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SINGULAR LIMITS IN PHASE DYNAMICS WITH PHYSICAL VISCOSITY AND CAPILLARITY

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ABSTRACT. Following pioneering work by Fan and Slemrod who studied the effect of artificial viscosity terms, we consider the system of conservation laws arising in liquid-vapor phase dynamics with *physical* viscosity and capillarity effects taken into account. Following Dafermos we consider self-similar solutions to the Riemann problem and establish uniform total variation bounds, allowing us to deduce new existence results. Our analysis cover both the hyperbolic and the hyperbolic-elliptic regimes and apply to arbitrarily large Riemann data.

The proofs rely on a new technique of reduction to two coupled scalar equations associated with the two wave fans of the system. Strong L^1 convergence to a weak solution of bounded variation is established in the hyperbolic regime, while in the hyperbolic-elliptic regime a stationary singularity near the axis separating the two wave fans, or more generally an almost-stationary oscillating wave pattern (of thickness depending upon the capillarity-viscosity ratio) are observed which prevent the solution to have globally bounded variation.

1. Introduction

The Navier-Stokes equations for van der Waals fluids with viscosity and capillarity effects included, allow one to model the dynamics of liquid-vapor flows. The associated set of first-order conservation laws is of hyperbolic or hyperbolic-elliptic type, and admits propagating discontinuities (shock waves). The singular limit corresponding to vanishing viscosity and capillarity coefficients allows one to select physically admissible, discontinuous solutions to the first-order conservation laws. In particular, this yields an approach to select solutions to the so-called Riemann problem, when the initial data consists of two constant states separated by a single jump.

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Our purpose in the present paper is to derive uniform (viscosity-capillarity independent) bounds and to justify this singular limit in the context of nonlinear elasticity and phase transition dynamics.

Attention will be concentrated on the following system of two conservation laws:

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\Big(\sigma(w) - \epsilon\,\frac{\partial v}{\partial x} + \delta\frac{\partial^2 w}{\partial x^2}\Big) = 0, \\ &\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} = 0, \end{aligned} \tag{1.1}$$

where $v = v(t,x)$ and $w = w(t,x)$ represent the velocity and deformation gradient of the fluid or solid material under consideration. We consider also the associated first-order system

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}(\sigma(w)) = 0, \\ &\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} = 0, \end{aligned} \tag{1.2}$$

and impose Riemann initial data

$$(v,w)(0,x) = \begin{cases} v_l, w_l, & x < 0, \\ v_r, w_r, & x > 0, \end{cases} \tag{1.3}$$

where $v_l, v_r, w_l, w_r \in \mathbb{R}$ are given constants. Solutions of (1.2)-(1.3) are known to be self-similar, that is, to depend only upon the variable $y := x/t$.

When viscosity and capillarity terms are taken into account, the corresponding set of differential equations reads

$$\begin{aligned} &-y\,v' - \sigma(w)' = \epsilon\, v'' - \delta w''', \\ &-y\,w' - v' = 0, \end{aligned} \tag{1.4}$$

supplemented with the boundary conditions

$$\lim_{y\to-\infty}(v,w)(y) = (v_l,w_l), \qquad \lim_{y\to+\infty}(v,w)(y) = (v_r,w_r). \tag{1.5}$$

Our main objective in this paper is to establish :

(1) the existence of a smooth, self-similar solution to (1.4)-(1.5) having uniformly bounded total variation,

$$TV(v_\epsilon, w_\epsilon) \lesssim |v_r - v_l| + |w_r - w_l|, \tag{1.6}$$

with the implied constant being *independent* of ϵ, δ within the range $\delta/\epsilon^2 << 1$, and

(2) the strong convergence of v_ϵ, w_ϵ toward a weak, self-similar solution v, w to the first-order conservation laws (1.2).

An outline of our main results is as follows.

Assuming first that the system (1.2) is *hyperbolic* we provide a rather direct proof of the above two properties, first in the viscosity-only case (Section 2) and then for general viscosity-capillarity (Section 3). In this regime, it is known that the limiting solutions in general contains nonclassical shocks, with depend upon the ratio δ/ϵ^2; see [16].

Next, we investigate the generalization to the hyperbolic-elliptic regime, and discover a concentration phenomena near the axis $x = 0$ separating the two wave fans. This seems to be consistent with numerical experiments with the model under consideration, but it would be interesting to check numerically the feature discovered here analytically. In the context of phase dynamics, it is also well-known that the limiting solutions contains subsonic phase boundaries which again depend on the capillarity to viscosity ratio; see [23, 25, 27, 1, 15, 21, 22].

Observe that our results cover arbitrary large Riemann data and physical viscosity and capillarity terms. Our results supplement the earlier, pioneering work by Fan and Slemrod [7, 8] where an artificial, "full" viscosity was used. Our technique of proof in the present paper is quite different from the one in [7], as we introduce a decomposition of the system of equations (1.4) into two coupled scalar equations.

Finally, we provide various generalization to the boundary value problem and more general classes of conservation laws.

Recall that the activity on self-similar vanishing viscosity limits started with an extensive research by Dafermos [2]–[5] (see also [6]) who advocated the use of self-similar regularizations to capture the whole structure of wave fans within solutions of the Riemann problem. Self-similar approximations in the context of phase transition dynamics were studied by Slemrod [23, 25], Fan and Slemrod [7], who covered large data solution and artificial regularization terms. Next, small data solutions to general systems were treated by Tzavaras [28] (conservative systems) and LeFloch and Tzavaras [18, 19] (nonconservative systems).

Self-similar diffusive-dispersive approximations for general systems were investigated in LeFloch and Rohde [17]. Joseph and LeFloch [9]–[12] extended the technique to cover boundary value problems, relaxation approximations, and general diffusion matrices. The present paper is the continuation of [9]–[12]. For issues related to the discretization of the viscosity and capillarity terms, we refer to [24, 14].

2. Elastodynamics with physical viscosity

We begin with the case that $\delta = 0$ and $\sigma_w > 0$. We will prove:

Theorem 2.1 (Vanishing viscosity limit in elastodynamics). *Assume that the system* (1.2) *is uniformly hyperbolic, in the sense that there exits a positive constant c_0 such that*

$$\inf_{w \in \mathbb{R}} \sigma_w(w) \geq c_0^2 > 0.$$

Then, given arbitrary Riemann data v_l, w_l and v_r, w_r, the viscous Riemann problem (1.4)-(1.5) *with $\delta = 0$ admits a solution v^ϵ, w^ϵ which has uniformly bounded total variation and converges pointwise to a limit v, w. This limit is a weak solution of the Riemann problem* (1.2)-(1.3)*.*

It is worth observing that the solutions w^ϵ, v^ϵ contain a mild singularity at $y = 0$: their derivatives up to order C/ϵ only are continuous at $y = 0$. Hence, this singularity vanishes in the limit $\epsilon \to 0$. The singularity is due to a factor y appearing in the key equation (2.2) below, in front of the term with the highest order derivative.

Proof. Step 1. Reduction to a scalar equation for the component w. To simplify the notation we suppress the subscript ϵ. We will first study the problem on a finite interval $[-L, L]$ with L chosen to be sufficiently large and we will show the existence of a solution to (1.4) satisfying the boundary conditions

$$(v, w)(-L) = (v_l, w_l), \qquad (v, w)(L) = (v_r, w_r). \tag{2.1}$$

First, we observe that the system can be reduced to a single scalar equation for the unknown w. Namely, using $v' = -y\, w'$ in the first equation of (1.4) we find

$$(y^2 + \epsilon - \sigma_w(w))\, w' = -\epsilon\, y\, w'', \tag{2.2}$$

which will be studied in two regions $[-L, 0]$ and $[0, L]$ with the following boundary conditions

$$w(-L) = w_l, \qquad w(0-) = w_*, \tag{2.3}$$

$$w(0+) = w_*, \qquad w(L) = w_r, \tag{2.4}$$

where w_* is a parameter to be determined so that the boundary condition on the variable v (which has been eliminated from (2.2)) is satisfied. This condition is obtained by integrating over $[-L, L]$ the second equation in (1.4):

$$v_r - v_l = \int_{-L}^{L} v'\, dy = -L\, w_r + L w_l + \int_{-L}^{L} w\, dy. \tag{2.5}$$

This is a global, integral condition on the solution w. We will now solve the problem (2.2)–(2.5).

Our boundary condition can be justified as follows. If we search for a piecewise smooth solution of (1.4), then clearly the measure $v' = -yw'$ can not have a mass point at $y = 0$, hence v is continuous. Similarly, by the first equation in (1.4) the function $\sigma(w) - \epsilon v'$ must be continuous at $y = 0$. But, using again the equation $\epsilon v' = -y\epsilon w'$ we see that $\sigma(w)$ itself must be continuous. Finally, since σ is assumed to be strictly increasing, w must be continuous.

