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nptotic analysis of some nonlinear problems using f-Cole transform and spectral theory

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Abstract. We consider initial boundary value problems for certain nonlinear scalar parabolic equations. A formula for the unique classical solution by Hopf-Cole transformations is obtained and the asymptotic behaviour of the solution as time goes to ∞ is studied.

Keywords. Nonlinear parabolic equation; initial boundary value problem; eigenvalue problem; asymptotic behaviour.

troduction

:onsider the following initial boundary value problem

$$u_t + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}u_{xx} + q(x) + \mu, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

$$u_x(0,t) = a, \tag{1.3}$$

$$u_x(1,t) = b, \tag{1.4}$$

strip $D = \{(x,t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$. μ is a real parameter and a and b are real ;tants. We assume $q(x)$ is continuous in $0 \leqslant x \leqslant 1$, $u_0(x)$ is twice continuously :rentiable in $0 \leqslant x \leqslant 1$, and $(u_0)_x(0) = a$, $(u_0)_x(1) = b$.

ʼenote by $D^0 = \{(x,t): 0 \leqslant x \leqslant 1, t > 0\}$. By a classical solution of (1.1)–(1.4) we .n a function $u(x,t)$ with $u(x,t)$ and $u_x(x,t)$ continuous in D and u_t and u_{xx} continuous)^0 which satisfies the partial differential equation (1.1) in D^0 and the initial and ndary conditions (1.2)–(1.4) in the usual sense.

Jsing the Hopf-Cole transformation, (see Hopf [2]) we linearize the problem)–(1.4) and obtain an expression for the solution in terms of the eigenvalues and :nfunctions of the eigenvalue problem

$$\tfrac{1}{2}\phi_{xx} = (q(x) + \lambda)\phi, \tag{1.5}$$

$$\phi_x(0) + a\phi(0) = 0, \tag{1.6}$$

$$\phi_x(1) + b\phi(1) = 0. \tag{1.7}$$

: prove the validity of the expression for $u(x,t)$ and study its asymptotic behaviour

as $t \to \infty$ using the following facts concerning the eigenvalues and eigenfunctions of (1.5)–(1.7), (see Birkhoff and Rota [1]).
(a) The spectrum of (1.5)–(1.7) is discrete and can be ordered

$$\lambda_0 > \lambda_1 > \cdots, \tag{1.8}$$

$$\lambda_n = -\tfrac{1}{2}(n^2\pi^2) + O(1) \quad \text{as } n \to \infty. \tag{1.9}$$

(b) Let $\phi_n(x)$ be the normalized eigenfunctions corresponding to λ_n, the set $\{\phi_n(x), n = 0, 1, 2, \ldots\}$ is a complete set for $L^2[0, 1]$.
Also $\phi_n(x)$ has the following estimates uniformly in $x \in [0, 1]$

$$\phi_n(x) = \sqrt{2}\cos\frac{n\pi}{\sqrt{2}}x + \frac{O(1)}{n} \quad \text{as } n \to \infty, \tag{1.10}$$

$$\phi_x'(x) = -n\pi\sin\frac{n\pi}{\sqrt{2}}x + O(1) \quad \text{as } n \to \infty. \tag{1.11}$$

(c) $\phi_0(x) \neq 0\ \forall x \in (0, 1)$.
Since $\phi_0(x) \neq 0\ \forall x \in (0, 1)$ we can assume $\phi_0(x) > 0$ for $x \in (0, 1)$. We claim that

$$\phi_0(x) > 0 \quad \forall x \in [0, 1]. \tag{1.12}$$

If $\phi_0(0) = 0$, then by the boundary condition (1.6) $(\phi_0)_x(0) = 0$, then $\phi_0(x) \equiv 0$, because $\phi_0(x)$ solves

$$\tfrac{1}{2}\phi_{xx} = (q(x) + \lambda)\phi,$$

$$\phi(0) = 0,$$

$$\phi_x(0) = 0,$$

and this has only one solution $\phi(x) \equiv 0$. By the same argument $\phi_0(1)$ is also not equal to 0. The claim is proved. In our discussion we always take $\phi_0(x)$ normalized so that

$$\int_0^1 \phi_0^2(x)\,\mathrm{d}x = 1$$

and

$$\phi_0(x) > 0 \quad \forall x \in [0, 1].$$

This paper is organized as follows. In §2 we prove the uniqueness of classical solution of (1.1)–(1.4) and obtain a valid expression for it. The asymptotic behaviour of the solution as $t \to \infty$ is analysed in §3. In §4 are given some comments on the stationary problem and the Dirichlet problem for the Burger's equation is studied in §5.

