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Explicit generalized solutions to a system of conservation laws

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Abstract. This paper studies a special 3 by 3 system of conservation laws which cannot be solved in the classical distributional sense. By adding a viscosity term and writing the system in the form of a matrix Burgers equation an explicit formula is found for the solution of the pure initial value problem. These regularized solutions are used to construct solutions for the conservation laws with initial conditions, in the algebra of generalized functions of Colombeau. Special cases of this system were studied previously by many authors.

Keywords. Conservation laws; Colombeau algebra; generalized solutions.

1. Introduction

In this paper we consider a system of partial differential equations of the form

$$\begin{aligned} &u_t + \left(\frac{u^2}{2}\right)_x = 0,\\ &v_t + (uv)_x = 0,\\ &w_t + \left(\frac{v^2}{2} + uw\right)_x = 0, \end{aligned} \tag{1.1}$$

in $-\infty < x < \infty, t > 0$, supplemented with an initial condition at $t = 0$,

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \tag{1.2}$$

where $u_0(x), v_0(x); w_0(x)$ are bounded measurable functions. The system is not strictly hyperbolic. In fact the eigenvalues of the Jacobian matrix of $F(u,v,w) = ((u^2/2), uv, (v^2/2) + uw)$ are equal, namely u and classical theory of conservation laws does not apply. Even for Riemann initial data (1.1) and (1.2) cannot be solved in the class of classical simple waves. For example when the initial data,

$$\begin{aligned} &u(x,0) = 1, x < 0, u(x,0) = -1, x > 0,\\ &v(x,0) = v_l, x < 0, v(x,0) = v_r, x > 0,\\ &w(x,0) = w_l, x < 0, w(x,0) = w_r, x > 0, \end{aligned} \tag{1.3}$$

where v_l, v_r, w_l, w_r are constants, a simple wave solution for (1.1) with initial data (1.3) can be found if and only if $v_l + v_r = 0$ and $w_l + w_r = 0$. This follows easily from the observation that the entropy weak solution of the first equation of (1.1) with the above data is $u(x,t) = 1$ if $x < 0$ and -1 if $x > 0$; a shock wave with speed $s = 0$. If $v_l + v_r$ is not equal to zero, then the v component contains a δ measure along $x=0$, see Joseph [9]. Though the product, uv, does not make sense in the classical theory of distributions, this product can be defined in the sense of Dalmaso–LeFloch–Murat [6], but not, v^2, a square of δ measure. To overcome such difficulties, Colombeau [2] introduced a new notion of generalized functions. In recent works [1, 3–5, 10, 13] and in many other references there, it is recognized by many authors that the generalized functions of Colombeau is a convenient setup to seek global solutions, where such difficulties arise. Further, this approach takes into account the microscopic structure of the shocks in the solutions. To do this we study (1.1) and (1.2) by the vanishing viscosity method. We obtain an explicit formula for the solution with viscous terms in the equation. We study the limit of these solutions as ϵ tends to 0 and construct a solution in the algebra of the generalized functions of Colombeau. Similar results based also on the viscous approximation to Riemann problem for a different system were recently published by Hu [8].

This paper is organized in the following way. In § 2, we recall the definition of the algebra of the generalized functions of Colombeau and the definition of association. In § 3 we get explicit formula for the solution with the viscous term and in § 4 we show that it is indeed in the algebra of generalized functions and is solution in the sense of association of Colombeau. The paper concludes with some remarks on a general system with viscous terms.

2. Colombeau algebras

We take the domain $\Omega=(x,t), -\infty<x<\infty, t>0$. Consider $C^\infty(\Omega)$, the class of infinitely differentiable functions in Ω and take the infinite product $\varepsilon(\Omega)=[C^\infty(\Omega)]^{(0,1)}$. Thus any element u of $\varepsilon(\Omega)$ is a map from (0,1) to $C^\infty(\Omega)$ and is denoted by $u=(u^\epsilon)_{0<\epsilon<1}$. We take a subclass $\varepsilon_{\mathcal{M}}(\Omega)$, called the moderate elements of $\varepsilon(\Omega)$. An element $u=(u^\epsilon)_{0<\epsilon<1}$ is called moderate if given a compact subset K of Ω and j and l nonnegative integers, there exists $N>0$ such that

$$||\partial_t^j\partial_x^l u^\epsilon||_{L^\infty(K)}=\mathcal{O}(\epsilon^{-\mathcal{N}}) \tag{2.1}$$

