

DOE/ER/40561-261-INT96-00-126
 TPI-MINN-96/06
 NUC-MINN-96/8-T
 May 1996

Lattice Computations of Small-x Parton Distributions in a Model of Parton Densities in Very Large Nuclei

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Abstract

Using weak coupling methods McLerran and Venugopalan [1] expressed the parton distributions in large nuclei as correlation functions of a two dimensional Euclidean field theory. The theory has the dimensionful coupling $g^2\mu$, where $\mu^2 \sim A^{1/3}$ is the valence quark color charge squared per unit area. We use a lattice regularization to investigate these correlation functions both analytically and numerically for the simplified case of $SU(2)$ gauge theory. In weak coupling ($g^2\mu L \ll 5$), where L is the transverse size of the nucleus, the numerical results agree with the *analytic* lattice weak coupling results. For $g^2\mu L \gg 5$, no solutions exist at $O(a^4)$ where a is the lattice spacing. This suggests an ill-defined infrared behavior for the two dimensional theory. A recent proposal of McLerran et al. [16] for an *analytic* solution of the classical problem is discussed briefly.

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1 Introduction

In Ref. [1] McLerran and Venugopalan proposed that weak coupling methods can be used to compute small x parton distribution functions in large nuclei. They wrote down a partition function for wee partons with $x \ll A^{-1/3}$ in the presence of external sources which are the valence quark charges. The only large component of the valence quark current is J^+ , which is modelled by

$$J_a^\mu = \delta^{\mu+} \rho_a(x^+, \vec{x}_\perp) \delta(x^-), \quad (1)$$

where ρ_a is the density (per unit area) of valence quark color charges. Their partition function is obtained by integrating the QCD partition function, coupled to the above static current, over all ρ_a 's with a Gaussian weight. The variance of this Gaussian distribution of valence quark charges, $\mu^2 \sim A^{1/3} \text{ fm}^{-2}$, the average valence quark color charge squared per unit area, is the only dimensionful parameter in the theory. If $\mu^2 \gg \Lambda_{QCD}^2$, $\alpha_S(\mu^2) \ll 1$ and weak coupling methods can be used. This model could then be studied as a toy model to understand both the rapid growth of structure functions at small x [2] and the eventual saturation of these structure functions as dictated by unitarity [3,4]. Note that the model of Ref. [1] is gauge invariant due to the Gaussian distribution for the valence quark densities.

In Ref. [5] the saddle point solution of the partition function in the presence of the Gaussian random source was obtained by solving the classical Yang–Mills equations $D_\mu F^{\mu\nu} = J^\nu$. Here D_μ is the covariant derivative, $F^{\mu\nu}$ the non–Abelian field strength tensor and J^ν is the current in Eq. 1. It was shown that the classical background field that satisfies the Yang–Mills equations has a simple structure. Consequently, the classical parton distributions can be expressed as correlation functions of a two dimensional Euclidean field theory. This is not too surprising since it is well known that at very high energies the longitudinal and transverse coordinates decouple. Indeed, it has been proposed recently that the limit of $x \rightarrow 0$ and color $N_c \rightarrow \infty$ is an exactly solvable two dimensional field theory [6]. In papers subsequent to Ref. [5], the problem of quantum fluctuations about the background field [7–9] and that of initial conditions in heavy ion collisions were addressed [10,11]. For a brief review of

these results, we refer the reader to Ref. [12]. An excellent introduction to all aspects of the low x problem is given in Ref. [13].

In this paper, we will discuss only the classical solutions of the Yang–Mills equations. As we shall see in Section 2, computing the correlation functions requires that we solve a stochastic differential equation for each color charge configuration. Since the equations are highly non-linear, no analytic solutions were found. However, it was claimed in Ref. [5] that the parton distributions have the Weizsäcker–Williams behavior in the weak coupling region $\alpha_S\mu \ll k_t$: $dN/dxd^2k_t \propto 1/xk_t^2$. It was conjectured that the solution of the stochastic differential equations in the strong coupling region of $\Lambda_{QCD} \ll \alpha_S\mu \ll \mu$ would reveal that the classical gluons generate a screening mass $m_{screen} \sim \alpha_S\mu$. If there is such a screening mass, its existence would strongly suggest that a mechanism for the restoration of unitarity at very small x exists already at the classical level.

We will address here the question of a screening mass in the classical theory quantitatively by solving stochastic difference equations on a two dimensional lattice. In Section 3, we will describe how we set up the problem and how one may use lattice perturbation theory to identify the weak coupling and strong coupling regimes of the theory. We define “reduced” correlation functions of gauge fields which are one dimensional projections of the original two dimensional fields. If a screening mass existed in the theory, these reduced correlation functions may be expected to have a very characteristic exponential fall off at large distances.

For simplicity, we will consider an SU(2) gauge theory in our numerical work. We use the conjugate gradient method to solve the difference equations on the lattice. Details of the numerical procedure are also discussed in Section 3. In Section 4, we describe lattice results for the reduced correlation functions and compare them to the results expected from lattice perturbation theory in weak coupling. It is observed that in the weak coupling region, the numerical results reproduce to high accuracy the results of lattice perturbation theory. However, as one approaches the strong coupling region on the lattice, the number of the stochastic difference equations to which solutions can be found decreases and eventually, in the strong coupling region,

no solutions of the lattice equations exist at the desired $O(a^4)$ accuracy. In Section 5, we will interpret these results and state our conclusions.