Based on the function w we can define $a(y) := y - (\sigma_w(w(y)) - \epsilon)/y$, consider the integral $\int_{\alpha^\pm}^y a(x)\,dx$ on $[-L, 0)$ and on $(0, L]$ with $\alpha^- \in [-L, 0)$ and $\alpha^+ \in (0, L]$, respectively. Let us show that the minimum with respect to y of this integral is attained at

$$\rho^\pm := \pm(\sigma_w(w(\rho^\pm)) - \epsilon)^{1/2}.$$

For definiteness, consider the case $\alpha = \alpha^- \in [-L, 0)$ with $\alpha^- < y$; then

$$\int_\alpha^y a(x)\,dx \geq \int_\alpha^y (x - \frac{(\sigma_w^M - \epsilon)}{x})\,dx \geq (y^2 - \alpha^2)/2 + \log(\alpha/y)^{\sigma_w^M - \epsilon},$$

which tends to infinity as $y \to 0$. Necessarily, the minimum is attained away from 0. As this integral has quadratic growth in y, for L large the minimum $\rho = \rho^-$ is attained in $(-L, 0)$ and is given by the equation $\rho^2 - \sigma_w(w(\rho)) + \epsilon = 0$.

This choice of $\rho^\pm$ gives $\int_{\rho^-}^y a(x)\,dx \geq 0$. By setting

$$\varphi_-(y) := \frac{e^{-(1/\epsilon)\int_{\rho^-}^y a(x)\,dx}}{\int_{-L}^0 e^{-(1/\epsilon)\int_{\rho^-}^y a(x)\,dx}} \qquad \varphi_+(y) := \frac{e^{-(1/\epsilon)\int_{\rho^+}^y a(x)\,dx}}{\int_0^L e^{-(1/\epsilon)\int_{\rho^+}^y a(x)\,dx}},$$

the problem under consideration is equivalent to

$$w(y) = \begin{cases} w_l + (w_* - w_l)\displaystyle\int_{-L}^y \varphi_-\,dx, & y < 0, \\ w_r + (w_* - w_r)\displaystyle\int_y^L \varphi_+\,dx, & y > 0, \end{cases} \tag{2.6}$$

together with the boundary condition (2.5).

To find w_*, we used the continuity of the solution at $y = 0$ namely, $w(0-) = w(0+) = w_*$ and $v(0-) = v(0+) = v_*$. Integrating the equation $v' = -yw'$ from $-L$ to 0 and from 0 to L and using (2.6), we get

$$v_* - v_l = (w_* - w_l)\int_{-L}^0 -y\varphi_-(y)dy,$$

$$v_r - v_* = (w_* - w_r) \int_0^L y\varphi_+(y)dy.$$

Adding these formulas and solving for w_*, we get

$$w_* = \frac{v_r - v_l + w_r \int_0^L y\,\varphi_+\,dy - w_l \int_{-L}^0 y\,\varphi_-\,dy}{\int_0^L y\,\varphi_+^\delta\,dy - \int_{-L}^0 y\,\varphi_-^\delta\,dy}, \tag{2.7}$$

which provides us with the value of w_*.

Observe that the denominator is bounded away from zero. To evaluate its minimum value (in terms of the constitutive function σ and the Riemann data) we now need to analyze the functions $\varphi_\pm$.

Step 2. Properties of the functions $\varphi_\pm$. We will show now that the support of the functions $\varphi_\pm$ is essentially concentrated away from the axis $y = 0$. Let $\sigma_w^m = \min_{-L\le y\le L} \sigma_w(w(y))$, $\sigma_w^M = \max_{-L\le y\le L} \sigma_w(w(y))$, $\lambda_\epsilon^m = (\sigma_w^m - \epsilon)^{1/2}$ and $\lambda_\epsilon^M = (\sigma_w^M - \epsilon)^{1/2}$. We claim that there exists a constant $C > 0$ such that

$$0 < \varphi_-(y) \le \frac{C}{\epsilon} \begin{cases} e^{-\frac{(y+\lambda_\epsilon^M)^2}{2\epsilon}}, & -L \le y < -\lambda_\epsilon^M, \\ 1, & -\lambda_\epsilon^M \le y < -\lambda_\epsilon^m, \\ e^{-\frac{(y+\lambda_\epsilon^M)^2}{2\epsilon}}, & -\lambda_\epsilon^m \le y < -\lambda_\epsilon^m/4, \\ (-2y/\lambda_\epsilon^m)^{\frac{3\lambda_\epsilon^m}{4\epsilon}}, & -\lambda_\epsilon^m/4 \le y \le 0, \end{cases} \tag{2.8}$$

$$0 < \varphi_+(y) \le \frac{C}{\epsilon} \begin{cases} (2y/\lambda_\epsilon^m)^{\frac{3\lambda_\epsilon^m}{4\epsilon}}, & 0 \le y \le \lambda_\epsilon^m/4, \\ e^{-\frac{(y-\lambda_\epsilon^m)^2}{2\epsilon}}, & \lambda_\epsilon^m/4 \le y < \lambda_\epsilon^m, \\ 1, & \lambda_\epsilon^m \le y < \lambda_\epsilon^M, \\ e^{-\frac{(y-\lambda_\epsilon^M)^2}{2\epsilon}}, & \lambda_\epsilon^M < y < L. \end{cases} \tag{2.9}$$

This is proved as follows. First, in the interval $0 \le y \le \lambda_\epsilon^m/4$ we find

$$\begin{aligned} \int_{\rho^+}^y \frac{(x^2 - \sigma_w(x) + \epsilon)}{x} dx &\ge \int_{\lambda_\epsilon^m/2}^y \frac{(x^2 - \sigma_w + \epsilon)}{x} dx \\ &= \int_y^{\lambda_\epsilon^m/2} \frac{(\sigma_w - \epsilon - x^2)}{x} dx \\ &\ge (3/4)(\lambda_\epsilon^m)^2 \int_y^{\lambda_\epsilon^m/2} \frac{dx}{x} = \log(\lambda_\epsilon^m/2y)^{(3/4)\lambda_\epsilon^{m2}}, \end{aligned}$$

and, next, for $y > \lambda_\epsilon^M$

$$\begin{aligned}\int_{\rho^+}^y (x^2 - \sigma_w(x) + \epsilon)dx &\geq \int_{\lambda_\epsilon^M}^y \frac{(x^2 - \sigma_w(x) + \epsilon)}{x} dx \\ &= \int_{\lambda_\epsilon^M}^y (x - (\sigma_w(x) + \epsilon)^{1/2})(1 + \frac{(\sigma_w(x) + \epsilon)^{1/2}}{x})dx \\ &\geq \int_{\lambda_\epsilon^M}^y (x - \lambda_\epsilon^M)dx = \frac{(y - \lambda_\epsilon^M)^2}{2}.\end{aligned}$$

The other cases are completely similar. Since, on the other hand, it is easy to check that

$$\int_{-L}^0 e^{-(1/\epsilon)\int_{\rho^-}^y a(x)\,dx} dy \geq C\epsilon, \qquad \int_0^L e^{-(1/\epsilon)\int_{\rho^+}^y a(x)\,dx} dy \geq C\epsilon,$$

this completes the derivation of the properties of $\varphi_\pm$.

Step 3. Existence result. Observe that w_* consists of the sum of

$$\frac{v_r - v_l}{\int_0^L y\,\varphi_+\,dy - \int_{-L}^0 y\,\varphi_-\,dy} = \frac{v_r - v_l}{\int_{-\lambda_\epsilon^M}^{-\lambda_\epsilon^m} -y\,\varphi_-\,dy + \int_{\lambda_\epsilon^m}^{\lambda_\epsilon^M} y\,\varphi_+\,dy + O(\epsilon^n)}$$

and a convex combination of w_l, w_r. In view of this observation, together with the estimates (2.8) and (2.9), we get

$$\begin{aligned}|w_*| &\leq \max(w_l, w_r) + \frac{|v_r - v_l|}{2\lambda_\epsilon^m + O(\epsilon^n)} \\ &\leq \max(w_l, w_r) + \frac{|v_r - v_l|}{c_0} =: \Lambda_0\end{aligned}$$

and we arrive at the uniform bound

$$|w_*| \leq \Lambda_0. \tag{2.10}$$

Replacing now w_* (by its value given in (2.7)) in the formula (2.6) we can define a mapping $w \in C^0([-L, L]) \mapsto T(w) \in C^0([-L, L])$. We claim that, for fixed ϵ, the function $T(w)$ is of class C^1 and

$$\begin{aligned}&\|T(w)\|_{C^0} \leq \Lambda_0, \\ &\|T(w)'\|_{C^0} \leq \frac{C}{\epsilon}\,(|w_r - w_*| + |w_r - w_*|).\end{aligned}$$

Indeed, these estimates follow immmediatly from (2.6), (2.8), and (2.9)

In consequence, the operator T is a compact map from the convex bounded set $\left\{w \,/\, \|w\|_{C^0([-L,L])} \leq \Lambda_0\right\}$ into itself. By Schauder's fixed point theorem, T has a fixed point which satisfies (2.6) and, clearly, is of class C^1. Furthermore, in view of (2.6) and (2.7), we have

$$\int_{-L}^{L} |w'(y)| dy \le |w_r - w_*| + |w_l - w_*| \le |w_r - w_l| + \frac{2}{c_0} |v_r - v_l| \tag{2.11}$$

and

$$\int_{-L}^{L} |v'(y)| dy \le \int_{-L}^{L} |y w'(y)| dy \le (\lambda_0^M + 1) \Big(|w_r - w_l| + \frac{2}{c_0} |v_r - v_l| \Big). \tag{2.12}$$

