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THE VELOCITY DISPERSION OF THE GIANT MOLECULAR CLOUDS: A VISCOUS ORIGIN

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ABSTRACT

We propose the energy source and study the details of the acceleration mechanism for the random motion of the Giant Molecular Clouds (GMCs) in the Galaxy. Gravitational scattering of the massive clouds off each other in the differentially rotating galactic disk constitutes an effective "gravitational" viscosity, which causes an increase in the random kinetic energy of the GMCs at the expense of their ordered, rotational kinetic energy in the galactic disk.

We calculate the rate of increase, due to this effect, of the random kinetic energy of a GMC with a nonzero initial random velocity. In order to do this, we treat an encounter between the test cloud and a field cloud in the sheared disk as a perturbed, coupled, two-dimensional harmonic oscillator problem, with the gravitational interaction between the two clouds being the time-dependent perturbation force. The equations are solved analytically to second (lowest significant) order in the small parameter.

In a steady state, the rate of energy input from the viscosity due to gravitational and physical interactions among the GMCs in the differentially rotating galactic disk equals the rate of energy loss due to the inelastic physical collisions among the GMCs; this yields the value for the equilibrium cloud velocity dispersion.

The resulting one-component velocity dispersion is determined by a fifth-order polynomial having approximate solution $V_{1\text{-D}} = 0.69[(Gm/r)\kappa H]^{1/3} = 0.38 V_{esc}(\kappa/\kappa_z)(V_z/V_{1.0})^{1/2}$; where m, r and V_{esc} are the cloud mass, radius and escape velocity, respectively, κ is the epicyclic frequency κ_z is the z oscillation frequency, and H is the total vertical scale height of the gas distribution. This result is independent of the cloud number density and depends only weakly (through κ/κ_z) on the galactocentric radial distance of a cloud. Note that the cloud velocity dispersion is an *increasing* function of m/r and V_{esc}^2. The derived value is $V_{1\text{-D}} = 5$–7 km s^{-1} and is nearly independent of cloud mass, in good agreement with current observations. Gravitational viscosity, therefore, can provide the main energy input for the random motion of the GMCs in the Galaxy.

Locally the fraction of the rotational kinetic energy lost in supporting inelastic cloud motions for ~10 billion years is small, ~0.1. Thus the rotational kinetic energy of the GMCs proves to be more than adequate for the long-term support of their random motion. As a result of the viscous interaction among the clouds, the clouds drift inward. The viscous evolution of the radial distribution of the GMCs, which will be treated in a future paper, will tend to evacuate clouds from within ~3 kpc.

Thus, the dynamics as well as the radial distribution in the Galaxy of the GMCs is determined by their gravitational viscous interaction, which operates because of their location in the differentially rotating galactic disk.

Subject headings: galaxies: internal motions — hydrodynamics — interstellar: molecules — nebulae: general

I. INTRODUCTION

A great deal of observational information has accumulated concerning motions of gas clouds in the galactic disk, but our understanding of how these motions are maintained in spite of energy losses is very poor. For the normal Spitzer H I clouds with masses of order 400 $M_\odot$ and typical diameters of 5 pc, the original idea of Spitzer (1968), buttressed by more recent calculations of McKee and Ostriker (1977), that supernovae can plausibly provide the energy input seems reasonable. This statement is independent of the geometry of these "clouds" whether quasi-spherical or cylindrical, since acceleration by supernova blast waves depends only on the mass per unit area, a quantity determined directly from absorption line studies.

But understanding the giant molecular clouds (GMCs) with typical masses of $\sim 5 \times 10^5$ $M_\odot$ and diameters of ~50 pc (see Sanders, Scoville, and Solomon 1985, hereafter SSS1985, Cohen *et al.* 1985 presents a much more serious challenge. The puzzling properties of the overall cloud distribution can be divided into three separate issues.

1. Observations of interstellar cloud motion show that the cloud velocity dispersion is nearly constant, to within a factor of 2, for clouds covering three orders of magnitude in mass (see, e.g., Stark 1984). For example, the Giant Molecular Clouds (GMCs) have a one-dimensional, planar, rms, cloud-cloud velocity dispersion ($V_{1\text{-D}}$) of ~3–4 km s^{-1} with lower and upper values in this range from Clemens (1985) and Liszt and Burton (1983), respectively. At the lower mass end of the cloud range, the H I clouds, of ~400 $M_\odot$ each, have a typical one-dimensional velocity dispersion of ~6 km s^{-1} (Spitzer 1978). Clearly, the clouds are not in kinetic energy equipartition (KEE). This non–KEE behavior is in contradiction to what one would expect for an isolated, three-dimensional cloud

system. Since both elastic and inelastic collisions lead to equipartition (McKee and Ostriker 1977), this behavior is surprising. What accounts for the very weak dependence of velocity on mass?

2. The spatial distribution of the GMCs in the galactic disk is *not* that of an isolated three-dimensional system; rather, the GMCs exhibit a very thin disk (nearly a monolayer) distribution, with the ratio of the diameter of a typical GMC to the vertical scale height of the GMC distribution (= 130 pc, see SSS1985) ~0.4. In this property the cloud ensemble resembles planetary rings (Goldreich and Tremaine 1982) more closely than the distribution of the H I clouds. Why is this?

3. The motions of GMCs contain over one half of the total kinetic energy in the interstellar cloud motions. Since the time between inelastic collisions is ~few × 10^8 yr, the required energy input is ~1% of the energy input required to sustain the motions of the H I clouds which have a typical loss time of ~10^7 yr. But the supernova shocks, which can accelerate the low-mass clouds, are extremely ineffective in accelerating the GMCs because of the much larger mass to area ratio of the GMCs. By direct application of results from McKee and Ostriker (1977), we find that supernova shocks can only provide less than 10% of the kinetic energy of the GMCs. What is the source of energy for the motion of the GMCs?

We try to answer these questions in this paper. The above points suggest that the GMCs do not constitute an isolated three-dimensional system—rather, they indicate that the dynamics of the GMCs is mainly determined because of their location in a differentially rotating galactic disk, and that, as for particles in planetary rings, "viscosity" is the primary energy input.

Specifically, we propose that gravitational scattering of the massive clouds off each other in the differentially rotating galactic disk constitutes an effective gravitational viscosity, which causes an increase in the random kinetic energy of the GMCs at the expense of their ordered, rotational kinetic energy in the galactic disk. This mechanism is developed for the first time in this paper. In the problem of the planetary rings, the gravitational interaction among the particles is negligible and in that case, physical collisions account for the viscosity (see Goldreich and Tremaine 1982, and references therein).

In § II we calculate the energy input due to this gravitational viscosity and compute the other terms contributing to the energy balance of the GMCs. In § III we use these results to determine the steady state cloud velocity dispersion and its dependence on the cloud mass and radius and on the galactocentric radial distance. Section IV contains a summary of our conclusions.

II. ENERGY BALANCE FOR THE GMCS IN THE GALAXY

In this section we discuss the various physical processes that affect the random kinetic energy of the GMCs in the Galaxy. The different terms considered in the energy balance for the GMCs are the following:

1. The energy input from the viscosity due to physical and gravitational interactions among GMCs in the differentially rotating galactic disk.
2. The energy loss due to the inelastic physical collsions among the GMCs.
3. Neglected are supernova input, collisions with less massive clouds, global instabilities in the cloud fluid, and interactions with the fluctuating spiral potential. All are important processes and we return to discuss them in §§ III*c*–*d*.

The rate of increase of the random kinetic energy at the expense of ordered motion, from viscosity due to physical collisions between the particles moving in a sheared flow, is known from standard fluid mechanics (see, e.g., Lamb 1932; Landau and Liftshitz 1959). The rate of energy loss due to inelastic physical collisions of the GMCs is discussed in § II*a*. In § II*b* we consider the gravitational scattering of the clouds off each other and obtain the corresponding resulting rate of increase of the random kinetic energy of the GMCs. A major portion of this paper deals with this calculation.

In § II*c*, we write down the expression for the net rate of change of energy (due to the processes described in §§ II*a*, *b*) and equate it to zero for a steady state case. We treat the GMCs as a monolayer and only consider planar motion. In a subsequent paper we will present the analysis for the motion normal to the disk.

We now introduce the notation used in this paper to describe the various physical quantities. The galactic coordinate system R, θ, z is chosen with R along the galactocentric radius, θ along the direction of rotation, and z along the normal to the galactic plane. The quantities n, m, and r, respectively, denote the number density, the mass, and the radius of an individual cloud. The subscripts t and f represent these quantities for the test and the field clouds, respectively.

As long as the cloud random velocities are much less than the rotational speed in the disk we may use standard first-order epicyclic theory for the test cloud (see Mihalas and Routly 1968). In the guiding center frame, let $(V_R)_{\rm epi}$ and $(V_\theta)_{\rm epi}$ denote the maximum values of the radial and the azimuthal components of the velocity of a test cloud. Then

$$(V_R)_{\rm epi} = \left(\frac{\kappa}{2\Omega}\right)(V_\theta)_{\rm epi} , \tag{1}$$

where κ and Ω denote the epicyclic frequency and the angular rotation speed, respectively. Similarly, the radial and the azimuthal amplitudes of the epicycle, call them a_R and a_θ, respectively, are related as follows:

$$a_R = \left(\frac{\kappa}{2\Omega}\right)a_\theta , \tag{2a}$$

where

$$a_R = (V_R)_{\rm epi}/\kappa , \qquad a_\theta = (V_\theta)_{\rm epi}/\kappa . \tag{2b}$$

The total energy for the test cloud in an epicyclic orbit is given by

$$E_{\rm epi} = \frac{1}{2}\, m_t[(V_R)^2_{\rm epi} + (V_\theta)^2_{\rm epi}] = \frac{1}{2}\, m_t\left[1 + \left(\frac{2\Omega}{\kappa}\right)^2\right](V_R)^2_{\rm epi} . \tag{3}$$

Now let $(V_R)_{\rm random}$ and $(V_\theta)_{\rm random}$ be, respectively, the *average* radial and the azimuthal components of the random (peculiar) velocity dispersion of a given cloud—the peculiar velocity components are measured with respect to the *local* circular speed and the average is taken over the epicyclic orbit of the cloud. The components of the random velocity dispersion are related as follows (see Mihalas and Routly 1968):

$$(V_R)_{\rm random} = \left(\frac{2\Omega}{\kappa}\right)(V_\theta)_{\rm random} . \tag{4}$$

The total planar, random kinetic energy of a test cloud is given by

$$E_{\rm random} = \frac{1}{2}\, m_t (V_{\rm plane})^2_{\rm random} \equiv \frac{1}{2}\, m_t [2(V_{\rm 1\text{-}D})^2] = \frac{1}{2}\, m_t [(V_R)^2_{\rm random} + (V_\theta)^2_{\rm random}] = \frac{1}{2}\, m_t \left[1 + \left(\frac{\kappa}{2\Omega}\right)^2\right](V_R)^2_{\rm random} . \tag{5}$$

But since, on average,

$$(V_R)^2_{\rm random} = (V_R)^2_{\rm epi} ,$$

it follows from equations (3) and (5) that

$$E_{\rm random} = \frac{\kappa^2}{4\Omega^2}\, E_{\rm epi} , \tag{6}$$

and

$$(V_{\rm 1\text{-}D})^2 = \frac{1}{2}\,(V_R)^2_{\rm epi}\left[1 + \left(\frac{\kappa}{2\Omega}\right)^2\right] . \tag{7}$$

a) Physical Collisions

Physical collisions cause both gains and losses in the random kinetic energy for the cloud fluid. The rate of conversion of the rotational kinetic energy into the random kinetic energy via the viscosity due to physical collision is given (for a single test cloud) by (see Lamb 1932)

$$\left(\frac{dE_{\rm random}}{dt}\right)_{\rm gain} = C\left(\frac{1}{2}\, m_t\right)\left[\lambda_R\, R\, \frac{d}{dR}\,(\Theta/R\right]^2 \omega_c , \tag{8}$$

where ω_c is the collision frequency between clouds, Θ is the rotational speed, C is a constant of order unity and is taken here to be equal to 2 (see § II*b*), and λ_R is the extent of the radial excursion of the test cloud:

$$\lambda_R \equiv (V_R)_{\rm epi}\, \min\,(\omega_c^{-1}, \kappa^{-1}) . \tag{9}$$

For our problem we shall see that the second case holds: $\lambda_R = a_R$ (see § III*a*). Finally, in the rest frame of the guiding center of the epicyclic motion of the test cloud, the local value of the radial gradient in the rotational speed, is given in terms of Oort's constant $Rd(\Theta/R)/dR = -2A$.

Next, the rate of loss of random kinetic energy of a test cloud due to inelastic collisions with the field clouds is given by

$$\left(\frac{dE_{\rm random}}{dt}\right)_{\rm loss} = \frac{1}{2}\, m_t (V_{\rm plane})^2_{\rm random}\, \omega_c (1 - \epsilon^2) , \tag{10}$$

where ϵ is the coefficient of restitution which varies from 0 to 1 for completely inelastic to completely elastic collisions, respectively. We shall assume the physical collisions between two GMCs to be completely inelastic; this is reasonable since the observed cloud velocity is much greater than max (sound speed, Alfvén speed) within a cloud ($\lesssim 0.7$ km s^{-1}; see Spitzer 1978). Hence, $\epsilon = 0$ in the above equation, when applied to the GMCs (neglecting the relatively small elastic magnetic interaction). Here we depart from the treatment of planetary rings where ϵ is determined by energy balance.

Finally, ω_c, the frequency of physical collisions between clouds is given by

$$\omega_c = n_f V' \left[\frac{1}{2}\,\pi (r_t + r_f)^2\right]\left[1 + \frac{2G(m_t + m_f) 2^{1/2}}{(r_t + r_f)V'^2}\right] , \tag{11}$$

where V' is the average relative random velocity dispersion between the test and the field clouds. We set the random velocity of the field cloud to be zero so as to be consistent with the assumption in § II*b*. Hence $V' = 2^{1/2} V_{\rm 1\text{-}D}$. A more proper but less consistent treatment with $V' = 2V_{\rm 1\text{-}D}$ would change ω_c very little as a result of the effect of the gravitational focusing term.

The first set of brackets in equation (11) contains the pure geometrical cross section, the factor $\frac{1}{2}$ in this term is approximate and is meant to represent the collisions in which there is a substantial overlap in the masses of the two clouds. The second set of brackets in equation (11) contains the gravitational focusing term.

Combining equations (8), (10), and (11) yields the net rate of increase of random kinetic energy of a test cloud due to processes involving physical collisions:

$$\frac{dE}{dt} = \frac{\sqrt{2}}{2}\, m_t [4A^2 \lambda_R^2 - V^2_{\rm 1\text{-}D}] n_f\, V_{\rm 1\text{-}D} [\pi (r_t + r_f)^2]\left[1 + \frac{\sqrt{2}G(m_t + m_f)}{(r_t + r_f) V^2_{\rm 1\text{-}D}}\right] , \tag{12}$$

where $(V_{\rm 1\text{-}D})$ and λ_R are defined by equations (7) and (9) respectively.

