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SETS OF PERIODS OF DYNAMICAL SYSTEMS

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Abstract In this article, we present a coherent, though not exhaustive, account of some well-known and some recent results of many mathematicians (including our own) on the following question: Given a "nice" class of dynamical systems, which subsets of $\mathbb{N}$ arise as the sets of periods of members of that class? While stating and explaining some elegant answers, proofs have been outlined or indicated occasionally.

Key words Sets of periods, Sarkovskii's theorem, Baker's theorem, trees, transive maps, toral automorphisms, cellular automata, subshift of finite type, linear operator.

1. Introduction

Dynamics is the study of eventual behaviour of orbits. The periodic orbits are the simplest kind of orbits. Therefore it is natural to investigate answers to the following general scheme of question: In a dynamical system (X, f), what are all the lengths of the cycles there? The answer to this question, gives a subset of $\mathbb{N}$, denoted by $\mathrm{Per}(f)$, and is called the set of periods of (X, f).

Given a 'nice' class of dynamical systems, we ask which subsets of $\mathbb{N}$ can arise as the sets of periods for members of this class? Apart from being a natural question, there are at least two other reasons why the problem is important. Many a time, we find that the knowledge of the set $\mathrm{Per}(f)$ enables us to decide some dynamical properties of the dynamical system (X, f). Here is an instance. Let f be a continuous real valued function on $\mathbb{R}$. If $\mathrm{Per}(f) = 2\mathbb{N} \cup \{1\} \setminus \{6\}$, then, we are sure that f cannot have a dense orbit. If $\mathrm{Per}(f) = 2\mathbb{N} \cup \{1\}$, then (though there can be a dense orbit), we are sure that $f \circ f$ cannot have. As another instance, if $\mathrm{Per}(T) = \mathbb{N} \setminus \{2\}$ for a toral automorphism T, we are sure that T is hyperbolic. One more reason for this study is that for many important classes of dynamical systems, it turns out that the family of their sets of periods, is elegantly describable and in fact is often totally unexpected. As we shall see, there is a lot of variety in the answers and the methods used to arrive at them.

Because of these various reasons, this problem has been extensively attacked by many. In this article only one kind of problem is discussed throughout. A subjective choice has been made, where the elegance of the result has received priority, rather than the exhaustiveness. The last five sections are devoted to recent results, some of which are yet to be published. Even the known results in the first five sections, have been reformulated and knitted in such a way that some of them are not available in literature in the same elegant form.

Notations: When X is a metric space and f is a continuous self map of X, we say that (X, f) is a dynamical system; we define $f^0(x) = x$ and recursively $f^{n+1}(x) = f(f^n(x))$ for each $n \in \mathbb{N}_0$. Here $\mathbb{N}_0$ denotes the set of all nonnegative integers. $\mathbb{N} = \mathbb{N}_0 \setminus \{0\}$. If $x \in X$ is such that $f^n(x) = x$ for some $n \in \mathbb{N}$, the least such n in $\mathbb{N}$ is called the period of the periodic point x. We let $\mathrm{Per}(f) = \{n \in \mathbb{N} : n$ is the period of some $x \in X\}$ and $\mathcal{PER}(X) = \{\mathrm{Per}(f) : f$ is a continuous self map of $X\}$.

Elementary terminology: f-orbit of x in X is the set $\{y \in X : y = f^n(x)$ for some $n \in \mathbb{N}_0\}$. A cycle in X is the f-orbit of a periodic point in it. The length of the cycle, is the cardinality of that orbit, and is same as the period of that periodic point.

Let $\mathcal{F}$ be a family of dynamical systems and $\mathcal{G}$ be a family of subsets of $\mathbb{N}$. When we say that $\mathcal{G}$ is the family of sets of periods of $\mathcal{F}$, we mean two things:

(i) If a dynamical system (X, f) is in $\mathcal{F}$, there is a set G in $\mathcal{G}$ such that $G = Per(f)$ (ii) Conversely, for every set G in $\mathcal{G}$ there is a dynamical system f in $\mathcal{F}$ such that $G = \mathrm{Per}(f)$.

2. Sharkovskii's Theorem

The following total order on $\mathbb{N}$ is called the Sharkovskii's ordering:

$3 \succ 5 \succ 7 \succ 9 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$
$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$
$...2^n \succ .... \succ 2^2 \succ 1$

We write $m \succ n$ if m precedes n (not necessarily immediately) in this order. In what follows, $n-$ cycle means a cycle of length n.

Theorem 1. ([35]) *Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self-map of $\mathbb{R}$, if there is an m-cycle, then there is an n-cycle.*

A converse of Sharkovskii's theorem : (See [22])
Let m and n be distinct positive integers. Let m not precede n in the above ordering. Then there is a continuous map f from $\mathbb{R}$ to $\mathbb{R}$, where there is an *m-cycle* but no *n-cycle*.

