

Work Fluctuations and Stochastic Resonance

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Abstract

We study Brownian particle motion in a double-well potential driven by an ac force. This system exhibits the phenomenon of stochastic resonance. Distribution of work done on the system over a drive period in the time asymptotic regime have been calculated. We show that fluctuations in the input energy or work done dominate the mean value. The mean value of work done over a period as a function of noise strength can also be used to characterise stochastic resonance in the system. We also discuss the validity of steady state fluctuation theorems in this particular system.

Key words: Stochastic Resonance, Fluctuation Theorems

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1. Introduction :

Stochastic resonance (SR) refers to the enhanced response of a system to a small deterministic periodic forcing in the presence of an optimal amount of noise [1]. It often occurs in bistable and excitable systems with subthreshold inputs. Noise plays a constructive role in this phenomenon. This is due to a cooperative interplay between nonlinearity of the system, input signal and noise. Thus the power from the whole spectrum is pumped into a single mode that is coherent with the external driving force. Because of its generic nature, this phenomenon, boasts applications in almost all areas of natural sciences. Different quantifiers of SR have been discussed [3,4,5,6,7] as regards to its validity as a bonafide resonance, i.e., the resonance phenomena as a function of noise strength as well as the driving frequency. Hysteresis loop area [3], input energy or the work done on the system per cycle [4,5] and area under the first peak in

the residence time distribution [2] have turned out to be good quantifiers characterizing SR as a bonafide resonance. Recently it has been shown that the different quantifiers of SR are mathematically related to each other [6]. Earlier studies on the work done in a periodically driven bistable system by an external agent has established that the average work peaks around the resonance, as expected [4,5]. In our present work, we study the fluctuations and nature of the probability distributions of work done over a period in a driven double-well system in the time asymptotic regime (where all averaged quantities become periodic in time with a period of the external drive).

Motivation for this study also comes from recent interest in the so called fluctuation theorems and transient violation of second law in systems driven to a nonequilibrium state [8,9]. Fluctuation theorems describe properties of the distribution of various nonequilibrium quantities, such as work, heat, entropy, etc. [8,9,10]. It is important to note that these theorems are rigorous and applicable to systems driven arbitrarily far away from equi-

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librium. These theorems can be extremely useful in analysing the role of fluctuations on the performance characteristics of engines at nanoscales (e.g., molecular motors). In these tiny systems interactions with the environment are dominated by thermal fluctuations[8]. Moreover fluctuations in physical quantities are more than the mean values and large fluctuations may occur which can lead to unexpected consequences. It has been shown recently that overdamped driven harmonic oscillator (linear system) satisfies a steady state fluctuation theorem (SSFT) for the work done over a single period (W_p) in the time asymptotic regime[11]. SSFT implies that the probability distribution $P(W_p)$ of work W_p satisfies the following relation[10,11,12,13,14], namely,

$$\frac{P(W_p)}{P(-W_p)} = e^{(\beta W_p)}. \quad (1)$$

We show that this is an artifact of the linearity of the model and in general does not hold for driven nonlinear systems. However, we show that SSFT for the work is indeed satisfied, if one instead considers the work done over a large number of cycles[14]. Our study of input energy fluctuations also reveals that the relative variance is larger than one across SR indicating that fluctuations dominate the mean value.

2. The Model:

The overdamped Langevin equation of a particle in a double-well potential in the presence of a time periodic force is given by[12]

$$\frac{dx}{dt} = -\frac{\partial U(x)}{\partial x} + \xi(t), \quad (2)$$

where

$$U(x) = \frac{x^4}{4} - \frac{x^2}{2} - Ax \sin \omega t. \quad (3)$$

We have set friction coefficient (γ) to unity. The random force field $\xi(t)$ is a zero mean Gaussian white noise, i.e,

$$\langle \xi(t)\xi(t') \rangle = D_0 \delta(t - t'). \quad (4)$$

where $D_0 = 2\gamma k_B T$ is the strength of the noise. The static double-well potential $V(x) = \frac{x^4}{4} - \frac{x^2}{2}$, has

