

# Wave propagation through a coherently amplifying random medium

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## Abstract

We report a detailed and systematic numerical study of wave propagation through a coherently amplifying random one-dimensional medium. The coherent amplification is modeled by introducing a uniform imaginary part in the site energies of the disordered single-band tight binding Hamiltonian. Several distinct length scales (regimes), most of them new, are identified from the behavior of transmittance and reflectance as a function of the material parameters. We show that the transmittance is a non-self-averaging quantity with a well defined mean value. The stationary distribution of the super reflection differs qualitatively from the analytical results obtained within the random phase approximation in strong disorder and amplification regime. The study of the stationary distribution of the phase of the reflected wave reveals the reason for this discrepancy. The applicability of random phase approximation is discussed. We emphasize the dual role played by the lasing medium, as an amplifier as well as a reflector.

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## I. INTRODUCTION

Wave propagation through a random passive medium is being studied intensively over several decades [1,2]. It is now well established that coherent interference effects, due to elastic scattering by the static disorder, induces Anderson localization for quantum as well as classical waves. Some physical examples are electron transport in disordered conductors, light or electro-magnetic wave propagation in random dielectric media, sound propagation in an inhomogeneous elastic medium, etc. These qualitatively different types of waves in an appropriate limit follow the same mathematical equation, namely, the Helmholtz equation. Thus studies on different types of wave propagation complement each other. It is basically the wave character leading to interference and diffraction which is the common operative feature. Coherent multiple scattering of a wave from a fixed spatial realization of randomly distributed scatterers generates an interference pattern in space which is very sensitive to the actual distribution of scatterers. Small relative displacement of scatterers, of the order of a fraction of a wavelength, can alter completely the interference pattern. Specifically in the context of quantum electron transport in one dimensional random media this will make the resistance ( or the transmittance ) a non-self averaging quantity [3,4] in that the resistance fluctuations over the ensemble of macroscopically identical samples dominate the ensemble average, *i.e.* root mean square variation of sample to sample fluctuations in the resistance over all the realizations of the macroscopically identical samples exceeds the mean value by orders of magnitude no matter how large the sample is. The inelastic scattering ( due to phonons or other quasi-particles ) lead to loss of phase memory of the wave function. Thus the motion of electrons becomes phase incoherent and sample to sample fluctuations become self-averaging in the high temperature limit leading to a classical behavior. In recent years the study of wave propagation in an active random medium [5–21], *i.e.*, in the presence of absorption or amplification, is being pursued actively. The light propagation in an amplifying (lasing) medium has its implications for stimulated emission from random media. In a stimulated emission photons emitted will have the same frequency, phase, direction, and polarization.

This will result in spatial and temporal coherence of laser light propagation.

The absorption in the medium corresponds to the actual removal of the particle ( or energy in the case of electro-magnetic wave propagation ) by re-combination processes. For example propagation of optical (excitons) or magnetic excitations in solids which terminate upon reaching trapping sites. To allow for the possibility of inelastic decay on the otherwise coherent tunneling through potential barriers several studies invoke absorption [22,23]. In the presence of inelastic scattering due to thermal excitations, electrons are scattered out of elastic channel to other inelastic channels. In these studies the absorption is identified as the spectral weight lost in the inelastic channels. As an example, in the case of one-dimensional double barrier structures the absorbed or attenuated part is assumed to tunnel through both the left and the right hand sides of the barriers in proportion to the transmission coefficient of each barrier, and this is added to the coherent transmission to get the overall transmission coefficient [23]. In the electro-magnetic wave propagation the bosonic nature of photons brings in both features, namely that of amplification as well as attenuation. Photons obey Bose statistics and their number is not conserved. Thus one can consider a problem of wave propagation in a coherently amplifying (or absorbing) optical medium. In the Schrödinger equation, to describe the absorption or amplification, one introduces the imaginary potentials. In that case the Hamiltonian becomes non-Hermitian and thus the particle number is not conserved. Such Hamiltonians are widely used in Nuclear physics literature and the corresponding imaginary potentials are called optical potentials. The absorption or amplification for the case of light propagation is simulated via the imaginary part of the dielectric constant with opposite signs. It should be noted that in quenched random systems with imaginary potentials the temporal coherence of the wave is preserved in spite of amplification or absorption which causes a particle non-conserving scattering process. Almost all the studies reported so far have considered a linear amplifying or absorbing medium, irrespective of the fact that real problem of laser oscillations and mode selection in an optically pumped random medium requires consideration of non-linearities [6]. In all these studies the basic issue is to understand the interplay of phase coherent multiple

scattering and amplification (or absorption).

Several new results have been obtained from the studies of wave propagation in an active medium. In earlier studies it has been widely thought that the effect of absorption on classical waves is analogous to that of inelastic scattering of electrons. Weaver [7] has shown that absorption does not provide a cut-off length scale ( similar to an inelastic scattering length ) for the renormalization of wave transport in the random medium. In other words, the absorption does not re-establish the diffusive behavior of the wave propagation by destroying the localization of eigenfunctions. The transport seems to remain non-diffusive even in the presence of absorption. This fact will have an important bearing on the physics at the mobility edge in higher dimensional (3-D) systems. In a related development it has been shown that absorption along with enhanced reflection induces coherence in quantum systems [24]. In a scattering problem, the particle experiences a mismatch from the real valued potential to the imaginary valued potential at the interface between the free region and the absorbing (or amplifying) medium, and hence it tries to avoid this region by enhanced back reflection. Thus a dual role is played by imaginary potentials as an absorber (or amplifier) and as a reflector. This point has been emphasized in earlier treatments [8,24,25]. One can readily show that, when the strength of the imaginary potential is increased beyond certain limit, both absorber and amplifying scatterer act as a reflector. Thus the reflection coefficient exhibits non-monotonic behavior as a function of the absorption (amplification) strength. Using the duality relations it has been shown that amplification suppresses the transmittance in the large length (L) limit just as much as absorption does [9]. This is somewhat contrary to the expectation. One would have expected that as a wave passes through a disordered amplifying medium it undergoes coherent multiple scattering and hence gets amplified before it escapes from the system. It turns out that coherent amplification in turn induces localization by enhancement of the coherent backscattering involving longer return paths, thereby cutting off transmission. Experimentally this reflects in a narrowing of the backscattering cone in random amplifying medium [5]. It has been noticed [10] that there exists a crossover length scale  $L_c$  below which the amplification enhances the

transmission and above which the amplification reduces the transmission which, in fact, vanishes exponentially in the  $L \rightarrow \infty$  limit. In contrast, super-radiant reflectance saturates to a finite value in the large length limit. Moreover, absorption and amplification of same strength (*i.e.*, differing only in the sign of the imaginary part) will induce same localization length [11]. Even for an ordered periodic system in the presence of coherent amplification, the transmittance always decreases in the asymptotic length limit [12]. This follows from the fact that the amplifier also acts as a back-scatterer (or reflector) as mentioned above. To obtain enhanced coherent transmittance, the synergy between wave confinement due to Anderson localization and coherent amplification by active medium is not necessary. By a proper choice of a length of an ordered amplifying medium one can achieve large transmittance. However, for a finite sample of length  $L$  to obtain enhanced reflection the synergy between disorder and amplification plays a major role.

In an amplifying medium even though the transmittance ( $t$ ) decreases exponentially with the length  $L$  in the large  $L$  limit, the average  $\langle t \rangle$  is shown [13] to be infinite due to the less probable resonant realizations corresponding to the non Gaussian tail of the distribution of  $\ln t$ . This result is based on the analysis using random phase approximation (RPA). Using duality argument Paasschens *et. al.* show that non-Gaussian tails in the distribution of  $\ln t$  contain negligible weight [9]. Thus one might expect finite value for  $\langle t \rangle$  in the asymptotic limit. It should be noted that even in the ordered periodic system all the states are resonant states and still the transmittance decreases exponentially for all the states in the large  $L$  limit. The above simple case may indicate that in the asymptotic limit  $\langle t \rangle$  is indeed finite. One of our objectives in this paper is to study (numerically) the behavior of the transmission probability as a function of length in the presence of coherent amplification. We show that the transmission coefficient is a non-self-averaging quantity. In the large length limit we do not find any resonant realization, which can give an enhanced transmission. We also study the behavior the of cross-over length  $L_c$  as a function of disorder and amplification strength. As mentioned earlier, upto the cross-over length  $L_c$  transmittance increases and after  $L_c$  it falls exponentially. We have studied the logarithm of the transmittance which

will have a maximum value  $\langle lnt \rangle_{max}$  at  $L_c$ . We have analyzed the behavior of  $\langle lnt \rangle_{max}$  as a function of disorder and amplification strength. We would like to emphasize that in the lasing medium the presence of disorder suppresses the average transmittance at all length scales in comparison with the ordered media having the same strength of amplification. For a given strength of amplification there exists a critical strength of disorder below which the average transmittance is always less than unity at all length scales and decreases monotonically. In this regime  $L_c$  and  $\langle lnt \rangle_{max}$  lose their physical significance. Yet in this regime we show that there exists a cross-over length scale  $\xi_c$  which diverges as the amplification strength is reduced to zero for a given strength of the disorder. In the case of super reflection in the presence of disorder we show that there exists a cross-over length  $L_1$  below which the averaged logarithm of reflectance,  $\langle lnr \rangle$ , is always less than  $lnr$  for the periodic ( $W = 0$ ) lasing ( $\eta \neq 0$ ) system.  $L_1$  depends on disorder and amplification strength. Below  $L_1$ ,  $\langle lnr \rangle$  is always larger than that for the ordered lasing medium. However, there is another disorder dependent length scale  $L_2 < L_1$ . For a system of size less than  $L_1$  disorder enhances the reflection whereas for sizes between  $L_2$  and  $L_1$  disorder suppresses the reflection.

In the work by Pradhan and Kumar [6], the analytical expression for the stationary distribution  $P_s(r)$  of a coherently backscattered reflection coefficient ( $r$ ) is obtained in the presence of both absorption and amplification using the method of invariant imbedding [26]. In the presence of a spatially uniform amplification in a random medium and with the help of random phase approximation, the expression for  $P_s(r)$  is given by

$$P_s(r) = \frac{|D| \exp(-\frac{|D|}{r-1})}{(r-1)^2} \quad \text{for } r \geq 1 \quad (1)$$

$$= 0 \quad \text{for } r < 1$$

where  $D$  is proportional to  $\eta/W$ ,  $\eta$  and  $W$  being the strength of amplifying potential and disorder respectively. One can readily notice from Eqn. (1) that  $P_s(r)$  does not tend to  $\delta(r-1)$  in the large  $\eta$  limit. In this limit, as mentioned earlier, an amplifying scatterer acts as a reflector. Instead, Eqn. (1) indicates that as  $D$  increases the distribution becomes broad and the most probable value of the reflection coefficient shifts to higher values. Since

the above expression for  $P_s(r)$  is obtained within the RPA, its validity is limited for small disorder and amplification strength. Even in the absence of amplification it is well known that RPA fails in the large disorder limit [27–29]. In the presence of small disorder it is shown that absorption suppresses phase fluctuations making the regime of validity of RPA still smaller in the parameter space of disorder and absorption [8,20]. We show that this is true also in the presence of spatially uniform amplification. From Eqn. (1) it follows that the average reflection coefficient  $\langle r \rangle$  is infinite.