A combined Statement: (See [23, 24])
$m \succeq n$ in the Sharkovskii's ordering if and only if for every continuous self-map of $\mathbb{R}$, the existence of an m-$cycle$ forces that of an n-$cycle$.

It is sometimes convenient to work with the reverse order $\prec$, instead of $\succ$.

A subset S of $\mathbb{N}$ is called an initial segment in this ordering $\prec$, if the following holds:

$m \in S$ and $m \succ n$ imply $n \in S$.

The main theorem of this section can be reformulated as follows:

Theorem 2. *(a) Initial segments in the ordering $\prec$ are precisely the sets of periods, for continuous self maps of R.*

(b) Nonempty ones among them, are precisely the sets of periods of interval maps.

Here and hereafter, an interval map means a continuous map from $[0, 1]$ to itself. We denote by $\mathcal{S}$ the family mentioned in (b) above. Accordingly, we have: $\mathcal{PER}(I) = \mathcal{S}$ and $\mathcal{PER}(\mathbb{R}) = \mathcal{S} \cup \{\emptyset\}$

The original paper of Sarkovskii is very long. The proof there, is involved and complicated, though elementary in the sense that nothing more than the intermediate value theorem is needed in the arguments of various cases divided. But later, shorter proofs have been obtained by others. The following two lemmas, needed for the proof, deserve a mention.

(1) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if I, J are intervals such that $f(I) \supset J$, then there is a subinterval K of I such that $f(K) = J$

(2) Each periodic f-orbit gives a finite partition of $\mathbb{R}$ into connected subsets. Taking these as vertices, we generate a directed graph; a vertex I is joined to a vertex J if $f(I) \supset J$. Then the graph theoretic cycles in this digraph ensure f-cycles in $\mathbb{R}$. This becomes a useful tool in finding more elements in $Per(f)$.

See [39] and [35]for these ideas.

3. Baker's Theorem

Theorem 3. ([10]) *Let p be a complex polynomial. Then the set of periods of p has to be one of the following subsets of $\mathbb{N}$:*

1. The whole set $\mathbb{N}$

2. $\mathbb{N} \setminus \{2\}$

3. $\{1, n\}$ for $n \in \mathbb{N} \setminus \{1\}$

4. $\{1\}$

5. Empty set.

Moreover, the following hold:

(a) Any polynomial p such that Per$(p) = \mathbb{N} \setminus \{2\}$ has to be topologically conjugate to $z^2 - z$.

(b) For all polynomials p of degree ≥ 2, Per$(p) \supset \mathbb{N} \setminus \{2\}$.

The following table gives some examples:

If p is	then Per(p) is
$z + 1$	Empty set
z	$\{1\}$
$-z$	$\{1, 2\}$
$z^2 - z$	$\mathbb{N} \setminus \{2\}$
z^2	$\mathbb{N}$

Refer to [43] for the following: Given a subset of $\mathbb{N}$ from the list of the above theorem, find all polynomials whose set of periods is the given set.

The chart below helps us to contrast four situations:

Complex polynomials	The subsets of $\mathbb{N}$ as in Theorem 3
Real continuous maps	A countably infinite family S of subsets of $\mathbb{N}$
Complex continuous maps	All subsets of $\mathbb{N}$
Real polynomials	An infinite proper subfamily of S

The third row means: Given any subset A of $\mathbb{N}$, there is a continuous map f from $\mathbb{C}$ to $\mathbb{C}$ such that $\mathrm{Per}(f) = A$.

The fourth row implies: There is a subset of $\mathbb{N}$, occurring as $\mathrm{Per}(f)$ for a continuous self map of $\mathbb{R}$, but not as $\mathrm{Per}(p)$ for a real polynomial. Explicitly, $\{2^k : k \in \mathbb{N}_0\}$ is one such set.

4. Trees

The theorem of Sarkovski is so appealing that one cannot resist the urge to generalise it. If X is a connected ordered space that is separable (like $I, \mathbb{R}$, etc), then most of the ideas of proof go through, and we have the same family $\mathcal{S}$ or $\mathcal{S} \cup \{\emptyset\}$ as $\mathcal{PER}(X)$. (See [36]). But the real difficulty crops up when X is not ordered. Alseda, Llibre and Misiurewicz [6] started with the simplest compact connected non-orderable space called triod, defined as $\{z \in \mathbb{C} : z^3 \in [0,1]\}$ which is geometrically in the shape of the letter Y. They succeeded in describing $\mathcal{PER}(Y)$. Its members are unions of three segments, in three different partial orders on $\mathbb{N}$ (of which, Sharkovski's total ordering is one). This led to the consideration of an indexed set of partial orders $\leq_n$ on $\mathbb{N}$, one for each positive integer n. Through a series of papers Baldwin, Alseda, Llibre and Misuirewicz were able to describe $\mathcal{PER}(T)$, where T is a general tree (See [12, 7]) (A tree is a connected graph without graph-theoretic cycles and a tree is viewed as a topological space, a subspace of the plane $\mathbb{R}^2$) and an expected generalisation for n-*od* ([11]). Here for each positive integer n, n-*od* denotes the space $\{z \in \mathbb{C} : z^n \in [0,1]\}$. Their description is completely in terms of the partial orders $\leq_n$, whose definition we prefer to skip here.