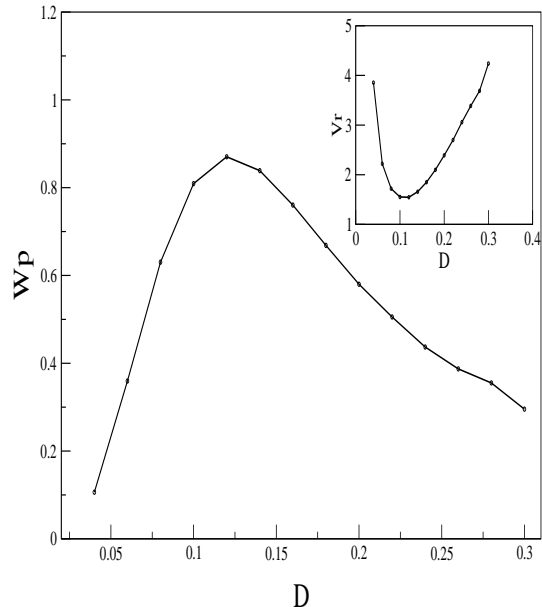


Fig. 1. Figure shows the variation of average work W_p done per period with temperature D . Inset shows the corresponding plot of relative variance V_r . The parameters are $A=0.1$, and $\omega = 0.1$.

a barrier height $\Delta V = 0.25$ between the two symmetrical wells (minima) located at distances $x_m = \pm 1$. We consider the case of weak forcing $A|x_m| < \Delta V$. All the physical quantities are taken in dimensionless units as prescribed in references[4,5].

The work done by the external drive on the system or the input energy ($\tau_\omega = \frac{2\pi}{\omega}$) is defined as[4,5]

$$\begin{aligned} W_p &= \int_{t_0}^{t_0+\tau_\omega} \frac{\partial U(x,t)}{\partial t} dt, \\ &= -A\omega \int_{t_0}^{t_0+\tau_\omega} x(t) \cos \omega t dt \end{aligned} \quad (5)$$

This follows from the stochastic energetics formalism developed by Sekimoto[16].

Numerical simulation of this model was carried out by using Huen's method [17]. To calculate the work done we first evolve the system and neglect initial transients and then work done over a period (W_p) is calculated (eqn.(5)). The values of W_p are different for different cycles. To get better statistics we have calculated W_p for 100000 different (sometimes even more) cycles.

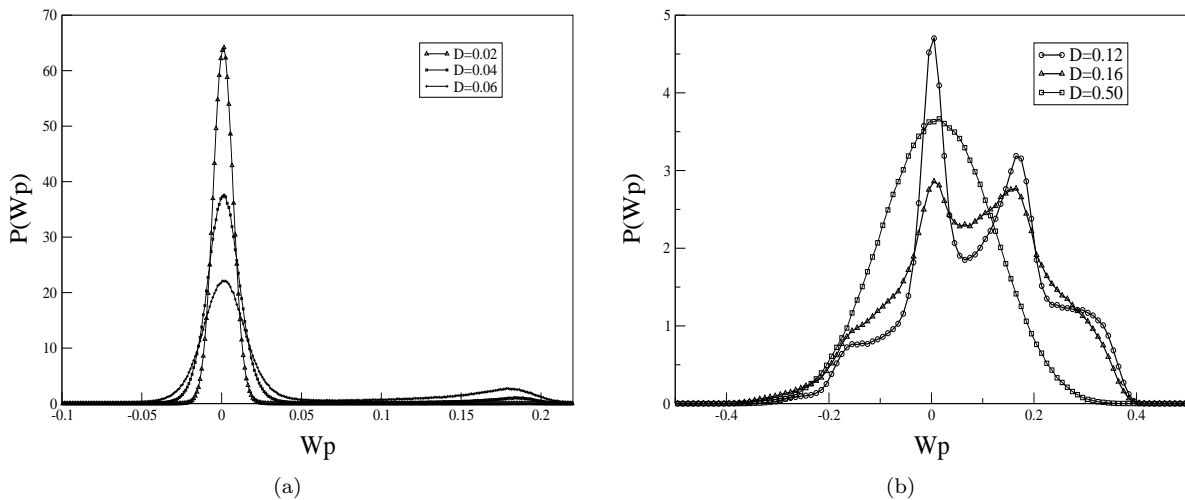


Fig. 2. Probability distribution for various values of D , other parameters being the same as in Fig.1.