2 Parton distributions as correlation functions of a 2-D field theory

In the model of McLerran and Venugopalan, the partition function which describes the ground state properties of wee partons with $x \ll A^{-1/3}$ and transverse momenta $q_t \ll A^{1/6} \text{ fm}^{-1}$, is [1]

$$Z = \int [dA_t dA_+] [d\psi^\dagger d\psi] [d\rho] \exp \left(iS + ig \int d^4x A_+(x) \delta(x^-) \rho(x) - \frac{1}{2\mu^2} \int d^2x_t \rho^2(0, x_t) \right). \quad (2)$$

In the above, ρ is the valence quark color charge density. Also, the parameter $\mu^2 \sim A^{1/3} \text{ fm}^{-2}$ is the average valence quark color charge squared per unit area. Since μ^2 is the only scale in the partition function above, the coupling constant will run as a function of this scale [9]. If $\mu^2 \gg \Lambda_{QCD}^2$, as will be true for *very* large nuclei, $\alpha_S(\mu^2) \ll 1$ and weak coupling methods can be used.

If we integrate over the ρ fields first, we obtain an effective action for the wee partons with non-local propagators and vertices. Instead, the procedure followed in Ref. [5] was to perform the ρ integrals last. In that approach, one needs to calculate the saddle point solution of the action for each ρ configuration to determine the classical background field. Any physical observable, such as a correlation function, is then obtained by evaluating it for the saddle point solution and then averaging it over all possible ρ configurations. The saddle point solution is nothing else but the solution to the classical Yang-Mills equations

$$D_\mu F^{\mu\nu} = gJ^\nu, \quad (3)$$

in the presence of the external source $J^\nu = \delta^{\nu+} \rho(x_t) \delta(x^-)$. It was shown in Ref. [5] that the solution of these classical equations of motion is

$$A^+ = 0$$

$$\begin{aligned}
A^- &= 0 \\
A^j &= \theta(x^-)\alpha^j(x_t)
\end{aligned}
\tag{4}$$

The transverse components A^j , where $j = 1, 2$ further satisfy the equations, $F_{12} = 0$ and $\nabla \cdot \alpha = g\rho$, where the latter equation follows from the Gauss' law. Since the fields A^j are thus gauge transforms of vacuum configurations with a gauge condition determined by the ρ -configuration, one can write $\alpha_j(x_t) = \frac{i}{g}U(x_t)\nabla_j U^\dagger(x_t)$, where U is a unitary $SU(3)$ matrix for QCD and an $SU(N)$ matrix for a theory with N colors. Substituting for α_j in the gauge condition results in the stochastic differential equation

$$\vec{\nabla} \cdot U \vec{\nabla} U^\dagger = -ig^2 \rho(x_t).
\tag{5}$$

Using the solutions of the equation above, which are the saddle point configuration of the partition function in Eq. 2, one can show that the *classical* correlation functions may be expressed (in matrix representation) as correlation functions of a two dimensional Euclidean field theory:

$$\begin{aligned}
\langle \alpha_i^{\alpha\beta}(x_t) \alpha_j^{\alpha'\beta'}(0) \rangle_\rho &= \frac{-1}{g^2} \int [d\rho] \left(U(x_t) \nabla U^\dagger(x_t) \right)_\rho^{\alpha\beta} \left(U(0) \nabla U^\dagger(0) \right)_\rho^{\alpha'\beta'} \\
&\times \exp \left(-\frac{1}{2\mu^2} \int d^2 x_t \rho^a(x_t) \rho^a(x_t) \right) / I,
\end{aligned}
\tag{6}$$

where the Gaussian random measure

$$I = \int [d\rho] \exp \left(-\frac{1}{2\mu^2} \int d^2 x_t \rho^a(x_t) \rho^a(x_t) \right),
\tag{7}$$

is all that is left from the original partition function. Note that the charges are highly localized in the transverse plane:

$$\langle \rho^a(x_t) \rangle = 0 ; \langle \rho^a(x_t) \rho^b(y_t) \rangle_\rho = \mu^2 \delta^{ab} \delta^{(2)}(x_t - y_t).
\tag{8}$$

In order to ensure that the valence quark color charge is confined to the transverse radius of the nucleus, we require that

$$\int d^2 x_t \rho^a(x_t) = 0.
\tag{9}$$

This constraint was not stated in Ref. [5]. In the momentum space this condition decrees that $\rho^a(k_t = 0) = 0$. Thus the $k_t = 0$ mode is excluded explicitly.

To compute the correlation function in Eq. 6, we need to solve Eq. 5 to determine $U \equiv U(\rho)$ for each ρ configuration. We were unable to find an analytic solution to this highly non-linear equation for all values of the coupling. In the following and subsequent sections we will discuss the analytic weak coupling solution and the numerical solutions of this equation on a two dimensional lattice.

For completeness, let us recall that the relation between distribution functions and the correlation functions above is straightforward and is discussed explicitly in Ref. [7]:

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2 k_t} = \frac{1}{(2\pi)^3} \frac{1}{x} \int d^2 x_t e^{i k_t x_t} \text{Tr} [\langle \alpha_i^{\alpha\beta}(x_t) \alpha_j^{\alpha'\beta'}(0) \rangle], \quad (10)$$

where the trace is over both Lorentz and color indices.

