Quantising Gravity Using Physical States Of A Superstring

B. B. Deo,[1, *] P. K. Jena,[2, †] and L. Maharana[1, ‡]

[1]*Department of Physics , Utkal University, Bhubaneswar-751004, India.*
[2] *Utkal University,Bhubaneswar-751004, India.*

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A symmetric zero mass tensor of rank two is constructed using the superstring modes of excitation which satisfies the physical state constraints of a superstring. These states have one to one correspondence with the quantised field operators and are shown to be the absorption and emission quanta of the Minkowski space Lorentz tensor, using the quantum field theory method of quantisation. The principle of equivalence makes the tensor identical to the metric tensor at any arbitrary space-time point. The propagator for the quantised field is deduced. The gravitational interaction is switched on by going over from ordinary derivatives to co-derivatives. The Riemann-Christoffel affine connections are calculated and the weak field Ricci tensor $R^0_{\mu\nu}$ is shown to vanish. The interaction part $R^{int}_{\mu\nu}$ is found out and the exact $R_{\mu\nu}$ of the theory of gravity is expressed in terms of the quantised metric. The quantum mechanical self energy of the gravitational field, in vacuum, is shown to vanish. By the use of a projection operator, it is shown that the gravitons are the quanta of the general relativity field which gives the Einstein equation $G_{\mu\nu} = 0$. It is suggested that quantum gravity may be renormalisable by the use of the massless ground state of this superstring theory for general relativity and a tachyonic vacuum creat and annihilate quanta of quantised gravitational field.

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1. INTRODUCTION

Attempts to quantize and to have a renormalised Einstein's theory of gravitation have been a disappointing failure. The incompatibility of relativity with quantum mechanics was first pointed out by Heisenberg who commented that the usual renormalisation programme is ruined by the dimensional gravitational coupling constant. However, it is quite possible that the theory of gravity is finite to every order of its coupling. In order to achieve this, supergravity was pursued vigorously, but no success is yet in sight. On the other hand, soon after a theory of dual resonance model explaining all the postulates of S-matrix by a host of workers was put in place, Nambu[1] and Goto [2] worked out a classical relativistic string, which was raised to quantum level by Goldstone [3], Goddard, Rebbi and Thorn[4] and also by Mandelstam[5]. The very bold suggestion of Scherk and Schwarz [6, 7] that the string theory carries quantum information for all the four interactions including gravity, did not make much head way till 1984. Green and Schwarz [8] formulated the superstring theory in ten dimensions which is still believed to be finite in all orders of perturbation theory. However, only later, it was found that the heterotic string theory of Gross, Harvey, Martinec and Rohm[9] is the best candidate to explain gravitational interactions.

Casher, Englert, Nicolai and Taormina[10] have made a much publicised proof that the 26-dimensional bosonic string contains closed 10-dimensional superstrings, the two $N = 1$ heterotic strings and two $N = 2$ superstrings. This group have followed this up by making further incisive attempts to find a mechanism which generates space-time fermions out of bosons. To make contact with real physical world, one has to make the usual unsuccessful and nonunique compactification from ten to four dimensions. Kaku [11] and Green, Schwarz and Witten [12] in their books have rightly and clearly spelt out that 'No one really knows how to break a 10-dimensional theory down to four'. In an earlier work [13] on supergravity, using the 4-d superstring theory given below, one of the authors(BBD) deduced the propagator for the graviton. It was proved that the vanishing of the Ricci tensor using the vielbeins in the tangent space formalism to go from flat to curved space time without using Riemann-Christoffel affine connections (called affines in short) which are essential for a Quantum Theory of Gravity. Here we demonstrate this essential aspect. The affines come out in a simpler form and will simplify calculations of other quantum gravity problems.