Next, since all of the estimates are uniform in L, we can let $L \to \infty$ and we conclude with existence of a solution (v^ϵ, w^ϵ) on defined on the whole real line $(-\infty, \infty)$. This function (v^ϵ, w^ϵ) is a bounded solution of (1.4)-(1.5) and has uniformly bounded total variation

$$\int_{-\infty}^{\infty} \Big(|w'(y)| + |v'(y)| \Big) dy \le (\lambda_0^M + 2) \Big(|w_r - w_l| + \frac{2}{c_0} |v_r - v_l| \Big).$$

By Helly's compactness theorem, it admits a subsequence converging pointwise to a limit (v, w), as ϵ goes to zero. Clearly, the limit is a weak solution of the problem (1.2)-(1.3). □

3. Elastodynamics with physical viscosity and capillarity

In this section, we consider the full system with physical viscosity and capillarity included. We will prove:

Theorem 3.1 (Vanishing viscosity-capillarity limit in elastodynamics). *Assume that* $\inf \sigma_w \ge c_0^2 > 0$*. Then, given arbitrary Riemann data* v_l, v_r, w_l, w_r*, the problem* (1.4)-(1.5) *with* $\delta = \gamma \epsilon^2$*,* $\gamma > 0$*, admits a solution* v^ϵ, w^ϵ *which has uniformly bounded total variation and converges to a limit* v, w*. This limit is a weak solution to the Riemann problem* (1.2)-(1.3).

Observe that this is a large data result.

Proof. **Step 1: reduction to a scalar equation.** We will first study the problem on a finite interval $[-L, L]$ with L chosen to be sufficiently large with boundary conditions

$$(v, w)(-L) = (v_l, w_l), \qquad (v, w)(L) = (v_r, w_r). \tag{3.1}$$

As before, the system can be reduced to a single scalar equation for the unknown w. Namely, using $v' = -y\, w'$ in the first equation of (1.4) we find

$$(y^2 + \epsilon - \sigma_w(w))\, w' = -\epsilon\, y\, w'' - \delta w''', \tag{3.2}$$

which will be studied in two regions $[-L, 0]$ and $[0, L]$ with the following boundary conditions

$$w(-L) = w_l, \qquad w(0-) = w_*, \tag{3.3}$$

$$w(0+) = w_*, \qquad w(L) = w_r, \tag{3.4}$$

where w_* is a parameter to be determined so that the boundary condition on the variable v (which has been eliminated from (2.2)) is satisfied. This condition is obtained by integrating over $[-L, L]$ the second equation in (1.4):

$$v_r - v_l = \int_{-L}^{L} v'\,dy = -L\,w_r + Lw_l + \int_{-L}^{L} w\,dy. \tag{3.5}$$

This is a global, integral condition on the solution w. We will now solve the problem (3.2)–(3.5).

We will follow a method originally introduced by LeFloch and Rohde [17] for small amplitude solutions of systems of conservation laws. Setting $\varphi = e^{\beta}H$, the differential equation under consideration becomes

$$\delta[H'' + 2H'\beta' + H\beta'' + H\beta'^2] + \epsilon y[H' + H\beta'] + (y^2 + \epsilon - \sigma_w)H = 0$$

Setting the coefficient of H' equal to 0, we get $\beta' = \frac{-\epsilon y}{2\delta}$, thus $\beta(y) = \frac{-\epsilon y^2}{4\delta}$, and

$$H'' = \frac{\mu(w(y), y)}{\delta} H, \tag{3.6}$$

where

$$\mu(w(y), y) = \sigma_w + y^2\left(\frac{\epsilon^2}{4\delta} - 1\right) - \frac{\epsilon}{2}. \tag{3.7}$$

We take $\delta = \gamma\epsilon^2$ and, in this case,

$$\mu(w, y) = \sigma_w + y^2\left(\frac{1}{4\gamma} - 1\right) - \frac{\epsilon}{2} > 0,$$

provided $\epsilon, \gamma > 0$ are sufficiently small. Upon integrating this equation for H and substituting, we get

$$\phi(y) = \frac{1 + \Phi(y)}{(4\gamma\mu)^{1/4}} e^{\frac{p(y,\rho)}{\epsilon}}, \tag{3.8}$$

where

$$|\Phi(y)| + \frac{\epsilon\gamma^{1/2}}{2\mu(y)^{1/2}}|\Phi'(y)| \leq \epsilon\gamma^{1/2}k,$$

$$k := \int_{-L}^{L} \mu^{-5/4}|\mu'|^2 dx + \int_{-L}^{L} \mu^{-3/2}|\mu''| dx$$

and

$$\begin{aligned} p(y,\rho) &:= \int_\rho^y \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\mu(w(x),x)}{\gamma}}\Big) dx \\ &= \int_\rho^y \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}} dx \\ &= \frac{1}{2\gamma} \int_\rho^y x(-1 + \sqrt{1 + 4\gamma(\sigma_w - \epsilon/2 - x^2)}) dx. \end{aligned} \tag{3.9}$$

Here, ρ is the maximizer of the the integral

$$\max_{-L<y<L} \int_\alpha^y \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\mu(w(x),x)}{\gamma}}\Big) dx.$$

Indeed ρ satisfies

$$\rho^2 - \sigma_w(w(\rho)) + \epsilon/2 = 0,$$

which has pairs of solutions $\rho_- \in [-L,0)$ and on $\rho_+ \in (0,L]$, satisfying

$$\rho^\pm = \pm(\sigma_w(w(\rho^\pm)) - \epsilon/2)^{1/2}.$$

We consider φ_- and φ_+ on $[-L,0]$ and $[0,L]$, defined by

$$\phi_-(y) = \frac{1+\Phi(y)}{(4\gamma\mu)^{1/4}} e^{\frac{p(y,\rho_-)}{\epsilon}} \tag{3.10}$$

$$\phi_+(y) = \frac{1+\Phi(y)}{(4\gamma\mu)^{1/4}} e^{\frac{p(y,\rho_+)}{\epsilon}} \tag{3.11}$$

and setting

$$\varphi_-(y) := \frac{\phi_-(y)}{\int_{-L}^0 \phi_-(x)\, dx} \qquad \varphi_+(y) := \frac{\phi_+(y)}{\int_0^L \phi_+(x)\, dx},$$

the problem under consideration is equivalent to

$$w(y) = \begin{cases} w_l + (w_* - w_l) \int_{-L}^y \varphi_-\, dx, & y < 0, \\ w_r + (w_r - w_*) \int_y^L \varphi_+\, dx, & y > 0, \end{cases} \tag{3.12}$$

together with the boundary condition (2.5). The latter can be expressed in its original form involving w' (see the second equation in (1.4)) and yields

$$w_* := \frac{v_r - v_l + w_r \int_0^L y\,\varphi_+\, dy - w_l \int_{-L}^0 y\,\varphi_-\, dy}{\int_0^L y\,\varphi_+\, dy - \int_{-L}^0 y\,\varphi_-\, dy}. \tag{3.13}$$

Observe that the denominator in the formula for w_* is bounded away from zero. To find the value of the minimum in terms of the constitutive function σ and the Riemann data we need the following properties of $\varphi_\pm$

Step 2: Properties of $\varphi_\pm$. With the same σ_ϵ^m and σ_ϵ^M as before, set $\lambda_\epsilon^m = (\sigma_w^m - \epsilon/2)^{1/2}$ and $\lambda_\epsilon^M = (\sigma_w^M - \epsilon/2)^{1/2}$. There exist constants $C_1 > 0$, and $C > 0$ depending only on $\gamma, \sigma_w^m, \sigma_w^M$ and such that

$$0 < \varphi_-(y) \le \frac{C_1}{\epsilon}\begin{cases} e^{\frac{C(y+\lambda_\epsilon^M)^2}{\epsilon y}}, & -L \le y < -\lambda_\epsilon^M, \\ 1, & -\lambda_\epsilon^M \le y < -\lambda_\epsilon^m, \\ e^{-\frac{C(y+\lambda_\epsilon^m)^2}{\epsilon}}, & -\lambda_\epsilon^m \le y < -\gamma^{1/2}\lambda_\epsilon^m, \\ e^{-\frac{C|y-\lambda_\epsilon^m\gamma^{1/2}|}{\epsilon}}, & -\gamma^{1/2}\lambda_\epsilon^m \le y \le 0, \end{cases} \tag{3.14}$$

$$0 < \varphi_+(y) \le \frac{C_1}{\epsilon}\begin{cases} e^{-\frac{C|y-\lambda_\epsilon^m\gamma^{1/2}|}{\epsilon}}, & 0 \le y \le \gamma^{1/2}\lambda_\epsilon^m, \\ e^{-\frac{C(y-\lambda_\epsilon^m)^2}{\epsilon}}, & \gamma^{1/2}\lambda_\epsilon^m \le y < \lambda_\epsilon^m, \\ 1, & \lambda_\epsilon^m \le y < \lambda_\epsilon^M, \\ e^{-\frac{C(y-\lambda_\epsilon^M)^2}{\epsilon y}}, & \lambda_\epsilon^M < y < L. \end{cases} \tag{3.15}$$

