Hartman effect and non-locality in quantum networks

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We study the phase time for various quantum mechanical networks having potential barriers in its arms to find the generic presence of Hartman effect. In such systems it is possible to control the 'super arrival' time in one of the arms by changing parameters on another, spatially separated from it. This is yet another quantum nonlocal effect. Negative time delays (time advancement) and 'ultra Hartman effect' with negative saturation times have been observed in some parameter regimes.

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I. INTRODUCTION

Quantum tunneling, where a particle has finite probability to penetrate a classically forbidden region is an important feature of wave mechanics. Invention of the tunnel diode [1], the scanning tunneling microscope [2] etc. make it useful from a technological point of view. In 1932 MacColl [3] pointed out that tunneling is not only characterized by a tunneling probability but also by a time the tunneling particle takes to traverse the barrier. There is considerable interest on the question of time spent by a particle in a given region of space [4, 5, 6]. The recent development of nanotechnology brought new urgency to study the tunneling time as it is directly related to the maximum attainable speed of nanoscale electronic devices. In a number of numerical [7], experimental [8, 9] and analytical study of quantum tunneling processes, various definitions of tunneling times have been investigated. These different time scales are based on various different operational definitions and physical interpretations. Till date there is no clear consensus about the existence of a simple expression for this time as there is no hermitian operator associated with it [4]. Among the various time scales, 'dwell time' [10] which gives the duration of a particle's stay in the barrier region regardless of how it escapes can be calculated as the total probability of the particle inside the barrier divided by the incident probability current. Büttiker and Landauer proposed [11] that one should study 'tunneling time' using the transmission coefficient through a static barrier of interest, supplemented by a small oscillatory perturbation. A large number of researchers interpret the phase delay time [5, 12, 13] as the temporal delay of a transmitted wave packet. This time is usually taken as the difference between the time at which the peak of the transmitted packet leaves the barrier and the time at which the peak of the incident Gaussian wave packet arrives at the barrier. Within the stationary phase approximation the phase time can be calculated from the energy derivative of the 'phase shift' in the transmitted or reflected amplitudes. Büttiker-Landauer [11] raised objection that the peak is not a reliable characteristic of packets distorted during the tunneling process. In contrast to 'dwell time' which can be defined locally, the 'phase time' is essentially asymptotic in character [14]. The 'phase time' statistics is intimately connected with dynamic admittance of micro-structures [15]. This 'phase time' is also directly related to the density of states [16]. The universality of 'phase time' distributions in random and chaotic systems has already been established earlier [17]. In the case of 'not too opaque' barriers, the tunneling time evaluated either as a simple 'phase time' [5] or calculated through the analysis of the wave packet behaviour [18] becomes independent of the barrier width. This phenomenon is termed as the Hartman effect [13, 18, 19]. This implies that for sufficiently large barriers the effective velocity of the particle can become arbitrarily large, even larger than the light speed in the vacuum (superluminal effect). Though this interpretation is a little far fetched for non-relativistic Schrd̈inger equation as velocity of light plays no role in it, this effect has been established even in relativistic quantum mechanics.

Though experiments with electrons for verifying this prediction are yet to be done, the formal identity between the Schrödinger equation and the Helmholtz equation for electromagnetic wave enables one to correlate the results for electromagnetic and microwaves to that for electrons. Photonic experiments show that electromagnetic pulses travel with group velocities in excess of the speed of light in vacuum as they tunnel through a constriction in a waveguide [20]. Experiments with photonic band-gap structures clearly demonstrate that 'tunneling photons' indeed travel with superluminal group velocities [8]. Their measured tunneling time is practically obtained by comparing the two peaks of the incident and