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*Electronic address: bdeo@iopb.res.in
†Electronic address: prasantajena@yahoo.com
‡Electronic address: lmaharan@iopb.res.in,lmaharan@yahoo.co.in

In the earlier paper [13], one of us (BBD) also noted the important works of Feynman who used gravity as a spin-2 field coupling to its energy momentum tensor. Mandelstam [5],Deser [14] and DeWitt [15] have done extensive work in deriving Feynman rules. The trouble is that gravity has too many constraints, inherent in the formal quantum gravity field theory. In the present approach, based on superstring, most of the constraints have been taken care of in constructing the superstring physical states. It is also important to realise that Riemann, Ricci, Weyl and all other tensors of any relevance with quantum gravity are in terms of general relativity metric tensor. This tensor is not traceless. So the quantisation of the metric field strength will have both the spin-2 graviton and spin-0 dilaton. Eventhough, one of us(BBD) worked with pure graviton in supergravity [13], we had to enlarge the scope, and approach the problem of quantum gravity by trying to quantise the metric tensor field. In all perturbative approaches, it is the general relativity metric tensor $g_{\mu\nu}(x)$ which is expanded around the flat metric $\eta_{\mu\nu}$ as indicated in our concluding section.

In our opinion, the best tool available to achieve a breakthrough is the superstring theory which, like gravity, needs to be formulated in the physical world of four dimensions. The simplest and the best way to descend directly from 26-dimensional bosonic string to 4-dimensional superstring is by using the Mandelstam equivalence between fermions and bosons in an anomaly free string theory. The bosons are four in nature. The fermions belong to SO(3,1) bosonic representation. They are divided into two groups. One group has 24 spinors placed right handedly in six ways and the other 20 placed left handedly in five ways.Thus the total number of the bosons is 4 and the fermions are 4x6 and 4x5 have opposite handedness. These will be relevant to gravity as much as the one of ref. [10]. It is worth mentioning that our present construction of the superstring bears resemblance to the attempts by Gates et al[16].

The supersymmetric action, with $SO(6) \otimes SO(5)$ world sheet symmetry, turns out to be

$$S_{ss} = -\frac{1}{2\pi} \int d^2\sigma \left[\partial^\alpha X^\mu(\sigma,\tau)\ \partial_\alpha X_\mu(\sigma,\tau) - i\sum_{j=1}^{6} \bar{\psi}^{\mu,j} \rho^\alpha \partial_\alpha \psi_{\mu,j} + i\sum_{k=7}^{11} \bar{\phi}^{\mu,k} \rho^\alpha \partial_\alpha \phi_{\mu,k} \right], \tag{1.1}$$

with

$$\partial_\alpha = (\partial_\sigma,\ \partial_\tau), \quad \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \bar{\phi} = \phi^\dagger \rho^0. \tag{1.2}$$

For the sake of completeness and to do justice to the subject of gravity and strings, we briefly outline some details which have already been published elsewhere.

The arrays $(e^j,\ \ e^k)$ are the rows of ten zeros with only '1' in the j^{th} place or '-1' in the k^{th} place. $e^j e_j = 6$ and $e^k e_k = 5$. The invariation of the action is under the SUSY transformations with constraints to lead to spatial translations on two successive applications,

$$\delta X^\mu = \bar{\epsilon}\left(e^j \psi^\mu_j - e^k \phi^\mu_k\right) = \bar{\epsilon}\Psi^\mu, \tag{1.3}$$

$$\delta\psi^{\mu,j} = -i\epsilon\ e^j \rho^\alpha \partial_\alpha X^\mu, \quad \psi^\mu_j = e_j \Psi^\mu, \tag{1.4}$$

and

$$\delta\phi^{\mu,k} = -i\epsilon\ e^k \rho^\alpha \partial_\alpha X^\mu, \quad \phi^\mu_k = e_k \Psi^\mu. \tag{1.5}$$