Theorem 4. *Let T be a tree.*

(a) Let $f : T \to T$ be a continuous map with all branching points fixed. Then $Per(f)$ is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$.

(b) Conversely, if S is a nonempty finite union of initial segments of $\{\leq_p : 1 \leq p \leq e(T)\}$ then there is a continuous map $f : T \to T$ with all branching points fixed such that $Per(f) = S$. (See [12]).

5. Circle Maps

Let S^1 be the unit circle. The family $\mathcal{PER}(S^1)$ has been completely described by Block and Coppel (See [16, 17]). But we do not reproduce that statement here. Instead, we prefer to state some subsidiary results in its proof, that are more elegant than the final result. $C(S^1, S^1)$ denotes the family of circle maps, i.e., continuous self map of S^1.

Theorem 5. *The following are equivalent for a subset S of $\mathbb{N}$:*

(1) $1 \in S \in \mathcal{PER}(S^1)$,

(2) $If\ n \in S$ *for some* $n > 1$*, (at least) one of the following should hold: (i) Every integer greater than* n *belongs to* S. (ii) *Every integer that comes later than* n *in the Sharkovski ordering, belongs to* S.

Corollary 1. *If* $\{1, 2, 3\} \subset$ *Per*(f) *for a circle map* f*, then Per*$(f) = \mathbb{N}$

(b) *Conversely, if* $S \subset \mathbb{N}$ *has the property that for any* $f \in C(S^1, S^1)$*,* $S \subset$ *Per*(f) *implies* $Per(f) = \mathbb{N}$ *then* $\{1, 2, 3\} \subset S$.

Contrast this with the following consequence of Sharkovskii's theorem, proved independently in ([31]): If 3 $\in$ Per(f), then Per(f) = $\mathbb{N}$. Moreover 3 is the only number with this property.

Theorem 6. ([18]) *Let* $f \in C(S^1, S^1)$ *and suppose that* $Per(f)$ *is finite. Then there are integers* m *and* n *(with* $m \geq 1$ *and* $n \geq 0$*) such that*

$Per(f) = \{m, 2.m, 2^2.m, ..., 2^n.m\}$.

Compare this with a corresponding result for interval maps, where Per(f) has to be $\{1, 2, 2^2, ..., 2^n\}$ for some $n \in \mathbb{N}$.

If $\mathcal{F}$ is the family of degree one maps of the circle, then $\mathcal{PER}(\mathcal{F})$ has been calculated in ([33]).

6. Transitive Maps on the Interval

The continuous selfmaps of the interval $[0, 1]$ are called interval maps. An important subclass of this class is that of topologically transitive interval maps. A dynamical system (X, f) is said to be (topologically) transitive if given any two nonempty open subsets V and W of X, some element of V, at some time, lands in W; more precisely there exist $x \in V$ and $n \in \mathbb{N}$ such that $f^n(x) \in W$. When X is a compact metric space without isolated points, this topological transitivity, (as can be proved by using Baire category theorem) is equivalent to the existence of a dense orbit. Now we seek to find the family $\{\text{Per}(f) : f \text{ is a transitive interval map}\}$. Its importance is evident from the following two reformulations:

(a) Which lengths of cycles should coexist with a dense orbit?

(b) Which lengths of cycles are available in all chaotic systems?

The answer is simple, but surprising. As a first step, we have:

Theorem 7. *(a) Every transitive interval map must have a cycle of length* 6 *(and therefore cycles of length* n *for all* n *with* $6 \succ n$ *in the Sharkovskii order).*

(b) Conversely if $n \in \mathbb{N}$ *has the property that every transitive interval map must have a cycle of length* n*, then* $6 \succ n$ *in the Sharkovskii order.*

Note: We are not saying that if $6 \in$ Per(f), then f is transitive. The above theorem proved in (1991) (See [34]) paves the way for a complete answer to our question. We have:

Theorem 8. *The following are equivalent for a subset S of $\mathbb{N}$:*

(a) $S = Per(f)$ for some transitive interval map.

(b) S has the following two properties: (i) $x \in S \setminus \{1\}$ implies $x + 2 \in S$. (ii) 1 and $2 \in S$.