It was argued in Ref. [5] that the distribution function has the general form

$$\frac{1}{\pi R^2} \frac{dN}{dx d^2 k_t} = \frac{(N_c^2 - 1)}{\pi^2} \frac{1}{x} \frac{1}{\alpha_S} H(k_t^2 / \alpha_S^2 \mu^2), \quad (11)$$

where $H(k_t^2 / \alpha_S^2 \mu^2)$ is a non-trivial function obtained by explicitly solving Eq. 5. The effective coupling constant of this theory was believed to be $\alpha_S \mu / k_t$ and that in the “weak coupling” limit $\alpha_S \mu \ll k_t$, $H(k_t^2 / \alpha_S^2 \mu^2) \rightarrow \alpha_S^2 \mu^2 / k_t^2$, recovering the Weizsäcker-Williams result scaled by μ^2 . It was also conjectured that the function H would have the form $\alpha_S^2 \mu^2 / (k_t^2 + M^2)$, where $M = c \alpha_S \mu$ is a screening mass which is a constant c times the dimensionful scale $\alpha_S \mu$.

Interestingly, the problem formulated above is analogous to the problem of the critical behaviour of Ising-like models coupled to a random magnetic field. As discussed by Parisi and Sourlas [15], the partition function in that case has a structure identical to Eq. 2 albeit they only discussed the case of a scalar theory. It was argued in Ref. [15] that the singular behavior of the theory near the critical point was best described by correlation functions analogous to those in Eq. 6. The remarkable result of Parisi and Sourlas was that their scalar version of Eq. 6 could be written as correlation functions of a theory which is identical to the original theory *without the random magnetic fields* but in $D - 2$ dimensions, where D is the dimensionality of the original

theory. This dimensional reduction is a consequence of a hidden supersymmetry of the expression analogous to Eq. 6. The gauge theory analogue of this symmetry is nothing other than the well known BRST symmetry. Unfortunately, Parisi–Sourlas dimensional reduction will not apply to Eq. 6 because the analog of their scalar field is the compact field U as opposed to the gauge field α .

Therefore, in order to test our conjecture about the existence of a screening mass in the the strong coupling domain, we address the question numerically by formulating the problem on a lattice. This also enables us to define the weak coupling limit more precisely. As we will discuss in the following section, the above statements about weak coupling are modified somewhat by the precise formulation of the problem on the lattice. The effective coupling of the theory is indeed $\alpha_S\mu/k_t$ but only for discrete multiples of $k_t = 2\pi/L$. Here L is the transverse size of the nucleus. Correspondingly, one obtains a discrete version of the Weizsäcker–Williams result for weak coupling ($g^2\mu L \ll 5$ as we will show) by using lattice perturbation theory. However, since the limit $L \rightarrow \infty$ is synonymous with strong coupling, the Weizsäcker–Williams result of Ref. [5] for continuous transverse momenta will not be recovered.

3 The 2–D Theory on the lattice

As discussed in the previous section, to compute correlation functions of the two dimensional field theory, we need to solve stochastic differential equations Eq. 5 for an arbitrarily large coupling. We intend to do this numerically by introducing a spatial lattice. The lattice spacing a serves as an ultra-violet regulator. Indeed, without such a regularization the functional integrals in Eq. 6 are not well defined since the correlations of the ρ -fields are proportional to a δ -function. Introducing the lattice, one sees from Eq. 7 that each $\rho^a(x)$ is μ/a times a Gaussian random number of unit variance. Approximating the circular transverse side of the nucleus of diameter L by a square of length L , one sees it to be a $N \times N$ grid of lattice points with $L = Na$. The continuum limit consists of taking $a \rightarrow 0$ and $N \rightarrow \infty$ such that L is held constant. The further removal of the infra-red regulator can be achieved

by taking $L \rightarrow \infty$, which was the limit in which the weak coupling considerations of Refs. [1] and [5] led to their computational scheme of the low-x parton distributions.

Using the unitarity condition on the U -matrices, the stochastic differential equation in Eq. 5 can be re-written as

$$\left(U \nabla^2 U^\dagger - \nabla^2 U \cdot U^\dagger \right) = -2ig^2 \rho, \quad (12)$$

On the lattice, finite differences replace the derivatives:

$$\nabla^2 U^\dagger = \sum_{j=1,2} \frac{\left(U^\dagger(x_t + a_j) + U^\dagger(x_t - a_j) - 2 U^\dagger(x_t) \right)}{a^2} + O(a^2). \quad (13)$$

The labels $j = 1, 2$ refer to the orthonormal directions on the lattice and a_j denotes a displacement by a single site, i.e., by distance a , in the j th direction. The resultant stochastic difference equation form (to $O(a^4)$ accuracy) of Eq. 5 is

$$\left[U(x_t) \sum_{j=1,2} \left(U^\dagger(x_t + a_j) + U^\dagger(x_t - a_j) \right) \right] - \text{h.c.} + 2ig^2 \mu a \rho(x_t) = 0. \quad (14)$$

In the equation above, h.c. denotes hermitean conjugate and we have scaled $\rho \rightarrow \mu\rho/a$. This has the advantage that the Gaussian random measure defined in Eq. 7 is now independent of the lattice spacing a and the dimensionful parameter μ . It is redefined to be

$$\int [d\rho^a] \exp \left(-\frac{1}{2} \sum_{x_t} \rho^a(x_t) \rho^a(x_t) \right). \quad (15)$$

The rescaled ρ on the lattice satisfy the following equations in analogy with their continuum version:

$$\langle \rho^a(x_t) \rangle = 0 ; \langle \rho^a(x_t) \rho^b(y_t) \rangle_\rho = \delta^{ab} \delta_{x_t, y_t}^{(2)}. \quad (16)$$

The zero net color charge constraint naturally becomes a sum on the lattice and $\rho^a(k_t = 0) = 0$ is true on the lattice as well. The only coupling this lattice theory has is the dimensionless $g^2 \mu a$ and the scale for the theory is provided by the nuclear transverse size L . Physical quantities can therefore be obtained as a function of $g^2 \mu a$ or equivalently $g^2 \mu L$.