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transmitted wave packets. Thus all these experiments directly or indirectly confirmed the occurrence of Hartman effect without violating 'Einstein causality' *i.e.*, the signal velocity or the information transfer velocity is always bounded by the velocity of light. This tunneling time could be interpreted as the 'time of passage' of the peak. Since the velocity of the 'peak' may exceed arbitrarily large numbers, this 'fast tunneling' has been frequently related to 'superluminal propagation' [20]. The 'Hartman effect' has been extensively studied both for nonrelativistic (Schrödinger equation) and relativistic (Dirac equation) [4, 5, 20] cases. Recently Winful [21] showed that the saturation of phase time is a direct consequence of saturation of integrated probability density under the barrier. Due to this saturation addition of a new particle in the incident side leads to an almost immediate release of another particle from the other side and these two particles are causally unrelated; *i.e.* even the superluminal tunneling does not violate causality. Equivalently, for electromagnetic waves the origin of the Hartman effect has been traced to stored energy. Since the stored energy in the evanescent field decreases exponentially within the barrier after a certain decay distances it becomes independent of the width of the barrier. The Hartman effect has been found in one dimensional barrier tunneling [18] as well as for cases beyond one dimension as in tunneling through mesoscopic rings in presence of Aharonov-Bohm flux [22]. In the current note we extend the study of phase times for branched networks of quantum wires. Such networks can readily be realized in optical wave propagation experiments.

In Section-II we give the description of typical quantum network systems under consideration. The theoretical framework used to analyze these systems is provided in Section-III. All the main results showing Hartman effect and the effects of quantum non-locality are discussed in Section-IV. Finally we summarize the results and draw conclusions in Section-V.

II. DESCRIPTION OF THE SYSTEM

As a model system, we choose a network of thin wires. The width of these wires are so narrow that only the motion along the length of the wires is of interest (a single channel case). The motion in the perpendicular direction is frozen in the lowest transverse subband. In a three-port Y-branch circuit (Fig. 1) two side branches of quantum wire S_1 and S_2 are connected to a 'base' arm S_0 at the junction J. In general one can have $N(\geq 2)$ such side branches connected to a 'base' wire.

We study the scattering problem across a network geometry as presented schematically in Fig. 1.

Such geometries are important from the point of view of basic science due to their properties of tunneling and interference [23, 24] as well as in applications such as wiring in nano-structures. In particular, the Y-junction carbon nanotubes are in extensive studies and they show

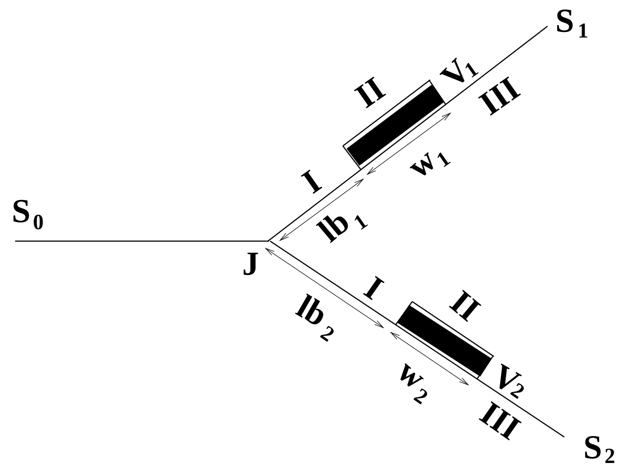

FIG. 1: Schematic diagram of a Y-junction or three-way splitter.

various interesting properties like asymmetric current voltage characteristics [25]. In our system of interest there are finite quantum mechanical potential barriers of strength V_n and width w_n in the n-th side branch. The number of side branches can vary from $n = 2, 3, \cdots N$. We focus on a situation wherein the incident electrons have an energy E less than V_n for all n. The impinging electrons in this subbarrier regime travels as an evanescent mode/wave and the transmission involve contributions from quantum tunneling and multiple reflections between each pair of barriers and the junction point. Here we are interested in a single channel case where the Fermi energy lie in the lowest subband. To excite the evanescent modes in the side branches one has to produce constrictions by making the width of the regions of wires containing barriers much thinner than that of other parts of the wires. The electrons occupying the lowest subband in the connecting wire on entering the constrictions experience a potential barrier (due to higher quantum zero point energy) and propagate as an evanescent mode [27]. In this work an analysis of the phase time or the group delay time in such a system is carried out.