(c) $6 \in S$ and $S = Per(g)$ for some interval map g.

The formulation (b) is as given in ([8]). As a major step for proving (c) implies (a), the following result deserves a mention:

Theorem 9. *For every integer $n > 1$, there is a transitive interval map for which there is a cycle of length $2n + 1$, but no cycle of length $2n - 1$.*

One can construct a transitive map whose set of periods is $2\mathbb{N} \cup \{1\}$.

$$f(x) = \begin{cases} 2x + \dfrac{1}{2} & \text{if } 0 \leq x < \dfrac{1}{4} \\[2ex] \dfrac{3}{2} - 2x & \text{if } \dfrac{1}{4} \leq x < \dfrac{3}{4} \\[2ex] 2x - \dfrac{3}{2} & \text{if } \dfrac{3}{4} \leq x < 1 \end{cases},$$

is one such example.

Here the subintervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$ are mapped to each other. It follows that these are invariant under $f \circ f$. Therefore $f \circ f$ is not transitive, though f is. This example leads us to another important dynamical property called total transitivity. A dynamical system is said to be totally transitive if for each positive integer n, the dynamical system (X, f^n) is transitive. Since the transitivity of $f \circ f$ is equivalent, among interval maps, to some well-known properties like total transitivity, weak mixing and topological mixing, we now consider this class of interval maps. We ask for a description of the family $\{\mathrm{Per}(f) : f \text{ is totally transitive}\}$. Obviously, this is contained in the family of all sets S satisfying the conditions of Theorem 8.

Actually we have:

Theorem 10. $2\mathbb{N} \cup \{1\}$ *is the only subset of $\mathbb{N}$ that arises as $Per(f)$ for some transitive interval map f, but does not arise as $Per(g)$ for any totally transitive interval map g.*

Theorem 11. *The following are equivalent for a subset of $\mathbb{N}$:*

(a) It is the set of periods of a totally transitive interval map;

(b) Its complement is a finite set of the form $\{$all odd positive integers strictly between 2 and $k\}$ for some $k > 1$. (Note that when $k = 2$ this set is empty).

Theorem 12. *The following are equivalent for an interval map f:*

(a) f is totally transitive;

(b) f is transitive and the complement of $Per(f)$ is finite.

Thus a knowledge of $\mathrm{Per}(f)$ is enough to distinguish totally transitive systems among transitive systems.

The above theorem is true in the more general setting of graph maps, (that include interval maps as a particular case).

Take a connected planar graph, with a finite set of vertices and edges. Provide it with the relative topology from the plane. Any continuous self map of it, is called a graph map (when there are only two vertices and one edge, these are nothing but interval maps).

Theorem 13. ([4]) *A transitive graph map is totally transitive if and only if its set of periods has a finite complement.*

7. For many more Compact Metric Spaces

For any compact metric space X, let $\mathcal{PER}(X) = \{\text{Per}(f)|\ f$ is a continuous self map of $X\}$. This is a family of subsets of $\mathbb{N}$. When X is the unit interval, this family has been neatly described in $Section$ 2. And in the sections 4 to 6, we have considered the same problem for some other spaces X.

It would be ambitious to expect to describe these families for each X in such a large uncountable family of topological spaces as the family of all compact subsets of $\mathbb{R}$. See problem 5.1 in Baldwin [11]. Saradhi [40] succeeded in obtaining exhaustive results that surpass this goal, and covers a still larger class of subsets of $\mathbb{R}$.

How many zero dimensional compact metric spaces are there? Each one of them can be embedded in the Cantor set as closed set and therefore there are at most c of them (Here c denotes the cardinality of the continuum). Actually, there are $\aleph_1$ of them that are countable, (Here $\aleph_1$ denotes the first uncountable cardinal number), and c that are uncountable. For each one of them, say X, we have a family $\mathcal{PER}(X)$. But it turns out that there is only a countable collection of subfamilies of $\mathcal{P}(\mathbb{N})$ that arise in this way. Moreover, these subfamilies form a chain under inclusion. This chain is of the type $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ...\mathcal{F}_n \subset ... \subset \mathcal{G}_1 \subset \mathcal{G}_2 \subset ... \subset \mathcal{G}_n \subset ... \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset ... \subset \mathcal{H}_n \subset ... \subset \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\} \subset \mathcal{P}(\mathbb{N})$, written as three increasing sequences of families. To put it precisely, this is of order-type $\omega.3 + 2$. Its members are described as follows:

For each positive integer n, let

$\mathcal{F}_n = \{A \subset \mathbb{N} :\ A$ is nonempty and the sum of all elements of A is $\leq n\}$,

$\mathcal{G}_n = \{A \subset \mathbb{N} : A$ is nonempty, there exists $F \in \mathcal{F}_n$ such that $F \subset A$ and all but finitely many elements of A are multiples of some element of $F\}$,

$\mathcal{G} = \{A \subset \mathbb{N}$: there exists a finite nonempty subset F of A such that every element of A is a multiple of some element of $F\}$,

$\mathcal{H}_n = \{A \subset \mathbb{N}$: either some element of A is $\leq n$ or there exists a finite nonempty subset F of A such that every element of A is a multiple of some element of $F\}$.