First, we prove the estimates for φ_+. Consider the region $y > \lambda_\epsilon^M$

$$\begin{aligned} p(y, \rho_+) &= \int_{\rho^+}^{y} \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)\, dx \\ &\le \int_{\lambda_\epsilon^M}^{y} \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)\, dx \\ &= \frac{1}{2\gamma}\int_{\lambda_\epsilon^M}^{y} x\Big(-1 + \sqrt{1 + 4\gamma\frac{(\sigma_w - \epsilon/2 - x^2)}{x^2}}\Big)\, dx \\ &\le \frac{\lambda_\epsilon^M}{2\gamma}\int_{\lambda_\epsilon^M}^{y} \Big(-1 + \sqrt{1 + 4\gamma\frac{(\sigma_w - \epsilon/2 - x^2)}{x^2}}\Big)\, dx, \end{aligned}$$

and, for $y > \lambda_\epsilon^M$,

$$p(y,\rho_+) =\leq \frac{\lambda_\epsilon^M}{2\gamma}\int_{\lambda_\epsilon^M}^y 2\gamma \frac{(\sigma_w - \epsilon/2 - x^2)}{x^2}dx$$
$$\leq \lambda_\epsilon^M \int_{\lambda_\epsilon^M}^y (\sigma_w - \epsilon/2)^{1/2} - x)\frac{(\sigma_w - \epsilon/2)^{1/2} + x)}{x^2}dx$$
$$\leq \lambda_\epsilon^M \int_{\lambda_\epsilon^M}^y \frac{(\sigma_w - \epsilon/2)^{1/2} - x)}{x}dx$$
$$\leq \frac{\lambda_\epsilon^M}{y}\int_{\lambda_\epsilon^M}^y (\lambda_\epsilon^M - x)dx = -\frac{\lambda_\epsilon^M}{y}\frac{(y-\lambda_\epsilon^M)^2}{2}.$$

Now, consider the region $\gamma^{1/2}\lambda_\epsilon^m/2 \leq y \leq \lambda_\epsilon^m$,

$$p(y,\rho_+) = \int_{\rho^+}^y \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx$$
$$\leq \int_{\lambda_\epsilon^m}^y \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx$$
$$\leq \int_y^{\lambda_\epsilon^m} \Big(\frac{x}{2\gamma} - \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx$$
$$= \frac{1}{2\gamma}\int_y^{\lambda_\epsilon^m} x\Big(1 - \sqrt{1 + 4\gamma\frac{(\sigma_w - \epsilon/2 - x^2)}{x^2}}\Big)dx$$
$$\leq \frac{\lambda_\epsilon^m}{4\gamma^{1/2}}\int_y^{\lambda_\epsilon^m} \Big(1 - \sqrt{1 + 4\gamma\frac{(\sigma_w - \epsilon/2 - x^2)}{\lambda_\epsilon^m}}\Big)dx,$$

and, for $\gamma^{1/2}\lambda_\epsilon^m/2 \leq y \leq \lambda_\epsilon^m$,

$$p(y,\rho_+) - \leq \frac{\lambda_\epsilon^m}{4\gamma^{1/2}}\int_y^{\lambda_\epsilon^m} 4\gamma\frac{(\sigma_w - \epsilon/2 - x^2)}{\lambda_\epsilon^m})dx$$
$$\leq -C\int_y^{\lambda_\epsilon^m} 4\gamma(\sigma_w - \epsilon/2)^{1/2} - x)(\sigma_w - \epsilon/2)^{1/2} + x)dx$$
$$\leq -C\int_y^{\lambda_\epsilon^m} (\sigma_w - \epsilon/2)^{1/2} - x)dx$$
$$\leq -C\int_y^{\lambda_\epsilon^M} (\lambda_\epsilon^m - x)dx = -C\frac{(y-\lambda_\epsilon^m)^2}{2}.$$

For $0 \le y \le \gamma^{1/2}\lambda_\epsilon^m/2$,

$$\begin{aligned}
p(y,\rho_+) &= \int_{\rho^+}^{y} \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx \\
&\le \int_{\lambda_\epsilon^m\gamma^{1/2}}^{y} \Big(\frac{-x}{2\gamma} + \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx \\
&\le \int_{y}^{\lambda_\epsilon^m\gamma^{1/2}} \Big(\frac{x}{2\gamma} - \sqrt{\frac{\sigma_w - \epsilon/2 + x^2(1/4\gamma - 1)}{\gamma}}\Big)dx \\
&\le \int_{y}^{\lambda_\epsilon^m\gamma^{1/2}} (\frac{\gamma^{1/2}\lambda_\epsilon}{2\gamma} - \frac{\lambda_\epsilon^m}{\gamma^{1/2}})dx \\
&= \int_{y}^{\lambda_\epsilon^m\gamma^{1/2}} -\frac{\lambda_\epsilon^m}{2\gamma^{1/2}}dx = -C|y - \lambda_\epsilon^m\gamma^{1/2}|.
\end{aligned}$$

The other cases are identical. One can also check (for some $c > 0$)

$$\int_{-L}^{0} \phi_-(y)dy \ge c\,\epsilon, \qquad \int_{0}^{L} \phi_+(y)dy \ge c\,\epsilon,$$

so that the desired properties of φ_+, φ_-, follow.

Existence arguments. Now w_* is sum of convex combination of w_l, w_r and the quantily

$$\frac{v_r - v_l}{\int_0^L y\,\varphi_+\,dy - \int_{-L}^0 y\,\varphi_-\,dy} = \frac{v_r - v_l}{\int_{-\lambda_\epsilon^M}^{-\lambda_\epsilon^m} -y\,\varphi_-\,dy + \int_{\lambda_\epsilon^m}^{\lambda_\epsilon^M} y\,\varphi_+\,dy + O(\epsilon^n)}.$$

From this we get

$$|w_*| \le \max(w_l, w_r) + \frac{|v_r - v_l|}{2\lambda_\epsilon^m + O(\epsilon^n)} \le \Lambda_0.$$

We can now replace w_* (given by (3.13)) in (3.12) and arrive at a mapping $w \in C^0([-L,L]) \mapsto T(w) \in C^0([-L,L])$. For fixed ϵ, $T(w)$ is of class C^1, and

$$\begin{aligned}
&\|T(w)\|_{C^0} \le \Lambda_0, \\
&\|T(w)'\|_{C^0} \le \frac{C}{\epsilon}\,(|w_r - w_*| + |w_r - w_*|).
\end{aligned}$$

These estimates follow from (3.12) (3.14) and (3.15)

Thus, T is a compact map from a convex bounded set $\{w : \|w\|_{C^0} \le \Lambda_0\}$ into itself. By Schauder's fixed point theorem, T has a fixed point in $C^0[-L,L]$ and satisfies (3.12) and so the solution is of class C^1. Furthermore, in view of (3.12) and (3.13),

$$\int_{-L}^{L} |w'(y)|dy \le |w_r - w_*| + |w_l - w_*| \le |w_r - w_l| + 2\frac{|v_r - v_l|}{c_0} \tag{3.16}$$

and

$$\int_{-L}^{L} |v'(y)|dy \le \int_{-L}^{L} |yw'(y)|dy \le (\lambda_0^M + 1)\Big(|w_r - w_l| + 2\frac{|v_r - v_l|}{c_0}\Big). \tag{3.17}$$

As the estimates are uniform in L, we have existence of solution (v^ϵ, w^ϵ) on $(-\infty,\infty)$ we have (v^ϵ, w^ϵ) bounded solution of (1.4) and (1.5) with uniform total variation

$$\int_{-\infty}^{\infty} (|w'(y)| + |v'(y)|dy \le (\lambda_0^M + 2)[|w_r - w_l| + 2\frac{|v_r - v_l|}{c_0}]$$

By compactness there exists a sequence converges in L^1 and pointwise a.e. to a function (v,w) as ϵ goes to zero and solves (1.2) and (1.3) with $\epsilon = 0$. □

4. Phase dynamics with physical viscosity

We turn to the model of phase transition dynamics when the system is not strictly hyperbolic, that is, when $\sigma_w(w)$ is only non-negative, or even is hyperbolic-elliptic when σ_w takes negative values.

To illustrate the key difficulty we will have to cope with, let us first consider the example

$$a(y) = y + c/y,$$

with c a constant. When $c \ge 0$, $\rho_+ = \delta$ and $\int_{\rho^+}^{y} a(x)dx = \frac{y^2-\rho_+^2}{2} + \log(|y/\rho_+|^c)$ and so

$$e^{-\frac{1}{\epsilon}\int_{\rho_+}^{y} a(x)dx} = e^{\frac{-(y^2-\rho_+^2)}{2\epsilon}}(y/\rho_+)^{-c/\epsilon} = e^{\frac{-(y^2-\delta^2)}{2\epsilon}}(y/\delta)^{-c/\epsilon}.$$

In this case

$$\varphi_+(y) = \frac{e^{\frac{-(y^2-\delta^2)}{2\epsilon}}(y/\delta)^{-c/\epsilon}}{\int_{\rho_+}^{L} e^{\frac{-(y^2-\delta^2)}{2\epsilon}}(y/\delta)^{-c/\epsilon}dy},$$

which is concentrated at $y = \delta$. For $c < 0$, we have, $\rho_+ = (-c)^{1/2}$ and

$$e^{-\frac{1}{\epsilon}\int_{\rho_+}^{y} a(x)dx} = e^{\frac{-(y^2+c)}{2\epsilon}}(y/(-c)^{1/2})^{-c/\epsilon},$$

so that

$$\varphi_+(y) = \frac{e^{\frac{-(y^2+c)}{2\epsilon}}(y/(-c)^{1/2})^{-c/\epsilon}}{\int_{\rho_+}^{L} e^{\frac{-(y^2+c)}{2\epsilon}}(y/(-c)^{1/2})^{-c/\epsilon}dy},$$

which is concentrated at $y = (-c)^{1/2}$.

