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Locating the minimum: Approach to equilibrium in a disordered, symmetric zero range process

MUSTANSIR BARMA and KAVITA JAIN
Department of Theoretical Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India

Abstract. We consider the dynamics of the disordered, one-dimensional, symmetric zero range process in which a particle from an occupied site k hops to its nearest neighbor with a quenched rate $w(k)$. These rates are chosen randomly from the probability distribution $f(w) \sim (w-c)^n$, where c is the lower cutoff. For $n > 0$, this model is known to exhibit a phase transition in the steady state from a low density phase with a finite number of particles at each site to a high density aggregate phase in which the site with the lowest hopping rate supports an infinite number of particles. In the latter case, it is interesting to ask how the system locates the site with globally minimum rate. We use an argument based on the local equilibrium, supported by Monte Carlo simulations, to describe the approach to the steady state. We find that at large enough time, regions with a smooth density profile are described by a diffusion equation with site-dependent rates, while the isolated points where the mass distribution is singular act as the boundaries of these regions. Our argument implies that the relaxation time scales with the system size L as L^z with $z = 2 + 1/(n+1)$ for $n > 1$ and suggests a different behavior for $n < 1$.

Keywords. Quenched disorder; approach to steady state; zero range process; diffusion; hydrodynamics.

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1. Introduction

The presence of quenched disorder is known to strongly affect the dynamical and steady state properties of many systems. For instance, for noninteracting particles moving in a random medium, disorder can lead to anomalous transport and change the manner in which the steady state is approached [1–3]. Moreover, when inter-particle interactions are present, disorder can induce new collective effects such as phase separation [4]. In this paper, we study the effect of quenched, site-wise disorder on a simple stochastic model of interacting particles known as the zero range process [5,6]. This process can be viewed as describing a system of interacting particles hopping in and out of wells with various depths. While the static properties of this model are known analytically, the temporal properties are not as well characterized. Here we study the approach to the steady state of a zero range process which undergoes a disorder-induced phase transition.

The study of stochastically evolving lattice models has played a central role to better understand interacting, statistical systems [7]. Such models are defined directly through sim-

ple stochastic rules, in contrast to the traditional route of defining the interactions through a Hamiltonian and then constructing dynamical rules consistent with it. The zero range process describes one such class of lattice models; other important classes include the asymmetric simple exclusion process which is a simple model of a current-carrying system [5,8], and the contact process whose steady state shows a phase transition from an active phase to a dead phase where no further evolution is possible [8].

The zero range process deals with a conserved number of particles hopping on a lattice with, in general, site-dependent rates. For almost all choices of these rates, the particles have on-site interactions; however, these rates do not depend on the particle occupation at other sites. In this sense, the range of interaction is zero. The steady state of this process is known to be of product measure form in all dimensions [5]. In recent past, the zero range process has been used to model several systems with quenched disorder, such as traffic flow in a system of cars with different preferred speeds [9] and activated flow down a rugged slope [10], besides several other applications [6]. A recent application is in a model of polymerization in a random medium with imperfect traps [11]. Interestingly, for several choices of disorder, as the density is increased, the steady state of the system shows a phase transition from a homogeneous phase in which the density is roughly uniform, to an infinite aggregate phase in which the density profile has a singularity at the site with the lowest hopping rate.

The latter state is particularly interesting since in this phase, starting from an initial random distribution of particles, an infinite aggregate is formed at the site with the lowest hopping rate. The question arises: what is the mechanism by which the system locates the site with *globally minimum* rate and transports a finite fraction of the total number of particles to it through the random medium? In this paper, we address this question for a symmetric zero range process in one dimension. Figure 1 shows how the density profile evolves in time starting from an initial random distribution of particles (figure 1a) to the steady state in which an aggregate builds up at the site with the lowest hopping rate (figure 1d).

The rest of the paper is organized as follows: We define the model and discuss its steady state properties in §2. In §3, we first give a simple, qualitative picture of the dynamics of relaxation to the steady state. The scaling properties of the relaxation time define a dynamic exponent which is determined using local equilibrium arguments, supported by Monte Carlo simulations. Finally, we conclude with a discussion of open questions.