Theorem 14. *Let X be any zero-dimensional compact metric space. Then the family $\mathcal{PER}(X)$ has to be one of the families listed above.*

These families form a chain under set inclusion, in the sense $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{G}_m \subset \mathcal{G}_{m+1} \subset \mathcal{H}_k \subset \mathcal{H}_{k+1}$ holds for all $m, n, k \in \mathbb{N}$. But no nontrivial member there, is a chain of subsets of $\mathbb{N}$. Contrast this with the fact that $\mathcal{S}$ (in Section 1) is a chain of subsets of $\mathbb{N}$, of the order-type $\omega^2 + \omega^*$.

We can specify which topological properties of X determine what $\mathcal{PER}(X)$ is. A sample theorem: Let $n \in \mathbb{N}$. Let X be a countable compact metric space having exactly n limit points. Then $\mathcal{PER}(X)$ is the family $\mathcal{G}_n$.

Let X' denote the set of all limit points of X and X'' denote the set of all limit points of X'. The following chart gives this information exhaustively.

If X is	then $\mathcal{PER}(X)$ is
a finite set with exactly n elements	$\mathcal{F}_n$
a compact metric space with a unique limit point	$\mathcal{G}$
a compact metric space X such that $\lvert X' \rvert = n$	$\mathcal{G}_n$
a compact metric space with $\lvert X'' \rvert = n$	$\mathcal{H}_n$
the Cantor set	$\mathcal{P}(\mathbb{N})$

Let X be a zero-dimensional metric space. Then X has to be in one of the six categories shown in the chart below:

If X is	then $\mathcal{PER}(X)$ is
not compact	$\mathcal{P}(\mathbb{N})$
not countable	$\mathcal{P}(\mathbb{N})$
countable, compact and X'' is infinite	$\mathcal{P}(\mathbb{N}) \setminus \emptyset$
compact and $\lvert X'' \rvert = n$	$\mathcal{H}_n$
compact and $\lvert X' \rvert = n$	$\mathcal{G}_n$
finite and $\lvert X \rvert = n$	$\mathcal{F}_n$

As a small part of determining $\mathcal{PER}(X)$, we are led to the following question: For which metric spaces X will the empty set be in $\mathcal{PER}(X)$? In other words, which metric spaces will admit a continuous self map without any periodic point? We have a neat answer to this question.

Theorem 15. ([26]) *(a) Let X be a compact subspace of the real line $\mathbb{R}$. Then the empty set occurs as the set of periods for some continuous self map of X, if and only if the boundary of X in $\mathbb{R}$ is uncountable.*

(b) *A zero dimensional metric space X admits a continuous self map without periodic points, if and only if X is either uncountable or noncompact.*

Another important class of spaces where we have the full knowledge of sets of periods, is that of convex subsets of Euclidean spaces.

Theorem 16. *Consider the following five families of subsets of $\mathbb{N}$:*

(1) $\mathcal{P}(\mathbb{N}) = \{$ all subsets of $\mathbb{N}\}$
(2) $\mathcal{U}_1 = \{A \subset \mathbb{N} : 1 \in A\}$
(3) $\mathcal{S} =$ the Sharkovskii-family as defined in section 1

(4) $\mathcal{S} \cup \{\emptyset\}$
(5) $\{1\}$.

If X is a convex subset of $\mathbb{R}^n$ (provided with relative topology), then $\mathcal{PER}(X)$ has to be one of the above five families. The next table of examples shows that all these five do occur in this way.

If X is	then $\mathcal{PER}(X)$ is
the whole $\mathbb{R}^n$	$\mathcal{P}(\mathbb{N})$
the closed unit disc	$\mathcal{U}_1$
the real line $\mathbb{R}$	$\mathcal{S} \cup \{\emptyset\}$
the line segment [0,1]	$\mathcal{S}$
singleton	$\{1\}$

See [27] for a proof and for a more detailed discussion to answer questions like: What are all the convex X whose $\mathcal{PER}(X)$ is $\mathcal{P}(\mathbb{N})$? etc.

This is not all. [40] gives complete description of $\mathcal{PER}(X)$ where X is any compact subset of $\mathbb{R}$. These answers are describable in terms of two binary operations on families.