At this point is is interesting to ask what would the steady state cloud velocity dispersion be if only the processes involving physical collisions were responsible in deciding the energy balance of the clouds. For this exercise to be meaningful, one has to retain the factor $(1 - \epsilon^2)$ on the right-hand side of equation (10). We use the flat rotation curve; hence $\kappa = 2^{1/2}\Omega$ and $A = (1/2)\Omega$ (see Mihalas and Routly 1968) and, locally, $\kappa = 35$ km s^{-1} kpc^{-1}. Using the typical cloud parameters (see § III*a*), $\omega_c = 24$ km s^{-1} kpc^{-1} $< \kappa$, so that $\lambda_R = a_R$ and hence equation (12) reduces to

$$2\left(\frac{2A}{\kappa}\right)^2 - \left[1 + \left(\frac{\kappa}{2\Omega}\right)^2\right](1 - \epsilon^2) = 0 \,, \tag{13}$$

or

$$(1 - \epsilon^2) = 2/3 \text{ or } \epsilon = (1/3)^{1/2} = 0.58 \,.$$

That is, if we were to follow the methods used by Goldreich and Tremaine for planetary rings, then ϵ would be determined to be equal to 0.58 corresponding to highly elastic collisions. This, in turn would implicitly fix the value of $V_{1\text{-D}}$ to give that value of ϵ in a typical collision requiring $V_{1\text{-D}}$ to be smaller than the Alfvén speed and the sound speed within the cloud gas. Thus, processes involving physical collisions can only give rise to a random one-dimensional velocity of $\lesssim 1$ km s^{-1}, significantly less than the observed value of $\sim$(3–4) km s^{-1}.

Clearly, an additional energy input is necessary to explain the observed velocity dispersion of the GMCs. This is the motivation for considering gravitational scattering between the clouds in a differentially rotating disk as an additional and the major source of energy input. The next subsection deals with this subject.

b) *Gravitational Viscosity in a Differentially Rotating Galactic Disk*

The gravitational interaction between the clouds at different radii acts as the viscous coupling between them, converting rotational kinetic energy into the random kinetic energy of the test and the field clouds. Thus, although each collision in the center of mass frame is elastic, the overall scattering process within a differentially rotating disk is *not* elastic, in that, random kinetic energy is not conserved.

We first obtain the change in the random kinetic energy of a test cloud due to a single gravitational encounter with a field cloud in the sheared disk; call this ΔE_{test}/encounter. We next assume that the consecutive encounters of a given test cloud with the field clouds are independent events and sum over encounters.

It is important to note that, even when the random velocity of the test cloud is zero, it experiences effective encounters with the other clouds due to the clouds being situated in a differentially rotating disk. In this case [of $(V_{1\text{-D}})_t = 0$], one can calculate the value of (ΔE_{test}/encounter) under the impulse approximation. Although such a calculation cannot be used for the general case (of nonzero random velocity for a test cloud), we give it in Appendix A for the sake of completeness and for its value in illustrating certain general ideas about gravitational viscosity.

i) *Formulation of the Equations*

For a more general and self-consistent case, we need to consider the initial random velocity of the test cloud to be nonzero but much smaller than the rotational velocity so that the epicyclic approximation is valid.

Consider a test cloud of mass, m_t, in an epicyclic orbit about galactocentric radius R interacting with a field cloud of mass, m_f, with zero random velocity, on a purely circular orbit, say at a galactocentric radius of $R - S$ with $S > 0$. Because of differential rotation, the field cloud has a higher circular rotation speed than the test cloud. The relative velocity, V_{rel}, between the field cloud and the guiding center rest frame for the test cloud motion is equal to $2AS$. Suppose we wished to neglect the random velocity and treat the problem as an encounter between the two clouds with impact parameter S. Then, the encounter time, t_{enc}, for these two clouds would be

$$t_{\text{enc}} \sim \frac{2S}{2AS} \sim \frac{1}{A} = \frac{t_{\text{epi}}}{2\pi}\frac{\kappa}{A} \sim t_{\text{epi}} \,. \tag{14}$$

Hence, in this case the effects of the encounter cannot be treated in the impulse approximation; one must take account of the (epicyclic) motion of the test cloud during its encounter with a field cloud.

Now, the unperturbed epicyclic motion of the test cloud can be treated as a coupled, two-dimensional harmonic oscillator. Assuming that ΔE_{test} during an encounter is very much less than E_{epi} (to be proved later in this subsection; see § II*b*[iv]), we can treat the encounter as a perturbed harmonic oscillator problem with the gravitational interaction between the clouds being the time dependent perturbation force imposed on the unperturbed epicyclic motion of the test cloud. In this case, in order to obtain ΔE_{test}/encounter, one has to evaluate the change in the *total energy* of the epicyclic motion of the test cloud, during an encounter lasting from a time, $t = -\infty$ to $+\infty$.

We write down the equations of motion for the test cloud in its guiding center rest frame. The Cartesian coordinate axes x, y, z are chosen to be along the $-R$, θ, z axes, respectively, of the galactic (cylindrical) coordinate system. The gravitational acceleration experienced by the test cloud due to the interaction with the field cloud can be written down most directly in a coordinate system frame (x', y', z') rotating about the z-axis with respect to the frame (x, y, z) with x' along the line joining the location of the guiding center of the test cloud on its epicyclic orbit and the instantaneous location of the field cloud.

Let d be the distance, at time t, between the field cloud and the guiding center, then

$$d^2 = S^2 + V_{\text{rel}}^2 t^2 = S^2(1 + 4A^2t^2) \,, \tag{15}$$

where $t = 0$ denotes the instant of closest approach.

In the (x', y', z') frame; the components of the force in the plane on the test cloud are given as

$$F_{x'} = \frac{2Gm_f x'}{d^3} m_t , \qquad F_{y'} = -\frac{Gm_f y'}{d^3} m_t , \qquad F_z = 0 . \tag{16}$$

For the moment we are using a tidal (quadrupole) expansion for the force about the guiding center. There are also nonnegligible terms independent of (x', y') due to the monopole force acting on the guiding center. We return to discuss these terms subsequently.

We have neglected terms that are second order or smaller in (a_R/d) = (the radial epicyclic excursion/d) in accordance with our assumption that the effect of an encounter is only a perturbation on the unperturbed epicyclic motion of the test cloud.

Equation (16) can be rewritten in the nonrotating frame (x, y, z axes) as follows:

$$x' \equiv x \cos\theta + y \sin\theta , \qquad y' \equiv -x \sin\theta + y \cos\theta , \tag{17}$$

where

$$\tan\theta = \frac{V_{\rm rel}\, t}{S} = 2At , \qquad \text{or} \quad \cos(\theta) = \frac{S}{d} . \tag{18}$$

Using equation (17) and the relation between $F(x', y')$ and $F(x, y)$, equation (16) reduces to

$$\frac{dv_x}{dt} = \frac{F_x}{m_t} = \frac{Gm_f}{d^3} [3y \sin\theta \cos\theta + x(2 - 3\sin^2\theta)] , \tag{19}$$

and

$$\frac{dv_y}{dt} = \frac{F_y}{m_t} = \frac{Gm_f}{d^3} [y(2 - 3\cos^2\theta) + 3x \sin\theta \cos\theta] . \tag{20}$$

With the use of equation (18), these can be rewritten as

$$\frac{dv_x}{dt} = \frac{\lambda S^3}{d^3} \left[x\left(1 - \frac{3S^2}{d^2}\right) - \left(\frac{3SV_{\rm rel}\, t}{d^2}\right) y \right] , \tag{21}$$

and

$$\frac{dv_y}{dt} = \frac{\lambda S^3}{d^3} \left[y\left(-2 + \frac{3S^2}{d^2}\right) - \left(\frac{3SV_{\rm rel}\, t}{d^2}\right) x \right] , \tag{22}$$

where

$$\lambda \equiv -\frac{Gm_f}{S^3} = \text{the perturbation parameter} . \tag{23}$$

Next, define

$$f_x(t) \equiv \frac{S^3}{d^3}\left(1 - \frac{3S^2}{d^2}\right) , \qquad f_y(t) \equiv \frac{S^3}{d^3}\left(\frac{3S^2}{d^2} - 2\right) , \tag{24}$$

and

$$f_2(t) = -\frac{3S^4 V_{\rm rel}\, t}{d^5} = \frac{-6S^5 At}{d^5} = -\frac{1}{2A}\frac{d}{dt}[f_x(t) + f_y(t)] . \tag{25}$$

Using equations (24)–(25), equations (21) and (22) reduce, respectively, to

$$\frac{dv_x}{dt} = \lambda[x f_x(t) + y f_2(t)] , \qquad \frac{dv_y}{dt} = \lambda[y f_y(t) + x f_2(t)] . \tag{26}$$

This derivation of the perturbation force and indeed the rest of the analysis in this subsection closely follows the paper by Spitzer (1958) where he studied the change of energy of stars in a cluster due to the encounter with a passing cloud. Our analysis is, however, fundamentally different from Spitzer's in several respects. First, Spitzer considers the unperturbed motion of a star in the cluster to be confined either to $x - z$ or to $y - z$ plane. Therefore, in Spitzer's calculation, the unperturbed motions along x and y are uncoupled. Also, the motions along x and z or those along y and z are uncoupled. This considerably simplifies the calculation. We on the other hand, must treat the unperturbed motions of the test cloud along x and y as being coupled. This is because, due to the location of the test cloud in a galactic disk, its unperturbed motion is an epicycle. This means that, in the present case, we must solve six coupled, second-order, homogeneous, linear differential equations, as shown next.

In addition $V_{\rm rel}$, the relative velocity between the test and the field clouds in our calculation is due to the differential rotation and is linearly proportional to the impact parameter, whereas in Spitzer's calculation, the relative velocity between the cluster center and the field cloud equals the random velocity of the cloud. Because of this difference and the different degree of coupling between the unperturbed motions as mentioned above, the actual value of $\Delta E_{\rm test}$/encounter is larger in our case than in Spitzer's calculation. And, finally, we attempt to derive the cloud velocities from first principles.

We next write down the final form for the equations of motion of a test cloud whose motion is perturbed by the perturbation force given by equation (26). First of all, recall that, the unperturbed motion of a test cloud of small nonzero random velocity can be written as a coupled, two-dimensional harmonic oscillator. The standard first-order epicyclic theory governs the coupling between its unperturbed motion along x and y directions, yielding the following equations of motion (see Mihalas and Routly 1968):

$$\frac{d^2x_0}{dt^2} + \kappa^2 x_0 = 0\,, \qquad \frac{d^2y_o}{dt^2} + \kappa^2 y_0 = 0\,, \tag{27}$$

which have the following solutions

$$x_0 = A_{0x}\cos\kappa t + B_{0x}\sin\kappa t = a_R \sin\kappa(t - t_0)\,, \tag{28}$$

$$y_0 = A_{0y}\cos\kappa t + B_{0y}\sin\kappa t = -\left(\frac{2\Omega}{\kappa}\right)a_R\cos\kappa(t - t_0)\,, \tag{29}$$

where

$$A_{0x} = -a_R\sin\kappa t_0\,, \qquad B_{0x} = a_R\cos\kappa t_0\,, \tag{30}$$

$$A_{0y} = -\left(\frac{2\Omega}{\kappa}\right)a_R\cos\kappa t_0\,, \qquad B_{0y} = -a_R\left(\frac{2\Omega}{\kappa}\right)\sin\kappa t_0\,, \tag{31}$$

and a_R is related to $(V_R)_{\rm epi}$, as in equation (2b).

We next present some relations between these coefficients which are used later on in this section:

$$\frac{A_{0x}A_{0y}}{B_{0x}B_{0y}} = -1\,, \qquad A_{0x}^2 + B_{0x}^2 = a_R^2\,, \tag{32a}$$

$$A_{0y}^2 + B_{0y}^2 = (A_{0x}^2 + B_{0x}^2)\left(\frac{2\Omega}{\kappa}\right)^2 = a_R^2\left(\frac{2\Omega}{\kappa}\right)^2\,, \tag{32b}$$

$$A_{0x}B_{0y} - B_{0x}A_{0y} = a_R^2\left(\frac{2\Omega}{\kappa}\right) = (A_{0x}^2 + B_{0x}^2)\left(\frac{2\Omega}{\kappa}\right)\,, \tag{32c}$$

$$E_x = \frac{1}{2}\,m_t\kappa^2 a_R^2\,, \qquad E_y = \frac{1}{2}\,m_t\kappa^2 a_R^2\left(\frac{2\Omega}{\kappa}\right)^2 . \tag{33}$$

The perturbed, coupled equations of motion along x and y are

$$\frac{d^2x}{dt^2} + \kappa^2 x = \frac{dv_x}{dt} = \lambda[xf_x(t) + yf_2(t)]\,, \tag{34}$$

and

$$\frac{d^2y}{dt^2} + \kappa^2 y = \frac{dv_y}{dt} = \lambda[yf_y(t) + xf_2(t)]\,. \tag{35}$$

Note that we require $\lambda \ll \kappa^2$ for the tidal force components to be considered perturbations on the unperturbed simple harmonic motion. The treatment based on equations (34) and (35) neglects the effect of the Coriolis force on the perturbed motion which neglect allows a considerable simplification in the already complex treatment. We have estimated that inclusion of this force (to be treated in work currently in progress) would affect the final results for cloud velocity dispersion by $\sim 10\%$. Because of the linear nature of the perturbation terms in these equations, the trial solution may be written as

$$x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots\,, \qquad y = y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots\,, \tag{36}$$

where x_0 and y_0 as given by equations (28) and (29) represent the unperturbed epicyclic motion of the test cloud in the plane. The values of x_1 and x_2 can be obtained by the method of variation of parameters. By substituting the above trial solution ([eq. 36]) in equations (34) and (35), and collecting terms of the same order in λ ($\neq 0$), we obtain

$$\frac{d^2x_1}{dt^2} + \kappa^2 x_1 = f_x(t)x_0 + f_2(t)y_0\,, \tag{37a}$$

$$\frac{d^2x_2}{dt^2} + \kappa^2 x_2 = f_x(t)x_1 + f_2(t)y_1\,, \tag{37b}$$

and

$$\frac{d^2y_1}{dt^2} + \kappa^2 y_1 = f_y(t)y_0 + f_2(t)x_0 , \tag{38a}$$

$$\frac{d^2y_2}{dt^2} + \kappa^2 y_2 = f_y(t)y_1 + f_2(t)x_1 , \tag{38b}$$

supplemented by equation (27).

