Asymmetric motion in a double-well under the action of zero-mean Gaussian white noise and periodic forcing

Mangal C. Mahato and A.M. Jayannavar

Institute of Physics, Sachivalaya Marg, Bhubaneswar-751005, India

Abstract

Residence times of a particle in both the wells of a double-well system, under the action of zero-mean Gaussian white noise and zero-averaged but temporally asymmetric periodic forcings, are recorded in a numerical simulation. The difference between the relative mean residence times in the two wells shows monotonic variation as a function of asymmetry in the periodic forcing and for a given asymmetry the difference becomes largest at an optimum value of the noise strength. Moreover, the passages from one well to the other become less synchronous at small noise strength as the asymmetry parameter (defined below) differs from zero, but at relatively larger noise strengths the passages become more synchronous with asymmetry in the field sweep. We propose that asymmetric periodic forcing (with zero mean) could provide a simple but sensible physical model for unidirectional motion in a symmetric periodic system aided by a symmetric Gaussian white noise.

PACS numbers: 82.20.Mj, 05.40.+j, 75.60.Ej

Several physical models have recently been proposed [1–7,9] to understand possible average asymmetric motion of a Brownian particle in a periodic potential. Living systems are manifestly nonequilibrium and quite understandably such an asymmetric motion has been observed recently in biological systems [8]. Though the quest for extracting useful work out of nonequilibrium systems is not new, the biological experimental observation has given enough motivation recently to renew effort in that direction. It has resulted in better understanding of the problem and also it has helped in inventing new devices for practical use [9]. In the present work we study a symmetric two-well system subjected to zero-mean Gaussian white noise. We apply an external field that is periodic in time. The external field is taken to be temporally asymmetric but with mean force zero in a period. We find that eventhough the mean deterministic force experienced by a particle due to the external field is zero, the Gaussian white noise (centered at zero) helps it to extract work while moving on a potential surface. This is reflected in the asymmetric passages between the two symmetrically connected wells modulated periodically by the external field.

We consider the symmetric two-well potential represented by $U(m) = -\frac{a}{2}m^2 + \frac{b}{4}m^4$ and consider the external field $h(t)$ to be periodic in time and assume an asymmetric saw-tooth form for it. The asymmetry in the saw-tooth form comes because the positive and negative slopes are taken to have different magnitude. The mean force in a period (because of the external field) is assured to be zero by taking the maxima and minima of the saw-tooth to have values $+h_0$ and $-h_0$, respectively. A Brownian particle will thus experience a combined (time dependent) potential

$$\Phi(m,t) = U(m) - mh(t). \tag{1}$$

We consider the time evolution of the particle coordinate $m(t)$ to be governed by the overdamped Langevin equation

$$\dot{m} = -\frac{\partial \Phi}{\partial m} + \hat{f}(t), \tag{2}$$

where $\hat{f}(t)$ is a randomly fluctuating force and is taken to be Gaussian with statistics

$$< \hat{f}(t) >= 0, \tag{3}$$

and

$$< \hat{f}(t)\hat{f}(t') >= 2D\delta(t - t'). \tag{4}$$

Here $< ... >$ represents average over a large number of realizations of the random forces.

Our calculation involves solving the Langevin equation numerically and to monitor the time evolution of $m(t)$ for a long time for given noise strength D. The calculation is done for a fixed subcritical $h(t)$ with amplitude $h_0 < h_c$, where h_c is the minimum value of $|h(t)|$ at which one of the two wells of $\Phi(m)$ becomes unstable and disappears. Since $h(t) < h_c$ always, the barrier between the two wells never vanishes. Therefore to pass from one well to the other a particle need necessarily have to surmount a nonzero potential barrier and therefore has to be noise aided. Along the time axis we record the events of passages between the two wells. We consider passage to take place from a given well as and when the trajectory $m(t)$, emerging from the given well, crosses the inflexion point on the other side of the potential barrier separating the two wells. From the markers recorded on the time axis we obtain the distribution $\rho_1(\tau)$ of residence times, τ, in the well 1 and (similarly for the well 2). And also the distribution $\rho_{12}(h)$ of field values $h(t)$ at which passages take place from the well 1 to the well 2 (and similarly from the well 2 to the well 1) are calculated from the same recordings.