2. The model and its steady state

In this section, we define the model and briefly discuss its steady state properties. We consider the unbiased or biased motion of a conserved number of particles M on a d dimensional lattice of length L with periodic boundary conditions. At any site k occupied by a nonzero number of particles, a single particle attempts to hop out at a rate $w(k)$, independent of the number of particles present at site k or its neighbors. Note that this choice of rates implies an attractive on-site interaction since the hop-out rate of these particles is lower than that of noninteracting particles. The zero range process with this particular choice of rates has appeared in several contexts, as in a particle-wise disordered, asymmetric exclusion process [9,12] and as a limiting case of a model of aggregation and fragmentation with site-wise disorder [11].

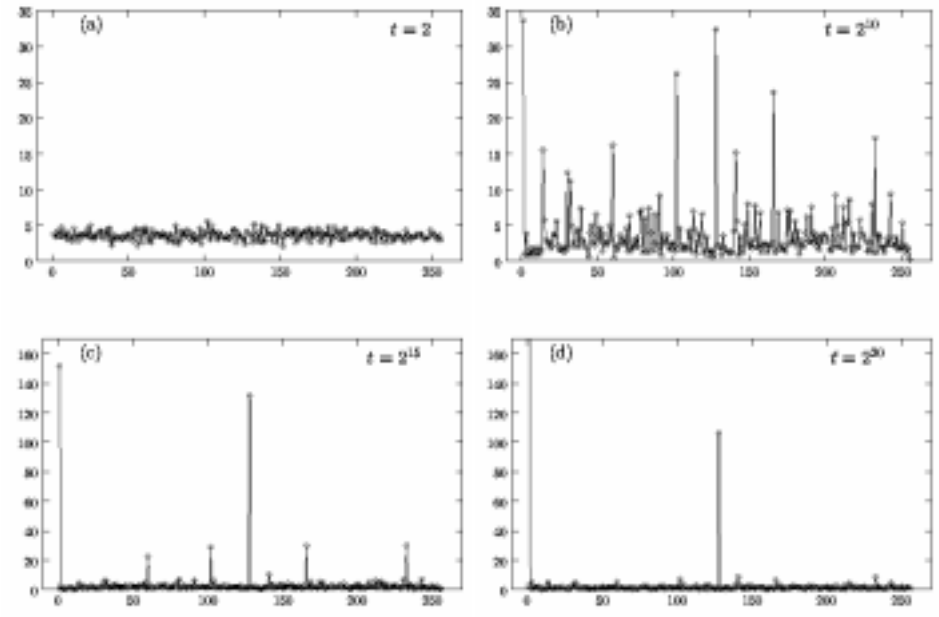

Figure 1. Density profile $\langle m(k,t)\rangle$ vs k in the aggregate phase for a given realization of disorder at time **(a)** $t = 2$, **(b)** $t = 2^{10}$, **(c)** $t = 2^{15}$, **(d)** $t = 2^{20}$. The site with the lowest hopping rate is located at $k = 1$ and the second lowest at $k = L/2$. At $t = 2^{20}$, the system is close to the steady state and $\langle m_1(1,t)\rangle = 426$. Parameters used: $L = 256$, $\rho = 4, n = 2, c = 0.5$.

As discussed in §1, the problem is essentially to understand how the system locates the site with the minimum hopping rate. For this reason, it is useful to assign two labels to the hopping rate. We denote the rates by $w_j(k)$ where k is the site index and j is the index when the rates are arranged in ascending order with $j = 1$ labelling the lowest rate. Correspondingly, $m_j(k)$ denotes the mass (or number of particles) at the site with hopping rate $w_j(k)$ at a particular instant. In the following, we will only display the labels which are pertinent to the discussion and suppress the rest. Also, unless the time dependence is explicitly specified, all the quantities will refer to the steady state.