3. Results and Discussion:

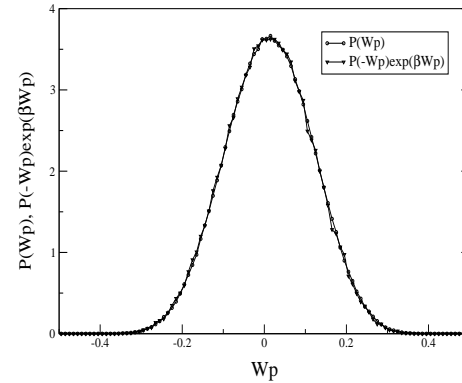
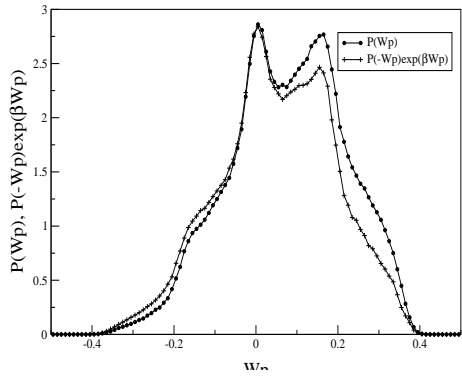
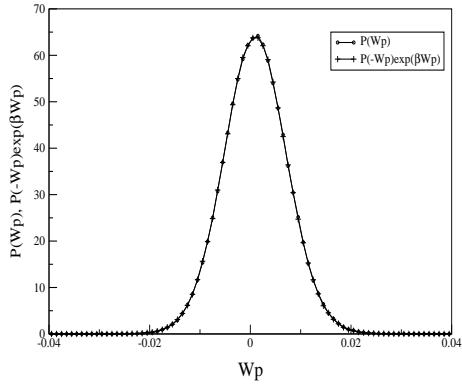
In Fig.1 we plot the average work done ($\langle W_p \rangle$) over a single period (in the time asymptotic regime) as a function of noise strength $D (= kT = \frac{1}{\beta}$, in dimensionless units) for low amplitude driving of strength $A = 0.1$ and for frequency $\omega = 0.1$. We clearly see that input energy exhibits a peak as a function of D . This can be attributed to the synchronized escape from the potential well with the external periodic drive. We reproduce exactly the same figure as in reference[5], which has been obtained using a different numerical method.

For the case where noise is small the particle remains in one of the potential wells for a longer time (intrawell motion dominates) with an occasional random jump to another well (Kramers' escape over the barrier) as a function of time. When the noise is very strong a large number of interwell random switches occur for each period of the sinusoid and the systems response is again random (interwell motion dominates). In between these two conditions, surprisingly, there exists an optimal value of the noise that cooperatively concurs with the periodic forcing to make almost exactly one switch per period. Hence interwell motion will be in synchrony with the input signal. Quantitatively this condition is determined by the matching of the two time scales [1,5], namely, the period of the input signal and the Kramers' escape rate r_k , $\left(\frac{1}{r_k} = \frac{\tau_\omega}{2} \rightarrow \frac{\omega}{\pi} = r_k, \text{ with } r_k = \frac{1}{2\pi} e^{\left(\frac{0.25}{D}\right)} \right)$. At the optimal value of noise the input energy is max-

imum. The nature of this maximum is a function of other physical parameters as discussed in detail in[4,5]. In the inset of Fig.1 we have plotted the relative variance $V_r = \sqrt{\frac{\langle W_p^2 \rangle - \langle W_p \rangle^2}{\langle W_p \rangle^2}}$ of work done W_p over a period as a function of noise strength.

We see that $V_r > 1$ in the parameter range we have chosen. This shows that fluctuations dominate the mean value. V_r exhibits a minimum near the value of D at which SR occurs. It may be noted that when V_r becomes much larger than one the average value ceases to be a good physical variable (non self-averaging quantity) and hence to characterise the behavior of work one has to resort to the study of the entire distribution function. In the time periodic state the average work done is dissipated into the system as heat. Thus one can identify $\langle W_p \rangle$ as a hysteresis loss in the medium. However, it may be noted that fluctuations of work done cannot be identified with heat fluctuations [11,12].

Now we will turn to the understanding of the nature of probability distribution $P(W_p)$ of W_p for various values of noise strength D across SR peak. For this we have plotted $P(W_p)$ as a function of W_p in Figs. 2a and 2b for various values of D . For low values of D (~ 0.02) particle dynamics is mostly confined to a small amplitude intrawell oscillatory motion. Hence the distribution is closer to the Gaussian. For slightly higher values of D ($\simeq 0.04$ and 0.06) the particle makes occasional random excursions into the other well as a function of time. This is clearly reflected as a small asymmetry and a hump (interwell motion) in the plot of $P(W_p)$. It is interesting to



(c)

Fig. 3.

