Study of transmission and reflection from a disordered lasing medium

Sandeep K. Joshi and A. M. Jayannavar

Institute of Physics, Sachivalaya Marg, Bhubaneswar 751 005, India

A numerical study of the statistics of transmission (t) and reflection (r) of quasi-particles from a one-dimensional disordered lasing or amplifying medium is presented. The amplification is introduced via a uniform imaginary part in the site energies in the disordered segment of the single-band tight binding model. It is shown that t is a non-self-averaging quantity. The cross-over length scale above which the amplification suppresses the transmittance is studied as a function of amplification strength. A new cross-over length scale is introduced in the regime of strong disorder and weak amplification. The stationary distribution of the backscattered reflection coefficient is shown to differ qualitatively from the earlier analytical results obtained within the random phase approximation.

In recent years the study of wave propagation in an active random medium [1–10] , in the presence of absorption or amplification, is being persued actively. The light propagation in an amplifying (lasing) medium has its implications for stimulated emission from random media. In the Schrödinger equation, to describe the absorption or amplification, one introduces the imaginary potentials. In that case the Hamiltonian becomes non-Hermitian and thus the particle number is not conserved. It should be noted that in quenched random systems with imaginary potentials the temporal coherence of the wave is preserved in spite of amplification or absorption which causes a particle non-conserving scattering process.

Some new results have been obtained in this area. In a scattering problem, the particle experiences a mismatch from the real valued potential to the imaginary valued potential at the interface between the free region and the absorbing (or amplifying) medium, and hence it tries to avoid this region by enhanced back reflection. Thus a dual role is played by imaginary potentials as an absorber (or amplifier) and as a reflector. When the strength of the imaginary potential is increased beyond certain limit, both absorber and amplifying scatterer act as reflectors. Thus the reflection coefficient exhibits non-monotonic behavior as a function of the absorption (amplification) strength. Using the duality relations it has been shown that amplification suppresses the transmittance in the large length (L) limit just as much as absorption does [4]. There exists a crossover length scale L_c below which the amplification enhances the transmission and above which the amplification reduces the transmission which, in fact, vanishes exponentially in the $L \to \infty$ limit. In contrast, reflectance saturates to a finite value. Moreover, absorption and amplification of same strength (*i.e.*, differing only in the sign of the imaginary part) will induce same localization length [6].

In an amplifying medium even though the transmittance (t) decreases exponentially with the length L in the large L limit, the average $\langle t \rangle$ is shown [7] to be infinite due to the less probable resonant realizations corresponding to the non Gaussian tail of the distribution of $\ln t$. This result is based on the analysis using random phase approximation (RPA). Using duality argument Paasschens *et. al.* show that non-Gaussian tails in the distribution of $\ln t$ contain negligible weight [4]. Thus one might expect finite value for $\langle t \rangle$ in the asymptotic limit. It should be noted that even in the ordered periodic system all the states are resonant states and still the transmittance decreases exponentially for all the states in the large L limit. The above simple case may indicate that in the asymptotic limit $\langle t \rangle$ is indeed finite. One of our objectives in this paper is to study the behavior of the transmission probability as a function of length in the presence of coherent amplification. We show that the transmission coefficient is a non-self-averaging quantity. In the large length limit we do not find any resonant realization, which can give an enhanced transmission. We also study the behavior the of cross-over length L_c as a function of amplification strength. We have analysed the behavior of logarithm of the transmittance which will have a maximum value $\langle \ln t \rangle_{max}$ at L_c, and its dependence on the amplification strength. For a given strength of amplification there exists a critical strength of disorder below which the average transmittance is always less than unity at all length scales and decreases monotonically. In this regime L_c and $\langle \ln t \rangle_{max}$ loose their physical significance. In this regime we show that there exists a new cross-over length scale ξ_c which diverges as the amplification strength is reduced to zero for a given strength of the disorder.