Because of the coupling between the unperturbed solutions x_0 and y_0, y_0 is nonzero for an arbitrary value of x_0, and vice versa; this explains the presence of the terms $f_2(t)y_0$ and $f_2(t)x_0$, respectively, on the right-hand sides of equations (37a) and (38a). These terms, not present in Spitzer's calculation, give rise to an additional contribution to ΔE_{test}/encounter as we shall show in the next subsubsection; their significance is not certain in the absence of a proper treatment of the Coriolis force.

ii) *Solution to the Equations of Motion*

The solution to the above six coupled, second-order, linear, homogeneous differential equations, required to obtain the perturbed solution for the motion of the test cloud in the galactic plane is obtained analytically.

We proceed iteratively. Given equations (28) and (29), we know the forcing terms in the equations for (x_1, y_1) and with these the forcing terms solve for (x_2, y_2) as functions of time. The solution for equation (37a), obtained using the technique of variation of parameters, is

$$x_1(t) = A_{1x}(t) \cos \kappa t + B_{1x}(t) \sin \kappa t , \tag{39}$$

where

$$A_{1x}(t) = -\frac{1}{\kappa}\int_{-\infty}^{t} [f_x(t')x_0 + f_2(t')y_0] \sin \kappa t' \, dt' , \tag{40}$$

and

$$B_{1x}(t) = \frac{1}{\kappa}\int_{-\infty}^{t} [f_x(t')x_0 + f_2(t')y_0] \cos \kappa t' \, dt' . \tag{41}$$

Similarly, the solution to equation (38a) is

$$y_1(t) = A_{1y}(t) \cos \kappa t + B_{1y}(t) \sin \kappa t , \tag{42}$$

where

$$A_{1y}(t) = -\frac{1}{\kappa}\int_{-\infty}^{t} [f_y(t')y_0 + f_2(t')x_0] \sin \kappa t' \, dt' , \tag{43}$$

and

$$B_{1y}(t) = \frac{1}{\kappa}\int_{-\infty}^{t} [f_y(t')y_0 + f_2(t')x_0] \cos \kappa t' \, dt' , \tag{44}$$

Now knowing (x_1, y_1), we repeat the above procedure to obtain (x_2, y_2) starting from equations (37b) and (38b):

$$x_2 = A_{2x}(t) \cos \kappa t + B_{2x}(t) \sin \kappa t , \tag{45}$$

where

$$A_{2x}(t) = -\frac{1}{\kappa}\int_{-\infty}^{t} [f_x(t')x_1 + f_2(t)y_1] \sin \kappa t' \, dt' , \tag{46}$$

and

$$B_{2x}(t) = \frac{1}{\kappa}\int_{-\infty}^{t} [f_x(t')x_1 + f_2(t)y_1] \cos \kappa t' \, dt' . \tag{47}$$

Similarly,

$$y_2 = A_{2y}(t) \cos \kappa t + B_{2y}(t) \sin \kappa t , \tag{48}$$

where

$$A_{2y}(t) = -\frac{1}{\kappa}\int_{-\infty}^{t} [f_y(t')y_1 + f_2(t')x_1] \sin \kappa t' \, dt' , \tag{49}$$

and

$$B_{2y}(t) = \frac{1}{\kappa} \int_{-\infty}^{t} [f_y(t')y_1 + f_2(t')x_1] \cos \kappa t' \, dt' . \tag{50}$$

Note that because of the form of $f_x(t), f_y(t)$, and $f_2(t)$, the functions $A_{1x}(t)$, $B_{1x}(t)$, $A_{2x}(t)$, $B_{2x}(t)$, and the corresponding ones for the y-case are all finite quantities as $t \to \pm\infty$. Hence if $\lambda x_1 \ll x_0$ at small $|t|$, it is also true at later times. The same argument holds for $\lambda^2 x_2 \ll \lambda x_1$.

The change in the total energy for the motion along the x-direction of the test cloud, resulting from an encounter with the field cloud, is given by

$$\Delta E_x = \Delta \left\{ \frac{1}{2} m_t \left[\left(\frac{dx}{dt} \right)^2 + \kappa^2 x^2 \right] \right\} = \frac{1}{2} m_t \kappa^2 [(A_{0x} + \lambda A_{1x} + \lambda^2 A_{2x} + \cdots)^2 - A_{0x}^2 + (B_{0x} + \lambda B_{1x} + \lambda^2 B_{2x} + \cdots)^2 - B_{0x}^2] ,$$

which is evaluated at $t = +\infty$. Hereafter, the upper limits in A_{1x}, B_{1x}, etc., will be taken to be $+\infty$, unless otherwise specified. Note that as $t \to \pm\infty$, $(f_x, f_y, f_2) \to 0$. On retaining only the terms up to second order in λ, ΔE_x becomes

$$\Delta E_x = \tfrac{1}{2} m_t \kappa^2 [(2A_{0x} A_{1x} + 2B_{0x} B_{1x})\lambda + (A_{1x}^2 + B_{1x}^2 + 2A_{0x} A_{2x} + 2B_{0x} B_{2x})\lambda^2] , \tag{52}$$

and similarly

$$\Delta E_y = \tfrac{1}{2} m_t \kappa^2 [(2A_{0y} A_{1y} + 2B_{0y} B_{1y})\lambda + (A_{1y}^2 + B_{1y}^2 + 2A_{0y} A_{2y} + 2B_{0y} B_{2y})\lambda^2] . \tag{53}$$

Before proceeding to calculate the different terms in the expressions for ΔE_x and ΔE_y above, it is useful to consider a few general points. First, note that the time $t = 0$ corresponding to the closest approach of the two clouds is arbitrary. In other words, t_0 (as defined in eqs. [28], [29]) is random. Hence, from equations (30), (31), we get

$$\langle A_{0x} B_{0x} \rangle = -\frac{a_R^2}{2} \langle \sin 2\kappa t_0 \rangle = \langle A_{0y} B_{0y} \rangle = \langle A_{0x} A_{0y} \rangle = \langle B_{0x} B_{0y} \rangle = 0 , \tag{54}$$

where the angle brackets denote the average over encounters with several different field clouds. Therefore, the terms proportional to those four quantities need not be retained while evaluating the various terms in $\langle \Delta E \rangle$. Also, from equations (30) and (2b) we obtain

$$\langle A_{0x}^2 \rangle = \langle B_{0x}^2 \rangle = \tfrac{1}{2} a_R^2 = \tfrac{1}{2} [(V_R)_{\rm epi}/\kappa]^2 . \tag{55}$$

Similarly,

$$\langle A_{0y}^2 \rangle = \langle B_{0y}^2 \rangle = \frac{1}{2} \left(\frac{2\Omega}{\kappa} \right)^2 a_R^2 = \frac{1}{2} \left(\frac{2\Omega}{\kappa} \right)^2 \left[\frac{(V_R)_{\rm epi}}{\kappa} \right]^2 . \tag{56}$$

The second point is that $f_x(t)$ and $f_y(t)$ are even functions of time, while $f_2(t)$ is odd (see eqs. [24], [25]). Also, $\sin 2\kappa t$ is an odd function of time, while $\sin^2 \kappa t$ and $\cos^2 \kappa t$ are even functions of time. Hence, he integral over $t = -\infty$ to $+\infty$ of $f_x(t) \sin 2\kappa t$ or of $f_y(t) \sin 2\kappa t$ is zero. The same is true for the integrals over $t = -\infty$ to $+\infty$ $f_2(t) \sin^2 \kappa t$ and of $f_2(t) \cos^2 \kappa t$ and of $f_2(t)$.

With this discussion in mind, consider ΔE_x first (see eq. [52]), then average and obtain $\langle \Delta E_x \rangle$.

Now equations (40), (28) and (29) give

$$A_{1x} = -\frac{1}{\kappa} \int_{-\infty}^{\infty} [f_x(t)(A_{0x} \cos \kappa t + B_{0x} \sin \kappa t) + f_2(t)(A_{0y} \cos \kappa t + B_{0y} \sin \kappa t)] \sin \kappa t \, dt .$$

Since, from symmetry, the first and the last terms are zero,

$$A_{1x} = -\frac{1}{\kappa} \int_{-\infty}^{\infty} [f_x(t) B_{0x} \sin^2 \kappa t + \frac{1}{2} f_2(t) A_{0y} \sin 2\kappa t] dt , \tag{57}$$

which, using equation (54), yields

$$\langle A_{0x} A_{1x} \rangle = 0 . \tag{58}$$

Similarly, from equations (41), (28), and (29) we get

$$B_{1x} = \frac{1}{\kappa} \int_{-\infty}^{\infty} [f_x(t) A_{0x} \cos^2 \kappa t + \frac{f_2(t)}{2} B_{0y} \sin 2\kappa t] dt , \tag{59}$$

and

$$\langle B_{0x} B_{1x} \rangle = 0 . \tag{60}$$

Hence, the lowest order nonzero terms in λ *in* $\langle \Delta E_x \rangle$ *and in* $\langle \Delta E_y \rangle$ *(i.e., averages of eqs. [52] and [53] respectively) are quadratic in* λ. As expected, the first-order perturbating forces, although larger than higher order terms, add and subtract equal amounts of energy on average from a test cloud, thus averaging to zero.

Next, using equation (57), we get

$$A_{1x}^2 = \frac{B_{0x}^2}{\kappa^2}\left[\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\right]^2 + \frac{A_{0y}^2}{4\kappa^2}\left[\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\right]^2 + \frac{A_{0y}B_{0x}}{\kappa^2}\left[\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\right]\left[\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\right]. \quad (61)$$

Similarly, from equation (59), we get

$$B_{1x}^2 = \frac{A_{0x}^2}{\kappa^2}\left[\int_{-\infty}^{\infty} f_x(t)\cos^2\kappa t\,dt\right]^2 + \frac{B_{0y}^2}{4\kappa^2}\left[\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\right]^2 + \frac{A_{0x}B_{0y}}{\kappa^2}\left[\int_{-\infty}^{+\infty} f_x(t)\cos^2\kappa t\,dt\right]\left[\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\right]. \quad (62)$$

To simplify these two equations, we now define the following quantities: Let

$$\int_{-\infty}^{\infty} f_x(t)dt \equiv I_x\,, \qquad \int_{-\infty}^{\infty} f_x(t)\cos 2\kappa t\,dt \equiv I_{cx}\,, \quad (63)$$

$$\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt \equiv I_s\,. \quad (64)$$

Hence,

$$\int_{-\infty}^{\infty} f_x(t)\cos^2\kappa t\,dt = \tfrac{1}{2}(I_x + I_{cx})\,, \quad (65)$$

and

$$\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt = \tfrac{1}{2}(I_x - I_{cx})\,. \quad (66)$$

Equations (61) and (62) can be rewritten, using equations (63)–(66), as follows:

$$A_{1x}^2 = \frac{B_{0x}^2}{4\kappa^2}(I_x - I_{cx})^2 + \frac{A_{0y}^2}{4\kappa^2}I_s^2 + \frac{A_{0y}B_{0x}}{2\kappa^2}(I_x - I_{cx})I_s\,, \quad (67)$$

and

$$B_{1x}^2 = \frac{A_{0x}^2}{4\kappa^2}(I_x + I_{cx})^2 + \frac{B_{0y}^2}{4\kappa^2}I_s^2 + \frac{A_{0x}B_{0y}}{2\kappa^2}(I_x + I_{cx})I_s\,. \quad (68)$$

Combining equations (67) and (68) we get

$$\langle(A_{1x}^2 + B_{1x}^2)\rangle = \frac{(I_x^2 + I_{cx}^2)}{4\kappa^2}\langle(A_{0x}^2 + B_{0x}^2)\rangle + \frac{I_s^2}{4\kappa^2}\langle(A_{0y}^2 + B_{0y}^2)\rangle + \frac{I_s}{2\kappa^2}\langle A_{0y}B_{0x}(I_x - I_{cx}) + A_{0x}B_{0y}(I_x + I_{cx})\rangle\,. \quad (69)$$

Next, consider $2A_{0x}A_{2x}$. Equations (46) and (39)–(44) give

$$2A_{0x}A_{2x} = -\frac{2A_{0x}}{\kappa}\int_{-\infty}^{\infty}\sin\kappa t\,dt\{[f_x(t)A_{1x}(t) + f_2(t)A_{1y}(t)]\cos\kappa t + [f_x(t)B_{1x}(t) + f_2(t)B_{1y}(t)]\sin\kappa t\}\,. \quad (70)$$

Using the expressions for A_{1x}, B_{1x}, A_{1y}, B_{1y} as given, respectively, by equations (40), (41), (43), and (44) and also substituting for x_0 and y_0 from equations (28) and (29), equation (70) becomes, after removing terms which will vanish on averaging due to relations given in equation (54),

$$2A_{0x}A_{2x} = \frac{A_{0x}}{2\kappa^2}\left[A_{0x}\int_{-\infty}^{\infty} f_x(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\sin 2kt'\,dt'\right. \quad (71a)$$

$$\left.+\,2B_{0y}\int_{-\infty}^{\infty} f_x(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin^2\kappa t'\,dt'\right] \quad (71b)$$

$$+\frac{A_{0x}}{2\kappa^2}\left[2B_{0y}\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_y(t')\sin^2\kappa t'\,dt'\right. \quad (72a)$$

$$\left.+\,A_{0x}\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin 2\kappa t'\,dt'\right] \quad (72b)$$

$$-\frac{A_{0x}}{\kappa^2}\left[2A_{0x}\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\cos^2\kappa t'\,dt'\right. \quad (73a)$$

$$\left.+\,B_{0y}\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin 2\kappa t'\,dt'\right] \quad (73b)$$

$$-\frac{A_{0x}}{\kappa^2}\left[B_{0y}\int_{-\infty}^{\infty} f_2(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_y(t')\sin 2\kappa t'\,dt'\right. \tag{74a}$$

$$\left.+2A_{0x}\int_{-\infty}^{\infty} f_2(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\cos^2\kappa t'\,dt'\right]. \tag{74b}$$

Now, the first term in equation (71) is equal to zero as shown next. Using the general formula for repeated integration, the integral in expression (71a) reduces to

$$\frac{1}{2}\left[\int_{-\infty}^{\infty} f_x(t)\sin 2\kappa t\,dt\right]^2 = 0\,,$$

since $f_x(t)$ is an even function of t.