Plot of $P(W_p)$ (line with circles), $P(-W_p)\exp(\beta W_p)$ (line with crosses) vs. W_p . Plots are for $D=0.02$ (a), $D=0.16$ (b) and $D=0.5$ (c) respectively. For all plots $\omega = 0.1$ and $A = 0.1$.

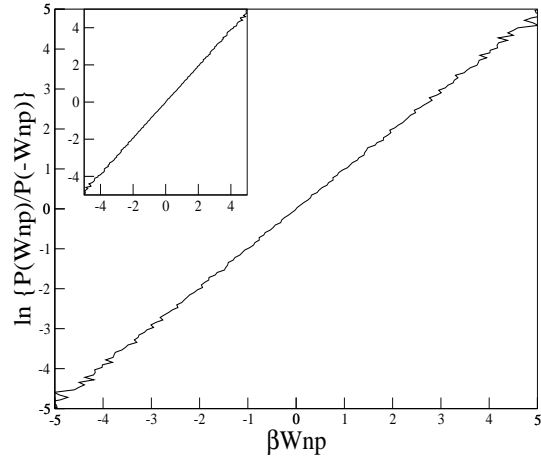


Fig. 4. Plot of $\ln\{P(W_{np})/P(-W_{np})\}$ vs. βW_{np} (over 10 periods). Inset shows the corresponding plot of W_{np} (over 5 periods). The plots are for $D=0.16, \omega = 0.1$ and $A=0.1$

note that there is substantial weight towards negative values of W_p . The negative values of W_p corresponds to transient second law violating hysteresis loops. As we increase D further the interwell dynamics starts playing a prominent role. It is interesting to note that position of the first and the second peak does not shift much, however, its width increases. The weight of the probability distribution towards higher negative values of W_p increases. On further increasing D (Fig.2b) finally a single peak structure appears. The shape of the single peak structure is closer to a Gaussian at high D (~ 0.5). For such high values of D particle makes several random excursions between the two wells during a single time period of an external drive. Hence total work can be treated as an addition of independent increments of work. Then the central limit theorem leads us to expect that the distribution of work will be approximately Gaussian. This is indeed the case. The parameter range for which we have explored the distributions are broad and the variance of fluctuations dominate over the average values (Fig.1, inset). This is one of our main observations.

Finally we discuss the applicability of SSFT for the work done over a single period. For this we have plotted in Fig.3, $P(W_p)$ and $P(-W_p)\exp(\beta W_p)$ for three different values of D . For small D ($\simeq 0.02$), Fig.3a and for large D ($\simeq 0.5$), Fig.3c $P(W_p)$ and $P(-W_p)\exp(\beta W_p)$ are almost equal, thus satisfying SSFT for work distribution. For the intermediate value of D ($\simeq 0.16$), Fig.3b we see that SSFT does not hold[18]. This clearly indicates that prediction of validity of SSFT for work done over a single period

is restricted only to a driven overdamped harmonic oscillator (linear problem)[11]. This will not be the case in the presence of nonlinearity. We consider a parameter regime where $D = 0.6$, $\omega = 0.01$, $A = 0.1$ of Fig. 3, where SSFT over a single period does not hold. However, we calculate the work distribution W_{np} for 5 and 10 periods.

In Fig.4 we have plotted $\ln(\frac{P(W_{np})}{P(-W_{np})})$ as a function of βW_{np} for periods 5 (inset) and 10. From the plots it is obvious that the slope is 1 and hence proving the validity of SSFT [14]. Moreover, the observed probability distribution is Gaussian and its variance V is related to the mean value $\langle W_{np} \rangle$, i.e., $V = \frac{2}{\beta} \langle W_{np} \rangle$, which is a requisite for SSFT if the distribution is Gaussian (eqn. 24 of[11]).

Thus we see that work distribution satisfies SSFT provided we consider work over a large number of periods. The number of periods above which SSFT holds depends on the physical parameters. The finite time corrections are very complex[14] which we have not discussed here.