If $\mathcal{F}$ and $\mathcal{G}$ are two families of subsets of $\mathbb{N}$, we let $\mathcal{F} * \mathcal{G}$ denote $\{\bigcup_{n\in B} nA_n : B \in \mathcal{F}$, each $A_n \in \mathcal{G}\}$ and $\mathcal{F} \vee \mathcal{G}$ denote $\{B \cup C : B \in \mathcal{F}, C \in \mathcal{G}\} \cup \mathcal{F} \cup \mathcal{G}$.

It so happens that using these operations $*, \vee$ on families that we have already encountered in this section so far, it is possible to describe $\mathcal{PER}(X)$ for all compact subspaces X of $\mathbb{R}$.

Theorem 17. *Let $n \in \mathbb{N}$. (a) If X is the disjoint union of n closed intervals, then $\mathcal{PER}(X) = \mathcal{F}_n * \mathcal{S}$.*

(b) If X has infinitely many nontrivial components and has only one non-open component, then $\mathcal{PER}(X) = \mathcal{G}_1 * \mathcal{S}$.

(c) If X is a compact subset of $\mathbb{R}$ with n nontrivial components and with only one non-open component, then $\mathcal{PER}(X) = (\mathcal{F}_n * \mathcal{S}) \vee \mathcal{G}_1$.

Theorem 18. *(a) Let $m, n, p \in \mathbb{N}$. If X has n non-open trivial components, m nontrivial open components, then $\mathcal{PER}(X) = \bigcup((\mathcal{G}_r \vee \mathcal{G}_s) \vee (\mathcal{F}_s * \mathcal{S}) \vee (\mathcal{F}_t * \mathcal{S}))$ where the union is taken over all triples (r, s, t) of positive integers satisfying the inequalities $s \le p$, $r + s \le n + p$, $s + t \le m + p$, and $r + s + t \le m + n + p$.*

(b) Let all the open components of X be nontrivial. If $|\tilde{X}| = n$ then $\mathcal{PER}(X) = \mathcal{G}_n * \mathcal{S}$(Note: Some non-open components may be trivial, some others not).

(c) If X has n non-open trivial components and infinitely many open nontrivial components and p nontrivial non open components, and among them r nontrivial non-open components such that every open set containing it intersects infinitely many nontrivial components. Then $\mathcal{PER}(X) = \bigcup_{0\le s\le r}(\mathcal{G}_s * \mathcal{S}) \vee \mathcal{G}_{n+p-s}$.

(d) If $|\tilde{X}''| = n$ then $\mathcal{PER}(X) = \mathcal{H}_n$.

$$\text{(e) } \mathcal{PER}(X) = \begin{cases} \mathbb{P}(\mathbb{N}) \setminus \{\emptyset\} & if\ the\ boundary\ of\ X\ is\ countable\ and \\ & |\tilde{X}"| = \infty \\ \mathbb{P}(\mathbb{N}) & if\ the\ boundary\ of\ X\ is\ uncountable. \end{cases},$$

Here $\widetilde{X}$ denotes the quotient space, namely the space of components of X.

8. Toral Automorphisms

The class of toral automorphisms, induced by 2×2 integer matrices of determinant ± 1, is an important class of dynamical systems, studied extensively (See [20, 22]).

Now we take up the natural question: Which subsets of $\mathbb{N}$ arise as the set of periods of a toral automorphism? It is easy to see that the number 1 should belong to such a set. The following theorem is surprising because it gives a short list of five finite subsets and three infinite subsets and asserts that there are no others.

Theorem 19. *Let T_A be the toral automorphism induced by a matrix A. Then $Per(T_A)$ is one of the following eight subsets of* $\mathbb{N}$.

(1) $\{1\}$
(2) $\{1, 2\}$
(3) $\{1, 3\}$
(4) $\{1, 2, 4\}$
(5) $\{1, 2, 3, 6\}$
(6) $2\mathbb{N} \cup \{1\}$
(7) $\mathbb{N} \setminus \{2\}$
(8) $\mathbb{N}$.

Open Problem: What is the analogue of this theorem for higher dimensional toral automorphisms?

Now some remarks on our method of proof would be appropriate. It is well-known that for every n in $\mathbb{N}$, the two integers (i) modulus of the determinant of $A^n - I$ and (ii) the number of solutions of the equation $A^n X = X$ in the torus, have to be equal. This is our starting point. Using Cayley-Hamilton theorem for easy calculation of the matrix powers A^n, we obtain a recurrence formula to calculate the sequence (p_n) where p_n is the number of solutions of $A^n X = X$. Then we could have resorted to the known method of convolution of (p_n) with the Mobius function μ and arithmetically arrived at those n that belong to the set P of periods. But we find that there is a simpler method. If the sequence (p_n) is found to be increasing so fast that every term is greater than the sum of all the previous terms, then we have an easier argument to conclude that all these n should be in P. The present situation is only slightly harder than this; certain elementary inequalities involving the terms of (p_n) do yield the desired result.

