Noise induced currents and reliability of transport in frictional ratchets

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We study the coherence of transport of an overdamped Brownian particle in frictional ratchet system in the presence of external Gaussian white noise fluctuations. The analytical expressions for the particle velocity and diffusion coefficient are derived for this system and the reliability or coherence of transport is analysed by means of their ratio in terms of a dimensionless Péclet number. We show that the coherence in the transport can be enhanced or degraded depending sensitively on the frictional profile with respect to the underlying potential.

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I. INTRODUCTION

The study of the interplay of noise and nonlinear dynamics in systems under nonequilibrium conditions has generated wide interdisciplinary interests in the last two decades. Noise or fluctuations, which are normally considered to be an hinderance, are found to play an active constructive role in nonequilibrium systems. Infact, this constructive role of noise (as opposed to the conventional wisdom of its destructive or its disorganising role) has become a new paradigm in the study of complex systems. In the so called ratchet systems the presence of spatial/temporal anisotropy in potential together with nonequilibrium perturbations enable the extraction of useful work from random fluctuations without the violation of the second law of thermodynamics [1, 2]. In such systems it is possible to induce directed motion from nonequilibrium fluctuations in the absence of bias. Much of the studies in different classes of ratchet models deal with the nature of currents and their reversals [1], stochastic energetics (thermodynamic efficiency) [3, 4] etc. However, transport of Brownian particle is always accompanied by a diffusive spread and this spread is intimately related to the question of reliability or quality of transport. The diffusive spread infact detriments the quality of transport. There exists very few studies which address the question of diffusion accompanying transport in ratchet systems [5, 6, 7]. In our present work we address this aspect of transport and study the coherence in transport in a frictional ratchet in the presence of an external parametric Gaussian white noise fluctuation. This is infact studied in terms of a dimensionless quantity called the Péclet number (Pe) which is the ratio of velocity to the diffusion constant. Higher the Pe, lesser is the diffusive spread and higher is the transport coherence. Infact, subcellular transport in biological systems amidst a noisy environment are modeled based on the principle of ratchet mechanism and the experimental studies on these molecular motors show them to have highly efficient and reliable transport with Péclet number ranging from 2 to 6 [8]. A value of Pe greater than 2 corresponds to coherent transport [5]. The Péclet numbers for some of the models like flashing and rocking ratchets were found to be ~ 0.2 and ~ 0.6 [5] respectively implying a less reliable transport. Another study on symmetric periodic potentials along with spatially modulated white noise showed a coherent transport with Péclet number less than 3. In the same study a special kind of strongly asymmetric potential is found to increase Pe to 20 in some range of physical parameters [6].

There exists many physical systems like flashing ratchets [9], rocking ratchets [10], time asymmetric ratchets etc., where different aspects of noise induced transport has been widely studied [1, 2]. In the above models, to generate unidirectional current the nonequilibrium fluctuations need to be correlated in time. There exists a possibility to get unidirectional current even in presence of symmetric ratchet potentials provided it is driven by a time correlated asymmetric force [11]. In our present work we consider yet another class of ratchets, namely the frictional ratchets, where the friction coefficient and subsequently the dif-

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fusion coefficient varies in space [12, 13]. In these frictional ratchets it is possible to get unidirectional currents even in a symmetric underlying potential but in presence of an external noise which need not be correlated in time unlike in earlier models.

Space dependent diffusion coefficient, $D(x)$, felt by the Brownian particle could arise either due to space dependent temperature or space dependent friction coefficient. In the frictional ratchet system which we consider in the present work the unidirectional current arises due to a combination of both space dependent friction coefficient and external parametric white noise. The temperature of the bath (or the environment) of the Brownian particle is characterised by a constant temperature T. In the presence of external parametric noise the overdamped Brownian particle on an average absorbs energy from the external noise source. The strength of the absorbed energy depends on the local frictional coefficient. Hence the problem of particle motion in an inhomogeneous medium in presence of an external noise becomes equivalent to the problem in a space dependent temperature [12, 13, 14]. Such systems are known to generate unidirectional currents. This follows as a corollary to Landauer's blow torch theorem that the notion of stability changes dramatically in the presence of temperature inhomogeneities [15]. In such cases the notion of local stability, valid in equilibrium systems, does not hold.