In computing correlation functions on the lattice, we will find it most convenient to study correlations of one dimensional projections of the two dimensional gauge fields. These “reduced” gauge fields are defined as

$$\alpha_j^r(x_2) = \frac{\sum_{x_1} \alpha_j(x_1, x_2)}{N}. \quad (17)$$

This one dimensional projection sets the momentum $k_1 = 0$. If there exists a mass gap M in the theory, then

$$\langle \alpha_\mu^r(x_2) \alpha_\mu^r(y_2) \rangle = A \exp(-M |x_2 - y_2|), \quad (18)$$

for sufficiently large $|x_2 - y_2|$ (to avoid influence of excited states, if any). An exponential fall off of the correlations of these reduced gauge fields would therefore be an unambiguous signature of a mass gap in the theory.

3.1 Weak Coupling Limit

Since the above mentioned function H , and therefore the correlation functions we wish to obtain, have earlier been obtained in weak coupling limit of the 2-D theory, it will be instructive to first calculate them in the weak coupling limit on the lattice. For small enough $g^2 \mu a$, Eq. 14 clearly admits a solution for $U(x_t)$ which is close to the identity matrix for all x_t modulo a *global* gauge rotation. Writing the matrices U in terms of the the generators τ^k of the $SU(N)$ gauge group as $U(x_t) = \exp(ig^2 \mu a \cdot \phi(x_t))$, with $\phi(x_t) = \sum_k \phi^k(x_t) \cdot \tau^k$, one sees that weak coupling implies that $g^2 \mu a \cdot \phi \ll 1$. One can therefore expand the field U as

$$U = 1 + ig^2 \mu a \cdot \phi - \frac{1}{2}(g^2 \mu a)^2 \cdot \phi^2 + \dots \quad (19)$$

Keeping terms of only lowest order, Eq. 14 becomes

$$\sum_{j=1,2} [\phi(x_t + a_j) + \phi(x_t - a_j) - 2\phi(x_t)] = \rho(x_t). \quad (20)$$

These equations can be solved by Fourier transforming the fields ϕ and the sources ρ : Let

$$\phi(x_t) = \frac{1}{N^2} \sum_{\vec{n}} \exp(2\pi i \vec{n} \cdot \vec{x}_t / L) \tilde{\phi} \left(\frac{2\pi \vec{n}}{L} \right), \quad (21)$$

with $\vec{n} = (n_1, n_2)$ and $-(N-1)/2 \leq n_j \leq (N-1)/2$ (assuming N to be odd). Defining similarly the Fourier transform of $\rho(x_t)$, we obtain the following solution of the Eq. 20:

$$\tilde{\phi}\left(\frac{2\pi\vec{n}}{L}\right) = \frac{\tilde{\rho}\left(\frac{2\pi\vec{n}}{L}\right)}{2\sum_{j=1,2}\left[\cos\left(\frac{2\pi n_j a}{L}\right) - 1\right]}. \quad (22)$$

Substituting back in the equation for $\phi(x_t)$, we have

$$\phi(x_t) = \frac{1}{2N^2} \sum_{\vec{n}} \frac{\exp(2\pi i\vec{n} \cdot \vec{x}_t/L)}{\sum_{j=1,2}\left[\cos\left(\frac{2\pi n_j a}{L}\right) - 1\right]} \tilde{\rho}\left(\frac{2\pi\vec{n}}{L}\right). \quad (23)$$

In this leading order of weak coupling, $\alpha_j(x_t) = g\mu[\phi(x_t + a_j) - \phi(x_t - a_j)]$, and the one dimensional projections of the α fields, are easily computed to be

$$\begin{aligned} \alpha_1^r(x_2) &= 0. \\ \alpha_2^r(x_2) &= \frac{g\mu i}{2N^2} \sum'_{n_2} \frac{\sin\left(\frac{2\pi n_2 a}{L}\right) \tilde{\rho}\left(\frac{2\pi n_2 a}{L}\right)}{\left[\cos\left(\frac{2\pi n_2 a}{L}\right) - 1\right]} \exp\left(\frac{2\pi i n_2 x_2}{L}\right). \end{aligned} \quad (24)$$

Here prime denotes the exclusion of the $n_2 = 0$ due to the total vanishing charge condition. To obtain the “reduced” correlators, we take the product of the α^r fields and take the average over the $\tilde{\rho}$ fields. Using the relation between ρ and $\tilde{\rho}$, and the Eq. 16, one can easily show that $\tilde{\rho}$ satisfies similar equations as well except that its two point correlation function has an extra factor of N^2 :

$$\langle \tilde{\rho}^a(k_t) \tilde{\rho}^b(l_t) \rangle_{\tilde{\rho}} = N^2 \delta^{ab} \delta_{k_t, l_t}^{(2)}. \quad (25)$$

Using the relation above and after some simple algebra, we obtain

$$\begin{aligned} \langle \alpha_1^r(x) \alpha_1^r(x') \rangle &= 0. \\ \langle \alpha_2^r(x) \alpha_2^r(x') \rangle &= \frac{g^2 \mu^2}{2N^2} \sum_{n_2=1}^{(N-1)/2} \frac{\sin^2\left(\frac{2\pi n_2 a}{L}\right)}{\left[\cos\left(\frac{2\pi n_2 a}{L}\right) - 1\right]^2} \cos\left(\frac{2\pi n_2 (x - x')}{L}\right). \end{aligned} \quad (26)$$