as ϵ tends to 0. An element $u=(u^\epsilon)_{0<\epsilon<1}$ is called null if for all compact subsets K of Ω and for all nonnegative integers j and l and for all $M>0$,

$$||\partial_t^j\partial_x^l u^\epsilon||_{L^\infty(K)}=\mathcal{O}(\epsilon^{\mathcal{M}}), \tag{2.2}$$

as ϵ goes to 0. The set of null elements is denoted by $\mathcal{N}(\Omega)$. It is easy to see that $\varepsilon_{\mathcal{M}}(\Omega)$ is an algebra with partial derivatives, the operations being defined pointwise on representatives and $\mathcal{N}(\Omega)$ is an ideal which is closed under differentiation. The quotient space denoted by

$$\mathcal{G}(\Omega)=\frac{\varepsilon_{\mathcal{M}}(\Omega)}{\mathcal{N}(\Omega)}$$

is an algebra with partial derivatives, the operations being defined on representatives. The algebra $\mathcal{G}(\Omega)$ is called the algebra of generalized functions of Colombeau. Two elements u and v in $\mathcal{G}(\Omega)$ are said to be associated, if for some (and hence all) representatives $(u^\epsilon)_{0<\epsilon<1}$ and $(v^\epsilon)_{0<\epsilon<1}$, of u and v, $u_\epsilon - v_\epsilon$ goes to 0 as ϵ tends to 0, in the sense of distribution and is denoted by $u\approx v$. Here we remark that this notion is different from the notion of equality in $\mathcal{G}(\Omega)$, which means that $u - v \in \mathcal{N}(\Omega)$, or in other words,

$$\|\partial_t^j \partial_x^l (u^\epsilon - v^\epsilon)\|_{L^\infty(K)} = \mathcal{O}(\epsilon^{\mathcal{M}})$$

for all M, for all j, l nonnegative integers and for all compact subsets K of Ω.

3. Explicit formula with viscous term

In this section we consider the system

$$\begin{aligned} u_t + \left(\frac{u^2}{2}\right)_x &= \frac{\epsilon}{2} u_{xx}, \\ v_t + (uv)_x &= \frac{\epsilon}{2} v_{xx}, \\ w_t + \left(\frac{v^2}{2} + uw\right)_x &= \frac{\epsilon}{2} w_{xx}, \end{aligned} \tag{3.1}$$

in $-\infty < x < \infty, t > 0$ with initial conditions

$$u(x,0) = u_0^\epsilon(x),\ v(x,0) = v_0^\epsilon(x),\ w(x,0) = w_0^\epsilon(x), \tag{3.2}$$

at $t = 0$. Let us denote

$$F^\epsilon(x,y,t) = U_0^\epsilon(y) + \frac{(x-y)^2}{2t}, \tag{3.3}$$

where

$$U_0^\epsilon(x) = \int_0^x u_0^\epsilon(y)\mathrm{d}y, \tag{3.4}$$

then we have the following theorem.

Theorem 3.1. *Let $u_0^\epsilon(x)$, $v_0^\epsilon(x)$ and $w_0^\epsilon(x)$ be bounded measurable functions on R^1 for each ϵ positive, then*

$$u^\epsilon(x,t) = \partial_x U^\epsilon(x,t),\ v^\epsilon(x,t) = \partial_x V^\epsilon(x,t),\ w^\epsilon(x,t) = \partial_x W^\epsilon(x,t), \tag{3.5}$$

where U^ϵ, V^ϵ and W^ϵ are given by

$$U^\epsilon(x,t) = -\epsilon \log\left[\frac{1}{(2\pi t\epsilon)(1/2)} \int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y\right], \tag{3.6}$$

$$V^\epsilon(x,t) = \frac{\int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}, \tag{3.7}$$

$$W^\epsilon(x,t) = \frac{\int_{-\infty}^{+\infty} \left(W_0^\epsilon(y) - \frac{V_0^\epsilon(y)^2}{2\epsilon}\right) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y} + \frac{1}{2\epsilon}\left[\frac{\int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{d}y}\right]^2 \tag{3.8}$$

is a solution for (3.1)–(3.2).