Applying the general formula for repeated integration to equation (74a) and noticing that $f_2(t)\sin^2\kappa t$ is an odd function of time, equation (74a) reduces to a form that combines easily with equation (71b). Similarly, expression (73b) can be written as

$$-\frac{A_{0x}B_{0y}}{\kappa^2}\left[\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\int_{-\infty}^{\infty} f_2(t')\sin^2\kappa t'\,dt' - \int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\sin^2 kt'\,dt'\right]$$

$$= -\frac{A_{0x}B_{0y}}{\kappa^2}\frac{I_s}{2}(I_x - I_{cx}) + \frac{A_{0x}B_{0y}}{\kappa^2}\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\sin^2\kappa t'\,dt'\,. \tag{75}$$

Clearly, the second term in this last equation combines easily with equation (72a). With these simplifications, expressions (71)–(74) combine to yield

$$\langle 2A_{0x}A_{2x}\rangle = \frac{\langle A_{0x}^2\rangle}{2\kappa^2}\left[\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin 2\kappa t'\,dt'\right. \tag{76a}$$

$$-4\int_{-\infty}^{\infty} f_x(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\cos^2\kappa t'\,dt' \tag{76b}$$

$$\left.-4\int_{-\infty}^{\infty} f_2(t)\sin^2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\cos^2\kappa t'\,dt'\right] \tag{76c}$$

$$+\frac{\langle A_{0x}B_{0y}\rangle}{2\kappa^2}\left\{2\int_{-\infty}^{\infty}[f_x(t)+f_y(t)]\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin^2\kappa t'\,dt'\right. \tag{77a}$$

$$+2\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t}[f_x(t')+f_y(t')]\sin^2\kappa t'\,dt' \tag{77b}$$

$$\left.-I_s(I_x - I_{cx})\right\}. \tag{77c}$$

In a completely analogous fashion, starting from equations (47) and (39)–(44), we find

$$\langle 2B_{0x}B_{2x}\rangle = \frac{\langle B_{0x}^2\rangle}{2\kappa^2}\left[-4\int_{-\infty}^{\infty} f_x(t)\cos^2\kappa t\,dt\int_{-\infty}^{t} f_x(t')\sin^2\kappa t'\,dt'\right. \tag{78a}$$

$$-4\int_{-\infty}^{\infty} f_2(t)\cos^2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin^2\kappa t'\,dt' \tag{78b}$$

$$\left.+\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\sin 2\kappa t'\,dt'\right] \tag{78c}$$

$$+\frac{\langle B_{0x}A_{0y}\rangle}{2\kappa^2}\left\{+2\int_{-\infty}^{\infty}[f_x(t)+f_y(t)]\sin 2\kappa t\,dt\int_{-\infty}^{t} f_2(t')\cos^2\kappa t'\,dt'\right. \tag{79a}$$

$$+2\int_{-\infty}^{\infty} f_2(t)\sin 2\kappa t\,dt\int_{-\infty}^{t}[f_x(t')+f_y(t')]\cos^2\kappa t'\,dt' \tag{79b}$$

$$\left.-I_s(I_x + I_{cx})\right\}. \tag{79c}$$

Now we have evaluated all the terms in $\langle\Delta E_x\rangle$. Combining equations (52), (58), (60), (69), (76), (77), (78), and (79), we get

$$\langle\Delta E_x\rangle = \tfrac{1}{2}m_t\kappa^2\lambda^2\ [\text{sum of right-hand side of eqs. (69), (76), (77), (78), (79)}] \tag{80}$$

This is simplified next by first combining equations (76) and (78). To simplify these, we first note that

$$\int_{-\infty}^{\infty} f_x(\tau) \sin^2 \kappa\tau \, d\tau \int_{-\infty}^{\tau} \cos^2 \kappa t \, f_x(t) dt + \int_{-\infty}^{\infty} f_x(\tau) \cos^2 \kappa\tau \, d\tau \int_{-\infty}^{\tau} f_x(t) \sin^2 \kappa t \, dt = \tfrac{1}{4}(I_x^2 - I_{cx}^2) . \quad (81)$$

Using equation (81) and the fact that $\langle A_{0x}^2 \rangle = \langle B_{0x}^2 \rangle$ (see eq. [55]), the equations (76b) and (78a) add to give

$$\frac{\langle A_{0x}^2 + B_{0x}^2 \rangle}{4\kappa^2} (-I_x^2 + I_{cx}^2) . \quad (82)$$

At this point is is necessary to define I_y, I_{cy}, and I_2, I_{c2}, in analogy with I_x, I_{cx}, respectively—see equations (63)–(64), with $f_y(t)$ and $f_2(t)$ replacing $f_x(t)$ in the respective equations. Define

$$I_y \equiv \int_{-\infty}^{\infty} f_y(t) dt , \qquad I_{cy} \equiv \int_{-\infty}^{\infty} f_y(t) \cos 2\kappa t \, dt , \quad (83)$$

$$I_2 \equiv \int_{-\infty}^{\infty} f_2(t) dt , \qquad I_{c2} \equiv \int_{-\infty}^{\infty} f_2(t) \cos 2\kappa t \, dt . \quad (84)$$

We note that both I_2 and I_{c2} are equal to zero by symmetry. Next, again using equation (81)—except now we replace $f_x(t)$ by $f_2(t)$—the terms (76c) and (78b) add to give

$$\frac{\langle A_{0x}^2 + B_{0x}^2 \rangle}{4\kappa^2} (-I_2^2 + I_{c2}^2) = 0 .$$

Next, equations (76a) and (78c) add to give

$$\frac{\langle A_{0x}^2 + B_{0x}^2 \rangle}{4\kappa^2} I_s^2 . \quad (85)$$

Therefore, equation (76) plus equation (78) reduce to equation (82) plus equation (85). Equations (69), (82), and (85) add to give

$$\frac{\langle A_{0x}^2 + B_{0x}^2 \rangle}{2\kappa^2} I_{cx}^2 + \langle A_{0x}^2 + B_{0x}^2 + A_{0y}^2 + B_{0y}^2 \rangle \frac{I_s^2}{4\kappa^2} + \langle A_{0y} B_{0x}(I_x - I_{cx}) + A_{0x} B_{0y}(I_x + I_{cx}) \rangle \frac{I_s}{2\kappa^2} . \quad (86)$$

Hence

$$\langle \Delta E_x \rangle = \tfrac{1}{2} m_t \kappa^2 \lambda^2 [\text{sum of right-hand side of eqs. (77), (79), (86)}] . \quad (87)$$

It is worth noting here that we are ultimately interested in obtaining $\langle \Delta E_x + \Delta E_y \rangle$. It is easier to obtain $\langle \Delta E_x + \Delta E_y \rangle$ than it is to obtain either term separately because of the symmetry in x and y in the basic equations (eqs. [37] and [38]). We can start from equation (87) and write down

$$\langle \Delta E_y \rangle = \frac{1}{2} m_t \kappa^2 \lambda^2 \left[\frac{\langle A_{0y}^2 + B_{0y}^2 \rangle}{2\kappa^2} I_{cy}^2 + \frac{\langle A_{0y}^2 + B_{0y}^2 + A_{0x}^2 + B_{0x}^2 \rangle}{4\kappa^2} I_s^2 + \langle A_{0x} B_{0y}(I_y - I_{cy}) + A_{0y} B_{ox}(I_y + I_{cy}) \rangle \frac{I_s}{2\kappa^2} \right] \quad (88a)$$

$$+ \frac{1}{2} m_t \kappa^2 \lambda^2 \frac{\langle A_{0y} B_{0x} \rangle}{2\kappa^2} \left\{ 2 \int_{-\infty}^{\infty} [f_y(t) + f_x(t)] \sin 2\kappa t \, dt \int_{-\infty}^{t} f_2(t') \sin^2 \kappa t' \, dt' \right.$$
$$\left. + 2 \int_{-\infty}^{\infty} f_2(t) \sin 2\kappa t \, dt \int_{-\infty}^{t} [f_x(t') + f_y(t')] \sin^2 \kappa t' \, dt' - I_s(I_y - I_{cy}) \right\} \quad (88b)$$

$$+ \frac{1}{2} m_t \kappa^2 \lambda^2 \frac{\langle B_{0y} A_{0x} \rangle}{2\kappa^2} \left\{ 2 \int_{-\infty}^{\infty} [f_y(t) + f_x(t)] \sin 2\kappa t \, dt \int_{-\infty}^{t} f_2(t') \cos^2 \kappa t' \, dt' \right.$$
$$\left. + 2 \int_{-\infty}^{\infty} f_2(t) \sin 2\kappa t \, dt \int_{-\infty}^{t} [f_y(t') + f_x(t')] \cos^2 \kappa t' \, dt' - I_s(I_y + I_{cy}) \right\} . \quad (88c)$$

Hence $\langle \Delta E_x + \Delta E_y \rangle$ = sum of right-hand side of equations (87) and (88).

Equations (88a) and (86) add to yield

$$\frac{1}{2} m_t \kappa^2 \lambda^2 \left[\frac{\langle A_{0x}^2 + B_{0x}^2 \rangle I_{cx}^2 + \langle A_{0y}^2 + B_{0y}^2 \rangle I_{cy}^2}{2\kappa^2} + \frac{I_s^2}{2\kappa^2} \langle A_{0x}^2 + B_{0x}^2 + A_{0y}^2 + B_{0y}^2 \rangle + \langle A_{0x} B_{0y} + A_{0y} B_{0x} \rangle \frac{I_s}{2\kappa^2} (I_x + I_y) \right.$$
$$\left. + \langle A_{0x} B_{0y} - A_{0y} B_{0x} \rangle \frac{I_s}{2\kappa^2} (I_{cx} - I_{cy}) \right] . \quad (89)$$

Next, equations (77), (79), (87), (88b), and (88c) combine to yield:

$$\frac{1}{2} m_t \kappa^2 \lambda^2 \frac{\langle A_{0x} B_{0y} + B_{0x} A_{0y} \rangle}{\kappa^2} \left\{ \int_{-\infty}^{\infty} [f_x(t) + f_y(t)] \sin 2\kappa t \, dt \int_{-\infty}^{t} f_2(t') dt' + \int_{-\infty}^{\infty} f_2(t) \sin 2\kappa t \, dt \int_{-\infty}^{t} [f_x(t') + f_y(t')] dt' \right\}$$
$$- \frac{1}{2} m_t \kappa^2 \lambda^2 \left\{ \langle A_{0x} B_{0y} + A_{0y} B_{0x} \rangle \left[\frac{I_s(I_x + I_y)}{2\kappa^2} \right] - \langle A_{0x} B_{0y} - A_{0y} B_{0x} \rangle \left[\frac{I_s}{2\kappa^2} (I_{cx} - I_{cy}) \right] \right\} . \quad (90)$$

Hence

$$\langle \Delta E_x + \Delta E_y \rangle = \text{right-hand side of eqs. (89) and (90)} . \tag{91}$$

Now recall the relations between A_{0x}, B_{0x}, A_{0y}, and B_{0y} (eqs. [30]–[32]). From these we get the following two relations. First,

$$\langle A_{0x} B_{0y} + A_{0y} B_{0x} \rangle = -a_R^2 \frac{2\Omega}{\kappa} \langle \cos 2\kappa t_0 \rangle = 0 , \tag{92}$$

as follows from the random phase assumption.

Second,

$$\langle A_{0x} B_{0y} - A_{0y} B_{0x} \rangle = a_R^2 \frac{2\Omega}{\kappa} \langle \cos^2 \kappa t_0 + \sin^2 \kappa t_0 \rangle = a_R^2 \frac{2\Omega}{\kappa} . \tag{93}$$

Substituting equations (92) and (93) into equation (91),and substituting the values for A_{0x}, etc., from equation (32), we get

$$\langle \Delta E_x + \Delta E_y \rangle = \frac{1}{4} m_t \lambda^2 a_R^2 \left[\left(I_{cx}^2 + \frac{4\Omega^2}{\kappa^2} I_{cy}^2 \right) + I_s^2 \left(1 + \frac{4\Omega^2}{\kappa^2} \right) + I_s \frac{4\Omega}{\kappa} (I_{cx} - I_{cy}) \right] . \tag{94}$$

The values of the integrals I_{cx}, I_{cy}, and I_s are derived in Appendix B and are given as

$$I_{cx} = -\frac{\beta}{A} [K_1(\beta) + \beta K_0(\beta)] , \qquad I_{cy} = \frac{\beta^2}{A} K_0(\beta) , \qquad I_s = -\frac{\beta^2}{A} K_1(\beta) , \tag{95}$$

where

$$\beta \equiv \frac{\kappa}{A} . \tag{96}$$

Here $K_0(\beta)$ and $K_1(\beta)$ are the modified Bessel functions of the second kind.

Substituting equation (95) in equation (94) it reduces to

$$\langle \Delta E_x + \Delta E_y \rangle = \frac{1}{4} m_t \lambda^2 a_R^2 \frac{1}{A^2} f(\beta) , \tag{97}$$

where

$$f(\beta) \equiv \beta^2 \left\{ [K_1(\beta) + \beta K_0(\beta)]^2 + \frac{4\Omega^2}{\kappa^2} [\beta K_0(\beta)]^2 + [\beta K_1(\beta)]^2 \left(1 + \frac{4\Omega^2}{\kappa^2} \right) + \beta K_1(\beta) \frac{4\Omega}{\kappa} [K_1(\beta) + 2\beta K_0(\beta)] \right\} . \tag{98}$$

Substituting the values for λ and a_R, from equations (23) and (2b), respectively, the above equation reduces to

$$\langle \Delta E_x + \Delta E_y \rangle = \frac{1}{4} m_t \left(\frac{G m_f}{A S^3} \right)^2 \left[\frac{(V_R)_{\rm epi}}{\kappa} \right]^2 [f(\beta)] . \tag{99}$$

This is the total change of energy of the test cloud resulting from an encounter (of impact parameter S) with the field cloud due to the quadrupole–tidal gravitational interaction. Note that β, and hence $f(\beta)$, is purely a function of the rotation curve, and for a flat rotation curve $\beta \equiv K/A = 2(2)^{1/2}$ and $f(\beta) = 1.847$. It should be pointed out that so far the derivation of $\langle \Delta E_x + \Delta E_y \rangle$ is general, that is, the above expression is valid for any arbitrary rotation curve. In this paper, we will concentrate mainly on a special case of the above expression: namely that for a flat rotation curve. This is a valid assumption because, over $R = 4$–10 kpc, where the GMCs are located, the rotation curve is nearly flat (see, e.g., Clemens 1985) and also because the resulting steady state velocity varies weakly with κ (see § III*b*) and hence does not depend sensitively on the small observed variation with R in the rotation speed (see §§ III*b*, *c*).