III. THEORETICAL TREATMENT

We approach this scattering problem using the quantum wave guide theory [26, 28]. In the stationary case the incoming particles are represented by a plane wave e^{ikx} of unit amplitude. The effective mass of the propagating particle is m and the energy is $E = \hbar^2 k^2/2m$ where k is the wave vector corresponding to the free particle. The wave functions, which are solutions of the Schrödinger equation, in different regions of the system considered (Fig. 1) can be written as,

$$\begin{aligned}
\psi_{in}(x_0) &= e^{ikx_0} + Re^{-ikx_0} \quad (\text{in } S_0)\,,\\
\psi_{(n)_I}(x_n) &= A_n\, e^{i\,k\,x_n} + B_n\, e^{-i\,k\,x_n} \quad (\text{region I in } S_n)\,,\\
\psi_{(n)_{II}}(x_n) &= C_n\, e^{-\kappa_n\,(x_n - lb_n)} + D_n\, e^{\kappa_n\,(x_n - lb_n)} \quad (\text{region II in } S_n)\,,\\
\psi_{(n)_{III}}(x_n) &= t_n\, e^{i\,k\,(x_n - lb_n - w_n)} \quad (\text{region III in } S_n)\,,
\end{aligned}$$

with $\kappa_n = \sqrt{2m(V_n - E)/\hbar^2}$ being the imaginary wave vector in presence of rectangular barrier of strength V_n. $\psi_{(n)_I}, \psi_{(n)_{II}}$ and $\psi_{(n)_{III}}$ denote wave functions in three regions I, II and III, respectively, on n-th side branch. x_0 is the spatial coordinate for the 'base' wire, whereas x_n is spatial coordinate for the n-th arm. All these coordinates are measured from the junction J. In n-th side branch, the barrier starts at a distance lb_n from the junction J.

We use Griffith's boundary conditions [29]

$$\psi_{in}(J) = \psi_{(n=1)_I}(J) = \psi_{(n=2)_I}(J) = \cdots = \psi_{(n=N)_I}(J), \tag{1}$$

and

$$\frac{\partial \psi_{in}(x_0)}{\partial x_0}\Big|_J = \Sigma_n \frac{\partial \psi_{(n)_I}}{\partial x_n}\Big|_J\,, \tag{2}$$

at the junction J. All the derivatives are taken either outward or inward from the junction [26]. In each side branch, at the starting and end points of the barrier, the boundary conditions can be written as

$$\psi_{(n)_I}(lb_n) = \psi_{(n)_{II}}(lb_n)\,, \tag{3}$$

$$\psi_{(n)_{II}}(lb_n + w_n) = \psi_{(n)_{III}}(lb_n + w_n)\,, \tag{4}$$

$$\frac{\partial \psi_{(n)_I}}{\partial x_n}\Big|_{(lb_n)} = \frac{\partial \psi_{(n)_{II}}}{\partial x_n}\Big|_{(lb_n)}\,, \tag{5}$$

$$\frac{\partial \psi_{(n)_{II}}}{\partial x_n}\Big|_{(lb_n + w_n)} = \frac{\partial \psi_{(n)_{III}}}{\partial x_n}\Big|_{(lb_n + w_n)}\,. \tag{6}$$

From the above mentioned boundary conditions one can obtain the complex transmission amplitudes t_n on each of the side branches.

IV. RESULTS AND DISCUSSIONS

Once t_n is known, the 'phase time' (phase time for transmission) can be calculated from the energy derivative of the phase of the transmission amplitude [5, 12] as

$$\tau_n = \hbar\, \frac{\partial Arg[t_n]}{\partial E}\,, \tag{7}$$

where, $v = \hbar k/m$ is the velocity of the free particle. The concept of 'phase delay time' was first introduced by Wigner [12] to estimate how long a quantum mechanical wave packet is delayed by the scattering obstacle.

In what follows, let us set $\hbar = 1$ and $2m = 1$. We now proceed to analyze the behavior of τ_n as a function of various physical parameters for different network topologies. We measure time at the far end of each barrier in the branched arms containing barriers and in the case of arms in absence of any barrier we measure the phase time at the junction points. We express all the physical quantities in dimensionless units *i.e.* all the barrier strengths V_n in units of incident energy E ($V_n \equiv V_n/E$), all the barrier widths w_n in units of inverse wave vector k^{-1} ($w_n \equiv kw_n$), where $k = \sqrt{E}$ and all the extrapolated phase time τ_n in units of inverse of incident energy E ($\tau_n \equiv E\tau_n$).