In the continuum limit of $a \rightarrow 0$ and $N \rightarrow \infty$,

$$\langle \alpha_2^r \alpha_2^r \rangle_{a \rightarrow 0} = \frac{g^2 \mu^2}{2\pi^2} \sum_{n_2=1}^{\infty} \frac{\cos\left(\frac{2\pi n_2 (x - x')}{L}\right)}{n_2^2}. \quad (27)$$

By constructing similar “reduced” correlators for the calculations of Ref. [1], one can easily see that our results are very similar to theirs, except that our expression above

still has a finite L . Consequently, only discrete momenta are allowed in our sum and the lowest allowed momentum is $2\pi/L$. A naive $L \rightarrow \infty$ yields identical results to those of Ref. [1] but it turns out that this limit is not allowed.

In order to see why it is so, it is necessary to go back to the weak coupling condition $g^2\mu a \cdot \phi \ll 1$. Using the solution for $\phi(x_t)$ obtained above, one can translate this condition into

$$g^2\mu L \left\{ \frac{1}{N^3} \sum_{\vec{n}} \frac{\exp(2\pi i \vec{n} \cdot \vec{x}_t/L)}{\sum_{j=1,2} [\cos(\frac{2\pi n_j a}{L}) - 1]} \tilde{\rho}\left(\frac{2\pi\vec{n}}{L}\right) \right\} \ll 1. \quad (28)$$

Taking further the continuum limit, one obtains

$$g^2\mu L \left\{ \frac{1}{4\pi^2 L} \sum_{\vec{n}} \frac{\exp(2\pi i \vec{n} \cdot \vec{x}_t/L)}{j_1^2 + j_2^2} \tilde{\rho}\left(\frac{2\pi\vec{n}}{L}\right) \right\} \ll 1. \quad (29)$$

One can now see that the $L \rightarrow \infty$ limit will violate the above condition even if one ignores the possibly logarithmically divergent factor in the curly bracket in that limit. The weak coupling condition thus constrains L to stay finite and small. The expression in the curly brackets can be evaluated numerically and typically the largest values are ~ 0.2 if one keeps L finite. The weak coupling condition for a finite size L is then (approximately)

$$g^2\mu L \ll 5. \quad (30)$$

The correlation function $\langle \alpha_1^r \alpha_1^r \rangle$ becomes non-zero in the next-to-leading order of the expansion, when

$$\alpha_\mu = g\nabla_\mu\phi + ig^3(\phi\nabla_\mu\phi - (\nabla_\mu\phi)\phi). \quad (31)$$

For the sake of brevity we have used here the continuum notation to denote the finite differences. Using this expression, the reduced correlator can be computed straightforwardly. The final expression is fairly tedious (involving the Gaussian average of four $\tilde{\rho}$ fields). The key result of this computation is that in the continuum limit $a \rightarrow 0$,

$$\langle \alpha_1^r \alpha_1^r \rangle \propto g^2\mu^2(g^2\mu L)^2, \quad (32)$$

i.e., the correlation function grows as $(g^2\mu L)^2$. In the next section, we will compare the results of our lattice computation with these lattice perturbation theory results in the weak coupling region $g^2\mu L \ll 5$.

The weak coupling calculations above were done by introducing an ultra-violet cut-off, the lattice spacing a . Since the final result shows sensitivity only to the infra-red regulator, namely the size L , one can ask whether these results can be derived without introducing the lattice in x -space at all. The answer turns out to be affirmative. One can easily show that the continuum problem can be formulated in the momentum space of a finite square box of length L . Due to the discrete momentum spectra, the corresponding $\tilde{\rho}$ -measure is then well defined. One then solves Eq. 5 by first expanding $U(x)$ and then Fourier transforming the resultant equation. The final results, of course, remain unchanged when compared with the $a \rightarrow 0$ limit above.

3.2 Numerical Method

In order to obtain a result for the correlation functions of the α^r -fields which is free of the infra-red cut-off, one has to take the limit $L \rightarrow \infty$. Since it thus necessarily takes one out of the weak coupling region, we now turn to the procedure we used to solve the stochastic difference equations in Eq. 14 numerically. To simplify our computations and as a test, we choose to work with the gauge group $SU(2)$. No qualitative differences are anticipated with regard to the existence of the mass gap as a result of this simplification. Writing an $SU(2)$ matrix U as $a_0I + i\tau^k a_k$, we can write the first term of Eq. 14 as

$$\left[U(x_t) \sum_{j=1,2} \left(U^\dagger(x_t + a_j) + U^\dagger(x_t - a_j) \right) \right] = b_0I + i\tau^k b_k. \quad (33)$$

Here I is the unit 2×2 matrix and $\tau^k, k = 1, 2, 3$ are the Pauli matrices. The coefficients a_k satisfy the unitarity condition, $\sum_{k=0}^3 a_k^2 = 1$ but the coefficients b_k do not. Eq. 14 can now be re-expressed as $b_k(x_t, x_t \pm a_j) + g^2\mu a \cdot \rho_k(x_t) = 0$. In order to solve these coupled nonlinear equations, we minimize the function F , defined by

$$F = \sum_{x_t} \left\{ \sum_k \left(b_k(x_t, x_t \pm a_j) + g^2\mu a \cdot \rho_k(x_t) \right)^2 + \left(\sum_k a_k^2(x_t) - 1 \right)^2 \right\}. \quad (34)$$

Minimizing F is equivalent to solving Eq. 14 for each lattice point and color charge ($3 N^2$ equations) while simultaneously imposing the unitarity condition $U U^\dagger = 1$ at each point on the lattice. The latter is done by the second set of terms in F . Note that F is a sum of squares of real numbers with zero as its possible absolute and desired minimum. A lack of solution will be signalled by large values of F_{\min} for the absolute minimum.