The distributions $\rho_{12}(h)$ and $\rho_{21}(h)$ determine the evolution of the fraction of population in a well as the external field $h(t)$ varies. For example, the fraction $m_2(h)$ of the population in the well 2 evolves [from $(n-1)$th step to nth step] as

$$m_2(n) = m_2(n-1) - m_2(n-1)\rho_{21}(h_{n-1})(h_n - h_{n-1}) + m_1(n-1)\rho_{12}(h_{n-1})(h_n - h_{n-1}), \tag{5}$$

where h_n is the field value at the nth subdivision point in a cycle of h. The interval of uniform subdivisions $(h_n - h_{n-1})$ are taken to be optimally small for better accuracy. In our calculation we take $(h_n - h_{n-1}) = \Delta h = 0.001h_c$. So the whole period is divided into

$N = \frac{2h_0}{\Delta h}$ equal segments. This evolution equation together with the periodicity condition, for instance, $m_2(n=0) = m_2(n = \frac{2h_0}{\Delta h})$ gives the closed hysteresis loop $\bar{m}(h) = m_2(h) - m_1(h)$. Also, throughout our calculation, we take $a = 2.0$, and $b = 1.0$. The hysteresis loop area is a good measure of degree of synchronization of passages between the two wells. For example, if the passages take place only when the potential barrier for passage is the least, *i.e.,* at $h = \pm h_0$, the distributions $\rho_{12}(h)$ and $\rho_{21}(h)$ will be sharply peaked at $h = h_0$ and $h = -h_0$, respectively. In this case the hysteresis loop will be nearly rectangular and therefore will have the largest area. On the other hand, if the passages take place all over and randomly (corresponding to the case of least synchronization) so that $\rho_{12}(h)$ and $\rho_{21}(h)$ are uniform the loop area becomes the least. We explore how the hysteresis loop area A changes as a function of the asymmetry of the field $h(t)$ and also as a function of the noise strength D. The asymmetry of field sweep, Δ, is defined as $\Delta = \frac{T_1 - \frac{T_0}{2}}{\frac{T_0}{2}}$, where T_1 is the time for the field to change from h_0 to $-h_0$ and T_0 is the period of oscillation of $h(t)$. Fig. 1 shows a typical hysteresis loop with $\Delta \neq 0$ as compared to one with $\Delta = 0$. Notice that the hysteresis loops do not saturate for the field amplitude $h_0 = .7h_c$ considered here.

Figure 2 shows the variation of hysteresis loop area A as a function of the asymmetry parameter Δ for fixed values of h_0, T_0, and various values of D. The area is the largest at $\Delta = 0$ for small D but is the lowest for relatively larger D. In both the cases, however, $A(-\Delta) = A(\Delta)$. This result is to be compared with our earlier work on two-well systems [7] where *first-passage times* instead of residence times were calculated. In that work the upper half of the hysteresis loop, corresponging to passages from well 2 to well 1, was obtained from the first-passage-time distribution $\rho(h)$ that spread between h_0 and $-h_0$ (and not over the whole period) and the other half was obtained by symmetry. Consequently the hysteresis loops, by construction, saturated to $\bar{m}(h_0) = 1.0h_c$ and $\bar{m}(-h_0) = -1.0h_c$, and were symmetric for all h_0 including $h_0 = .7h_c$. The variation of hysteresis loop area, however, showed asymmetry and attained a peak (usually not at $\Delta = 0$) as a function of Δ. In Fig. 3, we show how, in the present work, area changes as a function of D for fixed Δ,

T_0, and h_0. However, before discussing the physical significance of these results we consider the mean residence times $\bar{\tau}_1$ and $\bar{\tau}_2$ in the two wells as a function of Δ.

From the distributions $\rho_1(\tau)$ and $\rho_2(\tau)$ of the residence times we calculate the mean residence times in each of the two wells. We, then, calculate the fraction of times, f_1 and f_2, the particle spends, on the average, in the two wells. The difference, $M = f_2 - f_1$, gives a quantity analogous to magnetization (normalized) in magnetic systems. In Fig. 4 we plot M as a function of Δ. From the figure it is clear that $M(\Delta) = -M(-\Delta)$ (upto the order of accuracy of our numerical calculation) and vary roughly monotonically. This indicates that the particles will tend to accumulate in the well 2 if $\Delta > 0$ and in the well 1 if $\Delta < 0$. This conclusion is plausible because in the situation under consideration the mean residence times $\bar{\tau}_1$ and $\bar{\tau}_2$ add up to $\bar{\tau} = \bar{\tau}_1 + \bar{\tau}_2$ which is larger than T_0, the period of oscilation of $h(t)$. This simply indicates that passages do not take place in every cycle of $h(t)$. For $\Delta > 0$, for instance, there will be larger number of passages from well 1 to 2 than from 2 to 1 per cycle of $h(t)$, in an ensemble. This results in a net accumulation of particles in the well 2, asymptotically. This asymmetry in passages is also reflected indirectly in the hysteretic property of the system. For $\Delta \neq 0$ the hysteresis loops are asymmetric. Also, as Δ deviates from zero the hysteresis loop area decreases for small D (but increases for relatively larger D) [Fig. 2], which indicates that now the passages are less(more) synchronous with the input signal $h(t)$. This effect indicates as if a net average constant field is applied in a direction determined by Δ.