Here ϵ is a constant anticommuting spinor and

$$\Psi^\mu = e^j \psi^\mu_j - e^k \phi^\mu_k, \tag{1.6}$$

is the superpartner of X^μ. This emits quanta of ψ^μ_j or ϕ^μ_k while in the site j or k respectively. The string fields are quantised for the coordinates

$$X^\mu(\sigma,\tau) = x^\mu + p^\mu\tau + i\sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n\ e^{-in\tau}\ \cos(n\sigma). \tag{1.7}$$

In terms of complex coordinates $z = \sigma + i\tau$ and $\bar{z} = \sigma - i\tau$, we have,

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha^\mu_0 \ln|z| + i\sum_{m\neq 0} \frac{1}{m}\alpha^\mu_m\ z^{-m}. \tag{1.8}$$

Further,

$$\psi^{\mu,j}_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in Z+\frac{1}{2}} b^{\mu,j}_r\ e^{-ir(\sigma\pm\tau)}, \qquad \phi^{\mu,k}_\pm(\sigma,\tau) = \frac{1}{\sqrt{2}} \sum_{r\in Z+\frac{1}{2}} b'^{\mu,k}_r\ e^{-ir(\sigma\pm\tau)} \quad \text{for NS sector}, \tag{1.9}$$

and

$$\psi_{\pm}^{\mu,j}(\sigma,\tau)=\frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty} d_m^{\mu,j}\ e^{-im(\sigma\pm\tau)}, \qquad \phi_{\pm}^{\mu,k}(\sigma,\tau)=\frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty} d_m^{'\mu,k}\ e^{-im(\sigma\pm\tau)} \text{ for R sector.} \tag{1.10}$$

The bosonic quanta obey the commutation relation and the fermionic quanta obey the anticommutation relation with the only major difference for Majorana fermions $b_{-r}=b_r^{\dagger}$ but $b'_{-r}=-b_r^{'\dagger}$ and $d_{-r}=d_r^{\dagger}$ but $d'_{-r}=-d_r^{'\dagger}$. So the number level density N_B (for bosons) and N_F (for fermions) are

$$N_B=\sum_{\mu}\langle\phi|\alpha_{-1}^{\mu}\alpha_{1\mu}|\phi\rangle=4, \tag{1.11}$$

and

$$N_F=\sum_{\mu}\sum_{i=1}^{6}\langle\phi|b_{-i}^{\mu}b_{i\mu}|\phi\rangle+\sum_{\mu}\sum_{j=1}^{5}\langle\phi|b_{-i}^{'\mu}b_{i\mu}^{'}|\phi\rangle=\sum_{\mu}\sum_{i=1}^{6}\langle\phi|b_i^{\dagger\mu}b_{i\mu}|\phi\rangle-\sum_{\mu}\sum_{j=1}^{5}\langle\phi|b_i^{'\dagger\mu}b_{i\mu}^{'}|\phi\rangle=24-20=4. \tag{1.12}$$

These number level densities are as required by supersymmetry.

It may be apprehended that the assembly of forty four fermions may give rise to a spectrum in space time, will be highly pathological. However, of the 44, when 24 Majorana fermions are excited in one way, the other 20 are excited in the opposite way.Therefore only four Majorana fermions are effective. It will show less complexity in pathology than an assembly of ten fermions in 10-D superstrings.

In the light cone basis, the energy momentum tensors T_{++}, T_{--} and the currents J_+, J_- are given by

$$T_{++} = \partial_+X^{\mu}\partial_+X_{\mu}+\frac{i}{2}\bar{\psi}_+^{\mu,j}\partial_+\psi_{+\mu,j}-\frac{i}{2}\bar{\phi}_+^{\mu,k}\partial_+\phi_{+\mu,k}, \tag{1.13}$$

$$T_{--} = \partial_-X^{\mu}\partial_-X_{\mu}+\frac{i}{2}\bar{\psi}_-^{\mu,j}\partial_-\psi_{-\mu,j}-\frac{i}{2}\bar{\phi}_-^{\mu,k}\partial_-\phi_{-\mu,k}, \tag{1.14}$$