In the work by Pradhan and Kumar [2], the analytical expression for the stationary distribution $P_s(r)$ of a coherently backscattered reflection coefficient (r) is obtained in the presence of amplification in a RPA using the method of invariant imbedding. The expression for $P_{s(r)}$ is given by

$$P_s(r) = \frac{|D| exp(-\frac{|D|}{r-1})}{(r-1)^2} \quad for \ \ r \geq 1 \qquad (1)$$
$$= 0 \qquad\qquad for \ \ r < 1$$

where D is proportional to η/W, η and W being the strength of amplifying potential and disorder respectively. One can readily notice from Eqn. (1) that $P_s(r)$

does not tend to $\delta(r-1)$ in the large η limit. In this limit, as mentioned earlier, an amplifying scatterer acts as a reflector. The validity of above expression is limited to small disorder and amplification strength.

We consider a quasi-particle moving on a lattice. The appropriate Hamiltonian in a tight-binding one-band model can be written as

$$H = \sum \epsilon'_n |n\rangle \langle n| + V(|n\rangle \langle n+1| + |n\rangle \langle n-1|). \quad (2)$$

V is the off-diagonal matrix element connecting nearest neighbors separated by a lattice spacing a (taken to be unity throughout) and $|n\rangle$ is the non-degenerate Wannier orbital associated with site n, where $\epsilon'_n = \epsilon_n - i\eta$ is the site energy. The real part of the site energy ϵ_n being random represents static disorder and ϵ_n at different sites are assumed to be uncorrelated random variables distributed uniformly ($P(\epsilon_n) = 1/W$) over the range $-W/2$ to $W/2$. We have taken imaginary part of the site energy η to be spatially uniform positive variable signifying amplification. Since all the relevant energies can be scaled by V, we can set V to unity. The lasing medium consisting of N sites ($n = 1$ to N) is embedded in a perfect infinite lattice with all site energies taken to be zero. To calculate the transmission and reflection coefficients we use the well known transfer-matrix method , and the details are described in Ref. [3] .

In our studies we have set the energy of the incident particle at $E = 0$, *i.e.*, at a midband energy. Any other value for the incident energy does not affect the physics of the problem. In calculating average values in all cases we have taken 10,000 realizations of random site energies (ϵ_n). The strength of the disorder and the amplification are scaled with respect to V, *i.e.*, $W (\equiv W/V)$ and $\eta (\equiv \eta/V)$. The length $L = L/a$.

Depending on the parameters η, W and L the transmission coefficient can be very large (of the order of 10^{12} or more). Hence, we first consider behavior of $\langle lnt \rangle$ instead of $\langle t \rangle$. The angular brackets denote the ensemble average. The localization length ξ can be computed from the behaviour of transmittance in the large length limit. We denote the localization length l_a for an ordered medium (W=0) in the presence of uniform amplification. The localization length [11] for a disordered passive medium ($\eta = 0$) is given by elastic back scattering length $l = 48V^2/W^2$ at the center of the band ($E = 0$). We have verified that the localization length in the presence of both disorder and amplification, [5] ξ is related to l and l_a (for $\eta/V < 1$ and $W/V < 1$) as $\xi = ll_a/(l + l_a)$.

In Fig. 1 we have plotted $\langle lnt \rangle_{max}$ against η for a fixed value of $W = 1.0$ and the inset shows variation of L_c with η for $W = 1.0$. Initially $\langle lnt \rangle_{max}$ increases with η and after exhibiting a maxima it decays to zero for large η. This arises from the fact that the lasing medium acts as a reflector for large η as discussed in the introduction. Near the maximum, in a finite regime of η, $\langle lnt \rangle_{max}$ exhibits several oscillations. In this region sample to sample fluctuations of lnt are very large. Thus average over $10,000$

realizations may not represent the true ensemble averaged quantity. From the curve fitting of our numerical data for L_c, we find that L_c does not follow a power law, $(1/\sqrt{\eta})$, in the full parameter regime [5].