In the previous section, to show that w_* is bounded independent of ϵ it was crucial that φ^+ and φ^- be concentrated away from 0. In the above example, if $c \geq 0$, we have seen that this is not the case. This is the difficulty is getting estimates when $\sigma_w(w(y))$ oscillates, i.e. takes both negative and positive values.

We now prove the following result:

Theorem 4.1 (Vanishing viscosity limit in phase dynamics)**.** *Suppose that the first order system* (1.2) *is uniformly hyperbolic for large value of w but may admit elliptic regions in the phase space, that is for some constants $M, c_0 > 0$*

$$\inf_{|w| \geq M} \sigma_w(w) \geq {c_0}^2.$$

Suppose also that there exist constants $c_1 > 0$ and $\eta > 0$ such that $|\sigma_w(w)| \leq c_1 |w|^{2-\eta}$ for $|w| > 1$. Then for all Riemann data v_l, w_l and v_r, w_r in the hyperbolic region of the phase space, i.e.

$$w_l, w_r \in \mathcal{H} := \Big\{ w \ / \ \sigma_w > 0 \Big\}$$

the viscous Riemann problem (1.4)-(1.5)*, with $\delta = 0$, admits a solution v^ϵ, w^ϵ which has uniformly bounded total variation at least away from $y = 0$, and more precisely*

$$\int_{\mathbb{R}} \Big(|v'_\epsilon| + y \, |w'_\epsilon| \Big) \, dy \lesssim |v_r - v_l| + |w_r - w_l|.$$

The functions v_ϵ, w_ϵ converge pointwise at all $y \neq 0$ to a limit v, w, which is a solution of the Riemann problem (1.2)-(1.3) away from the axis $y = x/t = 0$*. Furthermore, v has bounded variation and so admits left- and right-hand limits at $y = 0$, while the function w and its variation measure dw/dy satisfy*

$$|w| \lesssim \frac{1}{|y|}, \quad y \neq 0,$$

$$\int_{\mathbb{R}} |y| \, |\frac{dw}{dy}| < \infty.$$

Hence, the component v only has globally bounded variation. In turn, the conservation laws (1.2) are satisfied in the two regions $x < 0$ and $x > 0$, but a singularity may arise on the axis.

Proof. Step 1. A priori estimates for the components v, w. To simplify the notation we suppress the subscript ϵ. We will first study the problem away from the axis $y = 0$. We consider the function

$$a(y) = y - (\sigma_w(w(y)) - \epsilon)/y, \tag{4.1}$$

and, given some $\delta > 0$, we study the problem in two regions $[-L, -\delta]$ and $[\delta, L]$, with the following boundary conditions

$$w(-L) = w_l, \qquad w(\delta-) = w_*, \tag{4.2}$$

$$w(\delta) = w_*, \qquad w(L) = w_r. \tag{4.3}$$

We set

$$\varphi_-^\delta(y) := \frac{e^{-(1/\epsilon)\int_{\rho_-}^y a^\delta(x)\,dx}}{\int_{-L}^{-\delta} e^{-(1/\epsilon)\int_{\rho_-}^y a^\delta(x)\,dx}}, \qquad \varphi_+^\delta(y) := \frac{e^{-(1/\epsilon)\int_{\rho_+}^y a^\delta(x)\,dx}}{\int_{\delta}^{L} e^{-(1/\epsilon)\int_{\rho_+}^y a^\delta(x)\,dx}},$$

on $[-L, -\delta]$ and $[\delta, L]$ where $\rho_\pm$ are the points where $\int^y a(x)dx$ attains its global minimum.

Let us consider $\lambda_\epsilon^{M+} = \sup_{1\le y\le L} (\sigma_w(w) - \epsilon)^{+1/2}$ and let $c > \max\{1, \lambda_\epsilon^{M+}\}$. For $y > c$

$$\begin{aligned}
\int_{\rho^+}^y (x^2 - \sigma_w(x) + \epsilon)dx &\ge \int_c^y \frac{(x^2 - \sigma_w(x) + \epsilon)}{x}dx \\
&= \int_c^y ((x - (\sigma_w(x) + \epsilon)^+)^{1/2})(1 + \frac{((\sigma_w(x) + \epsilon)^+)^{1/2}}{x})dx \\
&\ge \int_c^y (x - c)dx = \frac{(y-c)^2}{2}.
\end{aligned}$$

So, for $y > (1 + \sup_{1\le y\le L}((\sigma_w(w) - \epsilon)^+)^{1/2}$, we get

$$\varphi(y)dy \le \frac{1}{\epsilon} e^{-\frac{(y-c)^2}{2\epsilon}} \tag{4.4}$$

and

$$\int_\delta^L y\varphi(y)dy \le (2 + \sup_{1\le y\le L} ((\sigma_w(w) - \epsilon)^+)^{1/2}. \tag{4.5}$$

We then set

$$w(y) = \begin{cases} w_l + (w_* - w_l)\int_{-L}^y \varphi_-^\delta\,dx, & y < -\delta, \\ w_r + (w_* - w_r)\int_y^L \varphi_+^\delta\,dx, & y > \delta. \end{cases} \tag{4.6}$$

Here, we have taken $w(-\delta) = w(\delta) = w_*$ and we also take $v(-\delta) = v(\delta) = v_*$. Integrating the equation $v' = yw'$ from $-L$ to $-\delta$ and δ to L and using (4.6), we get

$$v_* - v_l = (w_* - w_l)\int_{-L}^{-\delta} -y\phi_-(y)dy$$

and

$$v_r - v_* = (w_* - w_r)\int_\delta^L y\phi_+(y)dy.$$

Adding these formulas, we get

$$w_* := \frac{v_r - v_l + w_r \int_\delta^L y\,\varphi_+^\delta\,dy - w_l \int_{-L}^{-\delta} y\,\varphi_-^\delta\,dy}{\int_\delta^L y\,\varphi_+^\delta\,dy - \int_{-L}^{-\delta} y\,\varphi_-^\delta\,dy} \tag{4.7}$$

and, subtracting,

$$v_* = v_l + \frac{[v_r - v_l + (w_r - w_l) \int_\delta^L y\,\varphi_+^\delta\,dy] \int_{-L}^{-\delta} -y\,\varphi_-^\delta\,dy}{\int_\delta^L y\,\varphi_+^\delta\,dy - \int_{-L}^{-\delta} y\,\varphi_-^\delta\,dy}. \tag{4.8}$$

Now, v can be expressed in terms of the boundary data w_l, w_r, v_l, v_r and δ:

$$v^\delta(y) = \begin{cases} v_l + \frac{[v_r - v_l + (w_r - w_l) \int_\delta^L y\,\varphi_+^\delta\,dy] \int_{-L}^{y} -x\,\varphi_-^\delta\,dx}{\int_\delta^L y\,\varphi_+^\delta\,dy - \int_{-L}^{-\delta} -y\,\varphi_-^\delta\,dy}, & -L \le y < -\delta \\ v_r + \frac{[v_l - v_r + (w_r - w_l) \int_- L^\delta - y\,\varphi_-^\delta\,dy] \int_y^L x\,\varphi_+^\delta\,dx}{\int_\delta^L y\,\varphi_+^\delta\,dy + \int_{-L}^{-\delta} -y\varphi_-^\delta\,dy}. & L > y > \delta. \end{cases} \tag{4.9}$$

Also, $w(y)$ can be written as

$$w^\delta(y) = \begin{cases} w_l + (v_* - v_l) \frac{\int_{-L}^{y} \varphi_-^\delta\,dx,}{\int_{-L}^{-\delta} y\varphi_-^\delta\,dx} & y < -\delta, \\ w_r + (v_r - v_*) \frac{\int_y^L \varphi_+^\delta\,dx,}{\int_\delta^L y\varphi_+^\delta\,dx} & y > \delta. \end{cases} \tag{4.10}$$

Now clearly v_* is bounded independent of $\epsilon > 0, \delta > 0$:

$$|v_* - v_l| \le |v_r - v_l| + L(|w_r - w_l|)$$

and from (4.9) and (4.10), we get the following estimates

$$|y|\,|w^\delta(y)| \le (|v_r| + |v_l|) + 2L(|w_r| + |w_l|),$$
$$|v^\delta(y)| \le 2(|v_r| + |v_l|) + L(|w_r| + |w_l|),$$

Also we have the following estimates on the derivatives of v^δ, w^δ.