$$J_+ = \partial_+X_{\mu}\Psi_+^{\mu}, \tag{1.15}$$

$$\text{and}\quad J_- = \partial_-X_{\mu}\Psi_-^{\mu} \tag{1.16}$$

There are also energy momentum tensors associated with conformal T_{++}^{FP} and superconformal ghosts T_{++}^{SC} with their generators. Their corresponding quanta are

$$T_{++}^{FP} = \frac{1}{2}c^+\partial_+b_{++}+\partial_+c^+b_{++}, \qquad L_m^{FP}=\sum_{m}^{FP}(m-n)b_{m+n}c_{-n} \tag{1.17}$$

$$T_{++}^{SC} = -\frac{1}{4}\gamma\partial_+\beta-\frac{3}{4}\beta\partial_+\gamma, \qquad \text{and} \qquad L_m^{gh,sc}=\sum_{m}^{SC}(\frac{1}{2}m+n):\beta_{m-n}\gamma_n: . \tag{1.18}$$

Here (b,c) obey the anticommutation rules and (β,γ) obey the commutation rules.

The superconformal ghost action follows from the local fermionic symmetry of the superconformal invariance of the action as used by Brink, De Vecchia, Howe, Deser and Zumino [17],

$$\delta\chi_{\alpha}=i\rho_{\alpha}\eta, \quad \text{and} \quad \delta e_{\alpha}^{a}=\delta\Psi^{\mu}=\delta X^{\mu}=0, \tag{1.19}$$

where η is an arbitrary Majorana spinor and e_{α}^{a} is the usual 'Zweibein', so that the gravitino can be gauged away using

$$\delta\chi_{\alpha}=i\rho_{\alpha}\eta+\nabla_{\alpha}\epsilon.$$

Using the variation of $\chi, X_{\pm}$ and following reference [17], one finds (1.18). The central charge 'c' is given by c $=1-3k^2$. The anomaly free total super Virasoro generator can be written in two equivalent ways, each having zero central charge

$$L_m^{total}=L_m^{FP}+\frac{1}{\pi}\int_{-\pi}^{\pi}e^{im\sigma}T_{++}d\sigma=L_m^{FP}+L_m^{SC}+\frac{1}{\pi}\int_{-\pi}^{\pi}e^{im\sigma}(T_{++}-k\partial_+T_+^F)d\sigma, \tag{1.20}$$

where

$$T_+^F=\frac{i}{2}\left(\psi_+^{\mu,j}\psi_{+\mu,j}-\phi_+^{\mu,j}\phi_{+\mu,j}\right). \tag{1.21}$$

The quantity k is related to the conformal dimension $J = \frac{1+k}{2}$ and equals $\frac{1}{\sqrt{6}}$ here. The normal ordering constant $a = -1$ in either case.

The super Virasoro generators L_m of energy momenta and the currents (G, F) are [13]

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_{m+n}: + \frac{1}{2}\sum_{r\in z+\frac{1}{2}} (r+\frac{1}{2}m) : (b_{-r}\cdot b_{m+r} - b'_{-r}\cdot b'_{m+r}) : \quad \text{NS}, \tag{1.22}$$

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_{m+n}: + \frac{1}{2}\sum_{n=-\infty}^{\infty} (n+\frac{1}{2}m) : (d_{-n}\cdot d_{m+n} - d'_{-n}\cdot d'_{m+n}) : \quad \text{R}, \tag{1.23}$$

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot(e^j b^\mu_{r+n,j} - e^k b'^\mu_{r+n,k}), \tag{1.24}$$

and

$$F_m = \sum_{n=-\infty}^{\infty} \alpha_{-n}\cdot(e^j d^\mu_{m+n,j} - e^k d'^\mu_{m+n,k}). \tag{1.25}$$

These satisfy the super Virasoro algebra with central charge $c = 26$. The second version of equation (1.20) can be easily written down[11, 12].