To study the nature of fluctuations in the transmission coefficient, in Fig. 2 we have plotted, on log-scale, $\langle t \rangle$, root-mean-squared variance $t_v = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}$ and root-mean-squared relative variance (or fluctuation) $t_{rv} = \sqrt{\langle t^2 \rangle - \langle t \rangle^2}/\langle t \rangle$ as a function of L for $\eta = 0.1$ and $W = 1.0$. For these parameters $l \approx 48, l_a = 10$, $\xi \approx 8$ and $L_c \approx 30$. We notice that both $\langle t \rangle$ and t_v exhibit maxima and decrease as we increase the length further. Except in the small length limit, variance is larger than the mean value. The relative variance is larger than one for $L > 10$ and remains large even in the large length limit. The t_{rv} fluctuates between values 50 to 300 in the large length ($L > 10$) regime, indicating clearly the non-self-averaging nature of the transmittance. This implies that the transmission over the ensemble of macroscopically identical samples dominates the ensemble average. In such a situation one has to consider the full probability distribution $P(t)$ of t to describe the system behavior.

We would now like to understand whether there exist any resonant realizations in the large length limit for which the transmittance is very large. This study calls for sample to sample fluctuations. It is well known from the studies in passive random media that the ensemble fluctuation and the fluctuations for a given sample as a function of chemical potential or energy are expected to be related by some sort of ergodicity [12], *i.e.*, the measured fluctuations as a function of the control parameter are identical to the fluctuations observable by changing the impurity configurations. In Fig. 3(a) we have plotted t versus incident energy E (within the band from -2 to $+2$) for a given realization of random potential with $\eta = 0$ and $L = 100$. The Fig. 3(b) shows the behavior of t versus E for the same realization in the presence of amplification $\eta = 0.1$ and $L = 100$. ¿From the Fig. 3(a) we observe that at several values of energy the transmittance exhibits the resonant behavior in that $t = 1$. ¿From Fig. 3(b) we notice that in the presence of amplification, transmittance at almost resonant realizations is negligibly small. Few peaks appear in the transmittance whose origin lies in the combined effect of disorder and amplification. However, we notice that the transmittance at these peaks is much smaller, where as one would have naively expected the transmittance to be much much larger than unity in the amplifying medium. We have studied several realizations and found that none of them shows any resonant behavior where one can observe the large transmittance. The peak value of observed transmittance is of the order of unity or less. This study clearly indicates that $\langle t \rangle$ is indeed finite contrary to the earlier predictions based on RPA [7].

So far our study was restricted to the parameter space of W and η for which L_c and hence $\langle lnt \rangle_{max}$ exist.

In Fig. 4 we have plotted $\langle lnt \rangle$ against L for ordered lasing medium ($W = 0,\eta = 0.01$), disordered passive medium ($W = 1.0,\eta = 0$) and disordered active medium ($W = 1.0,\eta = 0.01$). The present study is restricted to the parameter space of η and W such that $\eta \ll 1.0$ and $W \geq 1.0$. We notice that for an ordered lasing medium, the transmittance is larger than one. We have taken our range of L upto 300. For a disordered active medium ($W = 1.0,\eta = 0.01$), we notice that the transmittance is always less than one and monotonically decreasing. Initially, upto certain length, the average transmittance is, however, larger than that in the disordered passive medium ($W = 1.0,\eta = 0$). This arises due to the combination of lasing with disorder. In the asymptotic regime transmittance of a lasing random medium falls below that in the passive medium with same disorder strength. This follows from the enhanced localization effect due to the presence of both disorder and amplification together, i.e., $\xi < l$. It is clear from the figure that $\langle lnt \rangle$ does not exhibit any maxima and hence the question of L_c or $\langle lnt \rangle_{max}$ does not arise. We notice, however, from the figure that for random active medium initially $\langle lnt \rangle$ decreases with a well defined slope and in the large length limit $\langle lnt \rangle$ decreases with a different slope (corresponding to localization length ξ). Thus we can define a length scale ξ_c (as indicated in the figure) at which there is a cross-over from the initial slope to the asymptotic slope. In the inset of Fig. 4 we have shown the dependence of ξ_c on η. Numerical fit shows that ξ_c scales as $1/\sqrt{\eta}$, as we expect $\xi_c \to \infty$ with $\eta \to 0$. As one decreases η, the absolute value of initial slope increases and that of the asymptotic one decreases. Simultaneously, the cross-over length ξ_c increases. In the $\eta \to 0$ limit both initial as well as asymptotic slopes become identical.