With the help of transfer matrix approach we have studied the distribution and statistics of the transmittance  $t$  from which the non-self-averaging nature of  $t$  follows. We then study the behavior of  $L_c$ ,  $L_1$ ,  $L_2$  and  $\langle \ln t \rangle_{max}$  on the material parameters. The probability distribution of the reflection coefficient  $P(r, L)$  tends to a stationary distribution  $P_s(r)$  in the large  $L$  limit. For a small disorder  $W$  and small amplification  $\eta$ ,  $P_s(r)$  is qualitatively in conformity with Eqn. (1). As we increase  $\eta$ , a double peak structure appears in  $P_s(r)$  and as we increase  $\eta$  further,  $P_s(r)$  tends towards  $\delta(r - 1)$ . The average of  $\ln r$  obtained from  $P_s(r)$  exhibits maxima as a function of  $\eta$ . We also show that amplification suppresses the phase fluctuation of the complex reflection amplitude. In the next section we define our model Hamiltonian and transfer matrix approach. Later sections are devoted to results and conclusions.

## II. NUMERICAL PROCEDURE

We consider a quasi-particle moving on a lattice. The appropriate Hamiltonian in a tight-binding one-band model can be written as

$$H = \sum \epsilon'_n |n\rangle \langle n| + V(|n\rangle \langle n+1| + |n\rangle \langle n-1|). \quad (2)$$

$V$  is the off-diagonal matrix element connecting nearest neighbors separated by a lattice spacing  $a$  (taken to be unity throughout) and  $|n\rangle$  is the non-degenerate Wannier orbital associated with site  $n$ , where  $\epsilon'_n = \epsilon_n - i\eta$  is the site energy. The real part of the site

energy  $\epsilon_n$  being random represents static disorder and  $\epsilon_n$  at different sites are assumed to be uncorrelated random variables distributed uniformly ( $P(\epsilon_n) = 1/W$ ) over the range  $-W/2$  to  $W/2$ . Here  $W$  can be interpreted as the strength of the disorder. We have taken imaginary part of the site energy  $\eta$  to be spatially uniform and depending on whether the medium is amplifying or absorbing, it is set to positive or negative values. Since all the relevant energies can be scaled by  $V$ , we can set  $V$  to unity. The lasing medium consisting of  $N$  sites ( $n = 1$  to  $N$ ) is embedded in a perfect infinite lattice with all site energies taken to be zero.

To calculate the transmission and reflection coefficients we use the well known transfer-matrix method [30]. Here we describe the method very briefly. Let the sample be placed between two semi-infinite perfect leads. With the wave-function  $\psi$  inside the sample expressed as a linear combination of the Wannier orbitals  $|n\rangle$  with coefficients  $c_n$  the Schrödinger equation  $H\psi = i\dot{\psi}$  leads to

$$\begin{bmatrix} c_{n+1} \\ c_n \end{bmatrix} = \begin{bmatrix} \frac{(E-\epsilon'_n)}{V} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n-1} \end{bmatrix} \quad (3)$$

where  $E$  is the energy of the incident particle. Thus to obtain all the coefficients  $c_n$ , for  $n = 1$  to  $N$ , we just have to evaluate the product of  $N$   $2 \times 2$  matrices  $T_i$  of the type shown above. If a plane wave  $e^{ikn}$  is sent through the perfect lead from one side then the solutions on the two sides of the sample are related by a product matrix  $M$  *i.e.*,

$$M = \omega S^{-1} \prod_{i=1}^N T_i S,$$

where

$$\omega = \begin{bmatrix} e^{ik(N+1)} & 0 \\ 0 & e^{-ik(N+1)} \end{bmatrix}, \quad S = \begin{bmatrix} e^{-ik} & e^{ik} \\ 1 & 1 \end{bmatrix}$$

The transmission ( $t$ ) and reflection ( $r$ ) coefficients in terms of the matrix elements of  $M$  are

$$t = \frac{\det M}{|M_{11}|^2}, \quad r = \frac{|M_{12}|^2}{|M_{11}|^2}.$$

Since the Hamiltonian is non-hermitian we have  $t + r \neq 1$ .



### III. RESULTS AND DISCUSSION

In our studies we have set the energy of the incident particle at  $E = 0$ , *i.e.*, at a midband energy. Any other value for the incident energy does not affect the physics of the problem. In calculating average values in all cases we have taken 10,000 realizations of random site energies ( $\epsilon_n$ ). The strength of the disorder and the amplification are scaled with respect to  $V$ , *i.e.*,  $W$  ( $\equiv W/V$ ) and  $\eta$  ( $\equiv \eta/V$ ). The length  $L$  denotes the dimensionless length in the unit of lattice spacing  $a$ .

Depending on the parameters  $\eta$ ,  $W$  and  $L$  the transmission coefficient can be very large (of the order of  $10^{12}$  or more). Hence, we first consider behavior of  $\langle lnt \rangle$  instead of  $\langle t \rangle$ . The angular brackets denote the ensemble average. In Fig. 1 we have plotted  $\langle lnt \rangle$  as a function of the length  $L$  for a fixed value of amplification strength  $\eta = 0.1$  and for various values of the disorder strength  $W$  as indicated in the figure. In the absence of disorder ( $W = 0$ ) as one varies the length, initially the transmission increases to a very large value ( $t \approx 10^{12}$ ) through large oscillations and after exhibiting a maximum at the length  $L_c$ , and again through oscillations, it eventually decays exponentially to zero as  $L \rightarrow \infty$ . We denote the maximum in  $\langle lnt \rangle$  at  $L = L_c$  as  $\langle lnt \rangle_{max}$ ,  $L_c$  being the cross-over length. In the presence of disorder one readily notices that  $\langle lnt \rangle$  is suppressed at all lengths as compared to an ordered amplifying medium of same  $\eta$ . Both  $L_c$  and  $\langle lnt \rangle_{max}$  decrease as functions of the disorder strength. When the disorder strength is large (see Fig. 1 for  $W = 3.0$ ) the average transmittance becomes less than unity and decreases monotonically as a function of  $L$ . In this regime both  $L_c$  and  $\langle lnt \rangle_{max}$  lose their physical significance. We will show later that even in this regime one can still define a new cross-over length scale, say  $\xi_c$ . The existence of  $L_c$  can be attributed to the synergetic effect between the amplification and the localization. Eventually the strong back scattering arising due to both serial one dimensional disordered potential and amplification leads to an exponential decay of the transmittance. From the graphs of  $\langle lnt \rangle$  versus  $L$ , one can find the corresponding localization length. We denote the localization length by  $l_a$  for an ordered medium ( $W = 0$ ) in the presence of uniform

amplification. The localization length [31] for a disordered passive medium ( $\eta = 0$ ) is given by elastic back scattering length  $l = 48V^2/W^2$  at the center of the band ( $E = 0$ ). We have verified that the localization length in the presence of both disorder and amplification, [10]  $\xi$  is related to  $l$  and  $l_a$  (for  $\eta/V < 1$  and  $W/V < 1$ ) as  $\xi = ll_a/(l + l_a)$ . We have also verified that  $\xi(\eta) = \xi(-\eta)$  as shown by Paasschens *et. al.* using duality argument [9].

Fig. 2 illustrates the behavior of  $\langle lnt \rangle$  as a function of  $L$  for a fixed value of disorder  $W = 1.0$  and for various values of the amplification strength  $\eta$  as indicated in the figure. One finds that the cross-over length  $L_c$  is a monotonically decreasing function of  $\eta$ . However,  $\langle lnt \rangle_{max}$  initially increases with  $\eta$  and after exhibiting maxima,  $\langle lnt \rangle_{max}$  decreases with further increase in  $\eta$ .

We will now study the behavior of  $L_c$  and  $\langle lnt \rangle_{max}$  as functions of  $W$  and  $\eta$  in the parameter space where  $L_c$  and  $\langle lnt \rangle_{max}$  are well-defined. In Fig. 3 we have plotted  $\langle lnt \rangle_{max}$  versus  $W$  for a fixed  $\eta = 0.1$ . In the inset of Fig. 3 is shown the dependence of  $L_c$  on  $W$ . It is clear that both  $\langle lnt \rangle_{max}$  and  $L_c$  monotonically decrease with  $W$ . The cross-over length  $L_c$  does not follow the  $1/W$  prediction [10]. This comes out by curve fitting our numerical data in the full parameter range. The prediction that  $L_c \sim 1/W$  has the shortcoming that in the limit  $W \rightarrow 0$  it tends to infinity, but we know that for a perfect ordered lasing medium ( $W = 0$ )  $L_c$  is indeed finite. The validity of  $L_c \sim 1/W$  in the intermediate regime is not ruled out.

In Fig. 4 we have plotted  $\langle lnt \rangle_{max}$  against  $\eta$  for a fixed value of  $W = 1.0$  and the inset shows variation of  $L_c$  with  $\eta$  for  $W = 1.0$ . Initially  $\langle lnt \rangle_{max}$  increases with  $\eta$  and after exhibiting a maxima it decays to zero for large  $\eta$ . This arises from the fact that the lasing medium acts as a reflector for large  $\eta$  as discussed in the introduction. Near the maximum, in a finite regime of  $\eta$ ,  $\langle lnt \rangle_{max}$  exhibits several oscillations. In this region sample to sample fluctuations of  $lnt$  are very large. Thus average over 10,000 realizations may not represent the true ensemble averaged quantity. From the curve fitting of our numerical data for  $L_c$ , we find that  $L_c$  does not follow a power law,  $(1/\sqrt{\eta})$ , in the full parameter regime [10].