Proof. To prove the theorem first we note that this system (3.1) can be written as a matrix Burgers equation

$$A_t + \left(\frac{A^2}{2}\right)_x = \frac{\epsilon}{2} A_{xx}, \tag{3.9}$$

where A is the lower triangular matrix of the form

$$A = \begin{pmatrix} u & 0 & 0 \\ v & u & 0 \\ w & v & u \end{pmatrix} \tag{3.10}$$

with initial condition

$$A(x,0) = \begin{pmatrix} u_0^\epsilon(x) & 0 & 0 \\ v_0^\epsilon(x) & u_0^\epsilon(x) & 0 \\ w_0^\epsilon(x) & v_0^\epsilon(x) & u_0^\epsilon(x) \end{pmatrix}. \tag{3.11}$$

Now we use Hopf–Cole transformation, see [7], generalized to matrix equations (3.9), where we use the fact $AA_t = A_tA, AA_x = A_xA$,

$$C = \exp\left(-\frac{B}{\epsilon}\right), \tag{3.12}$$

with

$$B = \begin{pmatrix} U & 0 & 0 \\ V & U & 0 \\ W & V & U \end{pmatrix}, \tag{3.13}$$

where U, V, W are given by

$$U(x,t) = \int_0^x u(y,t)\mathrm{d}y, V(x,t) = \int_0^x v(y,t)\mathrm{d}y, W(x,t) = \int_0^x w(y,t)\mathrm{d}y. \tag{3.14}$$

Using (3.12)–(3.14) in (3.9)–(3.10), we see that C satisfies the equation

$$C_t = \frac{\epsilon}{2} C_{xx},$$
$$C(x,0) = C_0(x),$$

where $C_0(x)$ is the matrix

$$C_0(x) = \begin{pmatrix} a_0(x) & 0 & 0 \\ b_0(x) & a_0(x) & 0 \\ c_0(x) & b_0(x) & a_0(x) \end{pmatrix}$$

with

$$a_0(x) = \exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right),$$
$$b_0(x) = -\frac{V_0^\epsilon(x)}{\epsilon}\exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right),$$
$$c_0(x) = \left(\frac{V_0^\epsilon(x)^2}{2\epsilon^2} - \frac{W_0^\epsilon(x)}{\epsilon}\right)\exp\left(-\left[\frac{U_0^\epsilon(x)}{\epsilon}\right]\right).$$

On solving this explicitly we find that its solution takes the form of the following lower triangular matrix

$$C^\epsilon = \begin{pmatrix} a^\epsilon & 0 & 0 \\ b^\epsilon & a^\epsilon & 0 \\ c^\epsilon & b^\epsilon & a^\epsilon \end{pmatrix},$$

where $a^\epsilon, b^\epsilon; c^\epsilon$ are given by

$$a^\epsilon = \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y, \tag{3.15}$$

$$b^\epsilon = \frac{-1}{\epsilon} \cdot \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} V_0^\epsilon(y) \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y, \tag{3.16}$$

$$c^\epsilon = \frac{1}{(2\pi t\epsilon)^{(1/2)}} \int_{-\infty}^{+\infty} \left[\frac{V_0^\epsilon(y)^2}{2\epsilon^2} - \frac{W_0^\epsilon(y)}{\epsilon}\right] \exp\left[-\frac{F^\epsilon(x,y,t)}{\epsilon}\right] \mathrm{d}y. \tag{3.17}$$

Now to get back U^ϵ, V^ϵ and W^ϵ we use (3.12), namely

$$B^\epsilon = -\epsilon \log(C^\epsilon).$$

An easy calculation gives

$$B^\epsilon = \begin{pmatrix} U^\epsilon & 0 & 0 \\ V^\epsilon & U^\epsilon & 0 \\ W^\epsilon & V^\epsilon & U^\epsilon \end{pmatrix},$$

where

$$U^\epsilon = -\epsilon \log(a^\epsilon), V^\epsilon = -\epsilon \frac{b^\epsilon}{a^\epsilon}, W^\epsilon = \epsilon\left(-\frac{c^\epsilon}{a^\epsilon} + \frac{(b^\epsilon)^2}{2(a^\epsilon)^2}\right). \tag{3.18}$$

Now substituting the expressions (3.15)–(3.17) for a^ϵ, b^ϵ and c^ϵ in (3.18) we get the formulas (3.6)–(3.8) for $U^\epsilon, V^\epsilon, W^\epsilon$. Now it follows from (3.14) that

$$u^\epsilon = U_x^\epsilon, v^\epsilon = V_x^\epsilon, w^\epsilon = W_x^\epsilon.$$

The proof of the theorem is complete.

