrak{A} \mid B \setminus C \text{ finite}\}$.]

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Results.

THEOREM 1. *For any space Y, $\mathfrak{S}$ the following conditions are equivalent.*
(1) *Y, $\mathfrak{S}$ is hereditarily locally compact,*
(2) *$(Y, \mathfrak{S})$ is a simple local compactification of a discrete space,*
(3) *$(Y, \mathfrak{S})$ is homeomorphic to a space $(X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ for suitable A and $\mathfrak{B}$.*

PROOF. (1) $\Rightarrow$ (2). Let X be the set of all isolated points of $(Y, \mathfrak{S})$, let $\mathfrak{T}$ be the discrete topology on X, and let e: $X \to Y$ be the natural embedding, defined by $e(x) = x$ for each $x \in X$. We will show that the embedding e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is a simple local compactification. First, assume $e[X] = X$ were not dense in $(Y, \mathfrak{S})$. Then there would exist a nonempty, open subset A of $Y \setminus X$ with compact closure. Hence there would exist a sequence of pairwise disjoint open subsets A_n of A, a sequence of elements $a_n \in A_n$, and an adherence point y of $\{a_n | n \in \mathbf{N}\}$. Consequently the subspace of $(Y, \mathfrak{S})$, determined by the set $\{y\} \cup \bigcup \{A_n \setminus \{a_n\} | n \in \mathbf{N}\}$, would not be locally compact at y, contradicting condition (1). Hence e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is an open, dense embedding. It remains to show that the subspace $(Z, \mathfrak{R})$ of $(Y, \mathfrak{S})$, determined by the set $Z = Y \setminus X$, is discrete. To see this, let z be an element of Z. Since the subspace of $(Y, \mathfrak{S})$, determined by the set $X \cup \{z\}$, is locally compact there exists a neighbourhood U of z in $(Y, \mathfrak{S})$ such that $U \cap (X \cup \{z\})$ is compact. This implies $Z \cap \operatorname{int}_{(Y,\mathfrak{S})} U = \{z\}$, since otherwise $U \cap (X \cup \{z\})$ would not be closed in $(Y, \mathfrak{S})$ and hence could not be compact. Therefore z is isolated in $(Z, \mathfrak{R})$, hence $(Z, \mathfrak{R})$ is discrete.

(2) $\Rightarrow$ (3). Let $(X, \mathfrak{T})$ be a discrete space and e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ be a simple local compactification. For each $y \in Y \setminus e[X]$, the set $e[X] \cup \{y\}$ is a neighbourhood of y in $(Y, \mathfrak{S})$. Hence there exists a compact neighbourhood K_y of y in $(Y, \mathfrak{S})$ with $K_y \subset e[X] \cup \{y\}$. Since $e[X]$ is dense in $(Y, \mathfrak{S})$ and $(Y, \mathfrak{S})$ is a Hausdorff space, each set K_y is infinite. Since the subspace of $(Y, \mathfrak{S})$, determined by K_y, is compact and $e[X]$ consists of isolated points only, every neighbourhood of y meets every infinite subset of K_y. By the Hausdorffness of $(Y, \mathfrak{S})$ this implies that $K_y \cap K_z$ is finite for any two different elements y and z of $Y \setminus e[X]$. With $A = X$ and $\mathfrak{B} = \{K_y \setminus \{y\} | y \in Y \setminus e[X]\}$, the extensions $e_{(A,\mathfrak{B})}$: $(X, \mathfrak{T}) \to (X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ and $e(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ are obviously equivalent. In particular, the spaces $(X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ and $(Y, \mathfrak{S})$ are homeomorphic.

(3) $\Rightarrow$ (1). Straightforward.

COROLLARY. *Every hereditarily locally compact space is scattered, sequential, and an extension of a discrete space, which is simultaneously simple and strict* (*cf.* [1]).