At this point it is worth noting that if we had used Spitzer's results (instead of doing the detailed calculation as here), we would only have gotten the first three terms in equation (98), the only difference is that even in these three terms, instead of $4\Omega^2/\kappa^2$, we would have obtained 1 (see Spitzer 1958). The ratio of the last term in equation (98) to the first three terms is evaluated next for a flat rotation curve. Hence, $K_0(\beta) = 0.04245$ and $K_1(\beta) = 0.04946$ (see Gray, Mathews, and MacRobert 1922), and the above ratio is

$$\frac{\beta K_1(\beta)(4\Omega/\kappa)[K_1(\beta) + 2\beta K_0(\beta)]}{[K_1(\beta) + \beta K_0(\beta)]^2 + (4\Omega^2/\kappa^2)[\beta K_0(\beta)]^2 + [\beta K_1(\beta)]^2(1 + 4\Omega^2/\kappa^2)} = 0.99 .$$

This shows the importance of taking account of the coupling between the unperturbed x and y motions, while calculating the change in energy in an encounter. Addition of the first-order Coriolis terms will produce further comparable changes in $\langle \Delta E \rangle$.

iii) *Energy Input at the Guiding Center*

Equations (97) and (99) are quadratic in the epicyclic amplitude, implying that there would be no energy input to clouds in pure circular motion. This, as shown in Appendix A, is not true; equation (97) results from our expansion of the perturbing gravitational force about the guiding center and treatment only of the tidal, quadrupole terms. In this subsection we repeat the calculation for the monopole terms. Since the computation is analogous to that already presented but is simpler, most results will be simply stated without proof. The additional terms to be added to the right-hand side of equations (34) and (35) are, respectively,

$$\frac{F_{m,x}}{m_t} = \lambda \frac{S^3}{d^3} S \quad \text{and} \quad \frac{F_{m,y}}{m_t} = \lambda \frac{S^3}{d^3} 2AtS . \tag{100}$$

It can be shown that these produce no energy change to first order in λ, and that there are no cross terms. Thus, the only additional energy changes per encounter comes from the terms

$$\langle A_{1x}^2 + B_{1x}^2 \rangle_m \equiv P_+^2 = \frac{1}{\kappa^2}\left(S \int_{-\infty}^{+\infty} \frac{S^3}{d^3} \cos \kappa t\, dt\right)^2 , \tag{101}$$

$$\langle A_{1y}^2 + B_{1y}^2 \rangle_m \equiv P_-^2 = \frac{1}{\kappa^2}\left(S \int_{-\infty}^{+\infty} \frac{S^3}{d^3} 2At \sin \kappa t\, dt\right)^2 ,$$

giving

$$\langle \Delta E \rangle_m = \langle \Delta E_x + \Delta E_y \rangle_m = \tfrac{1}{2} m_t \kappa^{22}(P_+^2 + P_-^2) , \tag{102}$$

if, as assumed, succesive encounters occur with random phases.

Reducing the integrals in equation (101) to Bessel functions gives

$$P_+ = \frac{S}{2A^2} K_1\left(\frac{\beta}{2}\right), \qquad P_- = -\frac{S}{2A^2} K_0\left(\frac{\beta}{2}\right). \tag{103}$$

If we now define a function $g(\beta)$

$$g(\beta) \equiv \frac{\beta^2}{2}\left[K_0^2\left(\frac{\beta}{2}\right) + K_1^2\left(\frac{\beta}{2}\right)\right] , \tag{104}$$

which for $\beta = 2(2)^{1/2}$ has the value $g(\beta) = 0.626$, then we can replace equation (99) with

$$\langle \Delta E_x + \Delta E_y \rangle = \frac{1}{4} m_t \left(\frac{Gm_f}{AS^3}\right)^2 \left\{ S^2 g(\beta) + \left[\frac{(V_R)_{\rm epi}}{\kappa}\right]^2 f(\beta) \right\} . \tag{105}$$

iv) *Rate of Change of Energy due to Gravitational Viscosity*

The rate of change of the random kinetic energy due to the above process is therefore equal to

$$\frac{d(E_x + E_y)}{dt} = [\langle \Delta E_x + \Delta E_y \rangle] \times (\text{number of encounters s}^{-1}) = 2 \int_{S=S_{\min}}^{\infty} \langle \Delta E_x + \Delta E_y \rangle (2AS\, dS n_f) H , \tag{106}$$

where S is the impact parameter and $S_{\min}$ is its minimum value, H is the total vertical scale height of the cloud distribution in the disk. The factor of 2 in front of the integral is due to interaction of the test cloud with field clouds that are situated (rotating) at radii greater than R and less than R.

It can be shown (see Appendix C) that a gravitational encounter between the above test cloud and the field cloud (at a radius $R-S$, $S > 0$) leads to an increase in the random kinetic energy of *both*, the test *and* the field clouds. This increase is provided at the expense of a decrease in their rotational kinetic energy. For equally massive test and field clouds ($m_t = m_f$), the increase in the random kinetic energy following an encounter is equal. For $m_t \neq m_f$, the ratio of the increase in the random kinetic energies of the test and the field clouds is equal to m_f/m_t. The statements in this paragraph are most easily proved in the case of the impulse approximation (that is, when $V_{\text{1-D}} = 0$; see Appendix C for the details), although the results proved in Appendix C are valid even when $V_{\text{1-D}} \neq 0$.

Therefore, from equations (99) and (106) we have

$$\frac{d(E_x + E_y)_q}{dt} = \left\{\frac{1}{2} m_t \left[\frac{(V_R)_{\rm epi}}{\kappa}\right]^2 \left(\frac{Gm_f}{A}\right)^2 [f(\beta)]\right\} n_f H 2A \int_{S_{\min}}^{\infty} \frac{S\, dS}{S^6} , \tag{107}$$

where we examine first the most seriously divergent term, the quadrupole term.

Notice that the above expression is inversely proportional to the fourth power on $S_{\min}$. This strong dependence on $S_{\min}$ of the above expression is in a sharp contrast to the weak logarithmic dependence of $S_{\min}$ of the rate of change of energy due to the standard, random gravitational encounters. In the viscous case studied here, the effects of an encounter have been evaluated by treating it as a perturbed harmonic oscillator problem. Here, when the phase of the closest approach for subsequent encounters is random, the lowest order terms in $\langle \Delta E_x + \Delta E_y \rangle$ are proportional to $\lambda^2 = (Gm_f/S^3)^2$, to the square of the perturbation parameter leading to an inverse fourth-power dependence on $S_{\min}$. Treating the problem as three-dimensional rather than in a plane would lessen the divergence. Because of the strong dependence on $S_{\min}$ of the above expression, one has to be careful in assigning a value to $S_{\min}$.

First, we note that

$$S_{\min} = \max\, [H/2,\, r_t + r_f,\, a_R,\, S \text{ corresponding to } V_{\rm rel}/(V_R)_{\rm epi} > 1] . \tag{108}$$

Of these, the first two are absolute limits in the sense that unless $S_{\min}$ is greater than both of these, the very calculation of $d(E_x + E_y)/dt$ in the plane is meaningless. The last two terms arise because of the assumptions made in calculating the above rate of change of energy resulting from gravitational viscosity. The quantity $S_{\min}$ has to be greater than a_R if the approximation (namely $[a_R/S]^2 \ll 1$) made in calculating the tidal force is to remain valid. Lastly, only when $V_{\rm rel}$, the relative velocity between the field cloud on a circular orbit and the guiding center of the epicyclic motion executed by the test cloud, is greater than $(V_R)_{\rm epi}$, can one neglect the fact that the field cloud itself may be on an epicyclic orbit. In other words, only when $V_{\rm rel}/(V_R)_{\rm epi} > 1$, can one treat the field cloud as being on a purely circular orbit.

For the GMCs in the Galaxy, $H \sim 130$ pc and $r \sim 10$–40 pc (see, Sanders, Solomon, and Scoville 1984, hereafter, SSS1984; SSS1985). Next $a_R = (V_R)_{\rm epi}/\kappa = ([2]^{1/2} V_{\rm 1\text{-}D}/\kappa)[1 + (\kappa/2\Omega)^2]^{-1/2}$ from equations (2) and (7). Using the typical observed value of the one-dimensional random velocity dispersion of the clouds to be in the range of ~ 3–4 km s^{-1} (see the discussion in § I) and assuming the rotation curve to be flat (which gives $\kappa = 2^{1/2}\Omega$), and using the local value of the epicyclic frequency, $\kappa = 2^{1/2} \times 25 = 35$ km s^{-1} kpc^{-1} (see Schmidt 1965), we get $a_R = (0.098$–$0.131)$ kpc $> (H/2,\ r_t + r_f) = (0.065,\ 0.020$–$0.080)$ kpc. Finally, $V_{\rm rel}/(V_R)_{\rm epi} = 2AS/(V_R)_{\rm epi} > 1$ gives $S > (V_R)_{\rm epi}/2A = (V_R)_{\rm epi}/\kappa\,(\kappa/2A) = 2^{1/2} a_R$, for a flat rotation curve.

Note that this last condition is the most restrictive one, and therefore we set

$$S_{\min} = \frac{\kappa}{2A}\, a_R\,, \tag{109}$$

for a flat rotation curve.

Note that the last two limits arise not because of any intrinsic limitation in the applicability of the basic acceleration mechanism involving gravitational viscosity, but rather they arise solely because of the assumptions made using this viscous mechanism. Hence $d(E_x + E_y)/dt$, as calculated using the above value of $S_{\min}(=2^{1/2}a_R)$, is a lower limit on the actual rate of increase of random kinetic energy that is possible in the viscous acceleration mechanism.

We next estimate the contribution to the rate of change of energy during an encounter from encounters with impact parameters in the range of $S = S_{\min}\,[= (K/2A)a_R \approx 2^{1/2}a_R$, as obtained above] to $S = (r_t + r_f)$ (which is the lowest possible value for a calculation of this kind). For this purpose, we assume that for encounters with impact parameters in this range, the value of $\Delta E_{\rm test}$/encounter remains constant at its value at $S = S_{\min}$, a plausible assumption since $V_{\rm rel}$ is approximately constant in this range when both clouds are allowed to be on epicyclic orbits.

The integral over S in equation (107) is thus replaced by

$$\int_{S_{\min}}^{\infty} \frac{S\,dS}{S^6} + \frac{1}{S_{\min}^6}\int_{r_t+r_f}^{S_{\min}} S\,dS = \frac{3}{4}\frac{1}{S_{\min}^4}\left[1 - \frac{2}{3}\left(\frac{r_t + r_f}{S_{\min}}\right)^2\right] \approx \frac{3}{4}\frac{1}{S_{\min}^4}\,. \tag{110}$$

Since $S_{\min} = (\kappa/2A)a_R \approx 2^{1/2}a_R \gg (r_t + r_f)$, as shown earlier, hence $(r_t + r_f)^2$ can be ignored in comparison with $S_{\min}^2$. Substituting for $S_{\min}$ into equations (107) and (110) the rate of increase in the random kinetic energy of a test cloud arising from the tidal force term:

$$\left(\frac{dE}{dt}\right)_q = \frac{12A}{\beta^2} f(\beta) m_t \left[\frac{Gm_f}{(V_R)_{\rm epi}}\right]^2 n_f H\,, \tag{111}$$

where, we recall $f(\beta) = 1.847$ for a flat rotation curve. The monopole term which is less divergent by a factor of S^2 gives, on integration,

$$\left(\frac{dE}{dt}\right)_m = 4Ag(\beta) m_t \left[\frac{Gm_f}{(V_R)_{\rm epi}}\right]^2 n_f H\,, \tag{112}$$

with $g(\beta) = 0.626$ for a flat rotation curve, i.e., $\beta = 2(2)^{1/2}$.

The net rate of increase of the random kinetic energy of a test cloud due to gravitational viscosity is given by the sum of equations (11) and (112).

c) *Equation for Energy Balance of the GMCs*

In the previous subsections, we have obtained expressions for the rate of change of energy of a test cloud due to the various physical processes. For a cloud system in a steady state, the net rate of change of energy of a test cloud is zero, and we can write the following equation for energy balance for the GMCs:

$$\left.\frac{d(E_x + E_y)}{dt}\right|_{\rm tot} = 0 \tag{113}$$

= (rate of energy gain from gravitational viscosity) + (rate of energy gain from viscosity due to physical collisions)
− (rate of loss of energy due to inelastic physical collisions) ,

where the terms on the right-hand side of equation (113) are given by equations (12), (111), and (112). Writing the different terms in detail and using equation (7), we get

$$\frac{12A}{\beta^2} f(\beta)\left[\frac{Gm_f}{(V_R)_{\rm epi}}\right]^2 H\left[1 + \frac{\beta^2 g(\beta)}{3f(\beta)}\right] = \left[\frac{-4A^2}{\kappa^2} + \frac{(1 + \kappa^2/4\Omega^2)}{2}\right]\left(1 + \frac{\kappa^2}{4\Omega^2}\right)^{1/2} (V_R)_{\rm epi}^3\, \frac{\pi}{2}(r_t + r_f)^2$$
$$\times\left[1 + \frac{2^{3/2}G(m_t + m_f)}{(r_t + r_f)(1 + \kappa^2/4\Omega^2)(V_R)_{\rm epi}^2}\right]. \tag{114}$$

The ratio of the monopole to quadrupole terms (dE/dt) on the left-hand side of equation (114) does depend on the cutoff chosen. For the case adopted, the ratio is 0.90.

At this point it is worth noting that, for a flat rotation curve (with $\kappa = 2^{1/2}\Omega$ and $A = \Omega/2$), the first term on the right-hand side of equation (114) (physical viscosity gains) is equal to two-thirds of the second term (physical collision losses), and hence the first term is equal to twice the term on the left-hand side. That is, the rate of energy gain from viscosity due to physical collisions is equal to twice the rate of energy gain from gravitational viscosity. However, due to the different velocity dependence of these two terms, the gravitational viscosity is mainly responsible in determining the cloud velocity dispersion—while the viscosity due to physical collisions only increases the resulting velocity dispersion by $\sim 30\%$, as shown in § III*b*.

Specializing to the case of a flat rotation curve and replacing $(V_R)_{\rm epi}$ with the expression for $V_{\rm 1\text{-}D}$, the equation (114) reduces to

$$\frac{(Gm_f)^2\kappa H}{[(r_t + r_f)/2]^2} = 2.12V_{\rm 1\text{-}D}^5\left[1 + \frac{(1.414)G(m_t + m_f)}{V_{\rm 1\text{-}D}^2(r_t + r_f)}\right]. \qquad (115)$$

This is the equation of energy balance for the GMCs in the Galaxy. The velocity dispersion $V_{\rm 1\text{-}D}$ is determined by the requirement that energy gains and losses balance on average. Numerical solution of equation (115) for $V_{\rm 1\text{-}D}$ is presented in Tables 1 and 2.

iv) *General Discussion of the Mechanism*

A few general points about the viscous acceleration mechanism and the calculation presented in this subsection are described next.

1. Following is a list of restrictions on the applicability of the calculation given in this subsection:

First, the resulting change in energy per encounter, $\langle \Delta E_x + \Delta E_y \rangle$, call this ΔE, must be less than $E_{\rm epi}$ if the effect of the passing field cloud on the test cloud is to be treated as a second-order perturbation in the epicyclic energy of the test cloud. Second, only when ΔE following an encounter is less than $E_{\rm escape} = \frac{1}{2}m_t(2Gm_t/r_t)$, can the tidal effects on the test cloud be ignored. Last, $\Delta v_t = (2\Delta E/m_t)^{1/2}$ must be less than $(V_{\rm planar})_{\rm random} = 2^{1/2}V_{\rm 1\text{-}D}$, if the Fokker-Planck approximation is to be valid. Note that this approximation is implicitly assumed here, because we write the net rate of change of the energy of a test cloud as arising due to independent, subsequent encounters with the field clouds.