In order to study the limit of the functions $U^\epsilon, V^\epsilon, W^\epsilon$ given by (3.6)–(3.8) as ϵ goes to 0 we use the following result in the spirit of Hopf [7] and Lax [11].

PROPOSITION 3.2

Let $u_0(x)$ be bounded measurable and $p(x)$ Lipshitz continuous and both independent of ϵ. Let $F(x,y,t) = U_0(y) + ((x-y)^2/2t)$, where $U_0(x) = \int_0^x u_0(y)\mathrm{d}y$, then

(1) *For each $t > 0$ and $-\infty < x < \infty$, there exists at most a finite number of minimizers $y_0(x,t)$ for*

$$\min_{-\infty<y<+\infty} F(x,y,t). \tag{3.19}$$

For each (x,t) define maximum and minimum of these minimizers $y_0(x,t)$

$$y_0^+(x,t) = \max[y_0(x,t)],$$

$$y_0^-(x,t) = \min[y_0(x,t)],$$

then for each $t > 0$, except for a countable set of x, $y_0^+(x,t) = y_0^-(x,t)$.

(2) *For each $t > 0$, the limit,*

$$\lim_{\epsilon\to 0} \frac{\int_{-\infty}^{+\infty} p(y)\exp(-F(x,y,t)/\epsilon)\mathrm{d}y}{\int_{-\infty}^{+\infty} \exp(-F(x,y,t)/\epsilon)\,\mathrm{d}y} = p(y_0(x,t)),$$

exists except for a countable set of x. Also at every point $(x,t), p(y_0(x+,t))$ and $p(y_0(x-,t))$ exist.

Remark. Since the speed of propagation of the inviscid case is bounded by the L^∞ norm of u_0 we can restrict to initial data with compact support. We restrict to special initial data and using the above Proposition and from the explicit formula for U^ϵ, V^ϵ and W^ϵ given in theorem (3.1), the following is immediate.

PROPOSITION 3.3

Let $u_0^\epsilon(x) = (u_0 * \phi^\epsilon)(x)$, $v_0^\epsilon(x) = (v_0 * \phi^\epsilon)(x)$, $w_0^\epsilon(x) = (w_0 * \phi^\epsilon)(x)$ where u_0, v_0 and w_0 are bounded measurable functions with compact support and ϕ^ϵ is the usual Friedrichs mollifier with U^ϵ, V^ϵ and W^ϵ are as given by (3.6)–(3.8), then for each $t > 0$, the limits $\lim_{\epsilon\to 0} \epsilon u^\epsilon$, $\lim_{\epsilon\to 0} V^\epsilon$ and $\lim_{\epsilon\to 0} \epsilon W^\epsilon$ exists except for a countable x and is given by

$$\lim_{\epsilon\to 0} \epsilon u^\epsilon = 0, \quad \lim_{\epsilon\to 0} V^\epsilon = V_0(y_0(x,t)), \quad \lim_{\epsilon\to 0} \epsilon W^\epsilon = 0,$$

where $y_0(x,t)$ is a minimizer in (3.19).

Proof. First we notice that since

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^x u_0^\epsilon(y)\mathrm{d}y &= \int_0^x u_0(y)\mathrm{d}y, \lim_{\epsilon\to 0}\int_0^x v_0^\epsilon(y)\mathrm{d}y \\ &= \int_0^x v_0(y)\mathrm{d}y, \lim_{\epsilon\to 0}\int_0^x u_0^\epsilon(y)\mathrm{d}y = \int_0^x w_0(y)\mathrm{d}y,\end{aligned}$$

uniformly on R^1 as ϵ goes to zero, the conclusions follow from the expressions (3.6)–(3.8) and proposition (3.2).

4. Generalized solutions for (1.1) and (1.2)

In this section we solve the problem

$$\begin{aligned} &u_t + \left(\frac{u^2}{2}\right)_x \approx 0, \\ &v_t + (uv)_x \approx 0, \\ &w_t + \left(\frac{v^2}{2} + vw\right)_x \approx 0, \end{aligned} \tag{4.1}$$