$$\int_{L>|y|>\delta} |y w^\delta(y)'| dy \le (|v_r - v_l|) + L(|w_r - w_l|),$$
$$\int_{L>|y|>\delta} |v^\delta(y)'| dy \le (|v_r - v_l|) + L(|w_r - w_l|).$$

Furthermore, for all $L > |y| > \delta$

$$|v^\delta(y)'| \le \frac{1}{\epsilon\delta}\Big((|v_r - v_l|) + L(|w_r - w_l|)y\Big),$$
$$|w^\delta(y)'| \le \frac{1}{\epsilon\delta}\Big((|v_r - v_l|) + L(|w_r - w_l|)\Big).$$

The existence of solution for fixed L follows from these estimates as we will see. In order to pass $L \to \infty$, we need estimates independent of L. For this we use the growth condition on σ_w. From the expression (4.8) for v_*, and (4.5), we get

$$|v_* - v_l| \le |v_r - v_l| + (2 + \sup_{1\le y\le L} (\sigma_w(w) - \epsilon)^{+1/2})|w_r - w_l|.$$

Using this in (4.9) and (4.10), we get

$$\sup_{1\le y\le L} |w^\delta(y)| \le (|v_r| + |v_l|) + ((3 + \sup_{1\le y\le L} (\sigma_w(w) - \epsilon)^{+1/2})(|w_r| + |w_l|),$$
$$|v^\delta(y)| \le 2(|v_r| + |v_l|) + ((2 + \sup_{1\le y\le L} (\sigma_w(w) - \epsilon)^{+1/2})(|w_r| + |w_l|),$$

Now using our assumption $|\sigma_w| \le c_1|w|^{2-\eta}$ for $|w| > 1$ in the first inequality, we conclude that $\sup_{1\le y\le L} |w^\delta(y)|$ is independent of L. Similar estimate holds for $[-L \le y \le -1]$. We get there exists a constant $C = C(v_l, v_r, w_l, w_r)$, independent of L such that

$$\sup_{1\le |y|\le L} |w^\delta(y)| \le C \tag{4.11}$$

With this constant C, let $\lambda^M = sup_{|w|\le C}|\sigma_w(w)|^{1/2}$. From (4.4), we get φ^δ_+ is essentially supported in $[\delta, \lambda^M + 1]$. Similar arguments give φ^δ_- is essentially supported in the interval $[-\lambda^M - 1, -\delta]$. It follows from (4.9) and (4.10) that there exists a constant C_1

$$\begin{aligned}\int_{|y|>\delta} |yw^\delta(y)'|dy \le (|v_r - v_l|) + C_1(|w_r - w_l|),\\ \int_{|y|>\delta} |v^\delta(y)'|dy \le (|v_r - v_l|) + C_1(|w_r - w_l|).\end{aligned} \tag{4.12}$$

Step 2. Existence proof. For each $\delta > 0$, we can apply Schauder's fixed point theorem, as was explained earlier, and we obtain a solution (w^δ, v^δ) defined on the interval $[-L, -\delta] \bigcup [\delta, L]$ and satisfying uniform in $\epsilon > 0$ total variation estimate. The estimates (4.11) and (4.12) allows as to let L tend to infinity. By compactness, we have a solution (v, w) defined in the region $|y| > \delta$. Since $\delta > 0$ is arbitrary, we obtain a well-defined solution away from $y = 0$. Indeed, v^δ has a uniform total variation on R and v admits left- and right-limit at $y = 0$. One can check easily that the limit is a weak solution of the problem (1.2)-(1.3) away from the axis $y = 0$ at least. Without a control on $\sigma(w)$ near $y = 0$ we can not exclude a concentration term on the axis. □

5. Boundary Riemann problem and further generalizations

In this section, we outline how the method developed in previous sections can be generalized to the boundary-value problem and indicate several generalizations. It is well-known that in the strictly hyperbolic case with nondegenerate function σ, the boundary Riemann problem (in $x > 0, t > 0$)

$$v_t - \sigma(w)_x = 0, \qquad w_t - v_x = 0. \tag{5.1}$$

with initial and boundary conditions

$$w(0+, t) = w_b, \qquad v(x, 0) = v_r, w(x, 0) = w_r \tag{5.2}$$

is well posed. This easily follows from an analysis of the wave curves for the system corresponding to the left-moving and right-moving characteristic families. The physical regularizations considered in earlier sections is well-suited to handle this boundary value problem, without producing boundary layers in the limit.

Consider the nonlinear elastodynamics with physical viscosity

$$-yv' - \sigma_w(w))\, w' = \epsilon\, v'', \qquad -y\, w' - v' = 0, \qquad \text{on } [0, \infty), \tag{5.3}$$

with boundary conditions

$$w(0) = w_b, \qquad v(\infty) = v_r, w(\infty) = w_r. \tag{5.4}$$

Observe that, for the "full" viscosity approximation, we need to prescribe the component v at $y = 0$ and this generates a boundary layer at $y = 0$, after passage to the limit $\epsilon \to 0$; see [10, 11]. In the present case, we show that no boundary layer arises.

As in Section 2, the problem can be reduced to a scalar equation for the unknown w

$$(y^2 + \epsilon - \sigma_w(w))\, w' = -\epsilon\, y\, w'', \qquad \text{on } [0, L], \tag{5.5}$$

for sufficiently large L, with boundary conditions,

$$w(0) = w_b, w(L) = w_r. \tag{5.6}$$

Once we have w, we get the component v from the equation $yw' + v' = 0$ and the boundary condition $v(L) = v_r$.

The fixed point argument in Section 2 yields a solution w^ϵ of (5.5)-(5.6) which is of uniformly bounded variation. Furthermore, it can be represented by an integral formula in terms of the function φ_+ introduced in Section 2

$$w^\epsilon(y) = w_r + (w_b - w_r)\int_y^L \varphi_+ \, dx, \qquad y > 0.$$

Using that $\int_0^L \varphi_+(x)dx = 1$, we get

$$w^\epsilon(y) = w_b + (w_b - w_r)\int_0^y \varphi_+ \, dx, \qquad y > 0. \tag{5.7}$$

Using (5.7) in the equation $v' = -yw'$ and integrating from y to L, we get

$$v^\epsilon(y) = v_r + (w_b - w_r)\int_y^L x\varphi_+ \, dx, \qquad y > 0, \tag{5.8}$$

where we used $v(L) = v_r$. Note that we cannot prescribe boundary condition at $v(0)$ arbitrarily, since from the above equation it follows that

$$v(0) = v_r + (w_b - w_r)\int_0^L x\varphi_+ \, dx.$$

As in Section 2, using the fact that φ_+ decays exponentially we can let L tend to ∞. Again, using the properties (2.9) of φ_+ near $y = 0$ in (5.7), it follows easily that there exists $C > 0$, a constant independent of ϵ, such that for all $y > 0$ close to the origin

$$|w(y) - w_b| \le C\, y,$$

Hence, no boundary layer arises in this approximation at $y = 0$ and in the limit as $\epsilon \to 0$, the limit function $w = w(y)$ satisfies the boundary condition $w(0+) = w_b$. We summarize our results in the following theorem:

Theorem 5.1. *Assume that* $\inf \sigma_w \ge c_0^2 > 0$*. Then given arbitrary Riemann boundary data, the problem* (5.3)-(5.4) *admits a solution* v^ϵ, w^ϵ *which has uniformly bounded total variation and converges to a limit* v, w *which is a solution of the boundary Riemann problem* (5.1)-(5.2).

We now discuss a general system. The basic nature of the system (5.1) is that v appear linearly and the characteristic speeds are equal in magnitude and opposite in sign. Further using the second equation, the system can be reduced to uncoupled equations for w. After this reduction we solved for w first and then for v. Our results can be generalized for systems of first order equations for w and v vector valued functions, having same structure, using the ideas of the present work and the earlier work [10, 11] on boundary value problems.

For example let $F : R^N \to R^N$ be a smooth function with $A(w) = D_w F$ has real distinct positive eigenvalues $\lambda_1(w)^2 < \lambda_2(w)^2 < ... < \lambda_N(w)^2$ with a complete set of right-left normalized eigenvectors r_j, $j = 1, 2, ...N$, l_j, $j = 1, 2, ..., N$, $l_j.r_k = \delta_{jk}$.

We assume that there exists $c_0 > 0$ such that $\lambda_j(w) \geq c_0$ for all $j = 1, 2, ...N$ and for all $w \in B(\delta_0)$, a ball of radius δ_0 around a fixed state that we take to be 0. We consider the system of $2N$ equations

$$\begin{aligned} &\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\Big(F(w)\Big) = 0, \\ &\frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} = 0, \end{aligned} \tag{5.9}$$

where $v = v(t, x)$ and $w = w(t, x)$ are R^N valued functions. The system (5.1) is a special case of (5.9) with $N = 1$ and $\lambda(w)^2 = \sigma_w(w)$

We consider boundary value problem for (5.9) on $x > 0, t > 0$. When physical viscosity terms are taken into account, the corresponding set of differential equations becomes

$$\begin{aligned} &- y\, v' - F(w)' = \epsilon\, v'', \\ &- y\, w' - v' = 0, \end{aligned} \tag{5.10}$$

supplemented with the boundary conditions

$$\lim_{y\to\infty}(v, w)(y) = (v_r, w_r), \qquad w(0+) = w_b. \tag{5.11}$$

We first consider (5.10) on $[0, L]$ with boundary conditions

$$(v, w)(L) = (v_r, w_r), \qquad w(0+) = w_b. \tag{5.12}$$

As in the previous case the problem can be reduced to first solving a system for $w(y)$ namely

$$(y^2 + \epsilon - A(w))\, w' = -\epsilon\, y\, w'', \tag{5.13}$$

on $[0, L]$, with boundary conditions,

$$w(0) = w_b, w(L) = w_r. \tag{5.14}$$

Existence of uniformly BV solutions of this problem easily follows from the work of Joseph and LeFloch [10, 11]. We just outline the main steps omitting the details.