All these three criteria do *not* affect the choice of $S_{\rm min}$ (see § II*b*[iv]). Rather, they are checks that have to be satisfied by the expression for $\langle \Delta(E_x + E_y) \rangle$ (as given by eq. [105]). We next show that the above criteria are indeed satisfied for the typical cloud parameters. We assume $m_t = m_f = 5.8 \times 10^5\ M_\odot$ (SSS1985) and $V_{\rm 1\text{-}D}$ is the calculated value of the one-dimensional cloud-cloud velocity dispersion, ~ 5 km s^{-1} (see Table 1). From equation (7), we get $(V_R)_{\rm epi} = (4/3)^{1/2}V_{\rm 1\text{-}D} = 5.8$ km s^{-1}. We get $S_{\rm min} = 2^{1/2}(V_R)_{\rm epi}/\kappa = 0.23$ kpc. Using these values, equation (105) gives $(2\Delta E/m_t) = 10.8$ (km s)^2 for the typical encounter which is far less than $2E_{\rm epi}/m_t = 3(V_R)_{\rm epi}^2 = 100$ (km s^{-1})^2 or $2E_{\rm esc}/m_t = (2Gm_t/r_t) = 200$ (km s^{-1})^2, or, by a smaller margin, less than $2V_{\rm 1\text{-}D}^2 = 50$ (km s^{-1})^2.

Thus, our claculation obeys the above three constraints. In particular, our treatment of the gravitational encounter as a perturbation calculation is justified.

2. Finally, we would like to stress that the above calculation involving gravitational viscosity is different from the acceleration of a small mass object (star) by a larger mass cloud in a sheared galactic disk, as studied by Spitzer and Schwarzschild (1953 hereafter SS53).

In the scheme of SS53, the star is in an epicyclic motion and the scattering clouds are taken to be in circular motion (with no random velocity) at the local circular speed at each point. In our calculation the systematic relative velocity between the test cloud and the field cloud is due to the differential rotation in the disk and therefore depends linearly on the impact parameter, S. (In a future paper, we will discuss the rate of increase of the *stellar* velocity dispersion—arising as a result of the gravitational viscous acceleration mechanism presented here). Also in SS53 the impact parameter is assumed to be small compared to a_R and is treated locally, whereas we assume the opposite.

The simplest way to see the difference between these two methods–one involving local scattering with no account of shear and the other involving gravitational viscosity as in our calculation—is to consider the extreme case when the impulse approximation is

TABLE 1

STEADY STATE, ONE-DIMENSIONAL, PLANAR, RANDOM VELOCITY DISPERSION FOR EQUALLY MASSIVE GMCs ($m = 5.8 \times 10^5\ M_\odot$)[a]

R (kpc)	$V_{\rm 1\text{-}D}$ (km s^{-1})
4	6.7
5	6.3
6	6.0
7	5.7
8	5.5
9	5.3
10	5.1

[a] The rotation curve is taken to be flat. The constant rotation speed is 250 km s^{-1}, and the Sun is at $R = 10$ kpc (Schmidt 1965).

TABLE 2

STEADY-STATE, ONE-DIMENSIONAL, PLANAR, RANDOM VELOCITY DISPERSION FOR GMCs OF DIFFERENT MASSES

R (kpc)	$V_{\rm 1\text{-}D}$ (km s^{-1})[a]	$V_{\rm 1\text{-}D}$ (km s^{-1})[b]
4	8.3	5.1
5	7.8	4.7
6	7.5	4.5
7	7.2	4.2
8	6.9	4.1
9	6.7	3.9
10	6.5	3.8

[a] $m_t = 10^5\ M_\odot$, $m_f = 5.8 \times 10^5\ M_\odot$.
[b] $m_t = 1.8 \times 10^6\ M_\odot$, $m_f = 5.8 \times 10^5\ M_\odot$.

valid [that is, $(V_{1\text{-D}})_t = 0$. For this case, the "local" scattering mechanism (of SS53) would yield zero change in the energy of the test cloud, while our calculation involving gravitational viscosity would yield a nonzero increase in the random kinetic energy of the test cloud, as shown in Appendix A.

Further, when $(V_{1\text{-D}})_t \neq 0$, one cannot naively apply the results from SS53 for the case of the cloud-cloud encounter. In the SS53 calculation, the stellar energy increases as the star and the field clouds try to achieve kinetic equipartition with each other. Because the clouds are more than 10^5 times more massive than the stars, this process causes a very slow increase in the stellar energy, with $t_{\text{relaxation}}$ (= energy doubling time) $\approx 10^9$ yr and $t_{\text{equipartition}} \approx 10^{12}$ yr. For the cloud-cloud "local" scattering case, however, this equipartition time is much shorter (due to the equal masses of the test and the field particles and due to the lower velocity of the test particle in this case) and is $\sim 6 \times 10^6$ yr (obtained using the SS53 mechanism), which is much shorter than the epicyclic period ($\sim 10^8$ yr). Therefore, in this case, one has to apply the result for equipartition of gravitationally interacting particles (as derived in Spitzer 1941). For a field cloud of zero random velocity this yields an equipartition time of $\sim 10^7$ yr, beyond which this process involving "local" scattering cannot cause any change in the energy of the clouds. An interesting point to note is that the "local" cloud-cloud scattering would only cause a *decrease* in the random kinetic energy of the test cloud, that too only until the test cloud comes into equipartition with the field clouds.

In any case, the main point here is that the "local" cloud-cloud scattering (i.e., scattering involving no account of the shear in the disk), *can* be ignored while considering the long-term ($\gtrsim 10^8$ yr) energy balance of the GMCs.

III. STEADY STATE CLOUD VELOCITY DISPERSION: RESULTS AND DISCUSSION

In this section we obtain and discuss the results for the steady state planar velocity dispersion for the GMCs in the galactic disk. In § III*a*, we list the relevant observational data for the GMCs in the Galaxy. In § III*b*, we discuss the general properties of the equation of energy balance (see eq. [115]) that describes the behavior of the steady state cloud velocity dispersion. In § III*c*, we calculate the quantitative results for the cloud velocity dispersion for the observed cloud parameters and compare our results with the observations given in § III*a*. Section III*e* contains a brief discussion of the long-term ($\geq 10^{10}$ yr) availability of the rotational support for the inelastic cloud motions and a comment on the effect of long-term rotational support on the cloud distribution in the galactic disk.

a) Observational Properties of the GMCs

The data on the masses and the radial distribution of the GMCs are taken from SSS1985. Following the notation of SSS1985, we define GMCs to be the molecular clouds more massive than $10^5\ M_\odot$. In the inner Galaxy (R = 2–10 kpc), with the Sun at 10 kpc, most ($\sim 85\%$) of the H_2 mass is contained in the GMCs. There are ~ 6000 clouds more massive than $10^5\ M_\odot$ and having a radius greater than 10 pc. Over half of the H_2 mass is in clouds more massive than $5.8 \times 10^5\ M_\odot$, and over 90% of the H_2 mass is in clouds more massive than $10^5\ M_\odot$. Hence, the typical GMC has a mass of $5.8 \times 10^5\ M_\odot$ and a radius of 25 pc. The lower and the upper limits on their mass range are, respectively, $10^5\ M_\odot$ and $1.8 \times 10^6\ M_\odot$, with radii in the range 10–40 pc (see SSS1985).

The observations of the velocity dispersion of the molecular clouds have been recently studied by Clemens (1985) and Stark (1984).

By analyzing the data of SSS1985, Clemens (1985) obtains $V_{1\text{-D}}$, the one-dimensional, random, planar cloud velocity dispersion (along the lines of sight along the loci of tangents in the inner Galaxy) to be equal to 3 km s^{-1}. This is an average over cloud masses in the SSS1985 sample, over the inner Galaxy.

Earlier, Liszt and Burton (1983) and Burton and Gordon (1978) had estimated $V_{1\text{-D}}$ to be 4.2 km s^{-1} and 4.0 km s^{-1}, respectively. These values were obtained by doing model calculations so as to fit the limited observational data then available.

Stark (1984) has observed clouds with masses in the range of 10^4–$3 \times 10^5\ M_\odot$ which lie within 3 kpc of the solar neighborhood. For these clouds, Stark obtains $V_{1\text{-D}} \approx 6.6$ km s^{-1}. Clemens argues that this value is too high because Stark did not take account of the streaming motion of the clouds. Also the actual sample of clouds studied by Stark is much smaller than the sample studied by Clemens.

In summary, the observed values of $V_{1\text{-D}}$ for massive clouds ($10^4\ M_\odot \lesssim M_c \lesssim 10^6\ M_\odot$) in the inner Galaxy lie in a range of 3–7 km s^{-1}, with the smaller values in this range being the more accurate estimates for the GMCs ($M \geq 10^5\ M_\odot$).

At this point, we would like to stress that this observed range of values for the cloud velocity dispersion is an *average* over cloud masses, galactocentric radial distances, and radial and azimuthal random velocity components. This only allows a general comparison between the observations and our results. This is also the reason why the equation of energy balance (eq. [115]) is written in terms of $V_{1\text{-D}}$ as the variable. It is clear that one can easily obtain $(V_R)_{\text{random}}$ and $(V_\theta)_{\text{random}}$ in terms of $V_{1\text{-D}}$ (using eqs. [4] and [5])—these may then be compared with the corresponding direction-dependent random cloud velocities (if and) when the latter become available observationally.

b) Properties of the Equation of the Energy Balance

The solution of the equation of energy balance for the GMCs (eq. [115], § II*c*) corresponds to the steady state cloud velocity dispersion.

Following is a list of the important properties of the equation of energy balance (eq. [115]):

1. Note that the equation of energy balance is a fifth-order polynomial in $V_{1\text{-D}}$; to a good approximation $V_{1\text{-D}}$ depends very weakly as $\frac{1}{3}$–$\frac{1}{5}$ root of the input parameters.

2. For a gravitationally bound cloud, $Gm/r > V_{1\text{-D}}^2$—that is, the gravitational focusing term in the formula for the collision cross section (eq. [11]) is much greater than unity. Hence, the equation of energy balance simplifies to the following form:

$$(V_{1\text{-D}})^3 = 0.33\left(\frac{Gm}{r}\right)\kappa H \qquad (116a)$$

or

$$V_{1\text{-D}} = 0.38 V_{esc}(\kappa/\kappa)_z^{1/2}(V_z/V_{1\text{-D}})^{1/2} . \tag{116b}$$

where $V_{ex}^2 \equiv 2Gm/r$, κ_z is the z oscillation frequency and we have eliminated H using the condition for equilibrium in the z-direction. This denotes the main functional dependence of $V_{1\text{-D}}$.

3. Note that $V_{1\text{-D}}$ is an *increasing* function of m/r. This point clearly underlines the gravitational viscous nature of the cloud acceleration mechanism studied in this paper. This behavior of $V_{1\text{-D}}$ is also of relevance in explaining why the clouds in the Galaxy are not in kinetic energy equipartition. A detailed équipartition analysis will be presented in a future paper.

4. The resulting $V_{1\text{-D}}$ is independent of n_f, the number density of the field clouds; this is because all the input and loss terms in the equation of energy balance are linearly proportional to n_f. Also, $V_{1\text{-D}}$ is only moderately dependent on the ratio m_t/m_f, and hence on r_t/r_f.

5. $V_{1\text{-D}}$ depends only weakly (through κ/κ_z) on the galactocentric radial location, R, of a cloud. For example, $V_{1-\text{D}}$ increases by a factor of $\sim 2.5^{1/3} = 1.36$, from $R = 10$ kpc (the solar neighborhood) to $R = 4$ kpc (the inner edge of the galactic disk CO distribution).

6. Equation (116) can be used to illustrate the relative dominance of the gravitational viscosity over the viscosity due to physical collisions in deciding the resulting value of $V_{1\text{-D}}$, as discussed next. From equation (116) and the discussion following equation (114), it is clear that if one were to ignore the contribution from viscosity to equation (114) due to physical collisions, the constant in the right-hand side of eq. (116) would need to be multipled by a factor of $\frac{1}{3}$; hence $V_{1\text{-D}}$ would be lowered by a factor of $\sim(\frac{1}{3})^{1/3} \sim 0.69$. In other words, the inclusion of energy input from viscosity due to physical collisions only increases the final value of the cloud velocity dispersion by ~30%–35% over what it would be if the only energy input were to be due to gravitational viscosity. At the other extreme, if physical viscosity were the only energy source for random motion of the clouds, then the steady state velocity is obtained implicitly as being less than 1 km s^{-1} (see § II*a*). Thus, gravitational viscosity is the main process responsible for supporting the random velocity dispersion of the GMCs in the Galaxy. Without gravitational viscosity, an energy balance cannot be obtained.

c) Quantitative Results and Comparison with Observations

We consider for the sake of illustration a flat rotation curve, with Θ the constant rotational speed, equal to 250 km s^{-1} (Schmidt 1965). The total vertical gas scale height, H, is equal to 0.13 kpc, which is roughly constant with R to within the error bars (see SSS1984). For these parameters, we solve the equation of energy balance and thus obtain the resulting steady state cloud velocity dispersion as a function of R, for the different cases listed below.

First, consider encounters between equally massive test and field clouds, each having a mass of $5.8 \times 10^5 M_\odot$, which is the typical representative mass (see § III*a*) for a GMC in the galactic disk. For this case, the resulting values for $V_{1\text{-D}}$ are listed in Table 1. For these typical GMCs, located between R4 and 10 kpc, the resulting $V_{1\text{-D}}$ lies in the range 5–7 km s^{-1}. Note that these resulting values are within the observed range of values for $V_{1\text{-D}}$.

Two comments about the choice of Θ, the constant rotational speed, are in order. The first is that the rotational speed is not truly constant over $R = 4$–10 kpc but instead varies by ~40 km s^{-1}. However, even if one were to use this varying Θ, it would only slightly alter the results for $V_{1\text{-D}}$, because $V_{1\text{-D}} \propto \kappa^{1/3}$ (see eq. [116]). Hence, for the sake of simplicity, we adopt a flat rotation curve, with a constant Θ. The second point is that if one were to use the "new standard" galactic constants of $\Theta = 220$ km s^{-1} and $R_\odot = 8.5$ kpc (Kerr and Lynden-Bell 1985) while keeping the cloud parameters constant, then the resulting values of $V_{1\text{-D}}$ at the inner and outer edge of the molecular distribution in the Galaxy would be 6.8 and 5.2 km s^{-1}, respectively, instead of being 6.7 and 5.1 km s^{-1} as in Table 1. That is, because of the form of equation (116), the resulting values of $V_{1\text{-D}}$ are not too sensitively dependent on the actual choice of the rotation curve parameters. Now, the cloud parameters (m, n, R, etc.) have been obtained by SSS1985 assuming the old standard galactic constants (i.e., $\Theta = 250$ km s^{-1} and $R_\odot = 10$ kpc) and these cloud parameters do not scale easily for another choice of the galactic constants. Hence, we use the old standard galactic constants so as to be consistent with SSS1985.