To study the nature of fluctuations in the transmission coefficient, in Fig. 5 we have

plotted, on log-scale,  $\langle t \rangle$ , root-mean-squared variance  $t_v = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}$  and root-mean-squared relative variance (or fluctuation)  $t_{rv} = \sqrt{\langle t^2 \rangle - \langle t \rangle^2} / \langle t \rangle$  as a function of  $L$  for  $\eta = 0.1$  and  $W = 1.0$ . For these parameters  $l \approx 48, l_a = 10, \xi \approx 8$  and  $L_c \approx 30$ . We notice that both  $\langle t \rangle$  and  $t_v$  exhibit maxima and decrease as we increase the length further. Except in the small length limit, variance is larger than the mean value. The relative variance is larger than one for  $L > 10$  and remains large even in the large length limit. The  $t_{rv}$  fluctuates between values 50 to 300 in the large length ( $L > 10$ ) regime, indicating clearly the non-self-averaging nature of the transmittance. This implies that the transmission over the ensemble of macroscopically identical samples dominates the ensemble average. The transmissions across the sample is very sensitive to the spatial realizations of impurity configurations. Because  $\langle t \rangle$  being non-self-averaging, it does not represent a well defined physical quantity. In such a situation one has to consider the full probability distribution  $P(t)$  of  $t$  to describe the system behavior. In Fig. 6 we have plotted  $P(t)$  as a function of  $t$  for various values of  $L$  as indicated in the figures. We have chosen  $W = 1.0$  and  $\eta = 0.1$ . We see that for  $L < L_c (\approx 30)$   $P(t)$  is a peaked distribution with a negligible weightage at large  $t$ . As we increase  $L$  further the distribution broadens and the peak shifts to higher values of  $t$  with the emergence of a tail. For larger value  $L > L_c$  the peak again becomes sharper and starts shifting towards lower value of  $t$  with a small weightage at tails. For further increase in  $L$  a sharp peak appears around  $t = 0$  with a negligible weightage in the tail of the distribution.

We would now like to understand whether there exist any resonant realizations in the large length limit for which the transmittance is very large. This study calls for sample to sample fluctuations. It is well known from the studies in passive random media that the ensemble fluctuation and the fluctuations for a given sample as a function of chemical potential or energy are expected to be related by some sort of ergodicity [1,32], *i.e.*, the measured fluctuations as a function of the control parameter are identical to the fluctuations observable by changing the impurity configurations. In Fig. 7(a) we have plotted  $t$  versus incident energy  $E$  (within the band from  $-2$  to  $+2$ ) for a given realization of random

potential with  $\eta = 0$  and  $L = 100$ . The Fig. 7(b) shows the behavior of  $t$  versus  $E$  for the same realization in the presence of amplification  $\eta = 0.1$  and  $L = 100$ . From the Fig. 7(a) we observe that at several values of energy the transmittance exhibits the resonant behavior in that  $t = 1$ . These resonant states make the average of Landauer four probe conductance ( $G = (e^2/\pi\hbar)t/(1-t)$ ) infinite [33,34]. From Fig. 7(b) we notice that in the presence of amplification, transmittance at almost resonant realizations is negligibly small. Few peaks appear in the transmittance whose origin lies in the combined effect of disorder and amplification. However, we notice that the transmittance at these peaks is much smaller, where as one would have naively expected the transmittance to be much much larger than unity in the amplifying medium. We have studied several realizations and found that none of them shows any resonant behavior where one can observe the large transmittance. The peak value of observed transmittance is of the order of unity or less. This study clearly indicates that  $\langle t \rangle$  is indeed finite contrary to the earlier predictions based on RPA [13].

So far our study was restricted to the parameter space of  $W$  and  $\eta$  for which  $L_c$  and hence  $\langle lnt \rangle_{max}$  exist. In Fig. 8 we have plotted  $\langle lnt \rangle$  against  $L$  for ordered lasing medium ( $W = 0, \eta = 0.01$ ), disordered passive medium ( $W = 1.0, \eta = 0$ ) and disordered active medium ( $W = 1.0, \eta = 0.01$ ). The present study is restricted to the parameter space of  $\eta$  and  $W$  such that  $\eta \ll 1.0$  and  $W \geq 1.0$ . We notice that for an ordered lasing medium, the transmittance is larger than one. We have taken our range of  $L$  upto 300. Needless to say, in the asymptotic regime, for an ordered lasing medium, the transmittance tends to zero exponentially. For a disordered active medium ( $W = 1.0, \eta = 0.01$ ), we notice that the transmittance is always less than one and monotonically decreasing. Initially, upto certain length, the average transmittance is, however, larger than that in the disordered passive medium ( $W = 1.0, \eta = 0$ ). This arises due to the combination of lasing with disorder. In the asymptotic regime transmittance of a lasing random medium falls below that in the passive medium with same disorder strength. This follows from the enhanced localization effect due to the presence of both disorder and amplification together, *i.e.*,  $\xi < l$ . It is clear from the figure that  $\langle lnt \rangle$  does not exhibit any maxima and hence the question of

$L_c$  or  $\langle lnt \rangle_{max}$  does not arise. We notice, however, from the figure that for random active medium initially  $\langle lnt \rangle$  decreases with a well defined slope and in the large length limit  $\langle lnt \rangle$  decreases with a different slope (corresponding to localization length  $\xi$ ). Thus we can define a length scale  $\xi_c$  (as indicated in the figure) at which there is a cross-over from the initial slope to the asymptotic slope. In the inset of Fig. 8 we have shown the dependence of  $\xi_c$  on  $\eta$ . Numerical fit shows that  $\xi_c$  scales as  $1/\sqrt{\eta}$ , as we expect  $\xi_c \rightarrow \infty$  with  $\eta \rightarrow 0$ . As one decreases  $\eta$ , the absolute value of initial slope increases and that of the asymptotic one decreases. Simultaneously, the cross-over length  $\xi_c$  increases. In the  $\eta \rightarrow 0$  limit both initial as well as asymptotic slopes become identical.

In Fig. 9 we plot  $\langle lnr \rangle$  as a function of the length  $L$  for a fixed value of amplification strength  $\eta = 0.1$  and for various values of the disorder strength  $W$  as indicated in the figure. In the absence of disorder ( $W = 0$ ) as one varies length, initially the reflectance increases to a very large value through large oscillations and after exhibiting a maximum again through oscillations, it eventually saturates to a finite (large) value. In the presence of disorder one can readily notice that initially  $\langle lnr \rangle$  increases and has a magnitude larger than that for  $W = 0$  case and asymptotically beyond a disorder dependent length scale  $L_1(W)$ , it saturates to a value which is smaller than that for a  $W = 0$  case. The magnitude of the saturation value of  $\langle lnr \rangle$  decreases as one increases the disorder as a result of localization induced by combined effect of disorder and amplification. Below the length scale  $L_1(W)$  we can identify another disorder dependent length scale  $L_2(W)$ . Above  $L_2$  (but smaller than  $L_1$ ) further increase in disorder suppresses the reflectance whereas below  $L_2$  it enhances the reflectance. The length scale  $L_2$  being much smaller than the localization length  $l$  for the passive medium, increase in disorder causes multiple reflections in a sample of size smaller than  $L_2$  and consequently due to the increase in delay time we get enhanced back reflection. Beyond  $L_2$  due to disorder induced localization delay time decreases and as a consequence we obtain reduced reflectance.

The dependence of  $L_1$  and  $L_2$  on  $W$  for a fixed value of  $\eta = 0.1$  is shown in Fig. 10. Both these length scales decrease as we increase  $W$ . The magnitude of  $L_2$  is closer to the

value of  $L_c$  for a given disorder. In calculating  $L_1$  we encounter error bars as the reflectance of a perfect system ( $W = 0$ ) exhibits oscillations. Curve fitting for larger values of disorder indicates that  $L_1$  and  $L_2$  decreases as  $1/W^{0.187}$  and  $1/W$  respectively. From the existence of  $L_1$  and  $L_2$  one can readily infer that, if we have a sample of fixed length  $L$  and amplification  $\eta$ , then as we increase disorder  $W$  first due to multiple reflection (sample size  $L$  being less than  $L_2$ ) reflectance will increase. When the disorder strength becomes large such that  $L_2 < L$  disorder induced localization will reduce the reflectance. This is shown in Fig. 11 where we plot  $\langle \ln r \rangle$  versus  $W$  for a sample of fixed length  $L = 45$  and amplification  $\eta = 0.1$ .

In Fig. 12 we have plotted the stationary distribution  $P_s(r)$  of reflection coefficient  $r$  for different values of  $\eta$  (as shown in the figure) and a fixed value of  $W = 5.0$ . To obtain stationary distribution we have considered sample sizes much larger than the localization length  $\xi$  such that any increment in the length does not change the distribution. For small values of  $\eta = 0.05$  the stationary distribution  $P_s(r)$  has a single peak around  $r = r_{max} = 1$ . The peak ( $r_{max}$ ) shifts to higher side as we increase  $\eta$  (Fig. 12(b)). The behavior of  $P_s(r)$  for small  $\eta$  is in qualitative agreement with Eqn. (1). As we increase  $\eta$  further  $P_s(r)$  exhibit a double peaked structure (Fig. 12(c)). At first the second peak appears at higher value of  $r$  at the expense of the distribution at the tail. As we increase  $\eta$  the second peak becomes more prominent and shifts towards left, where as the height of the first peak decreases. The distribution at the tail has a negligible weight (see Fig. 12(c)). At still higher values of  $\eta$ , the second peak approaches  $r \approx 1$  whereas the first peak disappears. The now-obtained single peak distribution  $P_s(r)$  in the large  $\eta$  limit tends to  $\delta(r - 1)$ . In this limit the amplifying medium acts as a reflector and the disorder plays a sub-dominant role. The occurrence of the double peak structure along with  $P_s(r) \rightarrow \delta(r - 1)$  in the large  $\eta$  limit cannot be explained even qualitatively from Eqn. (1). This is due to the failure of RPA in this regime.

In Fig. 13 we have plotted  $\langle \ln r \rangle_s$ , obtained from  $P_s(r)$ , as a function of  $\eta$  for  $W = 1.0$  and  $W = 5.0$  as indicated in the figure. As we increase the strength of  $\eta$ ,  $\langle \ln r \rangle_s$  first increases and after exhibiting a maximum at  $\eta = \eta_{max}$ ,  $\langle \ln r \rangle_s$  decreases monotonically. The numerical value of  $\eta_{max}$  depends on the material parameters. For the value  $\eta > \eta_{max}$ , the amplifying

medium acts dominantly as a reflector. It should be noted that the double peak structure in  $P_s(r)$  appears for values of  $\eta$  close to  $\eta_{max}$ . For a given amount of disorder and for  $\eta < \eta_{max}$  the increase in reflectance (beyond unity) as function of  $\eta$  is due to the presence of disorder along with amplification. The randomness leads to multiple reflections and as a consequence particles spend large amount of time in the sample before getting coherently reflected. This enhances the total reflection and the peak value of  $P_s(r)$  shifts to higher values of  $r$ . Beyond  $\eta_{max}$ , the amplifying medium plays a dominant role as a reflector, leading to decrease in  $\langle \ln r \rangle_s$ . Physics of the double peak and overall shape of  $P_s(r)$  shows similarity with the stationary distribution obtained in the case of absorption (for details we refer to Ref. [8]).