We use a multi-dimensional conjugate gradient method, described in the subroutine *FRPRMN* and its associated subprograms in Ref. [14], to minimize F to an accuracy better than $O(a^4)$ as dictated by accuracy of the original lattice equations. The zero net charge condition compels us to use periodic boundary conditions in accordance with the Gauss' law. We investigated both ordered and random starts for the initial guesses for the U 's. Each iteration consisted of choosing the source distributions randomly over the entire lattice in the momentum space such that 1) $\tilde{\rho}(0,0) = 0$, 2) $\tilde{\rho}^*(\vec{k}) = \tilde{\rho}(-\vec{k})$ and 3) both the real and imaginary parts of each $\tilde{\rho}(\vec{k})$ were random Gaussian numbers with variance $1/\sqrt{2}$. The ρ -distribution was then obtained by an explicit Fourier transformation. Using the conjugate gradient method, the set of U 's for the absolute minimum was found. If the minimum was $O(a^4)$ or smaller, then the matrices $U \equiv U(\rho)$ were used to compute the correlation functions on the lattice. We obtain α_μ from the relation

$$\tau_k \cdot \alpha_\mu^k = -\frac{1}{ga} \text{Im} \left(U(x_t) U^\dagger(x_t + a_\mu) \right). \quad (35)$$

We have also checked that the symmetric difference definition for the derivative yields the same result. Just as in Eq. 17, we define the “reduced” gauge fields α^r and compute correlators by taking the product of these gauge fields. This procedure is repeated over several iterations, typically a few hundred, and the the correlation function is averaged over these iterations. The errors are determined in the usual way by computing the standard deviations. Note that the sets of ρ 's in successive iterations are totally independent and one thus has negligibly small auto-correlations.

4 Results

In view of the facts that that the coupling for the lattice theory above is $g^2\mu a$, and that none of the three quantities in this expression occur independently, we chose to set $g^2\mu = 1$ in our simulations and varied $g^2\mu a$ by varying the lattice spacing a . Simulations were performed for a range of lattice sizes, ranging from $N = 21$ to $N = 211$, and for values of $g^2\mu L$ ranging from 0.5 to 20. Typically 200 iterations were performed, each consisting of an independent set of the ρ -distributions, unless stated otherwise. Noting from Eq. 26 that $g^2\mu^2$ sets the scale of the correlation functions, and using the definition in Eq. 35, one can show that the factor $(g^2\mu a)^2$ relates the dimensionless lattice correlation function to the physical α^r -correlations. We therefore show the results for the latter in the units of $g^2\mu^2$.

In Fig. 1a we show the results of our computation for $\langle\alpha_j^r(x)\alpha_j^r(0)\rangle/g^2\mu^2$, $j = 1, 2$, plotted as a function of the dimensionless distance x/L for a small value of $g^2\mu L = 0.5$. The lattice size was 41×41 . Also shown are the analytic weak coupling results of Eq. 26 for this lattice size which agree rather well with the direct computation. One also sees clearly that the α_1^r -correlation is very small compared to the α_2^r -correlation. In fact, the former is consistent with zero on the scale of this plot. This, together with the excellent agreement with the weak coupling result, reassures us that 1) for small $g^2\mu L$ the assumptions made in deriving the weak coupling results are indeed justified and 2) our numerical procedure works fine.

Fig. 1b further shows that these results are indeed the continuum results. It displays the results for the same $g^2\mu L$ but on $N = 21$ and 41 lattices. The results for the $j = 2$ correlation are again displayed as a function of the dimensionless distance x/L and the results are seen to be lattice size independent. One may wonder why the correlation function is negative, given that it can now be thought of as a continuum property. The weak coupling result of Eq. 26 provides a hint for understanding this. The leading term in it is negative for $x - x' = L/2$ and the successive terms alternate in sign and become progressively smaller in magnitude. Thus for any finite L , the correlation function will be negative midway if one is in the weak coupling domain.

Although the $\langle \alpha_1^r(x) \alpha_1^r(0) \rangle$ correlation function appears to be zero at all x/L in Figs. 1a and 1b, it has an interesting structure as well. As Fig. 2 shows for $N = 21$ and 41 lattices, this correlation function decreases monotonically as x increases but remains positive all through. We will later compare this behavior for larger $g^2 \mu L$ but it is interesting to note this difference with the leading order weak coupling behavior. As remarked earlier in Section 3.1, we do expect a non-vanishing contribution to it from the next-to-leading order contribution and a detailed examination of it also reveals it to be positive definite.