with initial conditions

$$u(x,0) = u_0, v(x,0) = v_0, w(x,0) = w_0, \tag{4.2}$$

where $u_0 = (u_0^\epsilon(x))_{0<\epsilon<1}$, $v_0 = (v_0^\epsilon(x))_{0<\epsilon<1}$ and $w_0 = (w_0^\epsilon(x))_{0<\epsilon<1}$ are in $\mathcal{G}(R^1)$, the algebra of generalized functions. Here we assume that $u_0^\epsilon(x)$, $v_0^\epsilon(x)$ and $w_0^\epsilon(x)$ are obtained by mollifying compactly supported bounded measurable functions $u_0(x)$, $v_0(x)$ and $w_0(x)$ respectively with Friedrichs mollifiers so that we have the following estimates

$$\begin{aligned}
&\|\partial_x^l u_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}),\\
&\|\partial_x^l v_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}),\\
&\|\partial_x^l w_0^\epsilon\|_{L^\infty(R^1)} = \mathcal{O}(\epsilon^{-l}).
\end{aligned} \tag{4.3}$$

Now we state our main existence result.

Theorem 4.1. *Let $u = (u^\epsilon)_{0<\epsilon<1}$, $v = (v^\epsilon)_{0<\epsilon<1}$ and $w = (w^\epsilon)_{0<\epsilon<1}$, where u^ϵ, v^ϵ and w^ϵ are given by* (3.5)–(3.8) *with $u_0^\epsilon(x)$, $v_0^\epsilon(x)$, and $w_0^\epsilon(x)$ are as described above, then u, v and w are in the algebra of generalized functions of Colombeau, $\mathcal{G}(\Omega)$ and solve the problem* (4.1)–(4.2).

Proof. First we show that $u = (u^\epsilon)$, $v = (v^\epsilon)$ and $w = (w^\epsilon)$ are in $\mathcal{G}(\Omega)$. For this we have to verify the estimate (2.1), for $(u^\epsilon)_{0<\epsilon<1}$, $(v^\epsilon)_{0<\epsilon<1}$ and $(w^\epsilon)_{0<\epsilon<1}$. It is clear from the formulas (3.5)–(3.8) for u^ϵ, v^ϵ and w^ϵ that, they are $C^\infty(\Omega)$. Further a typical term in the expressions of these functions is of the form $\epsilon^{-k} H^\epsilon(x,t)$ for $k = 0,1,2$ with

$$H^\epsilon(x,t) = \frac{H_1^\epsilon(x,t)}{H_2^\epsilon(x,t)}$$

and $H_1^\epsilon(x,t)$ and $H_2^\epsilon(x,t)$ taking the form

$$\begin{aligned}
H_1^\epsilon(x,t) &= \int_{-\infty}^{+\infty} H_0^\epsilon(y)\exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{dy},\\
H_2^\epsilon(x,t) &= \int_{-\infty}^{+\infty} \exp\left(-\frac{F^\epsilon(x,y,t)}{\epsilon}\right)\mathrm{dy},
\end{aligned}$$

and H_0^ϵ satisfying estimates of the form (4.3). Now by Leibinitz's rule $\partial_x^{j_0} H^\epsilon$ is a finite linear combination of elements of the form

$$\frac{\partial_x^{j_k} H_1^\epsilon}{H_2^\epsilon}\cdot\frac{\partial_x^{(j_{k-1}-j_k)} H_2^\epsilon}{H_2^\epsilon}\cdots\frac{\partial_x^{j_0-j_1} H_2^\epsilon}{H_2^\epsilon},\quad j_k < j_{k-1} < j_1 < j_0,\ k = 0,1,\ldots,j_0.$$

Now making a change of variable $y = x - \sqrt{2t\epsilon z}$ in the integrals of H_1^ϵ and H_2^ϵ and using (4.3) we get,

$$\left\|\frac{\partial_x^j H_1^\epsilon}{H_2^\epsilon}\right\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}),\quad \left\|\frac{\partial_x^j H_2^\epsilon}{H_2^\epsilon}\right\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}).$$

These estimates together with our earlier observation on the form of u^ϵ, v^ϵ and w^ϵ leads to the estimates

$$\|\partial_x^j u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-j}),\ \|\partial_x^j v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+1)}),\ \|\partial_x^j w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+2)}). \tag{4.4}$$

Now from the PDE (3.1) and (4.4) we get

$$\|\partial_t u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-1}),\ \|\partial_t v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-2}),\ \|\partial_t w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-3}). \tag{4.5}$$