Fig. 14 shows the stationary distribution  $P_s(\theta)$  of the phase  $\theta$  of the complex reflection amplitude for different values of  $\eta$ . For the sake of convenience we have used the same parameters as in Fig. 12. It is clear from this figure that as we increase  $\eta$  phase fluctuations are suppressed. Double peak distribution is obtained even for small  $\eta$  (Fig. 14(a)). With increasing  $\eta$  the peaks become prominent and they move apart. In the large  $\eta$  limit  $P_s(\theta)$  tends to  $\delta(\theta)$  and  $\delta(\theta + 2\pi)$ . It should be noted that only in the limit  $W < 1$  and  $\eta < 1$  we obtain a uniform phase distribution over the range 0 to  $2\pi$ . It is this suppression of phase fluctuations in the presence of amplification which leads to the breakdown of RPA. Hence the results based on RPA cannot explain the observed distribution of  $P_s(r)$  at large  $\eta$  (Fig. 12(c,d)), even qualitatively.

#### IV. CONCLUSIONS

Our numerical study on the statistics of transmission coefficient in random lasing medium indicates that in the asymptotic regime the transmission coefficient is a non-self-averaging quantity, however, with a well defined finite average value. We have shown that disorder suppresses the transmittance at all length scales for a given fixed  $\eta$ . In some parameter space transmittance initially increases with  $\eta$  and falls off exponentially to zero in the asymptotic regime. In this regime there is a well defined cross-over length  $L_c$  at which the transmittance

is maximum, and it decreases monotonically with  $\eta$  as well as  $W$ . In the parameter range where  $\eta \ll 1$ , in the presence of disorder the average transmittance decreases monotonically and has a magnitude less than unity. In this regime  $L_c$  does not exist. However, one can still define a new length scale  $\xi_c$  which scales as  $1/\sqrt{\eta}$ . We have also shown that there are two more length scales  $L_1(W)$  and  $L_2(W)$  associated with reflectance. For a system size upto  $L_2(W)$  disorder increases reflectance, for  $L_2(W) < L < L_1(W)$  disorder suppresses the reflectance. However, in this regime the reflectance is larger than that for an ordered lasing medium. For  $L > L_1(W)$  disorder suppresses the saturated value of  $\langle \ln r \rangle$  to a value much less than that for the case of the ordered lasing medium. Our results clearly indicate that to obtain an enhanced back reflection for a sample of fixed length  $L$ , the synergy between the disorder and the amplification is necessary. The nature of the stationary distribution of reflection coefficient  $P_s(r)$  indicates that earlier analytical studies fail, even qualitatively, to explain the observed behavior in the large  $\eta$  limit. In this limit, one can show that amplification suppresses the phase fluctuations of complex reflection amplitude and thus RPA is no longer valid. Our study clearly brings out the dual role played by an amplifying medium, as an amplifier as well as a reflector.



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FIGURES

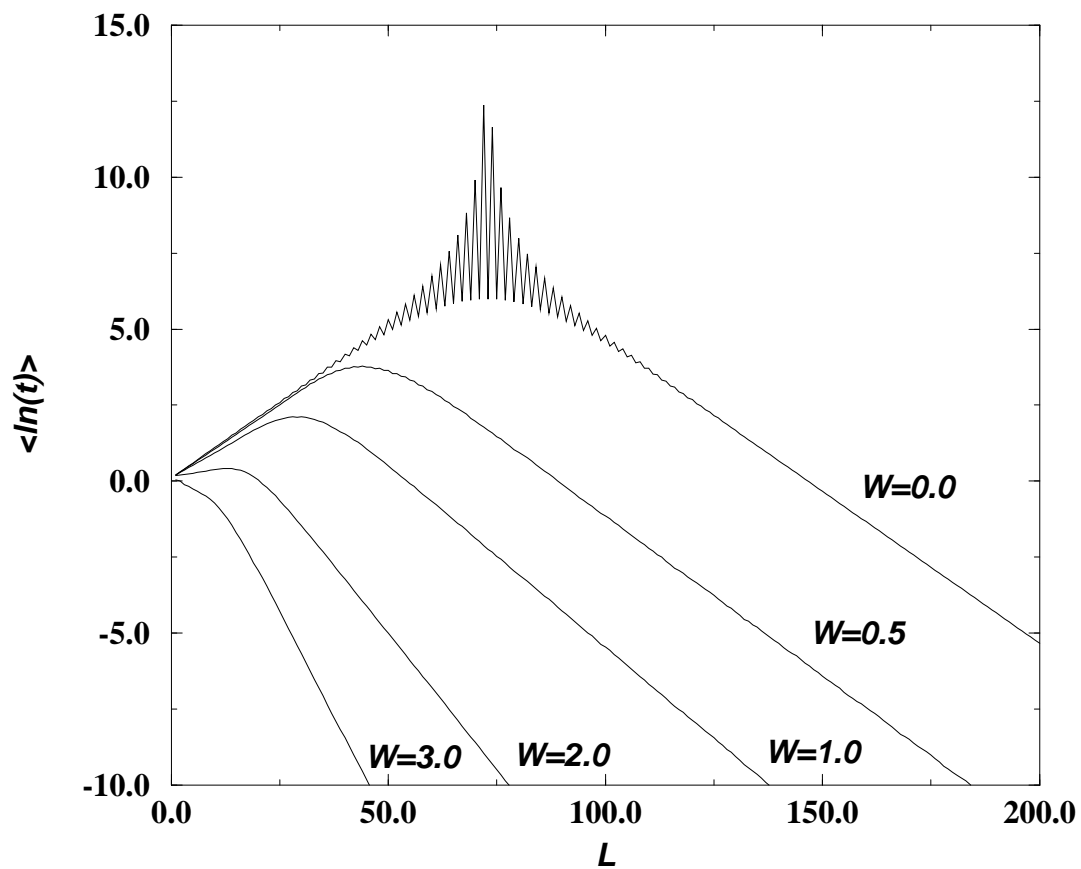


FIG. 1. Average of logarithm of transmission coefficient  $t$  versus length  $L$  for  $\eta = 0.1$  and different values of  $W$ .

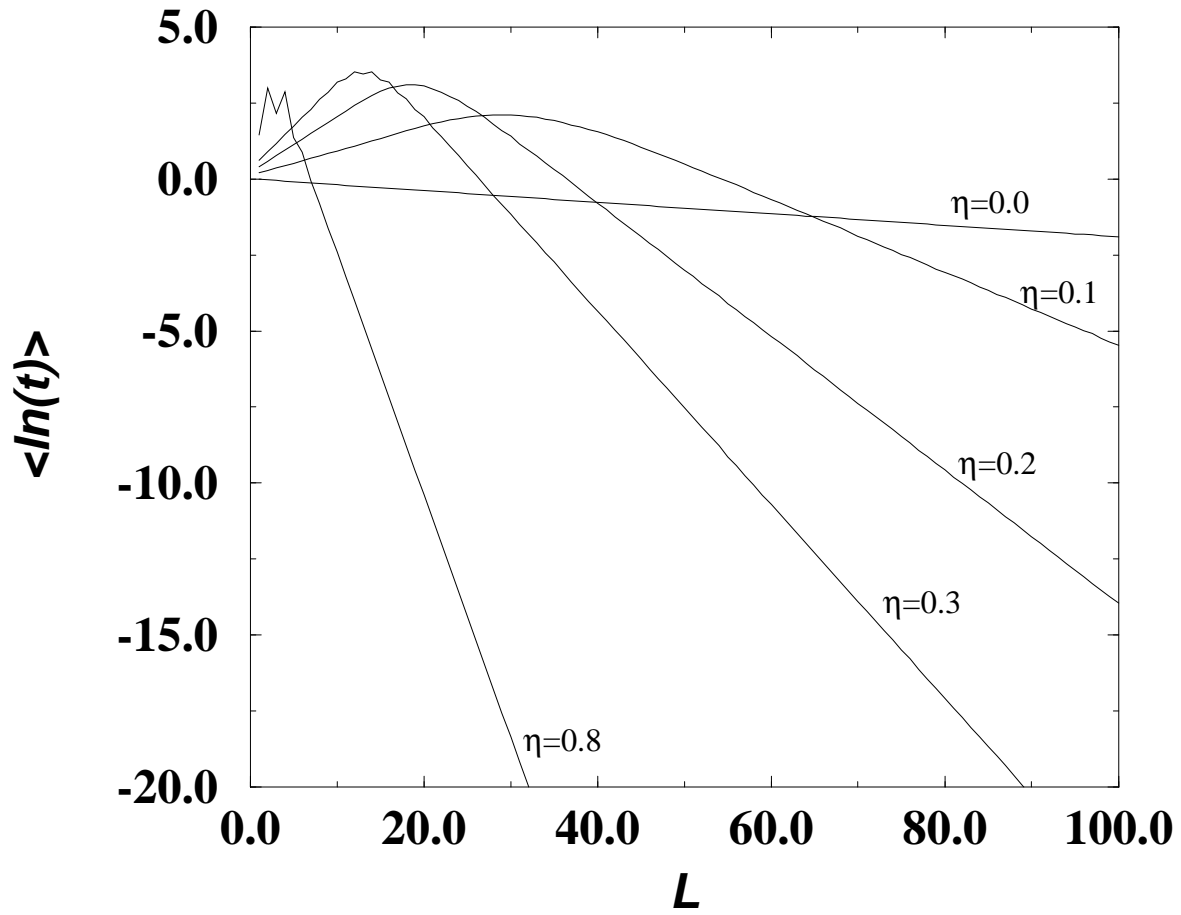


FIG. 2. Average of logarithm of transmission coefficient  $t$  versus length  $L$  for  $W = 1.0$  and different values of  $\eta$ .