In Fig. 5 we have plotted the stationary distribution $P_s(r)$ of reflection coefficient r for different values of η (as shown in the figure) and a fixed value of $W = 5.0$. For small values of $\eta = 0.05$ the stationary distribution $P_s(r)$ has a single peak around $r = r_{max} = 1$. The peak (r_{max}) shifts to higher side as we increase η (Fig. 5(b)). The behavior of $P_s(r)$ for small η is in qualitative agreement with Eqn. (1). As we increase η further $P_s(r)$ exhibit a double peaked structure (Fig. 5(c)). At first the second peak appears at higher value of r at the expense of the distribution at the tail. As we increase η the second peak becomes more prominent and shifts towards left, where as the height of the first peak decreases. The distribution at the tail has a negligible weight (see Fig. 5(c)). At still higher values of η, the second peak approaches $r \approx 1$ whereas the first peak disappears. The now-obtained single peak distribution $P_s(r)$ in the large η limit tends to $\delta(r - 1)$. In this limit the amplifying medium acts as a reflector and the disorder plays a sub-dominant role. The occurrence of the double peak structure along with $P_s(r) \to \delta(r - 1)$ in the large η limit cannot be explained even qualitatively from Eqn. (1). This is due to the failure of RPA in this regime [3].

Our study for $\langle lnr \rangle_s$, obtained from $P_s(r)$ indicate that

it initially increases with η and exhibiting a maxima at η_{max} decreases monotonically. The double peak in $P_s(r)$ appears for values close to η_{max}. For the value $\eta > \eta_{max}$ the amplifying medium acts dominantly as a reflector. In this regime we have verified that the stationary distribution of the phase θ of the complex reflection amplitude is not uniform. Physics of the double peak and overall shape of $P_s(r)$ shows similarity with the stationary distribution obtained in the case of absorption (for details we refer to Ref. [3].

In conclusion our numerical study on the statistics of transmission coefficient in random lasing medium indicates that in the asymptotic regime the transmission coefficient is a non-self-averaging quantity,however, with a well defined finite average value. In some parameter space transmittance initially increases with η and falls off exponentially to zero in the asymptotic regime. In this regime there is a well defined cross-over length L_c at which the transmittance is maximum, and it decreases monotonically with η. In the parameter range where $\eta \ll 1$, in the presence of disorder the average transmittance decreases monotonically and has a magnitude less than unity. In this regime L_c does not exist. However, one can still define a new length scale ξ_c which scales as $1/\sqrt{\eta}$. Our study on the stationary distribution of reflection coefficient $P_s(r)$ indicates that earlier analytical studies fail, even qualitatively, to explain the observed behavior in the large η limit. Our study clearly brings out the dual role played by an amplifying medium, as an amplifier as well as a reflector.