Having tested both the weak coupling limit and the conjugate gradient method on the lattice, we increased the $g^2 \mu L$, first by retaining the same lattice size of $N = 21$ and then increasing it as well up to $N = 71$ such that the lattice spacing stayed at $a \simeq 0.1$. This value was determined by making runs on the $N = 21$ lattice for various a and by checking that the errors due to finite a remained small. For the rest of our numerical work we have attempted to stay close to this value of a ; increasing thus the lattice size N in order to increase $g^2 \mu L$. Fig. 3 displays the results for the α_2^r -correlation functions in the units of $g^2 \mu^2$ as a function of x/L for $g^2 \mu L = 0.5, 1, 2, 3, 4, 5, 6,$ and 7 . It appears to remain almost independent of $g^2 \mu L$ until it reaches our estimated region of the validity of the weak coupling theory: for $g^2 \mu L \geq 5$ the correlation function tends to be less and less negative as $g^2 \mu L$ increases. This signals a departure from the leading order weak coupling result which, as seen in Eq. 27, is independent of $g^2 \mu L$ when viewed as a function of the dimensionless variable x/L . An obvious source of the departure from Eq. 26 are higher order contributions. If these tedious terms are indeed responsible for it then one expects a growth in the α_1^r -correlation as we argued in Section 3.1.

Fig. 4 exhibits the α_1^r -correlation function in the units of $g^6 \mu^4 L^2$ as a function of x/L for $g^2 \mu L$ up to 5. They do indeed group together to suggest a universal curve, and thus confirm the rise of this correlation function as $(g^2 \mu L)^2$. Fig. 5 demonstrates this in another way and also suggests that $g^2 \mu L \sim 5$ is the boundary of the weak coupling region. What is shown there is the $x = 0$ value for this correlation function in the units of $g^2 \mu^2$ as a function of $g^2 \mu L$, both before and after scaling out the factor

$(g^2\mu L)^2$. Note the scale of both the axes. A linear rising curve is thus an indication of the power law which seems to be consistent with the power two. What can also be inferred from this figure is a small trend to push this power up as one goes above $g^2\mu L \simeq 5$. A priori, such a behavior could also be due to yet more higher order terms. However, these results also suffer from a further defect.

For the larger values of $g^2\mu L$, one sees the F_{\min} slowly creep up and go beyond the $O(a^4)$ level. Indeed, typically one fails to obtain any acceptable minimum at that level for about 10-15 % of the iterations. This should be contrasted with the small $g^2\mu L$ case where F_{\min} was a lot smaller than $O(a^4)$ for *each* iteration. Increasing the $g^2\mu L$ even further, this becomes worse very quickly and by $g^2\mu L = 10$ no minimum exists at that accuracy.

In order to better understand the reason behind this, we show in Fig. 6 a normalized histogram plot for $g^2\mu L = 6, 7, 10$ and 20 . These runs were made on $N = 61, 71, 111$ and 211 lattices and the latter two have very few iterations, being 11 and 6 respectively. All the corresponding minima were too high compared to $O(a^4)$. The parameter R in Fig. 6 is defined as follows. Defining $R^k(x_t)$, $k=1, 2$ and 3 , to be the terms on the LHS of Eq. 14 divided by a^4 , one sees that $R^k(x_t) = 0$, for all x_t and k , is the desired solution. A value of $R^k(x_t) \neq 0$ measures how far away from the desired minimum (found by minimizing F) is the solution for that value of x_t and k . What Fig. 6 depicts, for different values of $g^2\mu L$, is the fraction of the $3N^2$ equations which have, for the best minimum of F , R given by the value on the x -axis. We have checked that the similar histogram plots for the weak coupling region have only the bin near zero occupied, i.e. they peak sharply at zero. What one sees in Fig. 6 though, are increasing deviations away from $R = 0$ – fewer and fewer of the $3N^2$ equations are being satisfied at the required level of accuracy. Noting that $R = 1/a$, which is ~ 10 for these runs, corresponds to the equations not being satisfied at $O(a^3)$ level, one finds that the minimum of F has increasingly many equations like that. This is thus an indication that for $g^2\mu L \geq 10$, no solutions to the stochastic equations exist at $O(a^4)$.

Our results therefore suggest the following: when $g^2\mu L < 5$, the correlation

functions computed directly agree very well with the expectations from lattice perturbation theory. For intermediate values, $g^2\mu L \approx 5$, the lattice results still agree reasonably well with the analytical lattice expressions but one notices an increasing trend of lack of solutions to more and more equations at the a^4 level. For larger values of $g^2\mu L > 10$, no solutions exist at that level.

The absence of solutions as we increase $g^2\mu L$ was unexpected. Increasing $g^2\mu L$ for a fixed value of a is equivalent to merely increasing the number of sites N . In other words, the number of equations has been increased but the structure of the equations and the coupling is unchanged. Why then are there no solutions as we go beyond $g^2\mu L \sim 5$? One way to understand this is as follows: since $U = \exp(ig^2\mu a \cdot \phi)$ and in weak coupling $\phi \propto N$, increasing $g^2\mu L$ will cause the U matrices to deviate increasingly away from identity. However, since $g^2\mu a$ is unchanged and it remains small, and since ρ remains $O(1)$, Eq. 14 will still prefer the U to be close to identity. The ensuing mismatch will thus result in a lack of solutions for large $g^2\mu L$. It is possible that our lack of solutions is because our boundary conditions are too restrictive. However, because we need to satisfy Gauss' law in two dimensions, periodic boundary conditions appear to be the appropriate physical choice.

The absence of solutions for $g^2\mu L > 5$ is a serious problem for the classical theory discussed in Ref. [5]. Not only because the conjectured scenario of a screening mass needed the coupling to be strong but also because the removal of the infra-red regulator pushes one in that region. Considering that even the very large nuclei will be finite in size, one could check whether the condition above is physically acceptable. If we take $L \approx 2 A^{1/3}$, then the weak coupling condition $g^2\mu L < 5$, holds only for very small A . This is due to the fact that $\mu \sim A^{1/6}$ and the coupling g^2 is also evaluated at the scale μ . Thus this condition contradicts the weak coupling assumption of the four dimensional theory in Ref. [1] which is expected to be valid only for very large A .