The rates $\{w\}$ are chosen independently for all $k = 1,\ldots,L^d$ from a common probability distribution

$$f(w) = \left[(n+1)/(1-c)^{n+1}\right](w-c)^n, \quad w \in [c,1], \quad c > 0, \quad n > 0. \tag{1}$$

For both the symmetric and the totally asymmetric cases, the probability $P(\{m(k)\})$ of a configuration $\{m(1), m(2), \ldots, m(L^d)\}$ is known to be given by [6]

$$P(\{m(k)\}) = \frac{1}{\mathcal{N}} \prod_{k=1}^{L^d} \left(\frac{v}{w(k)}\right)^{m(k)}, \tag{2}$$

where $\mathcal{N}$ is the normalization constant and v is the fugacity which can be determined using the conservation law $\sum_{k=1}^{L^d} m(k) = M$. This solution holds in all dimensions and for

all bias. In the presence of bias, eq. (2) describes a nonequilibrium steady state whereas in the absence of bias, it describes an equilibrium state as the condition of detailed balance holds. This equilibrium state is described by a Hamiltonian which is long ranged, thus allowing the system to have a phase transition even in one dimension [6]. The probability $P(m,k)$ that there are m particles at site k is given by

$$P(m,k) = \left(1 - \frac{v}{w(k)}\right)\left(\frac{v}{w(k)}\right)^m, \tag{3}$$

which implies that the average number of particles at site k is

$$\langle m(k)\rangle = \frac{v}{w(k)-v} = \frac{s(k)}{1-s(k)}, \tag{4}$$

where $s(k) \equiv \sum_{m=1} P(m,k) = v/w(k)$ is the average occupation probability of site k. Since the total number of particles is conserved

$$\rho = \frac{1}{L^d}\,\frac{v}{w_1 - v} + \int_c^1 \mathrm{d}w\,\frac{v}{w-v}\;\; f(w), \tag{5}$$

where $\rho = M/L^d$ is the total particle density and disorder averaging has been done. This equation is reminiscent of Bose–Einstein condensation in the ideal Bose gas where the lowest momentum state is macroscopically occupied beyond a critical density ρ_c [12]. At $\rho = \rho_c$, the fugacity v gets pinned to the lower cutoff of w, namely c (to which w_1 will tend in the thermodynamic limit) so that the site with the lowest hopping rate supports an aggregate with a finite fraction of total particles. The condition $\rho - I(c) = 0$ determines the critical point $\rho_c = c(n+1)/n(1-c)$ where $I(v)$ is the integral on the right hand side of eq. (5).

For $\rho < \rho_c$, the typical mass at all sites is of order unity. As the density is increased, there is a phase transition at $\rho = \rho_c$. In the high density phase, the slowest site supports mass of $O(L^d)$ whereas the site with rate w_j, $j \neq 1$ supports mass of $O((L^d/j)^{1/n+1})$. The latter can be seen via a simple argument. Consider a variable x distributed uniformly between 0 and 1. Since on average, L^d observations of x will be equally spaced, it follows that the jth lowest observation in L^d trials is typically at a distance j/L^d above zero. Since $\langle m_j\rangle$ is inversely proportional to the separation between the lowest and the jth lowest hopping rate and $f(w)$ can be related to the uniform distribution by a change of variables ($w = c + (1-c)\,x^{1/n+1}$), one obtains $\langle m_j\rangle \sim (L^d/j)^{1/n+1}$, $j \neq 1$.

3. Approach to the steady state

In this section, we will discuss the approach to the steady state for the symmetric, zero range process in one dimension with the hopping rates chosen from $f(w)$ defined in eq. (1). We first describe a simple picture of approach to the steady state. Then we present an argument based on local equilibrium which suggests that the relaxation time scales with system size L as L^z with $z = 2 + 1/(n+1)$ for $n > 1$. However, this treatment breaks down for $n < 1$.