Frictional inhomogeneities are common in superlattice structures, semiconductors or motion in porous media. Particles moving close to a surface experience space dependent friction [16]. It is believed that molecular motor protiens moving close along the periodic structures of microtubules experience a space dependent friction [17]. Frictional inhomogeneity changes the dynamics of the particle nontrivially as compared to the homogeneous case. This in turn has been shown to give rise to many counter intuitive phenomena like noise induced stability, stochastic resonance, enhancement in efficiency etc., in driven non-equilibrium systems [17, 18].

In our present work we show that system inhomogeneities may help in enhancing/degrading the coherence in the transport depending sensitively on the physical parameters. We emphasize mainly the case where the underlying potential is a simple sinusoidal symmetric potential. The role of spatial asymmetry in potential is also discussed. The external noise is found to play a constructive role in enhancing the coherence in transport. As opposed to this, temperature (internal fluctuations) degrades the coherence in transport.

II. MODEL:

We start with the Kramer's equation of motion for a Brownian particle of unit mass in contact with a heat bath in a medium with spatially varying friction coefficient $\eta(q)$ at temperature T. In addition an external parametric Gaussian white noise fluctuation $\xi(t)$ is also included. The equation of motion is given by

$$\ddot{q} = -\eta(q)\dot{q} - V'(q) + \sqrt{k_B T \eta(q)} f(t) + \xi(t) \qquad (1)$$

where $V(q)$ is the potential seen by the Brownian particle and $f(t)$ is an internal Gausian white noise fluctuation arising from the bath having the property that $< f(t) >= 0$, and $< f(t)f(t') >= 2\delta(t-t')$ where $< ... >$ denotes the ensemble average and q the coordinate of the particle. Also, $< \xi(t) >= 0$ and $< \xi(t)\xi(t') >= 2\Gamma\delta(t-t')$, where Γ is the strength of the external white noise $\xi(t)$. The above equation, Eq. 1, has been derived earlier from the microscopic consideration of system bath coupling [12, 13].

On time scales larger than the inverse friction coefficient, η^{-1}, one can in most practical cases consider the overdamped limit of the Langevin equation. This in turn correspond to the adiabatic elimination of the fast variable, velocity, from the equation of motion by putting $\dot{p} = \ddot{q} = 0$ for a homogeneous system. In contrast, for the case of inhomogeneous system the above method of elimination does not work and Sancho et al. [19] has given a proper prescription for the elimination of fast variables. The corresponding overdamped Langevin equation for the Brownian particle in a space dependent frictional medium is given by

$$\dot{q} = -\frac{V'(q)}{\eta(q)} - \frac{k_B T \eta'(q)}{2[\eta(q)]^2} + \sqrt{\frac{k_B T}{\eta(q)}} f(t) + \frac{\xi(t)}{\eta(q)}. \qquad (2)$$

Using van Kampen Lemma [20] and Novikov's theorem [21] we get the corresponding Fokker-Planck or Smoluchowski equation for the probability density $P(q,t)$ of a particle being at q at a time t as [22]

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial q}\left[\left\{\frac{V'(q)}{\eta(q)} - \frac{\Gamma\eta'(q)}{\eta^3(q)}\right\}P + \left\{\frac{k_B T}{\eta(q)} + \frac{\Gamma}{\eta^2(q)}\right\}\frac{\partial P}{\partial q}\right]. \qquad (3)$$

For periodic functions $V(q)$ and $\eta(q)$ with periodicity L a finite probability current is obtained and one readily gets the analytical expression for the particle velocity as [22]

$$v = L\frac{(1-\exp(-\delta))}{\int_0^{2\pi} dy \exp[-\psi(y)] \int_y^{y+2\pi} dx \frac{\exp[\psi(x)]}{A(x)}}. \quad (4)$$