5 Summary and Outlook

In Ref. [1], a QCD based model was formulated to study the properties of low x , wee partons in large nuclei. For very large nuclei, it was argued that the problem could be formulated as a weak coupling, many body problem. In Ref. [5], it was shown that the classical saddle point solution of the model could be expressed as a two dimensional Euclidean field theory with the dimensionful coupling $\alpha_S\mu$. Computing correlation functions in this 2-D theory required the solution of highly non-linear stochastic differential equations in the presence of a Gaussian random source. No analytic solution of these equations was found in Ref. [5].

However, it was argued that classical distribution functions had a Weizsäcker-Williams distribution at large momenta $k_t \gg \alpha_S\mu$. It was also conjectured that at smaller momenta, in “strong coupling”, the theory acquired a screening mass $M \sim \alpha_S\mu$, which regulated the growth of the distribution function at small k_t . The existence of a screening mass would be suggestive of a weak coupling, albeit non-perturbative, restoration of unitarity already at the classical level.

In this paper, we have discussed the analytic weak coupling and the numerical solutions of the stochastic differential equations on a two dimensional lattice. For our numerical work, we made the simplifying assumption of two colors and investigated an $SU(2)$ gauge theory. With lattice perturbation theory as our guide, we identify $g^2\mu L \ll 5$ as the weak coupling condition. Our numerical results on the lattice agree very well with lattice perturbation theory for these values. For larger values of $g^2\mu L$, no solutions are found which satisfy the stochastic equations at the required level of accuracy. Thus not only is a screening mass absent but a further implication of this result is that the classical theory is ill defined in the infrared. Furthermore, if we identify $L \sim 2 A^{1/3}$ fm, the lattice weak coupling condition is satisfied only for very small A . This limit appears to contradict the weak coupling limit in the full theory, which is expected to be valid only for very large nuclei. In sum, our work suggests that the classical theory in Ref. [5] is seriously flawed.

Recently, McLerran and collaborators [16] have proposed that the original clas-

sical theory is flawed because the authors in Ref. [5] failed to properly solve the Yang–Mills equations for the transverse components A^i of the classical field. These are determined through the equation

$$\nabla_i \partial^+ A^i + A_i \times \partial^+ A^i = gJ^+. \quad (36)$$

Ref. [5] argued for a solution of the form in Eq. 4: $A^i = \alpha^i \theta(x^-)$. If one then ignores the commutator terms, because it involves fields at the same x^- , one then obtains $\nabla \cdot \alpha = g\rho$, which gives us the stochastic equation we solved for U using $\alpha_i = U \nabla_i U^\dagger / (-ig)$. The authors of Ref. [16] argue that the cross product term above cannot be dropped because of its peculiar singular structure. They argue that the source term must be regularized so that instead of being a δ -function in x^- , the charge density ρ depends on the spacetime rapidity $y = -\log(x^-)$. The above equation is then re-written as

$$D_i \frac{dA^i}{dy} = g\rho(y, x_t), \quad (37)$$

where D_i is the covariant derivative. It is claimed in Ref. [16] that this equation can be solved exactly and the correlation functions computed analytically. The distribution functions have the Weizsäcker–Williams form for large transverse momenta, $dN/d^2k_t \sim 1/k_t^2$. For small transverse momenta, it has the logarithmic form $dN/d^2k_t \sim \log(k_t^2/\chi(y, k_t^2))$. Here $\chi(y, k_t^2) = \int_{max_{y,y'}}^{y_0} dy' \mu^2(y', Q^2)$. We refer the reader to Ref. [16] for the details of their calculation.

Acknowledgments

We would both like to thank Jamal Jalilian–Marian, Alex Kovner, Larry McLerran and Heribert Weigert for useful discussions. We would also like to thank the Theoretical Physics Institute, University of Minnesota and the Institute for Nuclear Theory, University of Washington for hospitality to R.V and R.V.G respectively. Research supported by the U.S. Department of Energy under grants No. DOE Nuclear DE–FG06–90ER–40561 and DE–FG02–87ER40328.

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Figure Captions

Figure 1: (a) The correlation functions $\Gamma^j(x) \equiv \langle \alpha_j^r(x) \alpha_j^r(0) \rangle$ as a function of x/L for $j = 1$ (circles) and $j = 2$ (crosses) for $g^2\mu L = L = 0.5$ and $L = 41a$. The continuous line is the weak coupling result of Eq. 26. (b) Same as Fig. 1a but for $L = 21a$ and $L = 41a$ and for Γ^2 only.

Figure 2: The α_1^r -correlation function as a function of x/L for $g^2\mu L = 0.5$ and $N = 21$ and 41 lattices.

Figure 3: The α_2^r -correlation function as a function of x/L for $g^2\mu L = 0.5$ and 1, 2, 3, 4, 5, 6, and 7. The lattice sizes can be found in the text.

Figure 4: The α_1^r -correlation function in the units of $g^6\mu^4L^2$ as a function of x/L for $g^2\mu L = 0.5$ and 1, 2, 3, 4, and 5. The lattice sizes can be found in the text.

Figure 5: The α_1^r -correlation function at $x = 0$ as a function of $g^2\mu L$ without (crosses) and with (squares) a division by $(g^2\mu L)^2$.

Figure 6: The histograms of the fraction of total equations solved at an accuracy $R * a^4$ as a function of R . These are plotted for the following values of $g^2\mu L$: 6, 7, 10 and 20. For details, see the text.