First we take up a system similar to the Y-junction shown in Fig. 1 in presence of a barrier V_1 of width w_1 in arm S_1 but in absence of any barrier in arm S_2. For a tunneling particle having energy $E < V_1$ we find out the phase time τ_1 in arm S_1 as well as τ_2 in arm S_2 as a function of barrier width w_1 (Fig. 2). From Fig. 2(a) it is clear that τ_1 evolves with w_1 and eventually saturates to τ_{s1} for large w_1 to show the Hartman effect. Fig. 2(b)

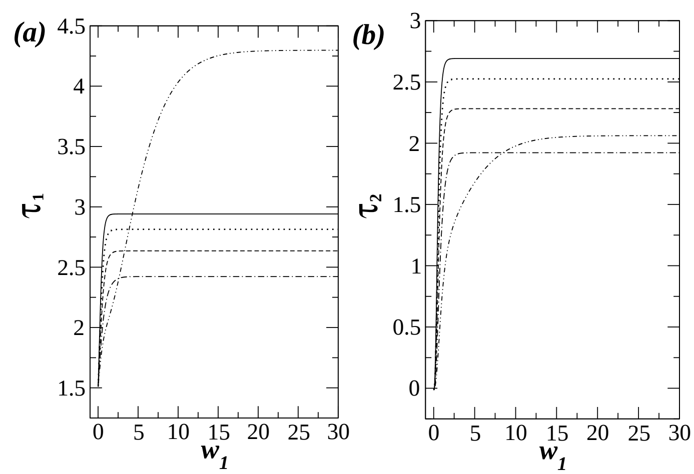

FIG. 2: For a 3-way splitter with a barrier in S_1 arm, the 'phase times' τ_1 and τ_2 are plotted as a function of barrier width 'w_1' in (a) and (b) respectively. The solid, dotted, dashed, dot-dashed and the dashed-double dotted curves are for $V_1 = 5, 4, 3, 2,$ and 1.05 respectively. Other system parameters are $E = 1, lb_1 = 3$.

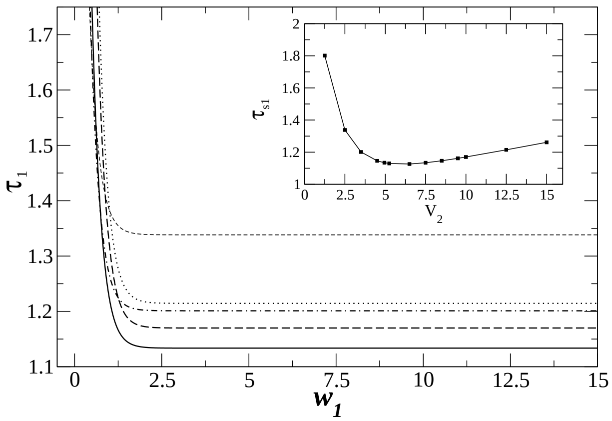

FIG. 3: Here for a 3-way splitter with one barrier in each branched arm S_1 and S_2, the 'phase times' τ_1 is plotted as a function of barrier width 'w_1' keeping $w_2(=1)$ and $V_1(=5)$ fixed and for different values of parameter V_2. The small dashed, dot-dashed, solid, long dashed and dotted curves are for $V_2 = 2.5, 3.5, 5.0, 10.0$ and 12.5 respectively. Other system parameters are $E = 1, lb_1 = lb_2 = 3$. In the inset τ_{s1} is plotted as a function of V_2 for the same system parameters.

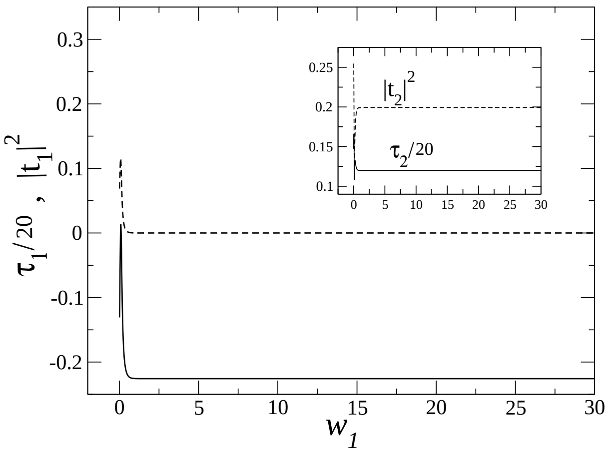

FIG. 4: Here for a 3-way splitter with one barrier in each side branch S_1 and S_2, the 'phase times' τ_1 (solid curve) and $|t_1|^2$ (dashed) are plotted as a function of 'w_1' for a very small $w_2(=0.5)$. Other system parameters are $E = 1$, $lb_1 = lb_2 = 2.5$, $V_2 = 5$ and $V_1 = 15$. In the inset, the solid and dashed curves represent τ_2 and $|t_2|^2$ respectively as a function of w_1. For better visibility we have plotted phase times scaled down by a factor of 20.