2. An expression for classical solution of (1.1)–(1.4)

First we prove the uniqueness of classical solution of (1.1)–(1.4) by standard energy estimates. Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions and let $Z(x, t) = u_1(x, t) - u_2(x, t)$.

Then from (1.1)–(1.4) we get, $Z(x,t)$ which is solved as

$$Z_t + \tfrac{1}{2}[(u_1)_x + (u_2)_x]Z_x = \tfrac{1}{2}Z_{xx}, \tag{2.1}$$

$$Z_x(0,t) = 0, \tag{2.2}$$

$$Z_x(1,t) = 0, \tag{2.3}$$

$$Z(x,0) = 0. \tag{2.4}$$

Multiplying (2.1) by Z and integrating by parts w.r.t x, we get using (2.2) and (2.3)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Z^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 (u_1+u_2)_x Z_x Z\,\mathrm{d}x = -\frac{1}{2}\int_0^1 Z_x^2(x,t)\,\mathrm{d}x. \tag{2.5}$$

Let

$$C_T = \sup_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant t\leqslant T}} |(u_1+u_2)_x|.$$

From (2.5) we obtain, for $0 \leqslant t \leqslant T$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Z^2(x,t)\,\mathrm{d}x\\
&\leqslant -\int_0^1 Z_x^2(x,t)\,\mathrm{d}x + 2C_T\left(\int_0^1 Z_x^2(x,t)\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 Z^2(x,t)\,\mathrm{d}x\right)^{1/2}\\
&\leqslant -\int_0^1 Z_x^2(x,t)\,\mathrm{d}x + \int_0^1 Z_x^2(x,t)\,\mathrm{d}x + C_T^2\int_0^1 Z^2(x,t)\,\mathrm{d}x\\
&= C_T^2\int_0^1 Z^2(x,t)\,\mathrm{d}x.
\end{aligned}$$

By Grownwall's lemma and (2.4) we get

$$\int_0^1 Z^2(x,t) \leqslant 0 \quad \forall < 0 \leqslant t \leqslant T.$$

Thus $Z^2(x,t) \equiv 0$ for $0 \leqslant t \leqslant T$; since T is arbitrary we get $Z(x,t) \equiv 0$ i.e. $u_1(x,t) \equiv u_2(x,t) \forall (x,t) \in D$.

Next we construct the unique classical solution of (1.1)–(1.4). First we need the following:

Lemma 2.1. *Let $v(x,t)$ be the solution of*

$$v_t = \tfrac{1}{2}v_{xx} - (q(x)+\mu)v, \tag{2.6}$$

$$v_x(0,t) + av(0,t) = 0, \tag{2.7}$$

$$v_x(1,t) + bv(1,t) = 0, \tag{2.8}$$

$$v(x,0) = \exp[-u_0(x)]. \tag{2.9}$$

Then

(i) $v(x,t)>0$.

(ii) *The function*

$u(x,t) = -\log v(x,t)$

is a solution of (1.1)–(1.4)

Proof. To prove (i), set

$$v(x,t) = \exp(-Cx + Mt)W(x,t), \tag{2.10}$$

where

$$C = \min(a,b)$$

$$M = 1 + \sup_{0\leqslant x\leqslant 1}\left|\frac{C^2}{2} - q(x) - \mu\right|.$$

To prove $v(x,t)>0$, it is enough to prove $W(x,t)>0$. We prove this for the case $C=a$, the other case being a similar one.