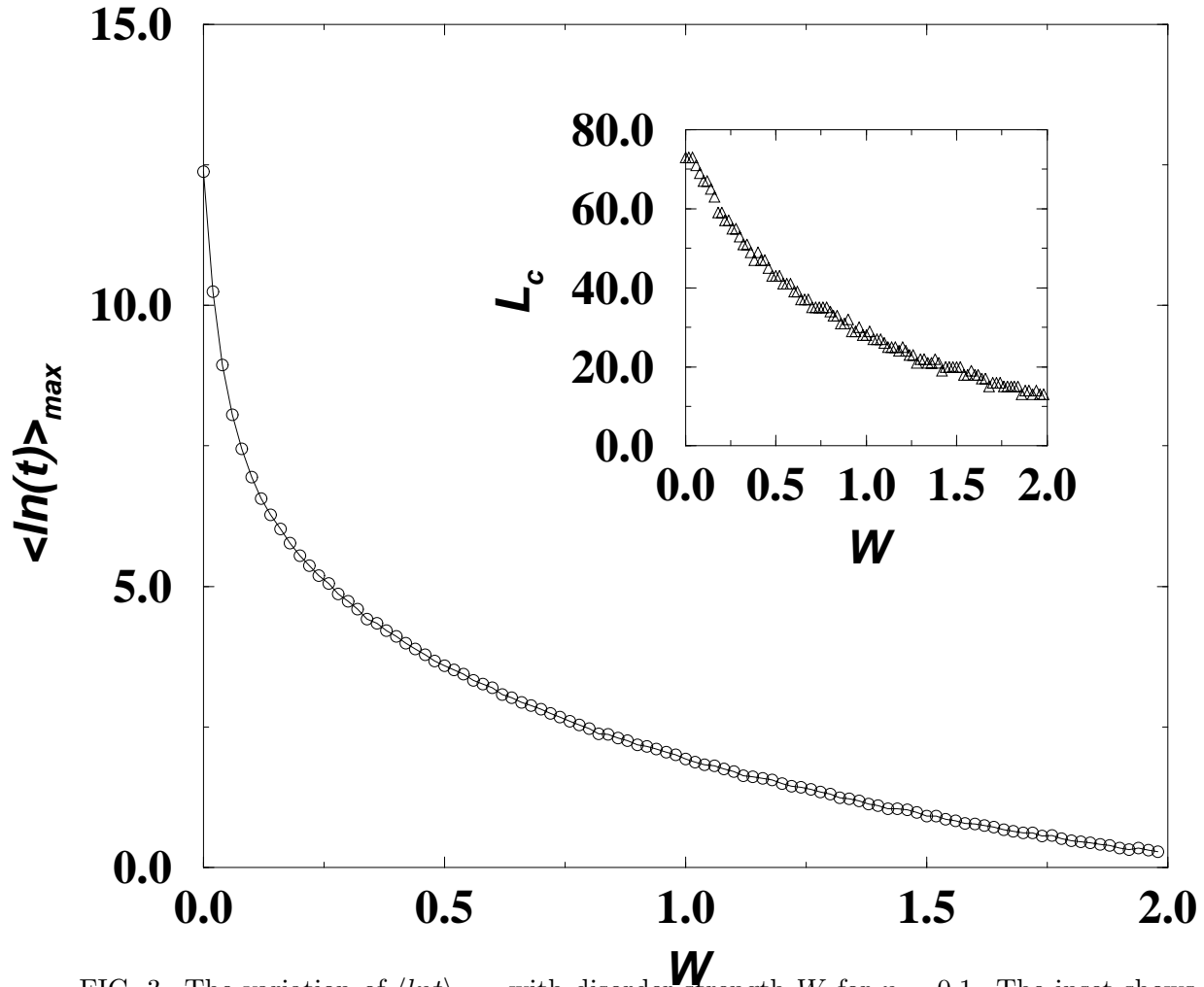


FIG. 3. The variation of  $\langle \ln t \rangle_{max}$  with disorder strength  $W$  for  $\eta = 0.1$ . The inset shows the variation of  $L_c$  with  $W$  for  $\eta = 0.1$ .

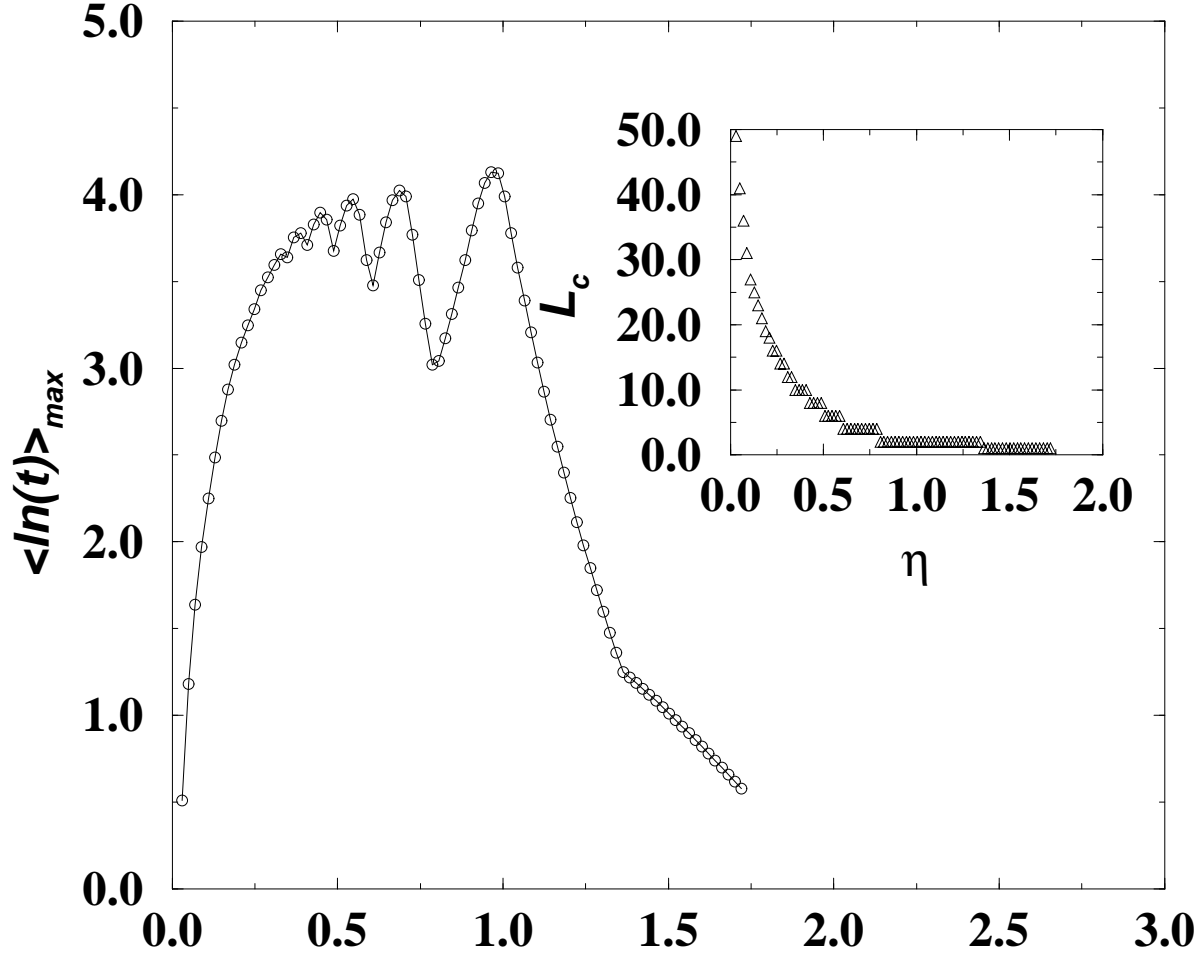


FIG. 4. The variation of  $\langle \ln t \rangle_{max}$  with amplification strength  $\eta$  for  $W = 1.0$ . Inset shows the variation of  $L_c$  with  $\eta$  for  $W = 1.0$ .

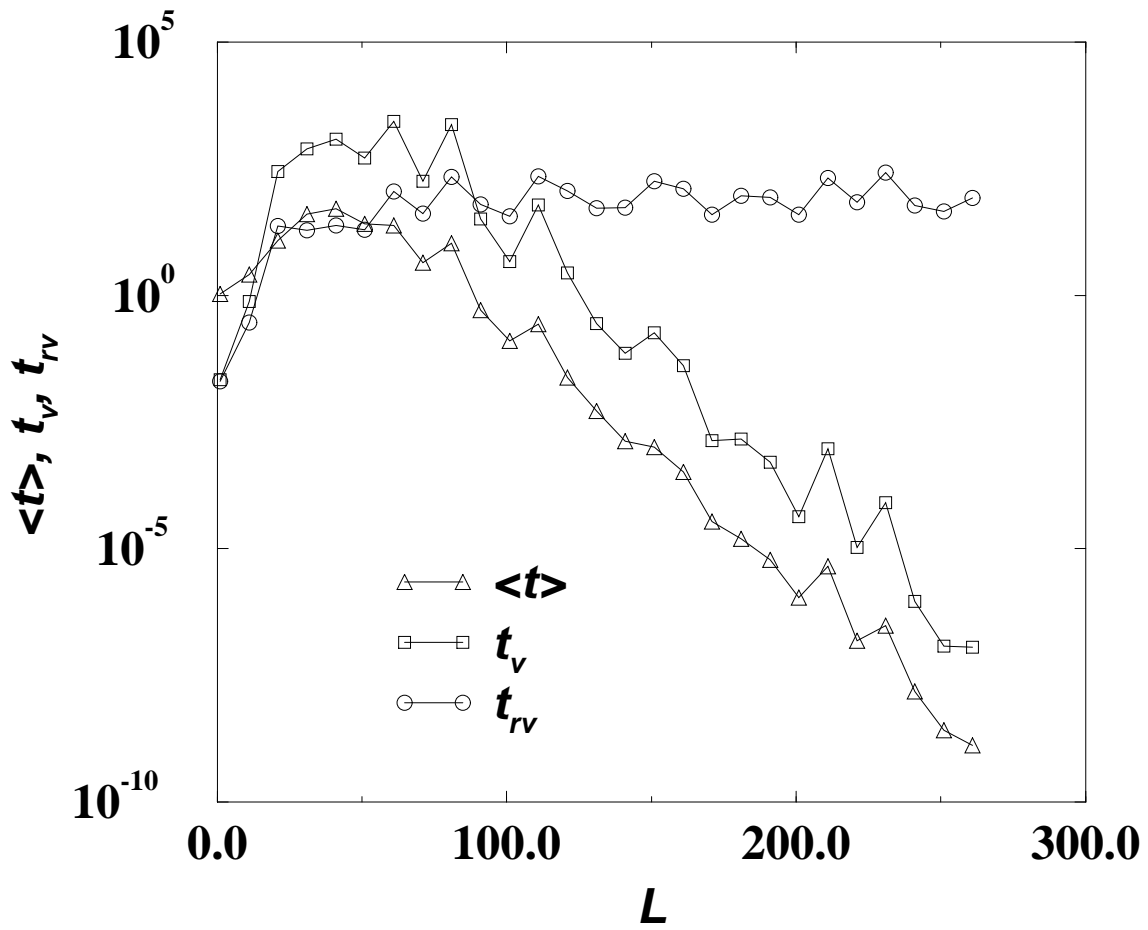


FIG. 5. The plot of  $\langle t \rangle$ , root-mean-squared variance ( $t_v$ ) and root-mean-squared relative variance ( $t_{rv}$ ) as a function of length  $L$  for  $\eta = 0.1$  and  $W = 1.0$ .