[1] N. M. Lawandy et. al. , Nature **368**, 436 (1994); D. S. Wiersma, M. P. van Albada and Ad Lagendijk, Phys. Rev. Lett. **75**, 1739 (1995).
[2] P. Pradhan and N. Kumar, Phys. Rev. B **50**, 9644 (1994).
[3] Abhijit Kar Gupta and A. M. Jayannavar, Phys. Rev. B **52**, 4156 (1995).
[4] J. C. J. Paasschens, T. Sh. Misirpashaev and C. W. J. Beenakker, Phys. Rev. B **54**, 11887 (1996).
[5] Z. Q. Zhang, Phys. Rev. B **52**, 7960 (1995).
[6] C. W. J. Beenakker, J. C. J. Paasschens and P. W. Brouwer, Phys. Rev. Lett. **76**, 1368 (1996).
[7] V. Freilikher, M. Pustilnik, and I. Yurkevich, preprint cond-mat/9605090.
[8] T. Sh. Misirpashaev, J. C. J. Paasschens and C. W. J. Beenakker, Physica A **236**, 189 (1997).
[9] V. Freilikher, M. Pustilnik, and I. Yurkevich, Phys. Rev. Lett. **73**, 810 (1994).
[10] S. John, Phys. Rev. Lett. **53**, 2169 (1984).
[11] E. N. Economou, *Green's Functions in Quantum Physics*, 2nd ed. (Springer-Verlag, Berlin, 1983), p. 174.
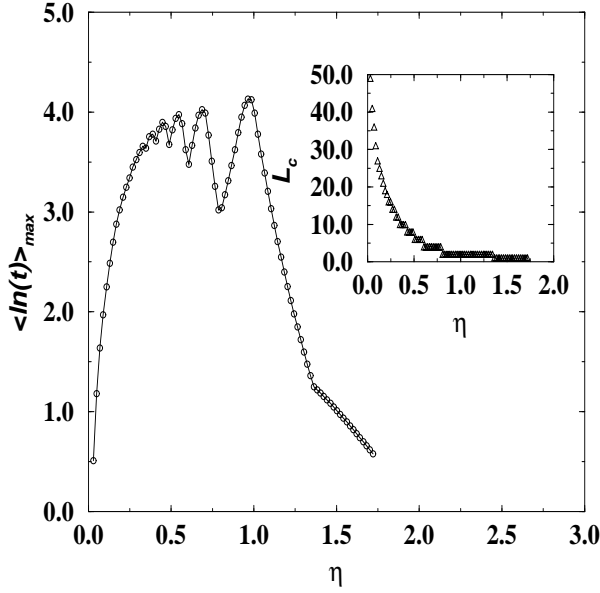[12] P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

FIG. 1. The variation of $\langle lnt \rangle_{max}$ with amplification strength η for $W = 1.0$. Inset shows the variation of L_c with η for $W = 1.0$.
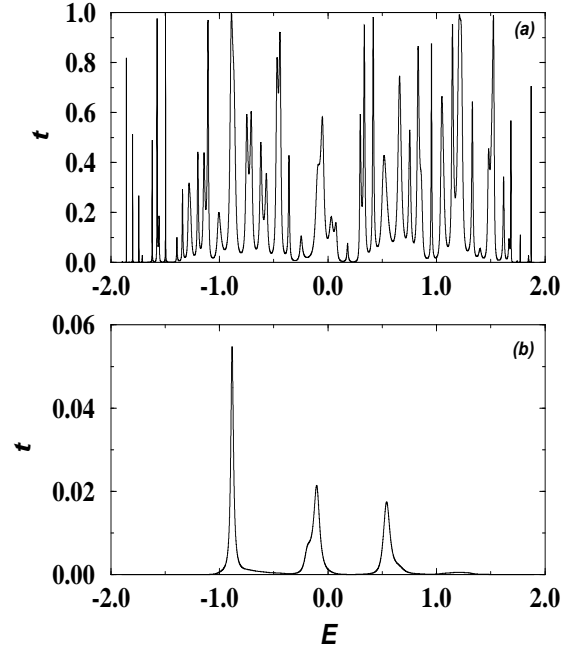

FIG. 3. Transmittance t as function of incident energy E for $W = 1.0$, $L = 100$ and (a) $\eta = 0$ and (b) $\eta = 0.1$
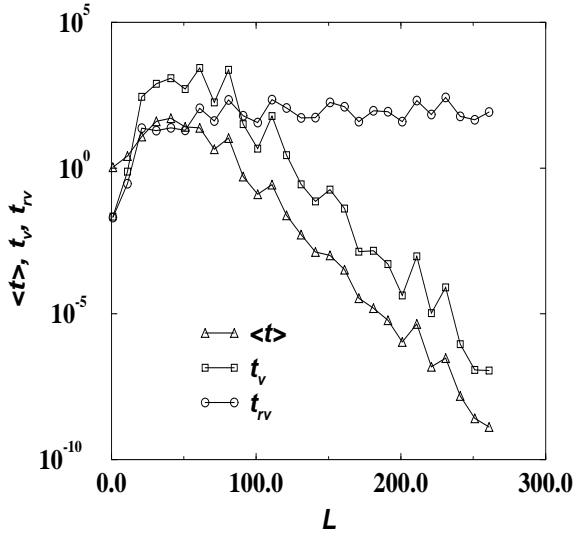