Here $\psi(q)$ is the dimensionless generalized effective potential given by

$$\psi(q) = \int^q dx \frac{V'(x)\eta^2(x) - \Gamma\eta'(x)}{\eta(x)[k_B T\eta(x) + \Gamma]} \quad (5)$$

and $A(q)$ is the effective space dependent diffusion coefficient given by

$$A(q) = \frac{k_B T\eta(q) + \Gamma}{\eta^2(q)}. \quad (6)$$

Also,

$$\delta = \psi(q) - \psi(q+2\pi) \quad (7)$$

determines the effective slope of the generalized potential $\psi(q)$. Thus the sign of δ gives the direction of current in Eq. 4.

In our present work we have taken the potential $V(q) = V_0\sin(q)$ and $\eta(q) = \eta_0[1-\lambda\sin(q-\phi)]$, $0 < \lambda < 1$. The phase lag ϕ between $V(q)$ and $\eta(q)$ brings in the intrinsic asymmetry in the dynamics of the system.

For the case with $T = 0$ we get a simple analytical expression for the effective potential as

$$\psi(q) = \frac{V_0\eta_0}{\Gamma}\Big(\frac{\lambda x\sin(\phi)}{2} + \frac{\lambda}{4}\big[\cos(2x)\cos(\phi) \quad (8)$$

$$+\sin(2x)\sin(\phi)\big] + \sin(x)\Big) - \ln[\eta(x)] \quad (9)$$

with δ given by

$$\delta = -\frac{V_0\eta_0\pi\lambda\sin(\phi)}{\Gamma}. \quad (10)$$

The first term in the right hand side of Eq. 9 represents the tilt in the effective potential. This tilt identically vanishes when $\phi = 0$ or π. Hence it is expected that the unidirectional current does not arise for the case when the phase lag is 0 or π. From Eq. 4 and Eq. 10 it is clear that for $0 < \phi < \pi$ current will be in the negative direction while for $\pi < \phi < 2\pi$ the current will be in the positive direction. Fig. 1 shows the plot of the effective potential as a function of coordinate q for a fixed noise strength $\Gamma = 0.36$ for two different values of phase lag ϕ between the potential and frictional profile for the above zero temperature case. The effective potential is scaled with respect to V_0. For the parameters chosen in the figure the current flows in the negative direction as has been mentioned earlier.

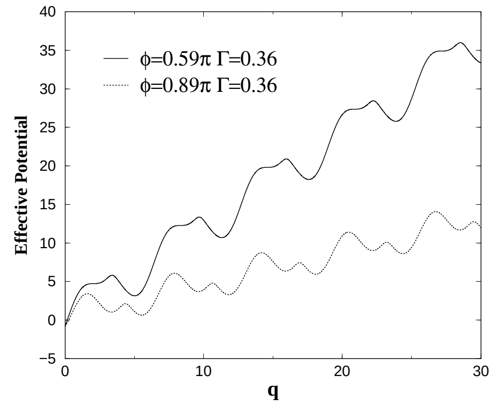

FIG. 1: Effective potential $\psi(q)$ as a function of q for $V_0 = 1$, $\Gamma = 0.36$ and $\lambda = 0.9$ at zero temperature for $\phi = 0.59\pi$ and 0.89π.

Following references [23, 24], one can obtain exact analytical expressions for the diffusion coefficient D as

$$D = \frac{\int_{q_0}^{q_0+L} \frac{dx}{L} A(x)\,[I_+(x)]^2 I_-(x)}{\Big[\int_{q_0}^{q_0+L} \frac{dx}{L} I_+(x)\Big]^3} \quad (11)$$

where $I_+(x)$ and $I_-(x)$ are as given below

$$I_+(x) = \frac{1}{A(x)}\exp[\psi(x)]\int_{x-L}^{x} dy\ \exp[-\psi(y)] \quad (12)$$

$$I_-(x) = \exp[-\psi(x)]\int_x^{x+L} dy\ \frac{1}{A(y)}\exp[\psi(y)] \quad (13)$$

L here represents the period of the potential ($= 2\pi$ in our case). The Brownian particle takes a time $\tau = L/v$ to traverse a distance L with a velocity v. The diffusive spread of the particle in the same time is given by $<(\Delta q)^2> = 2D\tau$. The criterion to have a reliable transport is that the diffusive spread should be less compared to the distance traversed, i.e., $<(\Delta q)^2> = 2D\tau < L^2$. This in turn implies that $Pe = Lv/D > 2$ for coherent transport.