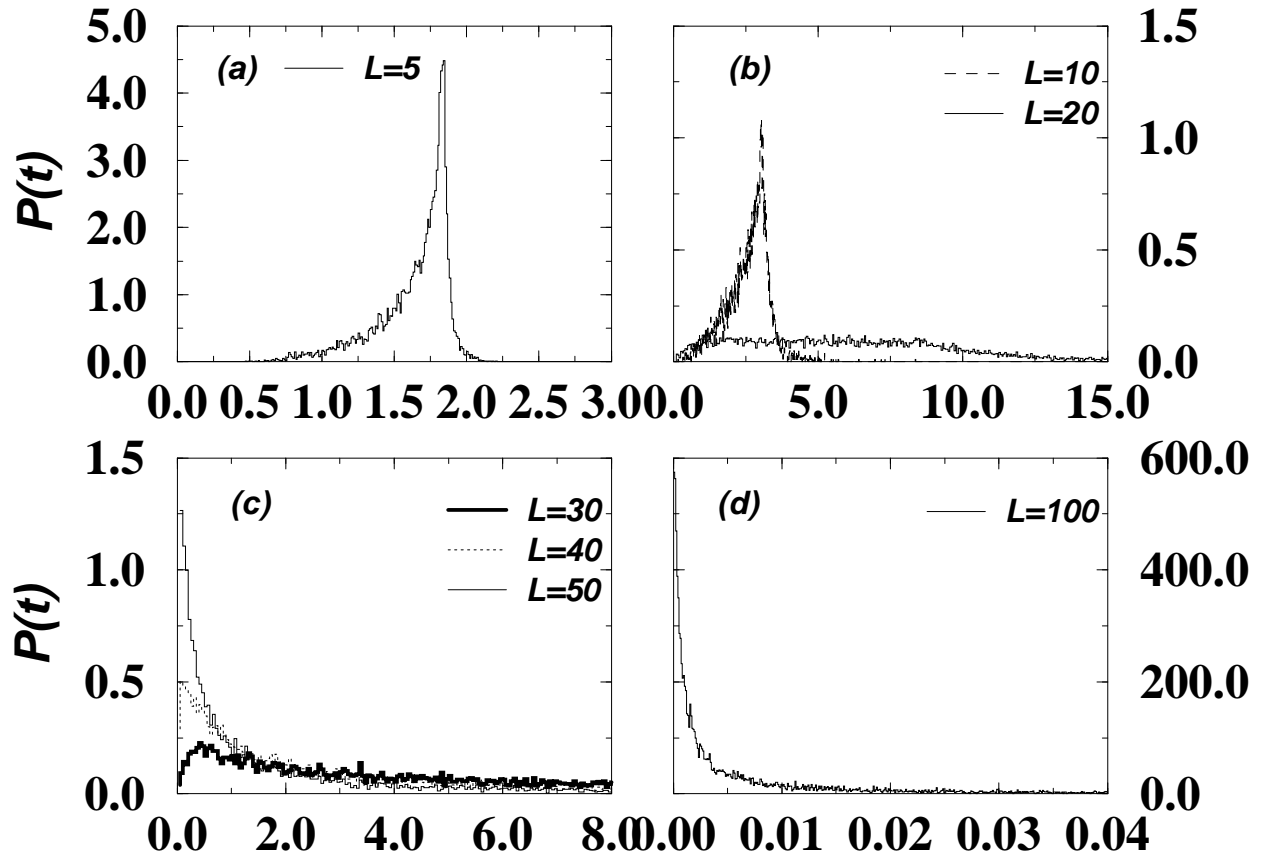


FIG. 6. Distribution of transmission coefficient  $t$  for  $W = 1.0$  and  $t_\eta = 0.1$  at different sample lengths, as indicated in the figure.

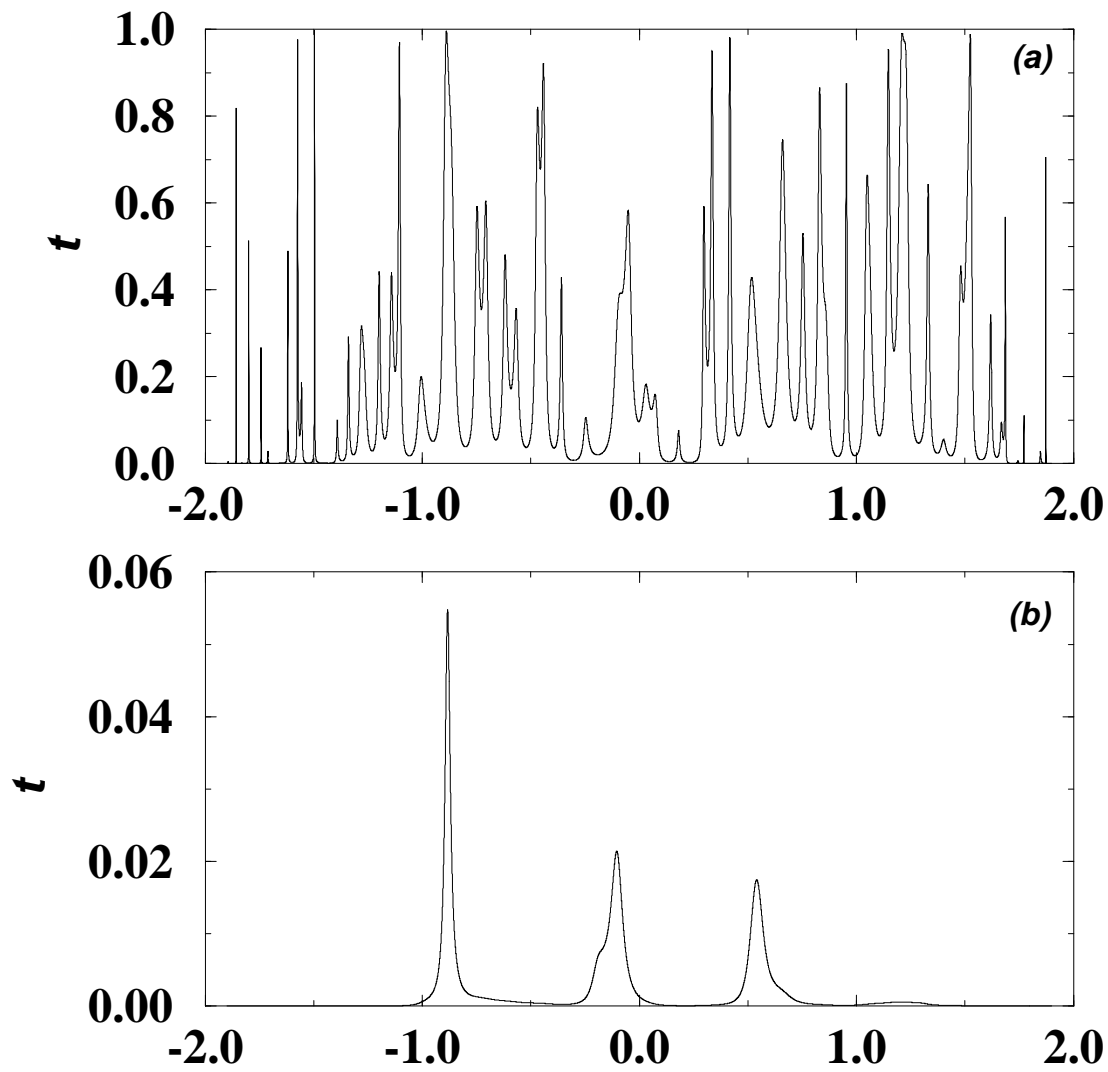


FIG. 7. Transmittance  $t$  as function of incident energy  $E$  for  $W = 1.0$ ,  $L = 100$  and (a)  $\eta = 0$  and (b)  $\eta = 0.1$

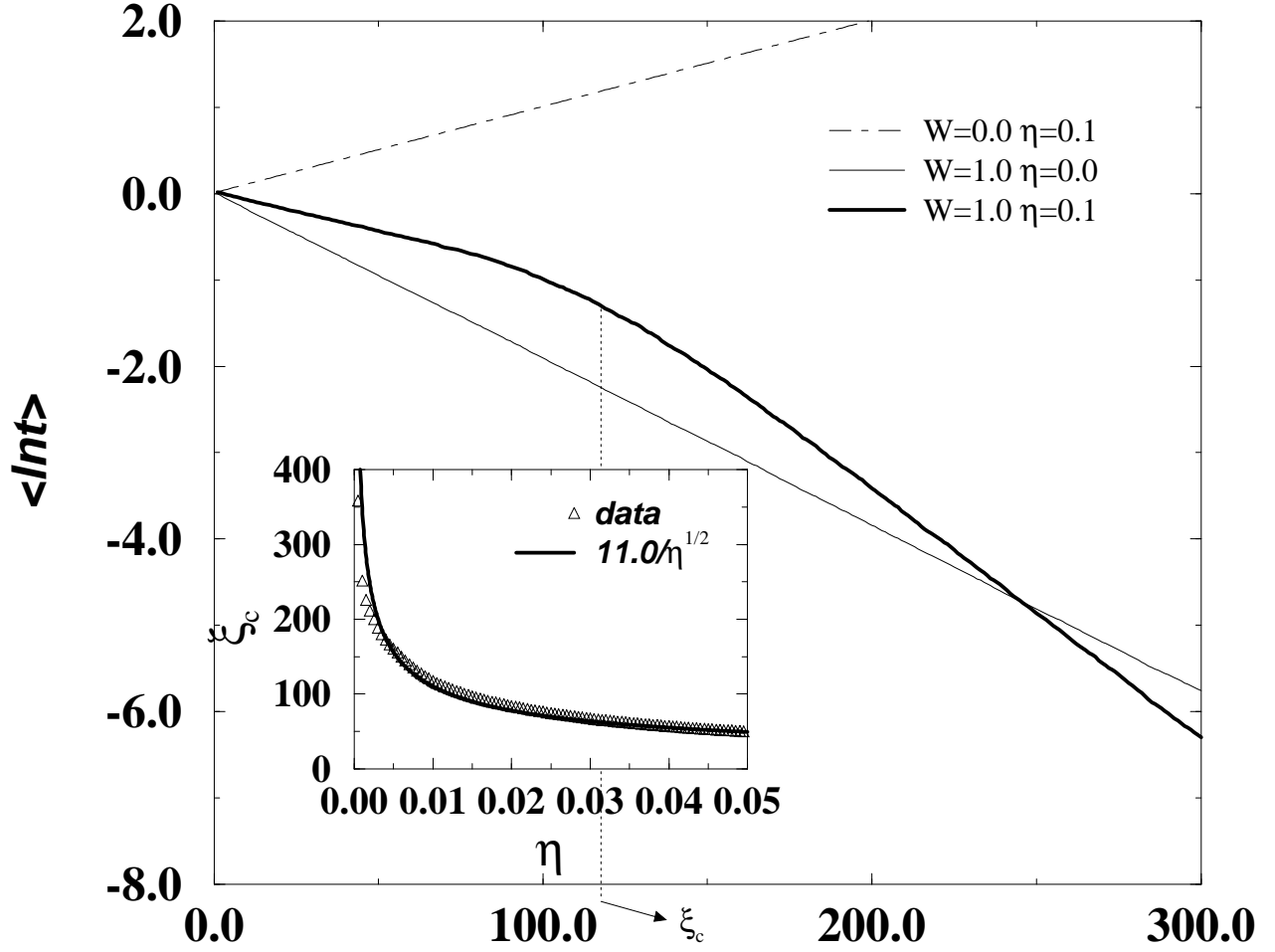


FIG. 8. Variation of  $\langle \ln t \rangle$  with  $L$ . The new length scale  $\xi_c$  which arises for  $\eta \ll 1.0$  is shown by a vertical dotted line. The inset shows the variation of  $\xi_c$  with  $\eta$  for  $W = 1.0$ . The numerical fit shown by the thick line indicates that  $\xi_c$  scales as  $\eta^{-1/2}$  in this regime.