The physical states are defined through the relations

$$(L_0 - 1)|\phi\rangle = 0, \quad L_m|\phi\rangle = 0, \quad G_r|\phi\rangle = 0, \qquad \text{for } (r,m) > 0, \text{ NS (bosonic)} \tag{1.26}$$

$$(L_0 - 1)|\psi\rangle_\alpha = (F_0^2 - 1)|\psi\rangle_\alpha, \quad L_m|\psi\rangle_\alpha = F_m|\psi\rangle_\alpha = 0,, \quad \text{for } m > 0, \text{ R (fermionic)} \tag{1.27}$$

and the mass spectrum is given as

$$\alpha' M^2 = -1, \ -\frac{1}{2}, \ 0, \ \frac{1}{2}, \ 1, \ \frac{3}{2}, ... \qquad \text{NS} \tag{1.28}$$

and

$$\alpha' M^2 = -1, \ 0, \ 1, \ 2, \ 3, ... \qquad \text{R}. \tag{1.29}$$

The G.S.O. project out [18] the half integral mass spectrum values, tachyonic or otherwise. The tachyonic bosonic energy of the NS sector $\langle 0|(L_0-1)^{-1}|0\rangle_S$ is cancelled by $\langle 0|(F_0-1)^{-1}(F_0+1)^{-1}|0\rangle_R$ of the Ramond sector. This is typical of supersymmetry and is well known. Just to remind, the SUSY charge is [19],

$$Q = \frac{1}{\pi}\int d\sigma \rho^o \rho^{+\alpha}\partial_\alpha X^\mu \Psi_\mu, \tag{1.30}$$

and since, the Hamiltonian is related as

$$\sum_\alpha \{Q^\dagger_\alpha, \ Q_\alpha\} = 2H, \quad \text{and} \quad \sum_\alpha |Q_\alpha|^2 = 2\langle\phi_0|H|\phi_0\rangle, \tag{1.31}$$

the ground state is massless and the physical Fock space is tachyonless. The references [19, 20] give a detailed derivation of the modular invariance and vanishing the partition function of the string. Thus with no anomaly in four dimensional superstring theory, we can go over to construct zero mass Fock space physical state functions. We shall demonstrate this in the next sections and attempt to quantise gravity using ideas based on superstring theory and its exact equivalence with quantum operators.

2. STRING STATES $\longleftrightarrow$ QUANTUM OPERATORS

In the bosonic superstring, the Neveu Schwarz bosonic sector contains the bosonic tachyon. This tachyon state is very useful to construct the ground state of zero mass. The vacuum state $|0,0\rangle$ of the string is the functional integral of the string theory over a semi-infinite strip. This could be conformally mapped to the unit circle. These ideas become clearer, if we consider a closed string where we have a second set of Virasoro algebraic equations. The following is a recipe, given by Polchinski [21], prescribing the link between superstring states and operators, which is very important for quantising gravity.

Radial quantisation has a natural isomorphism between the string state space of conformal field theory (CFT) in a periodic spatial dimension, and the space of local operators [22]. Let there be a local isolated operator $\mathcal{A}$ at the origin and no more inside the unit circle denoted by $|z| = 1$, but with no other specification outside the circle. Let us open a slit in the circle and consider the path integral on the unit circle, giving an inner product $\langle\psi_{out}|\psi_{in}\rangle$. Here, ψ_{in} is the incoming state given by the path integral $|z| < 1$ and ψ_{out} is the outgoing one at $|z| > 1$. Explicitly, a field ϕ is split into integrals outside, inside and on the circle. The last one will be called ϕ_B. The outside integral is $\psi_{out}(\phi_B)$ and the inside integral is $\psi_{in}(\phi_B)$, and the remainder is $\int[d\phi_B]\ \psi_{out}(\phi_B)\psi_{in}(\phi_B)$. The incoming state depends on the operator $\mathcal{A}$ and hence is denoted by $|\psi_A\rangle$. This is the needed important mapping from operators to states. Summarising, 'the mapping from operators to states is given by the path integral on the unit disk'. The inverse is also true.