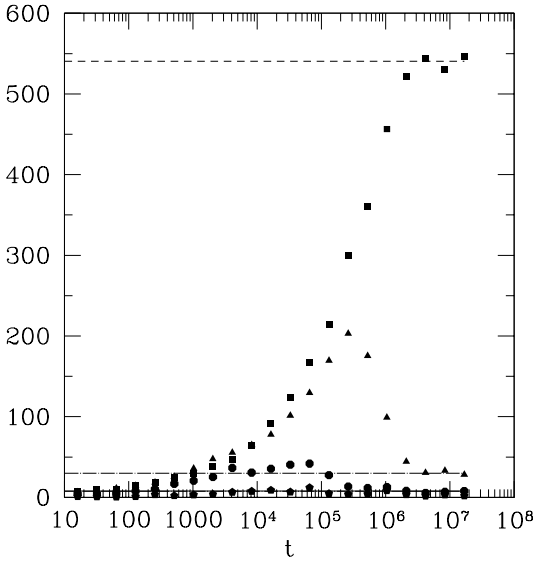

Figure 2. Time dependence of the average density $\langle m_j(t)\rangle$ for four slowest sites namely $j = 1$ (squares), 2 (triangles), 3 (circles) and 4 (pentagons) for a given disorder configuration. The horizontal lines denote the steady state value for $\langle m_j(t)\rangle$ calculated using eqs (4) and (5). Parameters used: $L = 256$, $\rho = 4$, $n = 2$, $c = 0.5$.

We first illustrate qualitatively the temporal sequence of events through which an aggregate with mass of $O(L)$ is formed at the site with the lowest hopping rate, starting from a random initial condition in which each site has mass of order unity (figure 1a). The behavior of the average mass $\langle m_j(t)\rangle$ for $j = 1,\ldots,4$ as a function of time is shown in figure 2. Here $\langle\ldots\rangle$ is to be understood as an ensemble average over evolution histories. We find that $\langle m_1(t)\rangle$ rises steadily and then saturates to its steady state value while each of $\langle m_2(t)\rangle$, $\langle m_3(t)\rangle$ and $\langle m_4(t)\rangle$ rise, attain a maximum and then decay to their respective steady state values. This nonmonotonic behavior is not hard to understand. At some finite time, each particle is able to move only a finite distance away from its initial position and tends to get temporarily trapped at the site with the lowest w within the neighborhood explored by it, rather than the global minimum w_1. A typical configuration at such an intermediate time thus has several large aggregates at such local minima while the rest of the system has masses of order unity (figure 1b). This explains why at short enough times, $\langle m_j(t)\rangle$ for $j = 1,\ldots,4$ are of the same order and increasing (figure 2). As time progresses, the particles are able to access larger regions in space and identify new local minima. Then the mass increases at these newly accessed local minima at the expense of the previous ones (figure 1c). This explains the drop in $\langle m_4(t)\rangle$ and $\langle m_3(t)\rangle$ after they have reached their respective peak values, though $\langle m_1(t)\rangle$ and $\langle m_2(t)\rangle$ continue to rise (figure 2). Finally, $\langle m_2(t)\rangle$ also starts dropping and the excess mass is transported to the location of the global minimum (figure 1d). Once the global minimum is recognized by all the particles, the system reaches a steady state and $\langle m_j(t)\rangle \to \langle m_j\rangle$ for all j.

We now turn to an analytical description of the mechanism of the mass transport. The exact time evolution equation obeyed by $\langle m(k,t)\rangle$ for a given $\{w\}$ can be written as

$$\frac{\partial \langle m(k,t)\rangle}{\partial t} = w(k-1)s(k-1,t) + w(k+1)s(k+1,t) - 2w(k)s(k,t). \tag{6}$$

At large enough times, the system is expected to be in *local* equilibrium. This allows one to assume the $s(k,t) - \langle m(k,t)\rangle$ relation to be approximately as in the steady state (eq. (4)). Substituting for $s(k,t)$ in the above equation, one obtains

$$\frac{\partial \langle m(k,t)\rangle}{\partial t} = G(k-1,t) + G(k+1,t) - 2G(k,t), \tag{7}$$

where $G(k,t) = w(k)\langle m(k,t)\rangle/(1+\langle m(k,t)\rangle)$.

The treatment so far is valid for any density, but we are primarily interested in densities for which the system is in the infinite aggregate phase in the steady state. For such densities, at large times, one needs to divide the system into two sets – set A composed of those sites which support rather large aggregates and set B at which the masses are small and close to their steady state values (see figure 2). The elements of these sets are not fixed in time; with the passage of time, the number of elements in set A reduce and that of set B increase by the mechanism described at the beginning of this section. Eventually, when the system is close to the steady state, set A is left with a few isolated sites that support aggregates whose masses scale as a nonzero power of L while the bulk of the sites belong to set B.