This proof proceeds by considering various cases that arise depending on what the minimal polynomial of A is. In fact, the following table shows that the final result implies that any two matrices having the same minimal polynomial, should also have the same period sets for their induced toral automorphisms. Note that for all nonhyperbolic automorphisms, the trace of the matrix has absolute value atmost 2.

TABLE

Minimal polynomial of A	$Per(T_A)$
$x^2 - 1, x + 1$	{1,2}
$x^2 + 1$	{1,2,4}
$x^2 + x + 1$	{1,3}
$x^2 - x + 1$	{1,2,3,6}
$x^2 - 2x + 1$	$\mathbb{N}$
$x^2 + 2x + 1$	$2\mathbb{N} \cup \{1\}$
$x - 1$	{1}

It is also noteworthy that for a hyperbolic toral automorphism, the period set has only two possibilities, namely $\mathbb{N} \setminus \{2\}$ and $\mathbb{N}$; where as for nonhyperbolic toral automorphisms, there are seven possibilities; and there is an overlap because $\mathbb{N}$ can arise as the period-set, in both the hyperbolic and nonhyperbolic cases.

9. Linear Operators

One important class of dynamical systems that has been well-studied is the class of linear operators on a Hilbert space. Therefore, we now consider our question for this class. In what follows, lcm is the abbreviation for least common multiple; l^2 is the Hilbert space of square summable sequences of complex numbers. A subset A of $\mathbb{N}$ is said to be closed under lcm if whenever m, n are in A, it is true that the lcm of m and n, is also in A.

Theorem 20. *The following are equivalent for a subset A of $\mathbb{N}$.*

(i) $1 \in A$ and A is closed under lcm;

(ii) $A =$ Per(T) for some bounded linear operator $T : l^2 \to l^2$;

(iii) $A =$ Per(T) for some linear operator $T : l^2 \to l^2$;

(iv) $A =$ Per(T) for some linear isometry $T : l^2 \to l^2$.

Theorem 21. *The following are equivalent for a finite subset A of $\mathbb{N}$.*

(i) $1 \in A$ and A is closed under lcm;

(ii) $A =$ Per(T) for some linear operator $T : l^2 \to l^2$ having finite rank.

In the next theorem, we characterize the sets of periods of finite rank operators on l^2, with a given rank n. Here $l^2(\mathbb{R})$ denotes the Hilbert space of square summable sequences of real numbers.

Theorem 22. *Let $n \in \mathbb{N}$ and let $A \subset \mathbb{N}$. Then the following are equivalent:*

(a) $A = Per(T)$ for some linear operator T on $l^2(\mathbb{R})$ having rank n;

(b) $A = \{1\} \bigcup \tilde{B}$, where either (i)$|B| \le \frac{n}{2}$ or (ii) $2 \in B$ and $|B| = \frac{n+1}{2}$ (Here $\tilde{B}$ denotes the smallest subset of $\mathbb{N}$ containing B and closed under lcm).

Our method of proof uses the primary decomposition theorem and the Jordan canonical form of matrices. The role for lcm is explained by the following observation: If T splits as $T_1 \oplus T_2$, then $Per(T) = \{lcm(m, n) : m \in Per(T_1), n \in Per(T_2)\}$. For further details and related results, see [1].

10. Subshifts

There are many reasons for the importance of subshifts in the study of topological dynamics. Therefore, it is very natural to seek the knowledge of sets of periods, for the subshifts in the following important families:

(i) Subshifts of finite type (abbreviated as SFT) (ii) all subshifts and (iii) chaotic subshifts.

In [2] the following results have been obtained.

Theorem 23. *A subset P of $\mathbb{N}$ arises as the set of all periods of a chaotic SFT if and only if P is of the form $k\mathbb{N} \setminus F$ for some positive integer k and for some finite subset F of $\mathbb{N}$.*

Theorem 24. *A subset P of $\mathbb{N}$ arises as the set of all periods of a SFT if and only if it is of the form $FN\Delta G$ where F and G are two finite subsets of $\mathbb{N}$.*

Here Δ stands for the symmetric difference of sets. FN stands for $\{mn : m \in F, n \in \mathbb{N}\}$. Further, $k\mathbb{N}$ is same as $\{k\}\mathbb{N}$.

Theorem 25. *Every subset of $\mathbb{N}$ arises as the set of all periods for some subshift.*

Our proof of these results needs the following lemmas, some of which may be new; and may be of independent interest also.