If Q is any conserved charge, $Q|\psi_A\rangle$ is the operator equivalent of $Q \cdot \mathcal{A}$. In particular, if $\mathcal{A}$ is the unit operator $\mathbb{1}$, and $Q = \alpha_m = \oint \frac{dz}{2\pi} z^m \partial X$, for $m \geq 0$, so that ∂X is analytic and the integral vanishes for $m \geq 0$, we get $\alpha_m|\psi_{\mathbb{1}}\rangle = 0, \quad m \geq 0$. This establishes the exact correspondence of the unit operator to the string vacuum,

$$\mathbb{1} \leftrightarrow |0, 0\rangle. \tag{2.1}$$

Similarly, one finds the operator equivalence,

$$: e^{ik.X(z)} : \ \leftrightarrow |0, k\rangle. \tag{2.2}$$

In the above, $X(z)$ is given by equation (1.8). $: e^{ikX}:$ implies normal ordering of the operators contained in it. The first number in the bra refers to m, and second to the eigenvalue of α^μ_o i.e. $\alpha^\mu_o|0, k\rangle = k^\mu|o, k\rangle$. So for the tachyon, $|0, k\rangle \leftrightarrow e^{ik.x}$, because on the circumference of the circle $|z|$=1, the tachyonic vacuum can not annihilate[11]. This equivalence is utilised to convert the string states tensor of rank two metric field of graviton and dilaton to quantum operators and vice versa. The CFT unitarity gives

$$\langle 0, k|0, k'\rangle = 2\pi\ \delta(k - k'). \tag{2.3}$$

The three spatial components would lead to

$$\langle 0, \vec{k}|0, \vec{k}'\rangle = (2\pi)^3\ \delta^{(3)}(k - k').$$

This is generalised to normalisation of massless states with $k_0 = |\vec{k}|$ and we use one like the massive vector meson renormalisation,

$$\langle 0, \vec{k}|0, \vec{k}'\rangle = (2\pi)^3\ (2k_0)\ \delta^{(3)}(k - k'). \tag{2.4}$$

3. QUANTISATION OF THE GRAVITATIONAL FIELD METRIC USING SUPERSTRING STATES

To construct the second rank Lorentz tensor of general relativity, we shall use the quanta b^μ_i of the NS bosonic sector which comes from the SO(3,1) ψ^μ_j's, belonging to the SO(6) group of the action (1.1). In the NS sector, the hidden tachyonic vacuum $|\phi\rangle = |0, k\rangle$ of momentum k is such that $(L_0 - 1)|0, k\rangle = 0$. This satisfies the superstring Virasoro physical constraint of equation (1.16). The ghost free physical Fock space states containing matter and radiation are built up by operating creation operators on this state. For quantum gravity, we need massless quanta of spin 2 and a metric tensor. The later when quantised, would have both massless quanta of spin 2 and spin 0. Departing from earlier approaches to quantum gravity, we first proceed to construct a metric tensor quantum operator. This should be a Lorentz tensor of rank 2. The Lorentz tensor, which is symmetric but not traceless, is simply given by the quantum operator

$$g_{\mu\nu}(k) = \ : \sum_{ij} c_{ij} b^{i\dagger}_\mu b^{j\dagger}_\nu e^{ik.X} : \ = \sum_{ij} c_{ij} b^{i\dagger}_\mu b^{j\dagger}_\nu : e^{ik.X} : \ \longleftrightarrow a^{\mu\nu}|0, k\rangle = a^{\mu\nu}\ e^{ik.x}; \ \ c_{ij} = -c_{ji}, \tag{3.1}$$