Assume the contrary, i.e. $\min W(x,t)\leqslant 0$. Now from (2.6)–(2.10) we have $W(x,t)$ which is solved as

$$\tfrac{1}{2}W_{xx} - CW_x + \left(\frac{C^2}{2} - q(x) - \mu - M\right)W - W_t = 0, \tag{2.11}$$

$$W_x(0,t) + (a-C)W(0,t) = 0, \tag{2.12}$$

$$W_x(1,t) + (b-C)W(1,t) = 0, \tag{2.13}$$

$$W(x,0) = \exp[Cx - u_0(x)]. \tag{2.14}$$

Notice that the coefficient of W, $(C^2/2) - q(x) - \mu - M < 0$ by the definition of M so that the maximum principle can be applied (see Smoller [3]). The maximum principle min of $W(x,t)$ has to occur at the boundaries and if min is achieved at $x=0$, then $W_x(0,t)>0$ and if it occurs at $x=1$, $W_x(1,t)<0$.

By assumption $\min W(x,t)\leqslant 0$. So evidently min is not achieved for $t=0$. Since $a=C$, $W_x(0,t)=0$ (by (2.12)) so again min cannot be achieved at $x=0$. From (2.13)

$$W_x(1,t) = (a-b)W(1,t).$$

But $a-b\leqslant 0$, which implies that $W(x,t)$ cannot achieve a non-positive minimum at $x=1$. So we have $\min W(x,t)>0$.

To prove (ii) notice that from (i)

$$u(x,t) = -\log v(x,t) \tag{2.15}$$

is well defined. A simple calculation gives

$$u_t = \frac{-\frac{1}{2}v_{xx} + (q(x)+\mu)v}{v},$$

where we used (2.6)

$$u_x = -v_x/v$$
$$u_{xx} = (-vv_{xx} + v_x^2)/v^2. \tag{2.16}$$

It follows that $u(x,t)$ given by (2.15) is solved as

$$u_t + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}u_{xx} + q(x) + \mu.$$

From (2.16), (2.7) and (2.8) we get

$$u_x(0,t) = a,$$
$$u_x(1,t) = b.$$

The proof of lemma is complete.

Let λ_n and $\phi_n(x)$ be as in §1. In the next lemma we get an expression for the solution of (1.1)–(1.4).

Lemma 2.2. *Let* $u(x,t)$ *be the classical solution of* (1.1)–(1.4) *then*

$$u(x,t) = -\log\left[\sum_0^\infty \exp\left[(\lambda_n - \mu)t\right]a_n\phi_n(x)\right] \tag{2.17}$$

where

$$a_n = \int_0^1 \exp\left[-u_0(x)\right]\phi_n(x)\,\mathrm{d}x. \tag{2.18}$$

Proof. It is easy to check that $v(x,t)$ is a solution of (2.6)–(2.9) iff

$$v(x,t) = \exp(-\mu t)V^*(x,t), \tag{2.19}$$

where $V^*(x,t)$ is the solution of

$$V_t^* = \tfrac{1}{2}v_{xx}^* - q(x)V^*,$$
$$V_x^*(0,t) + aV^*(0,t) = 0,$$
$$V_x^*(1,t) + bV^*(1,t) = 0,$$
$$V^*(x,0) = \exp\left[-u_0(x)\right] = V_0^*(x).$$

By separation of variable we obtain

$$V^*(x,t) = \sum_0^\infty \exp(\lambda_n t)a_n\phi_n(x) \tag{2.20}$$

where

$$a_n = \int_0^1 \exp\left[-u_0(x)\right]\phi_n(x)\,\mathrm{d}x.$$

From (1.5), (1.9), (1.10) and (1.11) it follows that for $n > N$, N sufficiently large

$$|\phi_n^{(k)}(x)| \leqslant Cn^k, \quad k = 0, 1, 2$$

$$|\exp(\lambda_n t| \leqslant 1 \quad \forall t \geqslant 0 \tag{2.21}$$

$$|\exp(\lambda_n t)| \leqslant C\exp[-\tfrac{1}{2}(n^2\pi^2)\delta] \quad \forall t \geqslant \delta > 0.$$