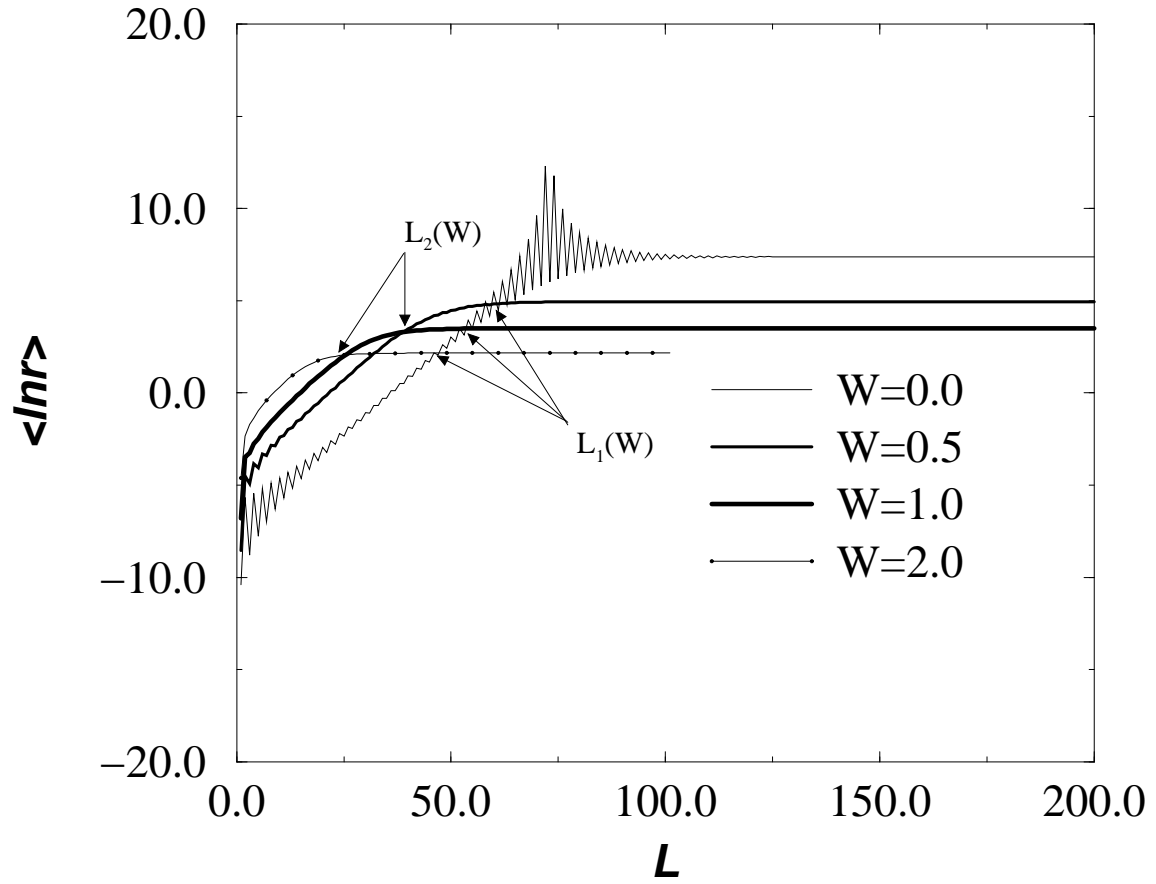


FIG. 9. Variation of  $\langle \ln r \rangle$  with  $L$  for values of  $W$  indicated in the figure. The two length scales  $L_1(W)$  and  $L_2(W)$  associated with the reflectance are shown with arrows.

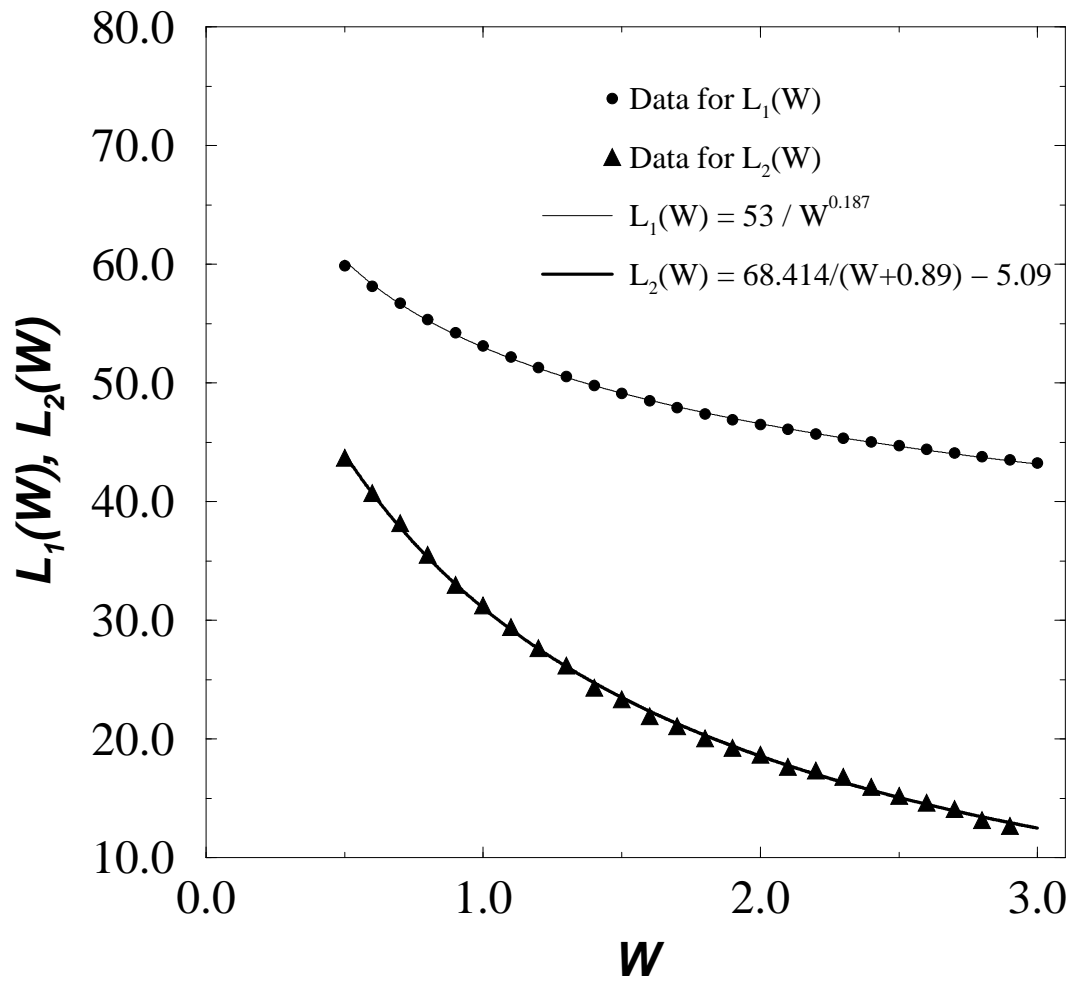


FIG. 10. The variation of  $L_1$  and  $L_2$  with disorder strength  $W$  for a sample with fixed amplification strength  $\eta = 0.1$ .

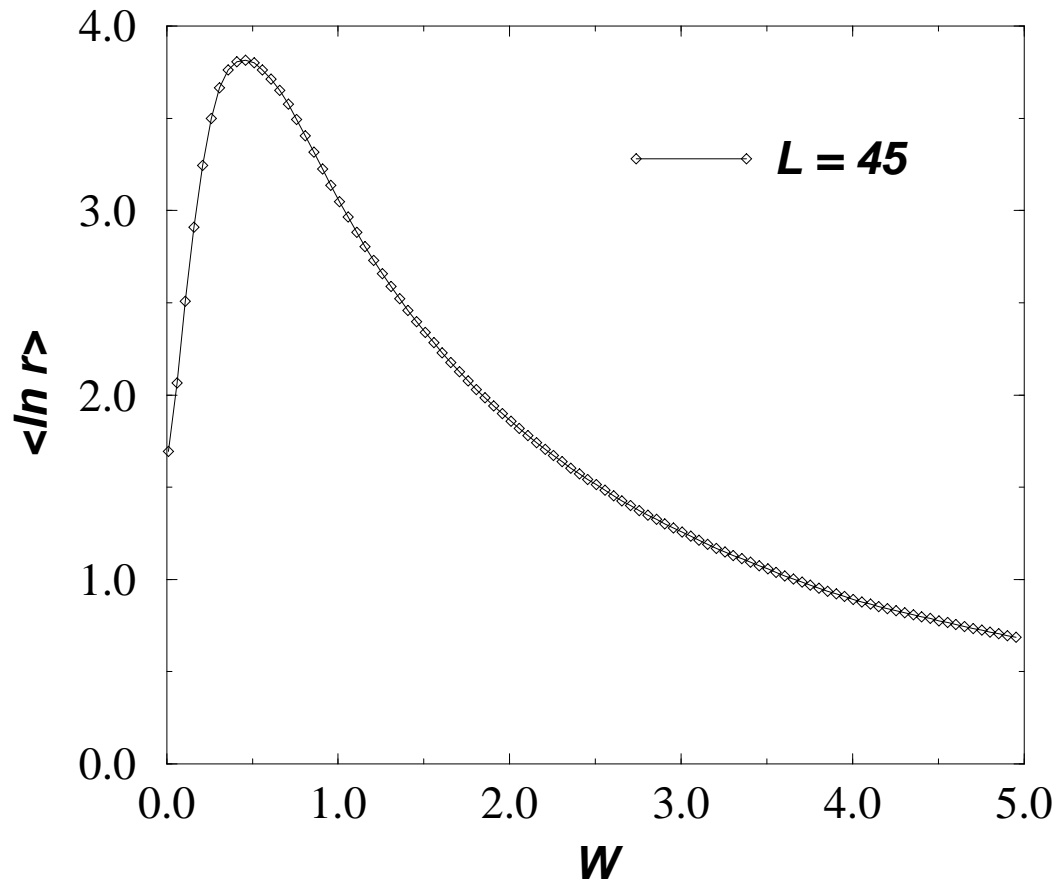


FIG. 11. The variation of  $\langle \ln r \rangle$  with disorder  $W$  for a sample of length  $L = 45$  and  $\eta = 0.1$ .