We next obtain the velocity dispersion for the GMCs having masses different from the typical values. Consider the lowest mass GMCs ($m = 10^5 M_\odot$). The direct application of equation (115)- for interaction between equally massive clouds—in this case gives very low values of $V_{1\text{-D}}$, since $V_{1\text{-D}} \propto (m/r)^{1/3}$. For example, this procedure gives $V_{1\text{-D}} = 3.6$ km s^{-1} at $R = 10$ kpc. At the other extreme end of the GMC mass scale, consider the more massive GMCs, each of mass $1.8 \times 10^6 M_\odot$. One cannot use equation (115) to obtain the velocity dispersion (resulting from encounters with similar clouds) for these clouds because their number density is too low ($n \sim 6.5$ clouds kpc^{-3}), and hence the effective S_{min} is higher than $2^{1/2}a_R$ (see § II*b*[iv] for the discussion of S_{min}). In other words, the use of equation (115) in this case would lead to an overestimate of the actual value of the cloud velocity dispersion.

In either of the above two cases, it is then necessary also to consider encounters with the intermediate-mass "typical" clouds. We first set $m_t = 10^5 M_\odot$ and $m_f = 5.8 \times 10^5 M_\odot$ and then evaluate $V_{1\text{-D}}$ for m_t using equation (115); the results are given in Table 2, column (2). Next, we repeat the procedure for $m_t = 1.8 \times 10^6 M_\odot$ and $m_f = 5.8 \times 10^5 M_\odot$; the results for this case are given in Table 2, column 3.

Note that the second column in Table 2 gives higher values for $V_{1\text{-D}}$ than the corresponding values for $V_{1\text{-D}}$ in Table 1. This is because m_f is the same in either case and the gravitational acceleration term is proportional to m_f^2 in either case, whereas the second term on the right-hand side of quation (115)—which is proportional to $m_t + m_f$—is smaller when m_t is smaller. The converse reasoning applies when $m_t > m_f = 5.8 \times 10^5 M_\odot$, and hence the values for $V_{1\text{-D}}$ in Table 2, column (3), are smaller than the corresponding values in Table 1, column (2).

Recall that these resulting values of the steady state cloud velocity dispersion are upper limits on the corresponding actual values. This is because of the tendency of the massive clouds to try to achieve kinetic energy equipartition with the lower mass H I clouds. The rate of loss of kinetic energy of a GMC due to its random, gravitational interaction with the lower mass H I clouds can be written as an additional loss term in the equation of energy balance for the GMCs. For the case of equally massive GMCs, the solution of this modified equation of energy balance gives $V_{1\text{-D}} = 5.5$ km s^{-1} and 4.1 km s^{-1} at $R = 4$ and 10 kpc, respectively (see Appendix D for details), instead of the earlier results of $V_{1\text{-D}} = 6.7$ and 5.1 km s^{-1} respectively, as given in Table 1. Note that these new results are in an even better agreement with the observed value of the velocity dispersion by Clemens (1985) than those given previously.

Similarly, the results for $V_{1\text{-D}}$ for GMCs with masses of $10^5\ M_\odot$ and $1.8 \times 10^6\ M_\odot$, as listed in Table 2, are only upper limits to the actual velocities for these clouds.

Another interesting point is that the GMCs, with a mass range of $\sim 10^5$–$1.8 \times 10^6\ M_\odot$, can interact with each other via random, gravitational encounters and can thus reach equipartition among themselves, in roughly a few $\times\ 10^7$ yr (as can be checked by the application of eq. [D1] in Appendix D). In fact, Stark (1983) and Scoville *et al.* (1987) have shown that the GMCs do indeed exhibit kinetic equipartition among themselves. However, their conclusion is based on the analysis of the z-velocity dispersion of the GMCs and for the clouds located at the peak of the molecular ring. Hence we cannot directly compare our results for a planar velocity dispersion, $V_{1\text{-D}}$, with the z-velocity values reported by Stark or by Scoville *et al.* A study of the z-velocity dispersion will be presented in a future paper. It is important to note that the above result regarding the establishment of kinetic energy equipartition among the GMCs in roughly a few $\times\ 10^7$ yr is valid irrespective of the cloud formation mechanism—that is, it is valid whether the clouds form as nonlinear condensations out of the two-fluid gravitational instabilities (see Jog and Solomon 1984*a*, *b*) or whether the clouds form via agglomeration of smaller mass clouds (see Kwan 1979; Cowie 1980).

Note that even when one takes account of the random, gravitational encounters of GMCs among themselves, the values for velocities as in Table 2 are still upper limits—even for the lower mass ($\sim 10^5\ M_\odot$) GMCs. This is because a GMC loses more kinetic energy per unit time due to its interaction with the H I clouds (of $\sim 400\ M_\odot$ each) than due to its interaction with the lower mass GMCs (see Appendix D).

Thus, because the $V_{1\text{-D}}$ as given by the gravitational viscous acceleration mechanism increases with mass (see eq. [115]), most of the kinetic energy in an ensemble of clouds is concentrated at the upper end of the cloud mass spectrum. Further random gravitational and physical interactions among the GMCs and the lower mass H I clouds would tend to increase the random kinetic energy of the H I clouds at the expense of the random kinetic energy of the GMCs, in an effort toward achieving equipartition, as discussed above. Analysis of velocity dispersion versus mass will be presented in a subsequent paper.

Actually, even for the analysis presented in this paper, the main restriction in comparing the results for the velocity dispersion with the observations is set by the lack of detailed observational data (as discussed in § III*a*)—in particular, by the fact that as yet only the average value of the velocity dispersion is known from observations (see § III*a*).

The main point of this analysis is that, for the typical GMCs, cloud-cloud gravitational interactions in the sheared galactic disk provides the main acceleration and that this process does give rise to the values of the cloud velocity dispersion that are in reasonable agreement with the observed values of the same. This agreement is even better when one takes into account the tendency of the GMCs toward achieving kinetic equipartition with the lower mass (H I) clouds.

d) Acceleration Time Scale and Competing Mechanisms

The typical gravitational collision at impact parameter $S_{\min}$ produces an energy change (from eq. [105]) of

$$\Delta E_{\text{epi}} = \frac{1}{2} m_t \left(\frac{Gm_f}{S_{\min}}\right)^2 \frac{8}{\beta^2} f(\beta) (V_R)_{\text{epi}}^{-2} \left[1 + \frac{\beta^2 g(\beta)}{4f(\beta)}\right] .$$

Comparing this with $E_{\text{epi}} = 3/2 m_t (V_R)^2_{\text{epi}}$, we see that the characteristic number of cloud-cloud gravitational collisions required to set up an equilibrium is of order

$$N_{\text{coll}} = 3f(\beta) \frac{8}{\beta^2} \left(\frac{Gm_f}{S_{\min} V^2_{R,\,\text{epi}}}\right)^2 \left[1 + \frac{\beta^2 g(\beta)}{3f(\beta)}\right]^{-1} = 4.1\left(\frac{H}{r}\right) \approx 10^2$$

for a monolayer with $r \approx H$. The mean free time between such collisions is

$$t_{\text{coll}}^{-1} = 2AS^2_{\min} nH ,$$

so the acceleration time is

$$t_{\text{acc}} = t_{\text{coll}} N_{\text{coll}} = 0.22 \frac{H\kappa}{nr^2 (V_{1\text{-D}})^2} \approx 2 \times 10^9 \text{ yr}$$

This is quite a bit slower than the acceleration time for the small clouds but sufficiently fast so it is reasonable to assume a steady state.

Supernovae are quite ineffective in accelerating such large clouds. But it is interesting to ask if the global instabilities of the cloud-star two-fluid system set a lower bound on the velocity dispersion. If the cloud fluid were treated alone, then the Toomre Q-factor was found to be $Q_g = 1.5$–2 in Jog and Solomon (1984*a*, *b*) indicating a stable fluid. A more appropriate calculation is to check what the critical value of $V_{1\text{-D}}$ would be (V_{crit}), to obtain $Q_g = 1$, marginal stability. We find the ratio of $V_{1\text{-D}}/V_{\text{crit}}$ to be 3.0 at $R = 4$ kpc and 1.5 at 10 kpc. The cloud fluid is always stable but not by a large factor. Acceleration of the clouds by interaction with the spiral pattern (Balbus 1987), has a comparable time scale.

e) Long-Term Rotational Support of the GMC Motion

The first question we would like to answer is: how long can the gravitational viscous acceleration mechanism operate? In order to answer this, recall that here the increase of random kinetic energy of the clouds is provided by the decay in their ordered, rotational kinetic energy in the galactic disk (see Appendix C). In a steady state, the rate of energy input due to viscosity is equal to the rate of loss of kinetic energy due to the inelastic cloud collisions.

The ratio of random kinetic energy thus lost over 10^{10} yr to the rotational kinetic energy of the clouds is equal to

$$(0.1 \text{ km s}^{-1} \text{ kpc}^{-1})^{-1}[\tfrac{1}{2}m_t\, 2(V_{1\text{-D}})^2\omega_c]/[\tfrac{1}{2}m_t(250 \text{ km s}^{-1})^2] .$$

For typical cloud parameters, $\omega_c = 24$ km s^{-1} kpc^{-1} and $V_{1\text{-D}} \approx 6$ km s^{-1} (see Table 1). For these parameters, the above ratio is equal to 0.1. Therefore, the rotational energy of the clouds in the galactic disk constitutes a vast energy reservoir, easily capable of supporting inelastic GMC motions, via the viscous acceleration process, for $\sim 10^{10}$ yr.

One can next ask how the cloud distribution in the galactic disk itself is altered as a result of having to support the inelastic cloud motions over 10^{10} yr. As a result of the viscous interaction among the clouds, the clouds drift inward with a local velocity of ~ 0.3 km s^{-1}, thus depleting the gas within the region of $R \lesssim 3$ kpc in the Galaxy in $\sim 10^{10}$ yr. This is in rough agreement with the observed minimum or the "hole" in the galactic CO distribution. The detailed analysis for the calculation of the radial distribution of the GMCs as a function of time will be presented in a future paper.

IV. CONCLUSIONS

In conclusion, the main acceleration for the GMCs in the galactic disk is provided by the effective viscosity that results from the gravitational interactions among these massive clouds while they are situated in a differentially rotating galactic disk.

In a steady state, the primary functional dependence of the cloud velocity dispersion is $V_{1\text{-D}}$ divided by $0.69[Gm/r)\kappa H]^{1/3} = 0.38 V_{\text{esc}}(\kappa/\kappa_z)^{1/2}(V_z/V_{1\text{-D}})^{1/2}$, where m, r and V_{esc} are the cloud mass, radius and escape velocity, respectively, κ is the epicyclic frequency in the disk κ_z the z oscillation frequency, and H is the total vertical scale height of the gas distribution. This result is independent of the number density of the (field) clouds, and it depends only weakly (through κ) on the galactocentric radial location of the cloud. Note that the cloud velocity dispersion is an *increasing* function of m/r—this point clearly underlines the gravitational viscous nature of the cloud acceleration mechanism considered in this paper. This behavior of $V_{1\text{-D}}$ is also important in explaining why the interstellar clouds in the Galaxy are not in kinetic energy equipartition.

For typical GMCs (each of mass $\sim 5.8 \times 10^5\, M_\odot$), located between the galactocentric radial distance of 4–10 kpc, the resulting steady state, one-dimensional, planar, random velocity dispersion, $V_{1\text{-D}}$, is $\sim$5–7 km s^{-1}. For GMCs covering masses from $\sim 10^5$–$1.8 \times 10^6\, M_\odot$, the resulting range of velocity dispersion is $\sim$4–8 km s^{-1}. These results are in good agreement with the observed values of $V_{1\text{-D}}$, especially when one takes note of the fact that all these resulting values of the velocity dispersion are upper limits—this is because the massive clouds tend to lose their random kinetic energy to the less massive (H I) clouds, in an attempt toward achieving equipartition with them. The detailed equipartition analysis yielding velocity dispersion as a function of mass will be presented in a future paper.

In the viscous acceleration mechanism, the ultimate energy source for the support of the random kinetic energy of the GMCs is their rotational kinetic energy in the galactic disk. The fraction of rotational kinetic energy lost in supporting the inelastic cloud motions over $\sim 10^{10}$ yr is only ~ 0.1, in the molecular ring ($4 \leq R \leq 8$ kpc). Hence the rotational energy of the GMCs in the disk proves to be more than adequate for the long-term support of their random motion. The viscous evolution of the radial distribution of the GMCs in the Galaxy will be treated in a future paper.

Thus, gravitational viscosity is the main process responsible for supporting the random motion of the GMCs in the Galaxy. The dynamics of massive clouds in the Galaxy as well as their radial distribution in the galactic disk is, therefore, mainly determined by their gravitational viscous interaction while they are located in a differentially rotating galactic disk. Hence, one cannot treat the GMCs as forming an isolated three-dimensional system; rather one must treat the GMCs as constituents of the galactic disk, insofar as their dynamics is concerned.

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APPENDIX A

THE CHANGE IN ENERGY PER ENCOUNTER FOR A TEST CLOUD WITH ZERO INITIAL RANDOM VELOCITY (IMPULSE APPROXIMATION)

We consider the special case when the initial random velocities of both the field and the test clouds are equal to zero. In this case the clouds are on purely circular orbits, say at radii $R - S$ and R respectively. Because of the differential rotation in the galactic disk, the relative speed of rotation of the above two clouds is equal to $2AS$, where A is the Oort constant. Hence the clouds undergo an effective gravitational encounter, as is clearly seen if one considers the motion of the two clouds in the rest frame of the test cloud. The impact parameter for this encounter is S, and the relative velocity between the two clouds before collision is given by $V_{\text{rel}} = 2AS$.

In the center of mass frame of the two clouds, the velocity of the test cloud is given by

$$V_t' = -\frac{m_f}{(m_t + m_f)} V_{\rm rel} = -\frac{m_f}{(m_t + m_f)} 2AS \, . \tag{A1}$$

As a result of the gravitational encounter between the two clouds, the random velocity of the test cloud changes by a magnitude ΔV_t and its direction is perpendicular to the direction of V_t'. Using impulse approximation, the magnitude of ΔV_t is given as

$$\Delta V_t = -V_t' \frac{2\alpha}{1+\alpha^2} \, , \tag{A2}$$

where

$$\alpha = \frac{G(m_t + m_f)}{V_{\rm rel}^2 S} = \tan\frac{\delta}{2} \, , \tag{A3}$$

where δ is the angle of deflection of the velocity vector.