$C > 0$ is a constant independent of x, t and n and $\phi_n^{(k)}$ denote the kth derivative w.r.t. x of $\phi_n(x)$. To get an estimate for a_n, notice that $V_0^*(x) = \exp[-u_0(x)]$ satisfies

$$\begin{aligned} V_x^*(0) + aV^*(0) &= 0 \\ V_x^*(1) + bV^*(1) &= 0. \end{aligned} \tag{2.22}$$

Multiplying (1.5) by $V_0^*(x)$, integrating w.r.t. x, integrating by parts and using (2.22), we get

$$\frac{1}{2}\int_0^1 \phi_n(x)(V_0^*)''(x)\,\mathrm{d}x = \lambda_n \int_0^1 \phi_n(x)V_0^*(x)\,\mathrm{d}x.$$

for $n \geqslant N$

$$a_n = \int_0^1 \phi_n(x)V_0^*(x)\,\mathrm{d}x = \frac{1}{2\lambda_n}\int_0^1 (V_0^*)''(x)\phi_n(x)\,\mathrm{d}x$$

so that, for $n > N$

$$|a_n| \leqslant \frac{C}{n^2}|b_n|, \tag{2.23}$$

where

$$b_n = \int_0^1 (V_0^*)''(x)\phi_n(x)\,\mathrm{d}x.$$

From (2.21) and (2.23) it follows that $V^*(x,t)$ is continuous in D. Also

$$\begin{aligned} V_x^*(x,t) &= \sum_0^\infty \exp(\lambda_n t)a_n\phi_n'(x) \\ &= \sum_0^{N-1} \exp(\lambda_n t)a_n\phi_n'(x) + \sum_N^\infty \exp(\lambda_n t)a_n\phi_n'(x). \end{aligned} \tag{2.24}$$

Now

$$\left|\sum_N^\infty \exp(\lambda_n t)a_n\phi_n'(x)\right| \leqslant C^2\sum_N^\infty \frac{1}{n^2}|b_n|\cdot n \tag{2.25}$$

and

$$\sum_0^\infty |b_n|^2 = \int_0^1 (V_0^*)''(x)^2\,\mathrm{d}x \tag{2.26}$$

by completeness of eigenfunctions. From (2.24)–(2.26) it follows that

$$|V_x^*(x,t)| \leqslant \sum_0^{N-1} \exp(\lambda_n t)a_n\phi_n'(x) + C^2\left(\sum_N^\infty \frac{1}{n^2}\right)^{1/2}\left(\sum_N^\infty |b_n|^2\right)^{1/2}$$

so that the series is absolutely convergent and hence $V_x^*(x,t)$ is continuous in D.

Using the estimates (2.21), (2.23) and (2.26) and using the same argument as before, $v_{xx}^*(x,t)$ and $V_t^*(x,t)$ are also continuous in $0 \leqslant x \leqslant 1$, $t \geqslant \delta > 0$. Now by (2.19)

$$v(x,t) = \exp(-\mu t)V^*(x,t) = \sum_0^\infty \exp[(\lambda_n - \mu)t]a_n\phi_n(x)$$

and by lemma 2.1

$$u(x,t) = -\log\left(\sum_0^\infty \exp\left[(\lambda_n-\mu)t\right]a_n\phi_n(x)\right)$$

is the classical solution of (1.1)–(1.4). The proof of lemma (2.2) is complete.

Next we study the asymptotic behaviour of the solution $u(x,t)$ constructed in this section.

3. Asymptotic behaviour of the classical solution of (1.1)–(1.4)

In this section $u(x,t)$ denotes the unique classical solution of (1.1)–(1.4) constructed in §2. We shall prove the following.

Theorem. (i) *Let* $\mu=\lambda_0$; *then*

$$\sup_{0\leqslant x\leqslant 1} |u(x,t)+\log(a_0\phi_0(x))| \leqslant C\exp\left[(\lambda_1-\lambda_0)t\right].$$

(ii) *Let* $\mu\neq\lambda_0$; *then for* $t\geqslant 1$

$$\sup_{0\leqslant x\leqslant 1}\left|\frac{u(x,t)}{t}+(\lambda_0-\mu)+\frac{\log(a_0\phi_0(x))}{t}\right| \leqslant \frac{C}{t}\exp\left[(\lambda_1-\lambda_0)t\right].$$

where λ_0 *and* $\phi_0(x)$ *are as in* §1, a_0 *is given by* (2.18) *and* C *a positive constant independent of* x *and* t.