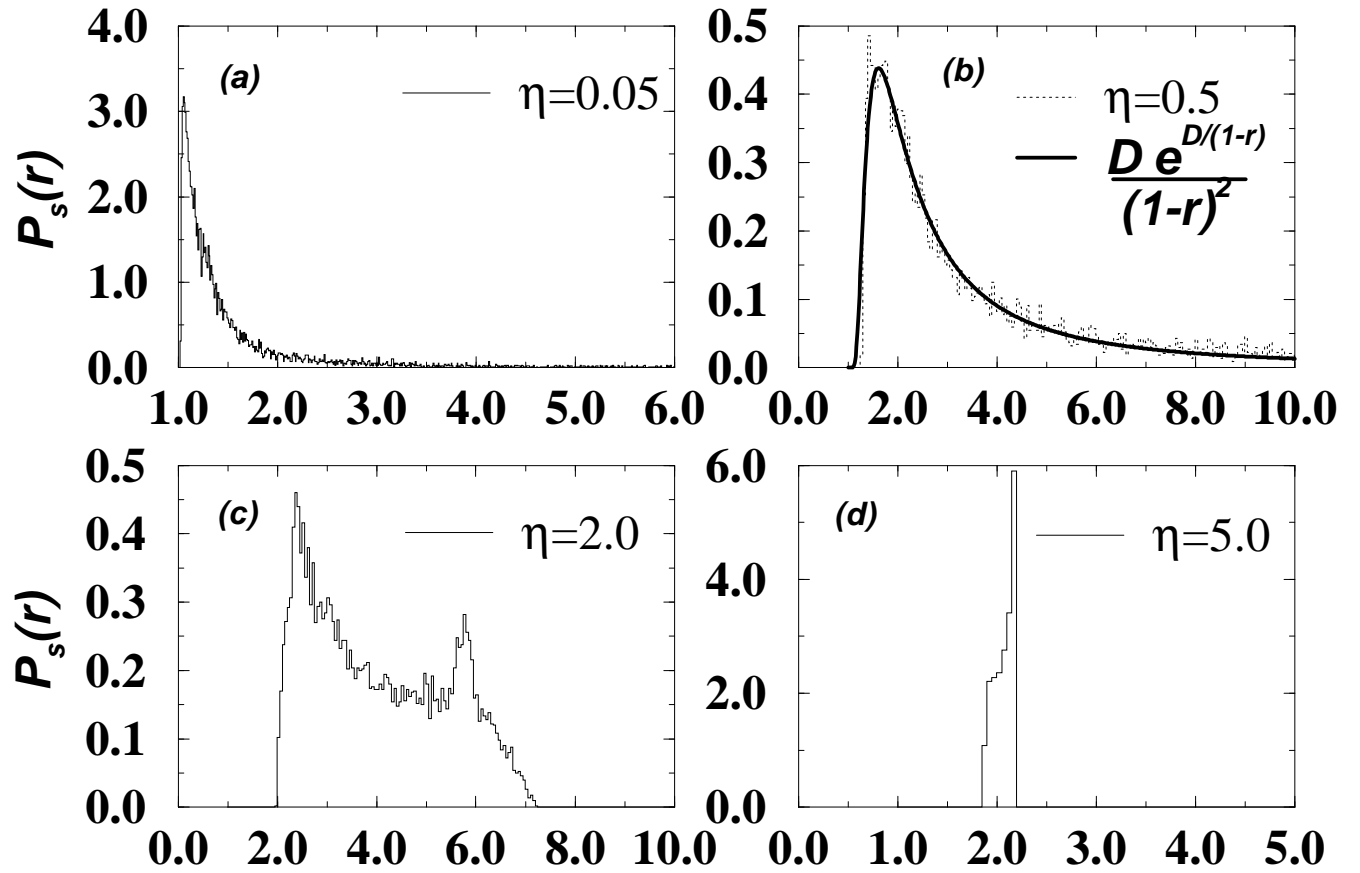


FIG. 12. Stationary distribution of reflection coefficient  $P_s(r)$  for  $W = 5.0$  and various values of  $\eta$ . The numerical fit shown in Fig. 12(b) with a thick line has  $D = 1.235$

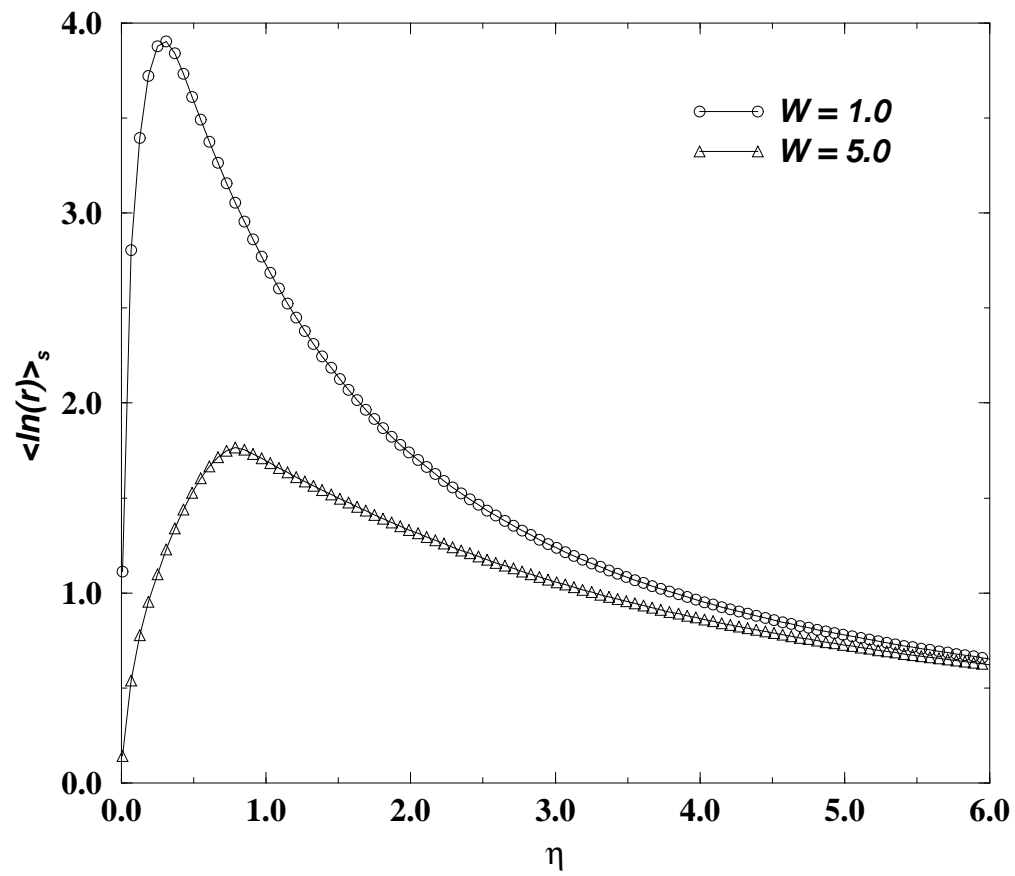


FIG. 13. Variation of  $\langle \ln(r) \rangle_s$  with  $\eta$  for two values  $W$  as indicated in the figure.



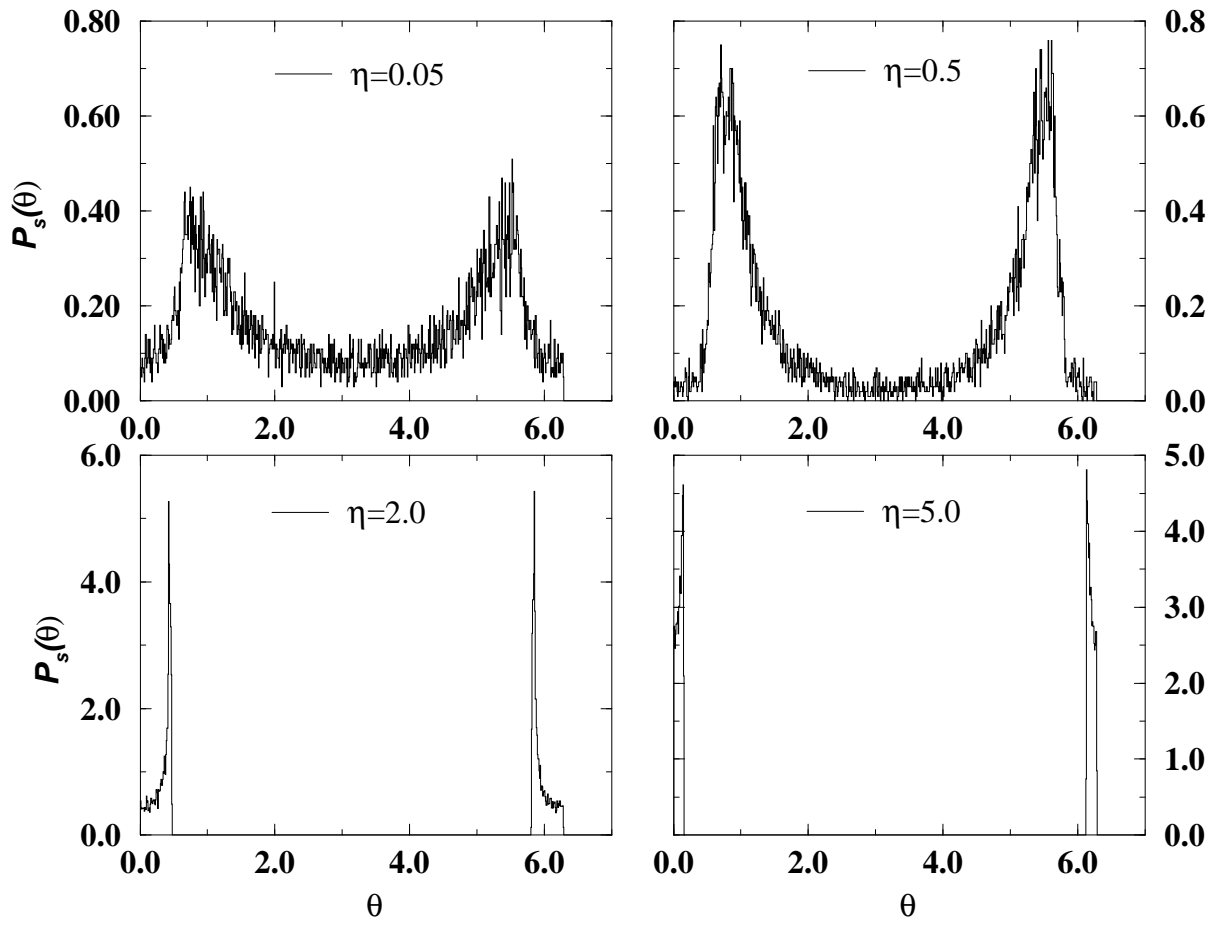


FIG. 14. Stationary distribution of the phase of reflection amplitude  $P_s(\theta)$  versus  $\theta$ .