III. RESULTS AND DISCUSSIONS

The velocity (v), diffusion constant (D) and the Péclet number (Pe) are studied as a function of different physical parameters. All the physical quantities are taken in dimensionless form. In particular, velocity and diffusion are normalized by $(V_0/\eta_0 L)$ and (V_0/η_0) respectively. Throughout our work we have set V_0 and η_0 to be unity. Similarly, Γ and T are scaled with respect to $V_0\eta_0$ and V_0 respectively. We have used the globally adaptive scheme based on Gauss-Kronrod rules for numerical evaluations [25].

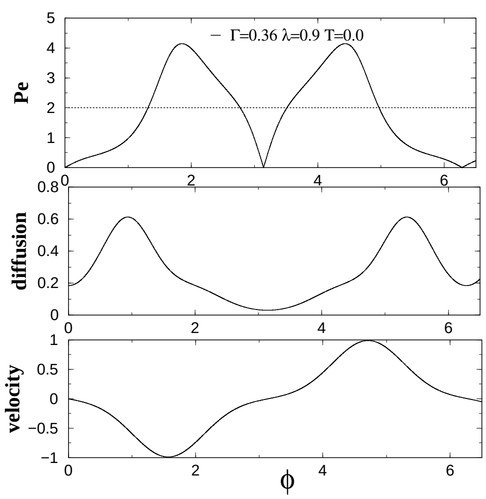

FIG. 2: Plot of v, D and Pe vs ϕ for $\Gamma = 0.36$, $\lambda = 0.9$ and $T = 0.0$.

In Fig. 2 we plot v, D and Pe as a function of phase difference ϕ at $T = 0$ for a fixed noise strength of $\Gamma = 0.36$. The values of the physical parameters are mentioned in the figure caption. All the physical quantities are periodic function of ϕ as expected and the velocity is zero at $\phi = 0\,,\pi$ and 2π. As can be seen from the plot of effective potential, Fig. 1, the direction of current is negative for ϕ upto π and then it becomes positive. The current is antisymmetric around the point $\phi = \pi$, $(V(\phi+\pi) = -V(\phi))$. This is expected on general grounds for the case of a simple sinusoidal symmetric potential. The absolute value of current exhibits a maxima between 0 to π and π to 2π. The nature of currents being positive or negative can be readily inferred from the slope of the effective potential (for example see Fig. 1). The diffusion coefficient is finite at all values of ϕ and exhibits minima wherever the currents are zero ($\phi = 0, \pi, 2\pi$). However, the magnitude of minima is more at the point where $\phi = \pi$. As expected, the diffusion coefficient is periodic in ϕ. However, it exhibits maxima at different values of ϕ than that for current. In the plot for Pe as a function of ϕ we have drawn a dotted line as a guide for eye. In regions where $Pe > 2$ the transport is said to be coherent. It is evident from the figure that there exists wide range of ϕ in which the transport is coherent ($Pe > 2$) along with adjoining regions where the transport is less coherent ($Pe < 2$). These regions are however sensitively dependent on other physical parameters. In the present case of sinusoidal potential Péclet number as high as 4 can be obtained by properly tuning the parameters.