Now applying the differential operator $\partial_t^j \partial_x^l$ on both sides of (3.1), first $l = 1, j = 0, 1, 2, \ldots$ and then $l = 2, j = 0, 1, 2, \ldots$ etc; proceeding successively we get the following estimate. For each j and l nonnegative integers,

$$\|\partial_t^j \partial_x^l u^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l)}),$$
$$\|\partial_t^j \partial_x^l v^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l+1)}),$$
$$\|\partial_t^j \partial_x^l w^\epsilon\|_{L^\infty(\Omega)} = \mathcal{O}(\epsilon^{-(j+l+2)}).$$

These estimates show that u, v and w are in $\mathcal{G}(\Omega)$. Now to show that u, v, and w satisfy eq. (1.1) in the sense of association we multiply (3.1) by a test function ϕ and integrate by parts to get

$$-\int_0^\infty \int_{-\infty}^\infty \left(u^\epsilon \phi_t + \frac{(u^\epsilon)^2}{2}\phi_x\right) \mathrm{d}x\,\mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty u^\epsilon \phi_{xx} \mathrm{d}x\,\mathrm{d}t,$$
$$\int_0^\infty \int_{-\infty}^\infty (v^\epsilon \phi_t + (u^\epsilon v^\epsilon)\phi_x) \mathrm{d}x\,\mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty V^\epsilon \phi_{xxx} \mathrm{d}x\,\mathrm{d}t,$$
$$\int_0^\infty \int_{-\infty}^\infty \left(w^\epsilon \phi_t + \left(\frac{(v^\epsilon)^2}{2} + u^\epsilon v^\epsilon\right)\phi_x\right) \mathrm{d}x\,\mathrm{d}t = \frac{\epsilon}{2}\int_0^\infty \int_{-\infty}^\infty W^\epsilon \phi_{xxx} \mathrm{d}x\,\mathrm{d}t.$$

It follows from the asumption (4.3) on the initial data and the formulas (3.5)–(3.8) for u^ϵ and V^ϵ and ϵW^ϵ that these are uniformly bounded. Further by Proposition (3.3) and an application of dominated convergence theorem it follows that the right hand side of each of the above equations goes to 0 as ϵ goes to 0. This completes the proof of the theorem.

5. Concluding remarks

In general we could use Hopf–Cole transformation to find explicit solutions for any system of equations of the form

$$A_t + \frac{(A^2)_x}{2} = \frac{\epsilon}{2} A_{xx}, \tag{5.1}$$

with initial condition at $t = 0$,

$$A(x, 0) = A_0(x), \tag{5.2}$$

where A is a lower triangular matrix of the form

$$A = \begin{pmatrix} u_1 & 0 & 0 & 0 & 0 & 0 \\ u_2 & u_1 & 0 & 0 & 0 & 0 \\ u_3 & u_2 & u_1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & u_1 & 0 \\ u_n & u_{n-1} & \cdot & \cdot & u_2 & u_1 \end{pmatrix}. \tag{5.3}$$

Component wise (5.1) and (5.3) gives a system of n equation for the unknowns $u_1, u_2, \ldots, u_n$ namely

$$(u_j)_t + \sum_{i=1}^{j} \left(\frac{u_i u_{j-i+1}}{2}\right)_x = \frac{\epsilon}{2}(u_j)_{xx}, \tag{5.4}$$

for $j = 1, 2, \ldots, n$. The case, $n = 1$, in (5.4) is the standard Burgers equation and Hopf [7], used the Hopf–Cole transformation to construct solutions for $\epsilon > 0$ and obtained global solution for the inviscid Burgers equation with bounded measurable initial data by passing to the limit ϵ tends to 0. The limit function remains to be bounded and hence the solutions are understood in the standard theory of distributions. If one considers more general initial data such as bounded Borel measures, standard distribution theory does not work and so Biagoni and Oberguggenberger [1] constructed global solutions, in the algebra of generalized functions of Colombeau, for the case of more general initial data. In the case, $n = 2$ even for Riemann data the limit function contains δ-measures and this case was treated by Joseph [9], see also LeFloch [12], for a more general 2 by 2 system where the theory of DalMaso–LeFloch–Murat [6], applies. The explicit solutions of the initial value problem for the system (5.4) is complicated for general n, however the general feature of the solution for $n = 3, 4, \ldots$ remains the same. In fact, as ϵ tends to zero, the order of 'singularities' in the solution of (5.4), increases as n increases and we cannot use the method of [6], to define some of the products which appear in the equation and get global solutions for the inviscid case. In the present paper we have shown that we can use Colombeau's theory to get a global existence result for the inviscid case for $n = 3$.

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