We decompose u' in the basis of eigenvectors of $A(u)$,

$$u'(y) = \sum_{j=1}^{N} a_j r_j(u), a_j = \langle l_j, u' \rangle \tag{5.15}$$

A straight forward calculation lead to the following system of non-liear equations for $a_j, j = 1, 2, ..., N$

$$\epsilon y a_j' + (y^2 + \epsilon - \lambda_j^2(w))a_j = D_1(a, a) \tag{5.16}$$

where

$$D_1(a, a) = -\epsilon y \sum_{k,i} a_i a_k (D_u r_i . r_k) \tag{5.17}$$

Observe that the linearized form of the equation (5.16) has the form

$$\epsilon \, y \, a_j'(y^2 + \epsilon - \lambda_j^2(w)) \, a_j' = 0,$$

and the corresponding wave measure φ_{j+} has exactly same properties (2.9) given in section 2. These wave measures naturally appear when we invert the linear part of the equation (5.16). Because of the quadratic righthand side this leads to interaction terms with different families of wave measures which is controlled by $\sum_{j=1}^{N} \varphi_{j+}$. Fixed point arguments give a uniform BV solution $w^\epsilon(y)$ for the system (5.13) and (5.14). The details are similar to Joseph and LeFloch [10] and, therefore, are omitted. Then the BV estimate for v follows from the second equation in (5.10) and

$$v(y) = v_r + \int_y^L x w'(x) dx.$$

As the estmate (2.9) shows that φ_{j+} is essentially supported in $[\lambda_j^m, \lambda_j^M]$ we can let L go to infinity. Here λ_j^m and λ_j^M denotes the minimum and maximum of $\lambda_j(w)$ on the ball $B(\delta_0)$. We have the following result.

Theorem 5.2. *Assume that* $\inf \lambda_j(w) \geq c_0 > 0$. *Then there exists* $\delta_1 > 0, \delta_2 > 0$ *such that, for* $w_b, w_r \in B(\delta_1)$ *and arbitrary* v_r *the boundary Riemann problem* (5.10)-(5.11) *admits a solution* v^ϵ, w^ϵ *which has uniformly bounded total variation with* $w^\epsilon(y) \in B(\delta_2)$ *and converges to a limit* v, w *which is a solution of the equation* (5.9) *with boundary conditions* $w(0+) = w_b, w(x, 0) = w_r, v(x, 0) = v_r$.

Remark 5.3. *This analysis is easily extended to the case with physical viscosity and capillarity as well namely*

$$\begin{aligned} &- y \, v' - F(w)' = \epsilon \, v'' - \gamma \epsilon^2 w''', \gamma > 0, \\ &- y \, w' - v' = 0, \end{aligned} \tag{5.18}$$

supplemented with the boundary conditions

$$\lim_{y \to \infty} (v, w)(y) = (v_r, w_r), \qquad w(0+) = w_b. \tag{5.19}$$

As in the previous case, the problem can be reduced to first solving a system for $w(y)$, namely

$$(y^2 + \epsilon - A(w))\, w' = -\epsilon\, y\, w'' - \gamma\epsilon^2 w''', \tag{5.20}$$

on $[0, L]$, for sufficiently large L, with boundary conditions

$$w(0) = w_b, w(L) = w_r. \tag{5.21}$$

We decompose u' in the basis of eigenvectors of $A(u)$,

$$u'(y) = \sum_{j=1}^{N} a_j r_j(u), a_j = \langle l_j, u' \rangle \tag{5.22}$$

and then the nonlinear equations for a_j, $j = 1, 2, ..., N$ take the form

$$\gamma\epsilon^2 a_j'' + \epsilon y a_j' + (y^2 + \epsilon - \lambda_j^2(w)) a_j = D_1(a, a) + D_2(a, a) + D_3(a, a, a), \tag{5.23}$$

where

$$D_1(a, a) = -\epsilon y \sum_{k,i} a_i a_k (D_u r_i . r_k)$$

$$D_2(a, a) = -\gamma\epsilon^2 (\sum_{k,i} a_i a_k' (D_u r_i . r_k) + \sum_{k,i} (a_i a_k)' (D_u r_i . r_k$$

$$D_3(a, a, a) = -\gamma\epsilon^2 \sum_{k,i,l} a_i a_k a_l D_u (D_u r_i . r_k) r_l$$

The linearized equation for (5.23) is

$$\gamma\epsilon^2 a_j'' + \epsilon y a_j' + (y^2 + \epsilon - \lambda_j^2(w)) a_j = 0,$$

and hence the wave measures φ_{j+} in this case has the same qualitative properties as (3.15).

Existence of uniformly BV solutions of the problem (5.20) and (5.21) can be easily deduced from [17]. Then, the BV estimate for v follows from the second equation of (5.18). The details are omitted. We get the following result:

Theorem 5.4. *Assume that* $\inf \lambda_j(w) \geq c_0 > 0$. *Then there exists* δ_1 *and* δ_2 *such that for* $w_b, w_r \in B(\delta_1)$ *and for arbitrary* v_r, *the problem* (5.18)-(5.19) *admits a solution* v^ϵ, w^ϵ, *which has uniformly bounded total variation with* $w^\epsilon(y) \in B(\delta_2)$ *and converges to a limit* v, w *which is a solution of the equation* (5.9) *with boundary conditions* $w(0+) = w_b, w(x, 0) = w_r, v(x, 0) = v_r$.

Now let us consider the phase transition case with physical viscosity and capillarity. Following previous sections , the system can be reduced to a single scalar equation for the unknown w,

$$(y^2 + \epsilon - \sigma_w(w))\, w' = -\epsilon\, y\, w'' - \gamma\epsilon^2 w''', \gamma > 0 \tag{5.24}$$

which can be studied in two regions $[-L, -\delta_0]$ and $[-\delta_0, L]$ away from $y = 0$, with the boundary conditions

$$w(-L) = w_l, \qquad w(-\delta_0) = w_*, \tag{5.25}$$

$$w(\delta_0) = w_*, \qquad w(L) = w_r, \tag{5.26}$$

where w_* is given by (4.7) with δ replaced by δ_0 chosen as follows. Following Sections 3 and 4, we can construct a BV solution w for (5.24), in $[-L, -\delta_0]$ and $[\delta_0, L]$ with the boundary conditions (5.25) (5.26), provided

$$\mu(w, y) = \sigma_w + y^2(\frac{1}{4\gamma} - 1) - \frac{\epsilon}{2} > 0.$$

This is the case if we impose the restriction $|y| > \delta_0$ where

$$\delta_0 > (\frac{4c\gamma}{1 - 4\gamma})^{1/2}. \tag{5.27}$$

Here, $c > 0$ is a constant such that $\sigma_w(w) \geq -c$ for all w.

We get the following result :

Theorem 5.5 (Vanishing viscosity-capillarity limit in phase dynamics)**.** *Suppose that the first-order system* (1.2) *is uniformly hyperbolic for large value of w but may admit elliptic regions in the phase space, that is for some constants $M, c_0 > 0$*

$$\inf_{|w| \geq M} \sigma_w(w) \geq {c_0}^2.$$

Suppose also that there exist positive constants c_1, η such that $|\sigma_w| \leq c_1 |w|^{2-\eta}$ for $|w| > 1$. Let $c > 0$ be such that $\sigma_w(w) > -c$ for all w and suppose δ_0 is chosen as in (5.27)*. Then, for all Riemann data v_l, w_l and v_r, w_r in the hyperbolic region of the phase space, i.e.*

$$w_l, w_r \in \mathcal{H} := \Big\{ w \,/\, \sigma_w > 0 \Big\}$$

the viscous-capillarity Riemann problem (1.4)-(1.5)*, with $\delta = \gamma \epsilon^2$, $\gamma > 0$ admits a solution v^ϵ, w^ϵ which has uniformly bounded total variation in $|y| > \delta_0$, and more precisely*

$$\int_{|y| > \delta_0} \Big(|v_\epsilon'| + |w_\epsilon'| \Big) \, dy \lesssim |v_r - v_l| + |w_r - w_l|.$$

The functions v_ϵ, w_ϵ converge pointwise at all $|y| > \delta_0$ to a limit v, w, which is a solution of the Riemann problem (1.2)-(1.3) *for $|y| = |x/t| > \delta_0$.*

For instance, the example $\sigma(w) = u(u^2 - 1)$ satisfies all the assumptions of the theorem. We can not exclude that the vanishing viscosity-capillarity approximations to the Riemann problem could contain a highly oscillating (i.e. weakly but not strongly converging) pattern near the axis; its thickness would be $2\delta_0$, at most. The existence of such stationary waves is not surprising as it has been observed numerically; they were never pointed out analytically until now.