where the operator $a_{\mu\nu}$ is,

$$a_{\mu\nu} = \sum_{ij} c_{ij} b^{i\dagger}_\mu b^{j\dagger}_\nu, \tag{3.2}$$

and commutes with α_μ's, we have used the operator $\longleftrightarrow$ state equivalence of equation (2.2). This operator creates a pair of fermions. Since

$$[L_0, b_j^{\dagger\nu}] = \frac{1}{2} b_j^{\dagger\nu} \tag{3.3}$$

and

$$[G_{\frac{1}{2}}, b^{j\nu\dagger}] = \alpha_0^\nu \; e^j = k^\nu e^j, \tag{3.4}$$

we have,

$$L_0 \; g_{\mu\nu}(k) = L_0 \sum_{ij} c_{ij} b_\mu^{i\dagger} b_\nu^{j\dagger} : e^{ik.X} :\longleftrightarrow \sum_{ij} c_{ij} b_\mu^{i\dagger} b_\nu^{j\dagger} (L_0 - 1)|0, k\rangle = 0, \tag{3.5}$$

since $(L_0 - 1)|0, k\rangle = 0$ as stated. So the quantum operator $g_{\mu\nu}(k)$ is the ground state which is a pair of vectorial quanta created, would lead to a massless state. In order to satisfy the G.S.O. condition, we examine equation (3.1)

$$\begin{aligned} \left(1 + (-1)^F\right) G_{\frac{1}{2}} g_{\mu\nu}(k) &= \left(1 + (-1)^F\right) \left[G_{\frac{1}{2}}, g_{\mu\nu}(k)\right] & (3.6) \\ &= \left(1 + (-1)^F\right) \frac{1}{\sqrt{2}} \sum c_{ij} \left(e^i k_\mu b_\nu^j - e^j k_\nu b_\mu^i\right) |0, k\rangle = 0 & (3.7) \\ &= k_\mu \sum c_{ij} \; e^i b_\nu^{j\dagger} |0, k\rangle. & (3.8) \end{aligned}$$

Since $G_{\frac{1}{2}}$ annihilates one fermion of the pair, the lone left out fermion is G.S.O. projected out. So $g_{\mu\nu}(k)$ is a ground state with zero energy.

Operator $a_{\mu\nu}$ defined in equation(3.2) is seen to satisfy the relations,

$$\left[a_{\mu\nu}, \; a_{\lambda\sigma}^\dagger\right] = f_{\mu\nu,\lambda\sigma} |c|^2, \quad [a_{\mu\nu}, \; a_{\lambda\sigma}] = \left[a_{\mu\nu}^\dagger, \; a_{\lambda\sigma}^\dagger\right] = 0, \tag{3.9}$$

where

$$f_{\mu\nu,\lambda\sigma} = \eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}. \tag{3.10}$$

Here the flat space metric $\eta_{\mu\nu}$ is, as usual,

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.11}$$

For the desired normalisation for $|c|^2$, in equation(3.9) will be suitably chosen.

In the Gupta-Bleuler method of quantisation [23] of only the ground state symmetric tensor $g_{\mu\nu}(k)$. The state satisfies the equation

$$L_0 \mid g_{\mu\nu}(k) \;\rangle = L_0 \; a_{\mu\nu}^\dagger : e^{ikX} : \;\; \leftrightarrow L_0 \; a_{\mu\nu}^\dagger |0, k\rangle = 0 \leftrightarrow L_0 \; a_{\mu\nu}^\dagger \; e^{ik.x}. \tag{3.12}$$

Likewise

$$: e^{-ikX} a_{\mu\nu}^\dagger : \quad \longleftrightarrow \quad \langle 0, k| a_{\mu\nu}.$$