We are interested in the fluctuations in the density profile about the steady state, $\Delta m(k,t) = \langle m(k,t)\rangle - \langle m(k)\rangle$. Since Δm is small for $k \in B$ and very large for $k \in A$, an expansion in $\Delta m(k,t)$ would be justified only for the background sites. Below we first analyze these background (B) sites and then treat the isolated sites in set A as the boundaries of the B regions. For the B sites, we may retain the lowest order terms in an expansion of eq. (7) in powers of Δm and obtain

$$\begin{aligned}\frac{\partial \Delta m(k,t)}{\partial t} &= D(k-1,t)\Delta m(k-1,t) + D(k+1,t)\Delta m(k+1,t) \\ &\quad -2D(k,t)\Delta m(k,t), \quad k \in B,\end{aligned} \tag{8}$$

with

$$D(k) = (w(k)-c)^2/w(k). \tag{9}$$

Here we have used the fact that v tends to c in the infinite aggregate phase.

Equation (8) describes a random walker in a random medium with site-dependent hopping rate $D(k)$. Using eqs (1) and (9), we find that these rates are distributed according to the probability distribution $g(D) \sim D^{(n-1)/2}$ for $D \to 0$. Note that whereas $f(w)$ has a nonzero lower cutoff, $g(D)$ has zero as the lower limit due to which it diverges as $D \to 0$ for $n < 1$. The problem of a particle moving with random, spatially inhomogeneous hopping rates is well studied and has been reviewed in [1]. For a configuration of randomly distributed $\{D\}$ in a large system observed on large enough time scales, the mean squared displacement of the random walker grows as $\mathscr{D}t$ where the diffusion constant $\mathscr{D} = 1/\langle\langle 1/D(k)\rangle\rangle$ and $\langle\langle ...\rangle\rangle$ stands for disorder average, provided $\langle\langle 1/D(k)\rangle\rangle$ is finite [2]. Thus for large spatial and time separations, the particle diffusion is described by a single, effective diffusion constant. In our case, one can calculate $\langle\langle 1/D\rangle\rangle$ by noting that $\langle\langle 1/D\rangle\rangle = \partial I/\partial v|_{v\to c}$. Expanding $I(v)$ for v close to c, we obtain

$$I(v) = \rho_c + O(c-v), \qquad n > 1 \tag{10}$$

$$= \rho_c + O((c-v)^n), \qquad 0 < n < 1 \tag{11}$$

which implies that $\langle\langle 1/D\rangle\rangle$ is finite for $n > 1$ and diverges for $n < 1$. The divergence in the latter case indicates anomalous diffusion, i.e. the mean squared displacement grows sublinearly. In the remainder, we will restrict ourselves to $n > 1$ [13].

On averaging over disorder configurations, for a system of size L, the mass $\langle\langle m_1(t,L)\rangle\rangle$ on the site with lowest hopping rate is expected to scale as

$$\langle\langle m_1(t,L)\rangle\rangle \approx t^{\beta} H\left(\frac{t}{L^z}\right) \tag{12}$$

where

$$H(x) \sim \begin{cases} \text{constant} & \text{for } x \ll 1 \\ x^{-\beta} & \text{for } x \gg 1 \end{cases}. \tag{13}$$

Since $\langle\langle m_1(t,L)\rangle\rangle \sim L$ in the steady state, it follows that $\beta z = 1$. We now determine the relaxation time of the system by estimating the time $T \sim L^z$ required for $\langle m_1(t,L)\rangle$ to reach its steady state value for a typical realization of disorder. At large enough times, the background sites in set B form the bulk of the system whereas only a few isolated sites with low hopping rates belong to set A. Let us now consider time scales above which the two sites with the lowest rates (i.e. w_1 and w_2) are the only elements left in set A. As shown in figure 2, $\langle m_2(t)\rangle$ drops after it has reached its peak while $\langle m_1(t)\rangle$ keeps rising at the expense of the former so that there is a net transfer of mass from the site with $j = 2$ to the site with $j = 1$. One can think of these two sites as the boundaries to the bulk background region with the former site feeding particles at a rate of w_2 to the bulk and the latter at a rate of w_1. On large time scales, a quasi-equilibrium is established, and in view of the discussion above, the particles diffuse through the bulk with an effective diffusion constant $\mathscr{D}$. Thus there is a transfer of mass at a rate $\mathscr{D}\Delta w_{12}/r_{12}$ where $\Delta w_{12} = w_2 - w_1$ and r_{12} is the spatial separation between the two sites. The relaxation time T_{12}, which is essentially the time taken to transfer the peak mass m_2 at the site with rate w_2, is given by