6. Effect of general viscosity in fluid dynamics

Following our earlier work [12] we now return to the general diffusion approximation for strictly hyperbolic systems with general diffusion matrix $B(u)$ of the form

$$-yu' + A(u)u' = \epsilon(B(u)u')', \tag{6.1}$$

and we consider an associated generalized eigenvalue problem.

Let μ_j, $\widehat{l_j}$, $\widehat{r_j}$, $j = 1, 2, \ldots, N$ be the eigenvalues and left and right-eigenvectors given by

$$\begin{aligned} \Big(-y + A(v)\Big)\widehat{r_j}(v,y) &= \mu_j(v,y)\,B(v)\,\widehat{r_j}(v,y), \\ \widehat{l_j}(v,y)\cdot\Big(-y + A(v)\Big) &= \mu_j(v,y)\,\widehat{l_j}(v,y)\,B(v). \end{aligned} \tag{6.2}$$

Let us impose the normalization

$$\widehat{l_i}(v,y)\,B(v)\,\widehat{r_j}(v,y) = 0 \ \text{ if } i \neq j; \quad \widehat{l_j}(v,y)\,B(v)\,\widehat{r_j}(v,y) = 1.$$

In the special case $B(u) = I$ we have simply

$$\mu_j(v,y) = -y + \lambda_j(v), \quad \widehat{r_j}(v,y) = r_j(v), \quad \widehat{l_j}(v,y) = l_j(v). \tag{6.3}$$

So, we expect the eigenvalues and eigenvectors in (6.2) to be close to to that of (6.3), at least in the case that $B(v) = I + \eta\,T(v,\eta)$ where $T(v,\eta) = \big(t_{ij}(v,\eta)\big)$ with $t_{ij}(v,\eta) = O(1)$ and $\eta << 1$.

Lemma 6.1. *If* $|B(v) - I| = O(\eta)$ *is sufficiently small, then*

$$\begin{aligned} &\mu_j(v,y) = -y + \lambda_j(u) + O(\eta), \quad \widehat{l_j}(v,y) = l_j(v) + O(\eta), \\ &\widehat{r_j}(v,y) = r_j(v) + O(\eta), \quad \widehat{l_i}(v,y)\,B(v)\,\partial_y\widehat{r_j}(v,y) = O(\eta). \end{aligned} \tag{6.4}$$

Proof. When B has the form $B(v) = I + \eta\,T(v)$ where the matrix $T(v)$ does not depend on η, the desired estimates follow from classical results. In the slightly more general case when $T = T(v,\eta)$ we may argue as follows. First, note that the eigenvalues under consideration are given by

$$\det\Big(\big(-y\,I + A(v)\big) - \mu\, B(v)\Big) = 0. \tag{6.5}$$

Clearly, (6.5) is a polynomial equation in μ whose coefficients are polynomials in the t_{ij}'s. The leading coefficient $det\big(I + \eta\,(t_{ij})\big)$ is different from zero. Since the roots of (6.5) are distinct for $t_{ij} = 0$, then for η sufficiently small the roots of (6.5) (namely the μ_j's) are distinct and depend smoothly upon the t_{ij}'s with, in addition,

$$\mu_j(v, y) = -y + \lambda_j(v) + \eta \sum_i \lambda_{ij}(v, y)\, t_{ij}(v, \eta) + \dots, \tag{6.6}$$

where the coefficients λ_{ij} are smooth functions in y.

Since the corresponding left- and right-eigenvectors (which need not be normalized at this stage) $\widehat{l_j}$ and $\widehat{r_j}$ are polynomials in a_{ij} and μ_j, it follows that they are also smooth in the a_{ij}'s. Substituting them in (6.3) we get

$$\begin{aligned} \widehat{l_j}(v, y) &= l_j(v) + \eta \sum_i l_{ij}(v, y)\, t_{ij}(v, \eta) + \dots, \\ \widehat{r_j}(v, y) &= r_j(v) + \eta \sum_i r_{ij}(v, y)\, t_{ij}(v, \eta) + \dots, \end{aligned} \tag{6.7}$$

where l_{ij} and r_{ij} depend smoothly upon y. From (6.7), it follows that for η sufficiently small the expansion holds for the normalized vectors $\widehat{l_j}(v, y)$ and $\widehat{r_j}(v, y)$. Finally, the required estimates (6.4) follow from (6.6) and (6.7). This completes the proof of (6.4). □

By the same technique as above we can also prove:

Theorem 6.2. *Under the conditions that $|D_u B(u)| < \eta$ and the characteristic fields associated with $A(u)$ are genuinely nonlinear, the generalized Lax shock inequalities*

$$\hat\lambda_j(u(y+), y) \le y \le \hat\lambda_j(u(y-), y)$$

are equivalent to the standard Lax shock inequalities,

$$\lambda_j(u(y+)) \le y \le \lambda_j(u(y-)),$$

where $\widehat{\lambda_j} = \langle \widehat{r_j}, A\,\widehat{r_j}\rangle$.

Proof. We assume that $\nabla\lambda_j \cdot r_j > 0$. The key observation is that $\nabla\widehat{\lambda_j} \cdot \widehat{r_j}$ is also positive and therefore the same part of the Hugoniot curve is selected by the standard and by the generalized Lax shock inequalities. This is so because $\nabla\widehat{\lambda_j} \cdot \widehat{r_j} = \nabla\lambda_j \cdot r_j + O(\eta)$ as we now check.

When $|D_u B(u)| < \eta$ we see that $D_v T(v,\eta)| \le C$ and that the expansion (6.7) is valid for the v-derivatives as well. This completes the proof. □

We end this section with a further discussion of systems of two equations, and provide some explicit calculations which lead to sufficient conditions on the diffusion matrix B allowing us to apply the techniques introduced earlier in this paper. Note first that (6.5) can be written as

$$\det\Big(\Big(-y\,I + \operatorname{diag}\big(\lambda_1(v), \lambda_2(v)\big) - \mu\, L(v)\, B(v)\, R(v)\Big) = 0, \tag{6.8}$$

where $L(u)$ and $R(u)$ are matrices of left- and right-eigenvectors associated with $A(u)$. Namely, $L(v) := \big(l_1(v), \dots, l_N(v)\big)$ and $R(v) := \big(r_1(v), \dots, r_N(v)\big)$. Setting $L(v)\, B(v)\, R(v) = \big(b_{ij}(v)\big)$, (6.8) becomes

$$\big(\det B(v)\big)\,\mu^2 - \big(b_{11}(v)\,(\lambda_2 - y) + b_{22}(v)\,(\lambda_1 - y)\big)\,\mu + (\lambda_1 - y)\,(\lambda_2 - y) = 0.$$

Solving this quadratic equation in μ and using the notation

$$\beta(v) := \frac{b_{12}(v)\, b_{21}(v)}{b_{11}(v)\, b_{22}(v)}, \quad a_i(v,y) := \frac{\lambda_i(v) - y}{b_{ii}(v)},$$

we arrive at

$$\mu_1(v,y) = \frac{\big(a_2(v,y) + a_1(v,y)\big) - \big(a_2(v,y) - a_1(v,y)\big)\Big(1 + 4\,\beta(v)\,\frac{a_2(v,y)\, a_1(v,y)}{a_2(v,y) - a_1(v,y)}\Big)^{1/2}}{2\,(1 - \beta(v)},$$

$$\mu_2(v,y) = \frac{\big(a_2(v,y) + a_1(v,y)\big) + \big(a_2(v,y) - a_1(v,y)\big)\Big(1 + 4\,\beta(v)\,\frac{a_2(v,y)\, a_1(v,y)}{a_2(v,y) - a_1(v,y)}\Big)^{1/2}}{2\,(1 - \beta(v)}.$$

It is easy to see that $\mu_1(v)$ and $\mu_2(v)$ are real if $0 < \beta(v) < 1$, which is guaranteed if

$$\begin{aligned} &b_{11}(v) > 0, \quad b_{22}(v) > 0, \\ &b_{12}(v)\, b_{21}(v) > 0. \end{aligned} \tag{6.9}$$

The first condition above is also a necessary condition for viscous shocks to be strictly stable in the sense of Majda and Pego [20].

Assuming (6.9), an easy calculation shows that the eigenvalues $\mu_i(v)$ are separated:

$$\mu_2(v) - \mu_1(v) \ge C > 0.$$

It is easy to compute the corresponding right- and left-eigenvectors, namely

$$\widehat{r_i}(v,y) = \Big(- b_{12}(v)\, \mu_i(v,y), \lambda_1(v) - y - b_{11}(v)\, \mu_i(v,y)\Big)^t,$$
$$\widehat{l_i}(v,y) = \Big(- b_{21}(v)\, \mu_i(v,y), \lambda_1(v) - y - b_{11}(v)\, \mu_i(v,y)\Big).$$

Finally, it is easily seen that the estimates (6.4) hold true when $|B(u) - I| = O(\eta)$.

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