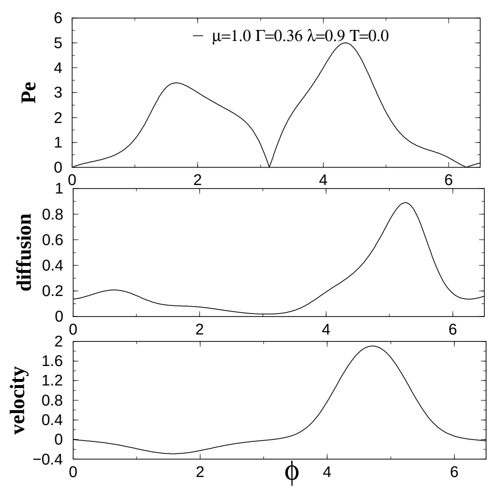

FIG. 3: Plot of v, D and Pe vs ϕ for $\Gamma = 0.36$, $\mu = 1.0$, $\lambda = 0.9$ and $T = 0.0$.

To understand the role of spatial asymmetry in the potential we have included an asymmetry in the potential such that $V(q) = V_0[\sin(q) - \mu/4\sin(2q)]$ where μ is the asymmetry parameter. Fig. 3 shows the behaviour of v, D and Pe for this simple asymmetric case at zero temperature with the other physical parameters kept the same as in Fig. 2. The potential asymmetry parameter μ affects the behaviour of velocity, diffusion and thereby Pe dramatically. We have considered the case of maximal asymmetry ($\mu = 1$). We first notice that the simple symmetry observed for the case of periodic potential no more holds true in the presence of spatial asymmetry in potential. The magnitude of velocity is nonzero for ϕ values $0\,,\pi$ and

2π. The velocity and diffusion constant in the region between 0 to π are suppressed while in the region between π and 2π are enhanced as compared with symmetric case. Consequently, the presence of asymmetry can enhance or suppress the coherence in the transport. For the chosen parameters, Péclet number as high as 5 is obtained. Moreover, for a given ϕ, Pe either increases or decreases monotonically with the asymmetry parameter μ $(0 < \mu < 1)$ which has been verified separately. In our further analysis we restrict to the case of symmetric potential alone.

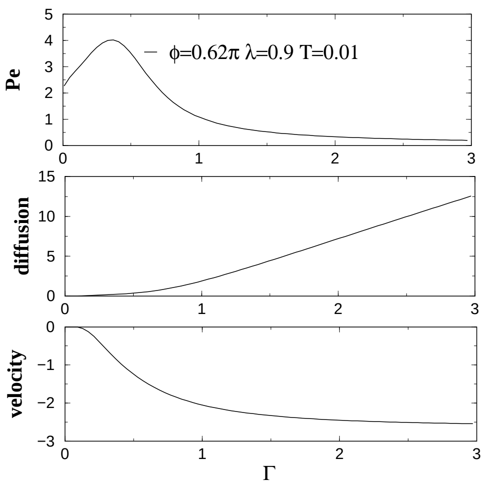

FIG. 4: Plot of v, D and Pe vs Γ for $\phi = 0.62\pi$, $\lambda = 0.9$ and $T = 0.01$.

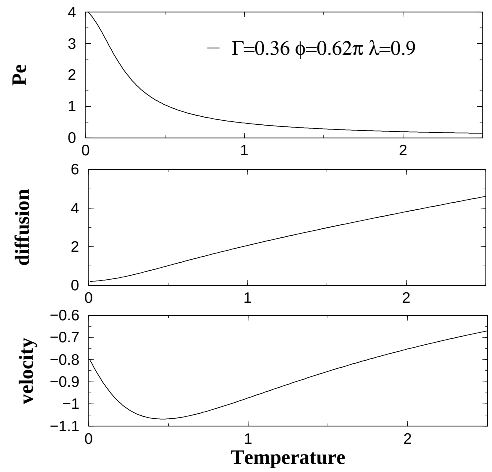

FIG. 5: Plot of v, D and Pe vs T for $\Gamma = 0.36$, $\phi = 0.62\pi$ and $\lambda = 0.9$.