$$T = T_{12} = \frac{r_{12}\, m_2}{\mathscr{D}\, \Delta w_{12}}. \tag{14}$$

Let us estimate the size dependence of the quantities that appear on the right hand side of the above equation. Typically the separation r_{12} is of order L in which case m_2 is also of order L. This is because for times earlier than when m_2 peaks, the site $j = 2$ is the slowest one encountered by the particles in a finite fraction of the system. Consequently, the numerator in the above equation is proportional to L^2. This result holds even in the exceptional case when the sites with $j = 1$ and 2 are separated by a distance of order unity, as they behave effectively as a single slow site and one can use the same reasoning as above on replacing the site with $j = 2$ by that with $j = 3$.

Further, using the argument given at the end of §2, we find that the inverse rate separation $(\Delta w_{12})^{-1}$ scales as $L^{1/n+1}$. Collecting all the dependences, it follows that the relaxation time T scales with the system size L as $T \sim L^z$ where

$$z = 2 + 1/(n+1), \quad n > 1. \tag{15}$$

We measured the growth of $\langle\langle m_1(t,L)\rangle\rangle$ using Monte Carlo simulations and find that it grows as t^{β}. As shown in figure 3, our expression for $\beta = 1/z = (n+1)/(2n+3)$ is consistent with the numerics.

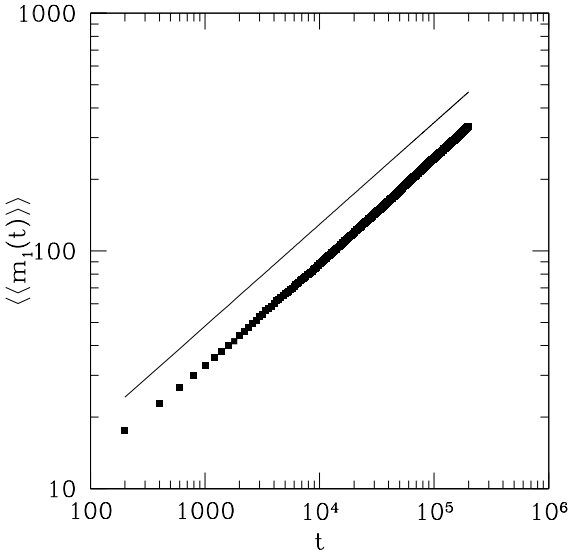

Figure 3. Log–log plot of $\langle\langle m_1(t,L)\rangle\rangle$ vs t to show the growth exponent β. The data for $\langle\langle m_1(t,L)\rangle\rangle$ has been averaged over 50 histories and 21 disorder configurations. The theoretical prediction for $\beta = 0.428$ for $n = 2$ is plotted as a solid line for comparison. Parameters used: $L = 16384$, $\rho = 4$, $c = 0.5$.

We conclude this paper with a discussion of two open questions: (i) We have seen that $\langle\langle 1/D\rangle\rangle$ diverges for $n < 1$ indicating anomalous diffusion, which suggests that the expression for z in eq. (15) may be invalid for $n < 1$. It would be interesting to find how the system relaxes in this case. (ii) For the case with asymmetric hopping rates, using the arguments analogous to those given above, we would expect the dynamic exponent to be $z = 1 + 1/(n+1)$ for $n > 1$. Once again, we may anticipate a different result for $n < 1$. In [9] and [14] the dynamics of the asymmetric version has been analyzed using numerical simulations and extremal statistics arguments for a related deterministic model. The expression for the dynamic exponent in [9,14] is the same as that quoted above but their arguments do not make a distinction between the regimes below and above $n = 1$. The elucidation of the anomalous regime for both the symmetric and asymmetric zero range process remains an interesting open problem.

We will not attempt a summary of this paper. Professor N Kumar once pointed out that since it is a part of the whole, a proper summary should include a summary of the summary and a summary of that summary, and so on. We would rather not try ! Kumar's own work has brought out many of the surprises that disordered systems have to offer, and we are very pleased that our article will appear in this special issue.

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