Proof. From lemma 2.2 we have

$$u(x,t) = -\log\left\{\sum_0^\infty \exp\left[(\lambda_n-\mu)t\right]a_n\phi_n(x)\right\}.$$

To prove (i), notice that when $\mu=\lambda_0$, the above expression for $u(x,t)$ becomes

$$u(x,t) = -\log\left[a_0\phi_0(x)+\sum_1^\infty \exp\left[(\lambda_n-\lambda_0)t\right]a_n\phi_n(x)\right]$$

$$= -\log\left[a_0\phi_0(x)\right]-\log\left[1+\sum_1^\infty \exp\left[(\lambda_n-\lambda_0)t\right]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n(x)}{\phi_0(x)}\right)\right]$$

$a_0>0$ and $\inf_{0\leqslant x\leqslant 1}\phi_0(x)>0$ by (1.12). Using the estimate (2.21) we have, for $t\geqslant 1$

$$\left|\sum_1^\infty \exp\left[(\lambda_n-\lambda_0)t\right]\left(\frac{a_n}{a_0}\right)\frac{\phi_n(x)}{\phi_0(x)}\right| \leqslant C\exp\left[(\lambda_1-\lambda_0)t\right]$$

so that we get

$$|u(x,t)+\log\left[a_0\phi_0(x)\right]| \leqslant \log\left\{1+C\exp\left[(\lambda_1-\lambda_0)t\right]\right\}.$$

But for $0<y<1$

$$\log(1+y)\leqslant y$$

we get

$$|u(x,t)+\log(a_0\phi_0(x)| \leqslant C\exp[(\lambda_1-\lambda_0)t].$$

To prove (ii) arguing as before, we get

$$u(x,t) = -\log\{\exp[(\lambda_0-\mu)t]a_0\phi_0(x)\} - \log\left[1+\sum_1^\infty \exp[(\lambda_n-\lambda_0)t]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n(x)}{\phi_0(x)}\right)\right] \tag{3.4}$$

As before, for $t \geqslant 1$,

$$\left|\log\left[1+\sum_1^\infty \exp[(\lambda_n-\lambda_0)t]\left(\frac{a_n}{a_0}\right)\frac{\phi_n(x)}{\phi_0(x)}\right]\right| \leqslant C\exp[(\lambda_1-\lambda_0)t]. \tag{3.5}$$

Also

$$\log\{\exp[(\lambda_0-\mu)t]a_0\phi_0(x)\} = (\lambda_0-\mu)t+\log(a_0\phi_0(x)). \tag{3.6}$$

Using (3.5) and (3.6) in (3.4) we get for $t \geqslant 1$

$$\sup_{0\leqslant x\leqslant 1}\left|\frac{u(x,t)}{t}+(\lambda_0-\mu)+\frac{\log(a_0\phi_0(x))}{t}\right| \leqslant \frac{C}{t}\exp[(\lambda_1-\lambda_0)t].$$

The proof of the theorem is complete.

4. Some remarks on the stationary problem

Consider the stationary problem

$$p_x^2/2 = \tfrac{1}{2}p_{xx}+q(x)+\lambda_0, \tag{4.1}$$

$$p_x(0) = a, \tag{4.2}$$

$$p_x(1) = b, \tag{4.3}$$

where λ_0 is as in §1. Consider the one-parameter family of functions

$$p_\alpha(x) = -\log(\alpha\phi_0(x)),\ \alpha>0 \tag{4.4}$$

$\phi_0(x)$ is as in §1 with condition (1.12). By a direct calculation it is easy to verify that $p_\alpha(x)$ is a solution of (4.1)–(4.3) for each $\alpha>0$. The theorem in §3, part (i) says that, in the case $\mu=\lambda_0$, the solution $u(x,t)$ of (4.1)–(4.4) converges to $p_{a_0}(x)$.