FIG. 2. The plot of $\langle t \rangle$, root-mean-squared variance (t_v) and root-mean-squared relative variance (t_{rv}) as a function of length L for $\eta = 0.1$ and $W = 1.0$.
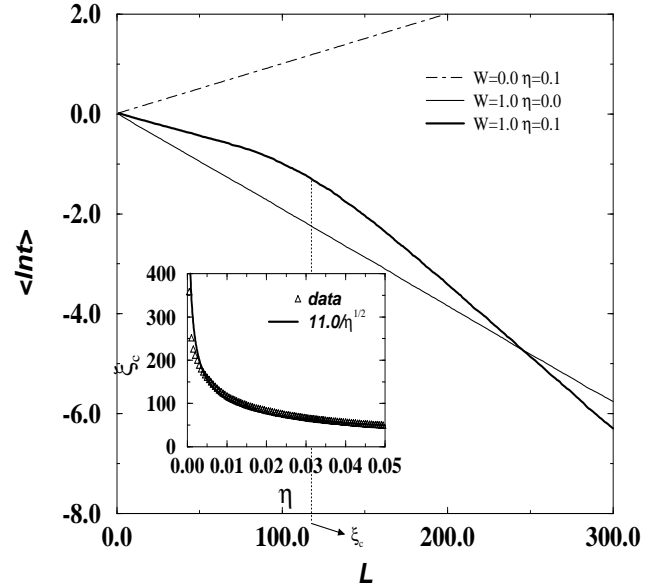

FIG. 4. Variation of $\langle lnt \rangle$ with L. The new length scale ξ_c which arises for $\eta \ll 1.0$ is shown by a vertical dotted line. The inset shows the variation of ξ_c with η for $W = 1.0$. The numerical fit shown by the thick line indicates that ξ_c scales as $\eta^{-1/2}$ in this regime.
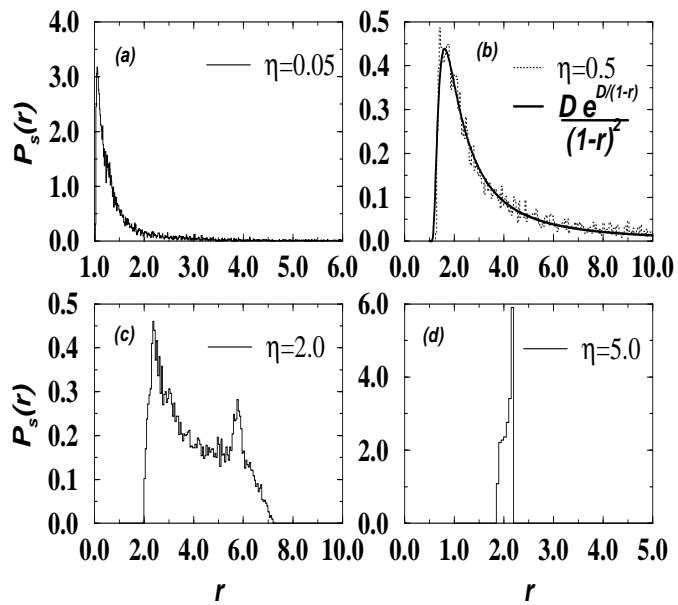
4

FIG. 5. Stationary distribution of reflection coefficient $P_s(r)$ for $W = 5.0$ and various values of η. The numerical fit shown in Fig. 5(b) with a thick line has $D = 1.235$