Therefore, we get

$$\Delta E_t/\text{encounter} = \tfrac{1}{2} m_t (\Delta V_t)^2 \, .$$

Using equations (A1)–(A3), this is equal to

$$\frac{2m_t(Gm_f)^2(2A)^6 S^8}{[G^2(m_t+m_f)^2 + (2A)^4 S^6]^2} \, . \tag{A4}$$

Thus, the change of energy of the test cloud per encounter is obtained trivially for the special case of $(V_{1\text{-D}})_t = 0$, as opposed to the more general case (studied in § II*b*) when the initial random velocity of the test cloud is finite.

APPENDIX B

INTEGRALS INVOLVING THE MODIFIED BESSEL FUNCTIONS

In this Appendix, we evaluate the values of the integrals I_{cx}, I_{cy}, and I_s (involving the modified Bessel functions of the second kind) which are defined by equations (63), (64), and (83), respectively.

Consider I_{cx} first. Recall from equation (63) that

$$I_{cx} = \int_{-\infty}^{\infty} f_x(t) \cos 2\kappa t \, dt = 2 \int_0^{\infty} f_x(t) \cos 2\kappa t \, dt \, , \tag{B1}$$

since the integrand is an even function of t. Now $f_x(t)$ can be expressed using equations (15) and (24) as follows

$$f_x(t) = \frac{S^3}{d^3}\left(1 - \frac{3S^2}{d^2}\right) = \frac{1}{(2A)^3[(1/4A^2) + t^2]^{3/2}} - \frac{3}{(2A)^5[(1/4A^2) + t^2]^{5/2}} \, . \tag{B2}$$

Substituting equation (B2) into equation (B1), we get

$$I_{cx} = \frac{2}{(2A)^3} \int_0^{\infty} \frac{\cos 2\kappa t \, dt}{[(1/4A^2) + t^2]^{3/2}} - \frac{6}{(2A)^5} \int_0^{\infty} \frac{\cos 2\kappa t \, dt}{[(1/4A^2) + t^2]^{5/2}} \, . \tag{B3}$$

Each of the integrals in the above equation can be solved by using the formula for the integral representation of the Bessel functions of the second kind (see eq. [8.432.5] from Gradshteyn and Ryzhik 1980). For the first integral we get

$$\int_0^{\infty} \frac{\cos 2\kappa t \, dt}{[(1/4A^2) + t^2]^{3/2}} = 4\kappa A K_1\left(\frac{\kappa}{A}\right) = 4\beta A^2 K_1(\beta) \, , \tag{B4}$$

where $\beta \equiv \kappa/A$, which is a measure of $t_{\rm enc}/t_{\rm epi}$, as seen in § II*b*(i).

Next, the second integral in equation (B3) is

$$\int_0^{\infty} \frac{\cos 2\kappa t \, dt}{[(1/4A^2) + t^2]^{5/2}} = \frac{16}{3} \beta^2 A^4 K_2(\beta) = \frac{16}{3} [2K_1(\beta) + \beta K_0(\beta)]\beta A^4 \, , \tag{B5}$$

where we have used the recurrence relation among Bessel functions (see Dwight 1961, eq. [804.3]):

$$K_2(\beta) = [2K_1(\beta) + \beta K_0(\beta)]/\beta \, , \tag{B6}$$

where $K_0(\beta)$, $K_1(\beta)$, and $K_2(\beta)$ are the modified Bessel functions of the second kind.

Combining equations (B3), (B4), and (B6), we get

$$I_{cx} = -\frac{\beta}{A}\,[K_1(\beta) + \beta K_0(\beta)]\ . \tag{B7}$$

Similarly, from equation (83) we define

$$I_{cy} = \int_{-\infty}^{\infty} f_y(t) \cos 2\kappa t\, dt = 2\int_0^{\infty} f_y(t) \cos 2\kappa t\, dt\ , \tag{B8}$$

since again the integrand is an even function of t.

Now equation (24) gives

$$f_y(t) = \frac{3S^5}{d^5} - \frac{2S^3}{d^3}\ .$$

Then by straightforward manipulations similar to those used to obtain I_{cx}, we find

$$I_{cy} = \frac{\beta^2}{A}\, K_0(\beta)\ . \tag{B9}$$

Finally, recall the definition of I_s from equation (64).

$$I_s \equiv \int_{-\infty}^{\infty} f_2(t) \sin 2\kappa t\, dt\ .$$

Using equation (25), this becomes

$$I_s = -\frac{1}{2A}\int_{-\infty}^{\infty} \left\{\frac{d}{dt}\,[f_x(t) + f_y(t)]\right\} \sin 2\kappa t\, dt = -\frac{1}{2A}\,\{[f_x(t) + f_y(t)] \sin 2\kappa t\}\Big|_{-\infty}^{\infty} + \frac{2\kappa}{2A}\int_{-\infty}^{\infty} [f_x(t) + f_y(t)] \cos 2\kappa t\, dt\ .$$

Note that the terms in the first expression in braces are both zero because $f_x(t)$ and $f_y(t) \to 0$ as $t \to \pm\infty$, while $\sin 2\kappa t$ remains finite as $t \to \pm\infty$. Hence, I_s reduces to

$$I_s = \beta\int_{-\infty}^{\infty} [f_x(t) + f_y(t)] \cos 2\kappa t\, dt = \beta[I_{cx} + I_{cy}] = -\frac{\beta^2}{A}\, K_1(\beta)\ , \tag{B10}$$

where we have evaluated I_{cx} and I_{cy} from equations (B7) and (B9).

APPENDIX C

ROTATIONAL SUPPORT OF THE RANDOM KINETIC ENERGY OF THE CLOUDS

In this Appendix we show for the special case of $V_t = 0$ (that is, zero initial random test cloud velocity) that the increase from gravitational viscosity in the random kinetic energy of the test cloud and the field cloud equals the loss of their rotational kinetic energy.

Here, as in Appendix A, the test and the field clouds execute purely circular orbits, say at radii R and $R - S$ ($S > 0$) respectively. Hence the relative speed of rotation, also equal to the magnitude of the relative velocity between the two clouds before the encounter, is given by $V_{\rm rel} = 2AS$. In the center of mass frame of the two clouds, the velocities before encounter of the test and the field clouds are given as

$$V_t' = -\frac{m_f}{(m_t + m_f)}\, V_{\rm rel}\ ,$$

and

$$V_f' = \frac{m_t}{(m_t + m_f)}\, V_{\rm rel}\ , \tag{C1}$$

respectively.

These two velocity vectors define a plane. Next, we choose a set of cylindrical coordinate axes R'', θ'', z'' such that z'' is along $V_{\rm rel}$, R'' is along the intersection of the above plane with the plane normal to z'', and θ'' is along the normal to R'' and is in a plane normal to z''. Further, let Δj_t, Δk_t, Δl_t denote the changes (resulting from the gravitational encounter) in the velocity of the test cloud, given in the center of mass frame, along the above three axes (z'', R'', θ''). Similarly, let Δj_f, Δk_f, Δl_f denote the corresponding changes in the velocity of the field cloud.

The values of these velocity components are obtained from Henon (1973, p. 201), and, on substituting $\alpha = \tan \delta/2$ (where δ is the angle of deflection of the velocity vector of the test or the field cloud), we get

$$\begin{pmatrix} \Delta j_t \\ \Delta k_t \\ \Delta l_t \end{pmatrix} = V_t' \begin{pmatrix} -\dfrac{2\alpha^2}{(1+\alpha^2)} \\ \left[\dfrac{2\alpha}{(1+\alpha^2}\right] \cos\theta \\ \left[\dfrac{2\alpha}{(1+\alpha^2)}\right] \sin\theta \end{pmatrix} \quad \text{(C2)}$$

Similar relations also hold good for the field cloud.

Now, in the above notation, the final rotational and the random kinetic energies of the test cloud are given by $\frac{1}{2}m_t(V_t' + \Delta j_t)^2$ and $\frac{1}{2}m_t(\Delta k_t^2 + \Delta l_t^2)$, respectively. For the field cloud, the similar quantities are given by $\frac{1}{2}m_f(V_f' + \Delta j_f)^2$ and $\frac{1}{2}m_f(\Delta k_f^2 + \Delta l_f^2)$, respectively.

Hence, the net change in the rotational kinetic energy of the test cloud and the field cloud is given by

$$\tfrac{1}{2}m_t(V_t' + \Delta j_t)^2 + \tfrac{1}{2}m_f(V_f' + \Delta j_f)^2 - \tfrac{1}{2}m_t V_t'^2 - \tfrac{1}{2}m_f V_f'^2 = \tfrac{1}{2}m_t(2V_t'\Delta j_t + \Delta j_t^2) + \tfrac{1}{2}m_f(2V_f'\Delta j_f + \Delta j_f^2)\,. \quad \text{(C3)}$$

On using equations (C1) and (C2), this reduces to

$$-\frac{2\alpha^2}{(1+\alpha^2)^2} V_{\rm rel}^2 \frac{m_t m_f}{(m_t + m_f)}\,. \quad \text{(C4)}$$

This is the net loss in the rotational kinetic energy, resulting from a gravitational encounter between the test and the field clouds.

Now the increase, following a gravitational encounter, in the random kinetic energy of the test cloud is given by

$$\tfrac{1}{2}m_t(\Delta k_t^2 + \Delta l_t^2)\,.$$

Using equations (C1) and (C2), this reduces to

$$\frac{2\alpha^2}{(1+\alpha^2)^2} \frac{m_t m_f^2}{(m_t + m_f)^2}\,. \quad \text{(C5)}$$

Similarly, the increase (change), following an encounter, in the random kinetic energy of the field cloud is given by

$$\frac{1}{2} m_f(\Delta k_f^2 + \Delta l_f^2) = \frac{2\alpha^2}{(1+\alpha^2)^2} \frac{m_f m_t^2}{(m_t + m_f)^2} V_{\rm rel}^2\,. \quad \text{(C6)}$$

First of all, note that equations (C4), (C5), and (C6) add to give a zero. *That is, in the case of gravitational viscosity, the increase in the random kinetic energy of the test and the field clouds is provided by the decrease in their rotational kinetic energy.* Although this has been shown here for the special case of zero initial test cloud velocity, it is also true for a general case of nonzero initial random velocity of the clouds (not proven here).

The second point to note is that *both* the test *and* the field clouds gain random kinetic energy as a result of the gravitational encounter. This fact is very important because it implies that, while obtaining the net increase in the random kinetic of a test cloud, one has to take account of its encounters with the field clouds orbiting at radii $R - S$ *and* at $R + S$ (see the discussion at the beginning of § II*b*[iii]). Further, the ratio of the increase in the random kinetic energy of the test cloud to that of the field cloud is equal to m_f/m_t; that is, it is inversely proportional to their mass ratio. An encounter between equally massive clouds leads to an equal increase in the random kinetic energy of both the clouds.

APPENDIX D

RANDOM, GRAVITATIONAL ENCOUNTERS BETWEEN THE GMCs AND THE LOWER MASS H I CLOUDS AND THE RESULTING LOWER VALUES OF $V_{1\text{-D}}$ FOR THE GMCs

The GMCs undergo random gravitational encounters with the H I clouds at the lower mass end of the cloud range—this process tends to increase the random kinetic energy of the H I clouds at the expense of the random kinetic energy of the GMCs, in an attempt toward achieving kinetic energy equipartition among the GMCs and the H I clouds.

The rate of loss of the random kinetic energy of a GMC due to its interaction with the H I clouds can be written as an additional loss term in the equation of energy balance for the GMCs; this then leads to lower values of $V_{1\text{-D}}$ than the ones reported earlier in Tables 1 and 2, as shown next. For gravitationally interacting clouds, the rate of loss of random kinetic energy of the massive clouds (which appears as the increase in the random kinetic energy of the less massive clouds) is given by Spitzer (1941):

$$\frac{dE_t}{dt} = \frac{\beta n_2(E_t - E_2)}{[(E_2/m_2) + (E_t/m_t)]^{3/2}}\,, \quad \text{(D1)}$$

where the subscript 2 denotes the various quantities for the lower mass clouds in the system. In the above equation,

$$\beta = 2(3\pi)^{1/2} G^2 m_t m_2 \log \left[1 + \frac{V^4 d_{max}^2}{G^2(m_t + m_2)^2} \right], \tag{D2}$$

where V^2 is the square of the larger of the mean square velocities of the test GMC and the less massive field cloud. The quantity d_{max} is the maximum distance at which a gravitational encounter can occur, we take this to be the size of the GMC distribution $\sim(10 - 4) \sim 6$ kpc.

It turns out that most of the loss of kinetic energy of the GMCs is due to their interaction with the H I clouds rather than due to their interaction with the less massive GMCs; this follows from the larger number density of the H I clouds and is also due to the higher energy difference $|E_t - E_2|$ for the H I clouds than for the less massive GMCs.

Now within the solar circle ($R \leq 10$ kpc), the total H_2 mass is ~ 3 times the total H I mass (SSS1984). If one were to consider all the GMCs to be equally massive, of $\sim 5.8 \times 10^5\ M_\odot$ each, and all the H I clouds to be equally massive, of $\sim 400\ M_\odot$ each, which is the mass of a "typical" Spitzer cloud (Spitzer 1978); then using equation (D1), one can write down the rate of loss of the random kinetic energy of a GMC due to its interaction with the H I clouds. On writing this as an additional loss term on the right-hand side of equation (113) and solving the resulting modified form of equation (115), we get $V_{1\text{-D}} = 5.5$ km s^{-1} and 4.1 km s^{-1} at $R = 4$ kpc and 10 kpc, respectively, instead of getting $V_{1\text{-D}} = 6.7$ km s^{-1} and 5.1 km s^{-1} as in Table 1. Note that these new results are in an even better agreement with the observed values of the velocity dispersion by Clemens (1985).

Recent observational evidence indicates (see, e.g., Kulkarni and Heiles 1986) that a large fraction of the total H I mass may be in a diffuse, filamentary form—rather than being in the form of spherical clouds. In that case, the ratio of the number density of the "standard" spherical H I clouds (of $\sim 400\ M_\odot$ each) to the number density of the GMCs is lower than described above. This means that the tendency toward equipartition between the GMCs and the (standard) H I clouds will be less apparent and the resulting values of $V_{1\text{-D}}$ would lie between the values calculated above and those given in Table 1.

An important point is that the time required for the full equipartition to be established among the GMCs and the H I clouds, due to the above process, is very large $\sim 10^{11}$ yr. Hence, in spite of their tendency toward equipartition, the GMCs are far from actually achieving full equipartition with the H I clouds. In fact, this is exactly what is seen observationally and was indeed a main motivation for the work presented in this paper (see § I).

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