shows the phase time τ_2 in arm S_2 which does not contain any barrier. This also evolves and saturates with w_1, the length of the barrier in the other arm S_1. This delay is due to the contribution from paths which undergo multiple reflection in the first branch before entering the second branch via junction point J. In absence of a barrier in the n-th arm the phase time τ_n measured close to the junction J should go to zero *i.e.* $\tau_n \to 0$ in the absence of multiple scatterings in the first arm. Note that τ_{s1} and τ_{s2} change with energies of the incident particle (Fig. 2). From Fig.2 it can be easily seen that τ_{s2} is always smaller than τ_{s1} for any particular V_1 *i.e.* the saturation time in the arm having no barrier is smaller. The phase time in both the arms show non-monotonic behavior as a function of V_1. As we decrease the strength of the barrier V_1 the value of τ_1 (τ_2) decreases in the whole range of widths of the barrier and also the saturated value of τ_{s1} (τ_{s2}) decreases until V_1 reaches 1.6 and with further decrease in V_1 the values of τ_1 (τ_2) as well as τ_{s1} (τ_{s2}) starts increasing.

As the second case we take up another Y-junction which contain potential barriers in both its side branches as shown in Fig. 1. We fix the values of $V_1(=5)$ and vary w_1 for each values of V_2 to study the w_1-dependence of τ_1 (Fig.3). From Fig. 3 we see that τ_1 decreases with increase in w_1 to saturate to a value τ_{s1} at each value of V_2 thereby showing 'Hartman effect' for arm 'S_1'. But now, we can tune the saturation phase time at arm S_1 non-locally by tuning strength of the barrier potential V_2 sitting on another arm S_2! *Thus 'quantum nonlocality' enables us to control the 'super arrival' time in one of the arms (S_1) by changing a parameter (V_2) on another, spatially separated from it.* In the inset of Fig. 3 we plot τ_{s1} as a function of V_2. It clearly shows that when the barrier strengths V_1 and V_2 are very close the 'phase time' reaches its minimum value. In all other cases *i.e.* whenever $V_1 \neq V_2$, the value of τ_{s1} is larger.

We will show now another interesting result related to the Hartman effect. For this we keep $V_2(=5)$ unaltered and reduce w_2. For very small $w_2(=0.5)$ we see from Fig. 4 that τ_1 is negative for almost the whole range of w_1-values showing 'time-advancement' and eventually after a sharp decrease saturates to a negative value of $\tau_{s1} = -4.514$ implying 'Hartman effect' with advanced time. It might be noted that, in principle, the 'time-advancement' (Fig. 4) can be measured experimentally as $|t_1|^2$ has a non-zero finite value for a small range of w_1 at lower w_1 regime where τ_1 is negative. In the inset we plot the corresponding τ_2 and $|t_2|^2$ as function of w_1. Again the values of τ_2 remains different from the one dimensional tunneling through a barrier of strength V_2 and width w_2 in the whole range of w_1 implying 'quantum nonlocality'. In the cases discussed so far τ_2 vary more sharply in small w_1 regime. Further the inset in Fig. 4 shows a dip in τ_2 at parameter regimes where $|t_2|^2$ has a minimum. For a wave packet with large spread in real space it is possible that the leading edge of the wave packet reaches the barrier much earlier than the peak of the packet. This leading edge in turn can tunnel through to produce a peak in the other end of the barrier much before the incident wave packet reaches the barrier region, sometimes referred to as pulse reshaping effect. This, in general, causes 'time advancement' [4].