THEOREM 2. *For any space $(Y, \mathfrak{S})$ the following conditions are equivalent*:
(1) *$(Y, \mathfrak{S})$ is a maximal simple local compactification of a discrete space*;
(2) *$(Y, \mathfrak{S})$ is pseudocompact and hereditarily locally compact*;
(3) *$(Y, \mathfrak{S})$ is homeomorphic to a space $(X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$, where A is a set and*

$\mathfrak{B}$ is a set of infinite subsets of A, which is maximal with respect to the property that any two of its members have finite intersection;

(4) *$(Y, \mathfrak{S})$ is regular, a simple extension of a discrete space, and every closed set of isolated points in $(Y, \mathfrak{S})$ is finite.*

PROOF. (1) $\Rightarrow$ (2). Let $(X, \mathfrak{T})$ be a discrete space and let e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ be a maximal simple local compactification of $(X, \mathfrak{T})$. According to Theorem 1, the space $(Y, \mathfrak{S})$ is hereditarily locally compact. If it were not pseudocompact, there would exist a sequence (y_n) in $e[X]$ and a continuous map f from $(Y, \mathfrak{S})$ into the reals with $\lim_{n\to\infty} f(y_n) = \infty$. This would, in contradiction to (1), allow the construction of a proper local compactification c: $(Y, \mathfrak{S}) \to (Z, \mathfrak{R})$ of $(Y, \mathfrak{S})$ such that $c \circ e$: $(X, \mathfrak{T}) \to (Z, \mathfrak{R})$ is simple. As Z one could choose the disjoint union of Y with a singleton set $\{z_0\}$, as c: $Y \to Z$ the natural embedding, as topology $\mathfrak{R}$ the set of all subsets R of Z satisfying the following two conditions:

(a) $R \cap Y \in \mathfrak{S}$, and

(b) if $z_0 \in R$ then $\{y_n | n \in \mathbf{N}\} \setminus R$ is finite.

(2) $\Rightarrow$ (3). According to Theorem 1, $(Y, \mathfrak{S})$ is homeomorphic to a space $(X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$ for suitable A and $\mathfrak{B}$. If $\mathfrak{B}$ would not be maximal, there would exist an infinite subset C of A, meeting each $B \in \mathfrak{B}$ in at most finitely many points. Hence C would determine an infinite, clopen, discrete subspace of $(X_{(A,\mathfrak{B})}, \mathfrak{T}_{(A,\mathfrak{B})})$, contradicting the pseudocompactness of the latter.

(3) $\Rightarrow$ (4). Straightforward.

(4) $\Rightarrow$ (1). Let $(X, \mathfrak{T})$ be a discrete space and let e: $(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ be a simple extension of $(X, \mathfrak{T})$. For any $y \in Y$, the set $e[X] \cup \{y\}$ is a neighbourhood of y in $(Y, \mathfrak{S})$. Hence there exists a closed neighbourhood U of y with $U \subset e[X] \cup \{y\}$. For any neighbourhood V of y, the set $U \setminus V$ is a closed set of isolated points in $(Y, \mathfrak{S})$, and hence finite. Consequently U is a compact neighbourhood of y. Thus e:$(X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is a simple local compactification of $(X, \mathfrak{T})$. To show maximality, let c: $(Y, \mathfrak{S}) \to (Z, \mathfrak{R})$ be a local compactification of $(Y, \mathfrak{S})$ such that $c \circ e$: $(X, \mathfrak{T}) \to (Z, \mathfrak{R})$ is a simple extension of $(X, \mathfrak{T})$. Then c: $(Y, \mathfrak{S}) \to (Z, \mathfrak{R})$ must be improper, since otherwise there would exist an element $z \in Z \setminus c[Y]$ and a compact neighbourhood K of z in $(Z, \mathfrak{R})$ with $K \subset c \circ e[X] \cup \{z\}$. Consequently $c^{-1}[K]$ would be an infinite, closed subset of isolated points in $(Y, \mathfrak{S})$, contradicting condition (4).

REFERENCES

1. B. Banaschewski, *Extensions of topological spaces*, Canad. Math. Bull. 7 (1964), 1–22.

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