The second part of the theorem says that if $\mu\neq\lambda_0$, $u(x,t)\to\infty$ as $t\to\infty$. The stationary problem

$$p_x^2/2 = \tfrac{1}{2}p_{xx}+q(x)+\mu, \tag{4.5}$$

$$p_x(0) = a, \tag{4.6}$$

$$p_x(1) = b \tag{4.7}$$

does not have any solution if $\mu\neq\lambda_0$. In fact as in lemma 2.1 one can easily check that $p(x)$ is a solution to (4.5)–(4.7) iff the function

$$h = \exp(-p) \tag{4.8}$$

is a solution to

$$\tfrac{1}{2}h_{xx} = [q(x) + \mu]h, \tag{4.9}$$

$$h_x(0) + ah(0) = 0, \tag{4.10}$$

$$h_x(1) + bh(1) = 0. \tag{4.11}$$

But (4.9)–(4.11) has a non-zero solution iff $\mu = \lambda_n$, $n = 0, 1, 2, \ldots$, and λ_n is as in §1. Further the corresponding solution has to be positive, by (4.8). This happen iff $\mu = \lambda_0$.

5. Burger's equation

The method presented in previous sections can be used to get a closed form expression and asymptotic behaviour of the solution of the Burger's equation in $D = \{|x, t): a \leqslant x \leqslant 1, t \geqslant 0\}$. Let $u(x, t)$ be the unique solution to

$$u_t + (u^2/2)_x = \tfrac{1}{2}u_{xx}, \tag{5.1}$$

$$u(x, 0) = u_0(x), \tag{5.2}$$

$$u(0, t) = a, \tag{5.3}$$

$$u(1, t) = b, \tag{5.4}$$

a and b are constants, $u_0(x)$ is $C^1[0, 1]$ and $u_0(0) = a$, $u_0(1) = b$. Then it can be easily seen as in lemmas 2.1 and 2.2 that

$$u(x, t) = -\frac{\sum_0^\infty \exp(\lambda_n t) a_n \phi_n^1(x)}{\sum_0^\infty \exp(\lambda_n t) a_n \phi_n(x)} \tag{5.5}$$

$\lambda_0 > \lambda_1 > \cdots \to -\infty$ are the eigenvalues of

$$\tfrac{1}{2}\phi_{xx} = \lambda\phi,$$

$$\phi_x(0) + a\phi(0) = 0,$$

$$\phi_x(1) + b\phi(1) = 0$$

and $\phi_n(x)$, $n = 0, 1, \ldots$ are the normalized eigenfunctions corresponding to λ_n with $\phi_0(x) > 0$ and

$$a_n = \int_0^1 \exp\left(-\int_0^x u_0(y)\,\mathrm{d}y\right)\phi_n(x)\,\mathrm{d}x. \tag{5.6}$$

As in §3, it is easy to prove the following.

Theorem. *Let $u(x,t)$ be the solution of* (5.1)–(5.4); *then*

$$u(x,t) = -[\log \phi_0(x)]_x + O[\exp[(\lambda_1 - \lambda_0)t]$$

uniformly in $x \in [0,1]$.

Proof. From (5.5) we have,

$$u(x,t) = -\frac{\left[\phi_0^1(x) + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\phi_n^1(x)\right]}{\phi_0(x)}$$

$$\times\left[1 + \sum_1^\infty \exp[(\lambda_n - \lambda_0)^t]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n}{\phi_0}\right)\right]^{-1}$$

$$= -\frac{\left[\phi_0^1(x) + \sum_1^\infty (\exp[(\lambda_n - \lambda_0)t])\right]}{\phi_0(x)}$$

$$\times [1 + O(\exp[(\lambda_1 - \lambda_0)t])]^{-1} \text{ uniformly in } x$$

$$= -\frac{\phi_0^1(x)}{\phi_0(x)}[1 + O(\exp[(\lambda_1 - \lambda_0)t])]$$

uniformly in $x \in [0,1]$. The proof of theorem is complete. q.e.d.

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