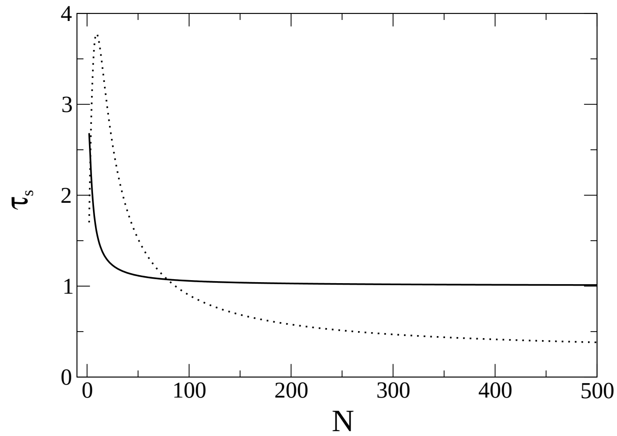

FIG. 5: Here $N(\geq 2)$ number of side branches each with a barrier of strength $V(=5)$ and width $w(=100)$ are connected with the incident arm S_0. In each arm barriers start at the same point $lb(=1)$. Thus all the side branches being identical the 'phase time' for transmission through these arms $S_n, n = 2, 3, \cdots N$ saturate to the same value τ_s. The saturated 'phase time' τ_s is plotted as a function of the total number of side branches N in the system. The dotted and solid curves are for $V = 5.0$ and 1.25 respectively. The incident energy is kept at $E = 1$.

This negative delay does not violate causality, however, the time is bounded from the below. In the presence of square wells in one dimensional systems negative time delays have been observed. This effect is termed as 'ultra Hartman effect' [see for details [30]].

Finally consider a similar system as that shown in Fig. 1, but in presence of $N(\geq 2)$ identical side branches and study phase time as a function of increasing N. All the side branches being identical the 'phase times' for transmission through each of these arms $S_n, n = 2, 3, \cdots N$ saturate to the same value τ_s for very large w_n. In Fig.5 we plot the saturation value τ_s as a function of the total number of side branches N present in the system. From the figure we see that for $V = 5$, τ_s first increases with N to a maximum value of 3.776 at $N = 9$ and thereafter keeps on decreasing with the increase of N. As we start reducing the strength of the barriers from 5 we see that for $V = 1.49$ the increasing nature of τ_s in small N range vanishes. In general, at larger N, the decreasing nature of τ_s with N persists, *e.g.*, note the solid curve in Fig.5 plotted at $V = 1.25$, but the initial increase in τ_s is not a generic feature. For larger N transmission amplitude in each side branch reduces with increase in N and hence the corresponding peaks of wave packets reach the far end at earlier times thereby reducing τ_s.

V. CONCLUSIONS

We have studied Hartman effect and non-locality in quantum network consisting of a main one dimensional arm having $N(\geq 2)$ side branches. These side branches may or may not have barriers. In presence of barrier the 'phase time' for transmission through a side branch shows the 'Hartman effect'. In general, as the number of side branches N increases, the saturated 'phase time' decreases. Due to quantum nonlocality the 'phase time' and it's saturated value at any side branch feels the presence of barriers in other branches. Thus one can tune the saturation value of 'phase time' and consequently the superluminal speed in one branch by changing barrier strength or width in any other branch, spatially separated from the former. Moreover Hartman effect with negative saturation times (time advancement) has been observed for some cases. In conclusion generalization of Hartman effect in branched networks exhibits several counter-intuitive results due to quantum non-locality. System parameters such as number of side branches N and barrier widths w_n, strengths V_n, distance lb_n (from J) and incident energy E etc. play very sensitive roles in determining delay times. The delay times are also sensitive to the junction S-matrix elements used for a given problem. In our present problem junction S-matrix is determined uniquely by the wave guide transport methods. Depending on lb_n there may be one or several bound states located between the barriers in different branched arms and as a consequence saturated delay time can be varied from the negative (ultra Hartman effect) to positive and vice-versa. We have verified this by looking at the transmission coefficient in the second arm S_2 which exhibits clear resonances as a function of lb_2. Around these resonances, saturated delay time in the first arm τ_{s1} changes the sign [the details of which will be published elsewhere [31]]. Moreover the reported effects are amenable to experimental verifications in the electromagnetic wave-guide networks.

VI. ACKNOWLEDGMENTS

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