Lemma 1. *The following are equivalent for a subshift of finite type:*

(i) *It is topologically transitive.*

(ii) *There is an element x in it with the "universal" property that all words occurring in any element of the subshift, do occur in x.*

(iii) *The associated digraph is strongly connected.*

Lemma 2 *Any finitely generated additive subsemigroup of $\mathbb{N}$ differs from some singly generated additive subsemigroup of $\mathbb{N}$, only in a finite set. In other words, for any finite set $a_1, a_2, ..., a_k$ of elements in $\mathbb{N}$, there is $a \in A$ (namely gcd of $a_1, a_2, ..., a_k$) such that for the two subsets $\{ma : m \in \mathbb{N}\}$ and $\{m_1a_1 + m_2a_2 + ... + m_ka_k : m_1, m_2, ..., m_k \in \mathbb{N}\}$ the difference is a finite set.*

Lemma 3. ([20]) *Every SFT is topologically conjugate to one where all forbidden words are of length 2.*

Lemma 4. *Every SFT with a dense set of periodic points can be written as a finite union of chaotic SFTs.*

11. Cellular Automata

Let A be a finite set having at least two elements. Let $r \in \mathbb{N}_0$. A function $f : A^{2r+1} \to A$ is called a local rule. It induces a function $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by the rule $(F(x))_n = f(x_{n-k}, x_{n-k-1}, ..., x_{-1}, x_0, x_1, ..., x_{n_k})$ for all $n \in \mathbb{Z}$. Then F is automatically a continuous function, commuting with the shift map σ. The dynamical system $(A^{\mathbb{Z}}, F)$ is called a cellular automaton (abbreviated as CA). The cellular automata play an important role in various contexts such as computer graphics, parallel computing and cell biology.

While it is natural to ask for a neat description of the sets of periods of cellular automata, unfortunately we do not have a complete answer for this. Moothathu

(See [41, 42]) has given a partial answer in the following way:

A CA is said to be additive if $A = \{0, 1, \ldots, m-1\}$ for some positive integer $m \geq 2$ and its local rule $f : A^{2k+1} \longrightarrow A$ can be expressed as $f(x_{-k}, \ldots, x_k) = \sum_{i=-k}^{i=k} \lambda_i x_i (\bmod m)$, where $\lambda_i \in A$.

Theorem 26. *Let F be an additive CA, where the addition is done modulo a prime p. Then, $Per(F)$ has only four possibilities: $\{1, m\}$ for some m where $1 \leq m < p$, $\mathbb{N} \setminus \{p^m : m \in \mathbb{N}\}$, $\mathbb{N} \setminus \{2p^m : m \in \mathbb{N} \cup \{0\}\}$ or the whole set $\mathbb{N}$.*

His method is combinatorial. For instance, he makes use of the following lemma.

Lemma 5. *Let p be a prime, let $k \in \mathbb{N}$, and let $a_0, a_1, \ldots, a_k$ be integers such that a_0 and a_k are not divisible by p. Also, let $l \geq 1$ be the smallest integer such that a_l is not divisible by p. Fix $n \in \mathbb{N}$ and write $n = p^m r$, where $m \geq 0$ and $p \nmid r$. Let β_t be the coefficient of x^t in the polynomial $(a_0 + a_1 x + \cdots + a_k x^k)^n$. Then, the smallest integer $t \geq 1$ such that β_t is not divisible by p, is $t = lp^m$.*

Admittedly, the class considered above, is a narrow one, not even exhausting all the additive CA. Hence it is good to mention some other partial unpublished results like the following:

Theorem 27. *Let F be any additive CA. Then $Per(f)$ has to be closed under lcm.*

On one hand, this theorem states that the fact that the subsets of $\mathbb{N}$ that have been listed previously in Theorem 26 happen to be closed under lcm, is not accidental; it has to be so for all additive CA. On the other hand, its proof does not even make use of the hypothesis that we are working with CA, and remains true in the following more general version:

If ϕ is any endomorphism of an abelian group (as every additive CA is), Per(ϕ) has to be closed under lcm. In this context, we have a more satisfactory result:

Theorem 28. *The following are equivalent for a subset S of $\mathbb{N}$:*

(1) *$1 \in S$ and S is closed under lcm;*

(2) *$S = Per(\phi)$ for some automorphism ϕ of an abelian group.*

But none of these results becomes applicable in the case of a general CA which may not be additive. The following result gives a little progress in this general context.

Theorem 29. *For every finite subset S of $\mathbb{N}$, there is a CA F such that $Per(F) = S$.*

In our proof, we take the given finite subset S as a part of the alphabet, for constructing the CA. We make use of the simple fact that every finite subset S of $\mathbb{N}$ appears as $Per(\phi)$ for some permutation ϕ on a finite set A; and it is actually this A that is taken as the alphabet.

Open Problem: Find which subsets of $\mathbb{N}$ arise as sets of periods of cellular Automata. It is worth-mentioning here that the answer has to be a countable family of subsets of $\mathbb{N}$, that includes all finite subsets, and also all the sets listed in Theorem 26, and is at present evading a good guess.

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