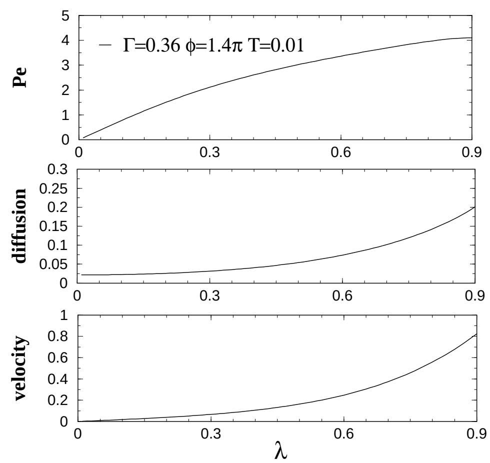

FIG. 6: Plot of v, D and Pe vs λ for $\Gamma = 0.36$, $\phi = 1.4\pi$ and $T = 0.01$.

In Fig. 4 we plot velocity, diffusion and Pe as a function of external noise strength Γ with $\phi = 0.62\pi$ and $\lambda = 0.9$ at finite temperature $T = 0.01$. It should be noted that velocity is negative in the entire range. The velocity is initially zero for Γ equal to zero and then increases with Γ and saturates to a constant value at higher values of noise strength. On the contrary, the diffusion constant keeps increasing monotonically with Γ. It is also clear that the external noise play a constructive role in optimizing the coherence in transport i.e., the Pe exhibits a peak as a function of Γ.

In Fig. 5 we plot velocity, diffusion and Pe as a function of temperature T with $\phi = 0.62\pi$, $\lambda = 0.9$ and $\Gamma = 0.36$. This value of noise strength corresponds to an optimal value for Pe in Fig. 4. The noise induced current is negative. The current exhibits a peak and then decreases to zero with temperature. This is expected because higher temperatures overshadows the effect of potential and frictional inhomogenieties thereby suppressing the ratchet effect. In contrast, the diffusion constant increases monotonically. As opposed to the case of Fig. 4 we observe that internal fluctuations or temperature degrades the coherence in the transport i.e., Pe decreases with increase in temperature.

In Fig. 6 we plot velocity, diffusion and Pe as a function of λ, the amplitude of oscillation of the friction coefficient $(0 < \lambda < 1)$ for $\phi = 1.4\pi$, $\Gamma = 0.36$ and $T = 0.01$. All the physical quantities, namely, velocity, diffusion and Pe increases monotonically with λ. Thus the increase in λ makes the transport more coherent.

IV. CONCLUSIONS

We have studied the coherence or reliablity of transport of an overdamped Brownian particle in a frictional ratchet system with an underlying sinusoidal potential in the presence of external Gaussian white noise fluctuations. The frictional inhomogeneities along with external fluctuations lead to a noise induced current or transport. The attained noise induced transport is always accompanied by a diffusive spread which inturn makes the transport to be less reliable. We have shown that frictional inhomogeneities with respect to the underlying potential can make the transport coherent or incoherent. While the external noise (Γ) optimizes the coherence in transport the internal noise (T) degrades the coherence.

In our present case, as mentioned in the beginning, the transport can be associated with an effective potential and an effective space dependent diffusion constant. The effective potential exhibits a tilt as a function of system parameters (Fig. 1). By looking at this effective potential one can infer only the direction of current and not its magnitude. The particle motion in this effective potential is determined by two time scales, (i) escape from the potential minima over the barrier along the effective bias followed by (ii) the relaxation into next minima. The coherent transport in homogeneous medium is obtained when the relaxation time dominates the transport [6, 24] as compared with the escape time. For details see reference [6, 24]. We would like to emphasize that the dynamics of the particle in our present problem arises due to the complex interplay between the potential, internal fluctuations, frictional profile and external fluctuations. This is amply reflected in the fact that both the effective potential and space dependent diffusion constant change their nature as we change the system parameters. Thus apriori analysis or prediction of the behaviour of the system in regard to transport coherence is a difficult task. It may not be surprising that by choosing appropriate asymmetric potential and frictional profile along with other parameters one may obtain much higher transport coherence as observed in [6].

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