4. Conclusion:

We have studied a problem of work fluctuations in a driven double well system which exhibits SR. We have shown that across SR fluctuations of work calculated over a single period in the asymptotic time region dominates the mean value. The distributions exhibit significant weight towards negative values of work indicating transient second law violating trajectories. As opposed to the case of a linear problem of driven harmonic oscillators SSFT for work distribution is not satisfied for a work calculated over a single period. However, if the work is calculated over a large number of periods SSFT is indeed satisfied. We have also calculated the distributions of internal energy and dissipated heat over a cycle so as to verify the extended form of SSFT [12]. The work will be reported elsewhere.

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References

- [1] L. Gammaitoni, P. Hanggi, P. Jung and F. Marchesoni, *Rev. Mod. Phys.* 70 (1998) 223.
- [2] L. Gammaitoni, F. Marchesoni and S. Santucci, *Phys. Rev. Lett* 74 (1995) 1052 ; M.H. Choi, R.F. Fox and P. Jung, *Phys. Rev. E* 57 (1998) 6335; G. Giacomelli, F. Marin, and I. Rabbiosi, *Phys. Rev. Lett.* 82 (1999) 675; F. Marchesoni, L. Gammaitoni, F. Apostolico and S. Santucci, *Phys. Rev. E* 62 (2000) 146.
- [3] M.C. Mahato and S.R. Shenoy, *Phys. Rev. E* 50 (1994) 2503; M.C. Mahato and A.M. Jayannavar, *Phys. Rev. E* 55 (1997) 6266; M.C. Mahato and A.M. Jayannavar, *Mod. Phys. Lett. B* 11 (1997) 815; M.C. Mahato and A.M. Jayannavar, *Physica A* 248 (1998)138; J.C. Phillips and K. Schulten, *Phys. Rev. E* 52 (1994) 2473; M. Thorwart and P. Jung, *Phys. Rev. Lett.* 78 (1997) 2503; S.W. Sides, P.A. Rikvold and M.A. Novotny, *Phys. Rev. Lett.* 81 (1997) 834, W. Sides, P.A. Rikvold and M.A. Novotny, *Phys. Rev. E* 57(1998) 6512; M. Thorwart, P. Reimann, P. Jung and R.F. Fox, *Chem. Phys.* 235 (1998) 61; J.S. Lim, M.Y. Choi and B.J. Kim, *Phys. Rev. B* 68 (2003) 012501,
- [4] T. Iwai, *Physica A* 300 (2001) 350; T. Iwai, *J. Phys. Soc. Jpn.* 70 (2001) 353.
- [5] D. Dan and A.M. Jayannavar, *Physica A* 345 (2005) 404.
- [6] M. Evstigneev, P. Riemann and C. Bechinger, *J. Phys. C* 17 (2005) S3795.
- [7] V. Berdichevsky and M. Gitterman, *Physica A* 249 (1998) 88.
- [8] C. Bustamante, J. Liphardt and F. Ritort, *Physics Today* 58(2005) 45.
- [9] F. Ritort, *Poincare Seminar* 2 (2003) 195.
- [10] D.J. Evans and D.J. Searls, *Adv. Phys.* 51 (2002) 1529.
- [11] A.M. Jayannavar and M. Sahoo, *cond-mat/0611516*;
- [12] R. van Zon and E.G.D. Cohen, *Phys. Rev. E* 67 (2003) 046102; R. van Zon and E.G.D. Cohen, *Phys. Rev. E* 69 (2004) 056121.
- [13] O. Mazonka and C. Jarzynski, *cond-mat/9912121*.
- [14] F. Douarche, S. Joubaud, N.B. Garnier, A. Petrosyan and S. Ciliberto *Phys. Rev. Lett* 97 (2006) 140603; R. von Zon, S. Ciliberto and E.G.D. Cohen, *Phys. Rev. Lett* 92 (2004) 130601; S. Joubaud, N.B. Garnier and S. Ciliberto, *cond-mat/0610031*
- [15] H. Risken, *The Fokker-Planck Equation*, Springer-Verlag, Berlin, 1984.
- [16] K. Sekimoto, *J. Phys. Soc. Jpn.* 66 (1997) 6335.
- [17] R. Mannella, *Stochastic Processes in Physics, Chemistry, and Biology*, J.A. Freund and T. Shoshel, *Lecture Notes in Physics*, vol. 557 (2000) 353.
- [18] R. Marathe and A. Dhar, *Phys. Rev. E* 71 (2005) 036126.