The net accumulation M in a two-well system for given Δ changes with the noise strength D. Fig. 5 shows that M increases initially, reaches a maximum, and then decreases gradually as D is increased. We, thus, have an optimum value of D at which the accumulation in the well 2 (when $\Delta > 0$) is the largest after a large number of cycles of $h(t)$. All the results obtained in the present work are susceptible to experimental verification by the recently developed optical interferometric techniques [10].

All the calculated results discussed so far are valid for a double-well potential. However, it is not difficult to envisage the situation in case of a periodic potential. In a two-well potential, for $\Delta > 0$, as mentioned earlier, the number of passages taking place from well 1 to well 2 per cycle of field sweep is larger than the number of passages from well 2 to well 1. One may justifiably extrapolate this result to state that in a periodic potential (that may even be symmetric), in a given number of cycles of $h(t)$ there will be more $1 \rightarrow 2$ passages than $2 \rightarrow 1$ passages, and hence there will be a net current of particles in the right direction ($1 \rightarrow 2$) when $\Delta > 0$ and in the reverse direction when $\Delta < 0$. This current will increase with the magnitude of Δ. However, for a given Δ we can find an optimum value of the Gaussian white noise strength D at which the current will be maximum. This is an important observation because here we have a physical model for unidirectional motion of a particle in a nonratchetlike symmetric periodic potential aided by symmetric Gaussian white noise (fluctuating forces).

REFERENCES

[1] M. Magnasco, *Phys. Rev. Lett.* **71**, 1477(1993).

[2] R. Bartussek, P. Hänggi, and J.G. Kissner, *Europhys. Lett.* **28**, 459(1994); R. Bartussek, P. Reimann, and P. Hänggi, *Phys. Rev. Lett.* **76**, 1166(1996); P. Hänggi, R. Bartussek, P. Talkner, and J. Luczka, *Europhys. Lett.* **35**, 315(1996); P. Jung, J.K. Kissner, and P. Hänggi, *Phys. Rev. Lett.* **76**, 3436(1996).

[3] R.D. Astumian and M. Bier, *Phys. Rev. Lett.* **72**, 1776(1994).

[4] C.R. Doering, N. Horsethemke, and J. Riordan, *Phys. Rev. Lett.* **72**, 2984(1994).

[5] M.M. Millonas, and M.I. Dykman, *Phys. Lett.* A**185**, 65(1994); M.M. Millonas, *Phys. Rev. Lett.* **74**, 10(1995).

[6] A.M. Jayannavar, *Phys. Rev.* E**53**, 2957(1996); M.C. Mahato, T.P. Pareek, and A.M. Jayannavar *Int. J. Mod. Phys.* **B**, (in Press).

[7] M.C. Mahato, and A.M. Jayannavar, *Phys. Lett.* A**209**, 21(1995); D.R. Chialvo, and M.M. Millonas, *ibid* A**209**, 26(1995); For motion induced by high amplitude ($h_0 > h_c$) deterministic *ac* forces see A. Ajdari, D. Mukamel, L. Peliti, and J. Prost, *J. Phys. I France* **4**, 1551(1994).

[8] K. Svoboda, C.F. Schmitt, B.J. Schnapp, and S.M. Block, *Nature* **365**, 721(1993).

[9] J. Prost, J.F. Chauwin, L. Peliti, and A. Ajdari, *Phys. Rev. Lett.* **72**, 2652(1994); J. Rousselet, L. Salome, A. Ajdari, and J. Prost, *Nature* **370**, 446(1994); J.F. Chauwin, A. Ajdari, and J. Prost, *Europhys. Lett.* **32**, 373(1995).

[10] A. Simon, and A. Libchaber, *Phys. Rev. Lett.* **68**, 3375(1992).

FIG. 1. Hysteresis loops $\bar{m}(h)$, for $h_0 = 0.7h_c$, $D = 0.15$, and $T_0 = 28.0$, are plotted for (a) $\Delta = 0$ (solid line) and (b) $\Delta = 0.5$ (dotted line).

FIG. 2. Plots of hysteresis loop area A versus Δ for (a) $D = 0.1(\circ)$, (b) $D = 0.15(\Box)$, and (c) $D = 0.2(\diamond)$, (d) $D = 0.5(\triangle)$, and (e) $D = 0.7(\bigtriangledown)$.

FIG. 3. Plot of hysteresis area as a function of D for (a) $\Delta = 0.5(\circ)$, and (b) $\Delta = \frac{13}{14}(\Box)$.

FIG. 4. Shows the variation of accumulation M in a well as a function of Δ for (a) $D = 0.15(\Box)$, (b) $D = 0.2(\diamond)$, (c) $D = 0.5(\triangle)$, and (d) $D = 0.7(\bigtriangledown)$.

FIG. 5. Plot of M versus D for (a) $\Delta = 0.5(\circ)$, and (b) $\Delta = \frac{13}{14}(\Box)$.