Consider the expansion of $g_{\mu\nu}(x)$, given by

$$g_{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \left(a_{\mu\nu}^\dagger \; e^{ikx} + a_{\mu\nu} \; e^{-ikx}\right). \tag{3.13}$$

which satisfies the zero mass Klein-Gordon equation $\Box g_{\mu\nu}(x) = 0$. The collection of states (3.12, 3.13), due to the isomorphism with operators, is the field quantisable as, for instance, in Gupta-Bleuler formalism [23]. When $g_{\mu\nu}(x)$ is eventually quantised, the associated creation and annihilation operators correspond to massless quanta. It may be pointed out that this is same as the Gupta-Bleuler formalism which is covariant and hence used in most string quantisations to maintain manifest Lorentz covariance. Thus the string Fock states $g_{\mu\nu}(k)$ and the hermitian conjugates will turn out to be related to the quantum creation and annihilation quanta of spin-2 and spin-0 for the field tensor. In the instance case, $a^\dagger_{\mu\nu}(k)$ and $a_{\mu\nu}(k)$ are the creation and annhilation quanta of these objects.

Before proceeding further, we must ensure that $g_{\mu\nu}(k)$ is a special type of symmetric Lorentz tensor [25] which, under a proper Lorentz transformation from x to x', does not behave as the usual second rank tensor,

$$g_{\mu\nu} \;\rightarrow\; \Lambda^\rho_\mu \Lambda^\sigma_\nu g_{\rho\sigma} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu . \tag{3.14}$$

Since the 2nd and 3rd terms are vectorial, they ruin the Lorentz invariance and general covariance in quantum gravity, as has been specifically pointed out by Weinberg [25] and discussed in [13]. The arguement for spin-2 case as considered in [13] is emphasised again , not only as to its importance, but also to show the absence of spin-1 vectors which complicate calculations. Only spin-2 and spin-0 objects present in the theory have propagators and vertices. In string theory $\alpha^\mu_0 = k^\mu$. So, writing

$$a_{\mu\nu} \mid 0, k\rangle \;\rightarrow\; \Lambda^\rho_\mu \Lambda^\sigma_\nu a^\dagger_{\rho\sigma} \mid 0, k\rangle + \; O_{\mu\nu}|0, k\rangle, \tag{3.15}$$

where

$$O_{\mu\nu}|0, k\rangle = (k_\mu \epsilon_\nu + k_\nu \epsilon_\mu) \mid 0, k\rangle, \tag{3.16}$$

are additional tachyons due to supersymmetry. Since $L_0 = F_0^2$, we have

$$O_{\mu\nu} \mid 0, k\rangle = L_0 O_{\mu\nu} \mid 0, k\rangle = F_0^2 O_{\mu\nu} \mid 0, k\rangle, \tag{3.17}$$

in Ramond sector. So,

$$F_0 O_{\mu\nu} \; |0, k\rangle_\alpha = \pm O_{\mu\nu} \; |0, k\rangle_\alpha . \tag{3.18}$$

In general, one can construct spinorial states $\mid 0\rangle_\alpha$ such that

$$F_0 \mid 0\rangle_\alpha = \; \mid 0\rangle_\alpha \; , \quad {}_\alpha\langle 0 \mid F_0 = -{}_\alpha\langle 0 \mid \quad \text{and} \quad \sum_\alpha \; \mid 0\rangle_\alpha \;\; {}_\alpha\langle 0 \mid \; = 1. \tag{3.19}$$

Again,

$$\begin{aligned} O_{\mu\nu}|0, k\rangle \;&=\; L_0 O_{\mu\nu} \; |0, k\rangle = \sum_\alpha F_0 \mid 0\rangle_\alpha \;\; {}_\alpha\langle 0 \mid F_0 O_{\mu\nu} \mid 0, k\rangle \\ &=\; -\sum \mid 0\rangle_\alpha \;\; {}_\alpha\langle 0 \mid O_{\mu\nu} \mid 0, k\rangle = -O_{\mu\nu}(k) = 0. \end{aligned} \tag{3.20}$$

This is due to the tadpole cancellation mechanism noticed also by Casher et al [10]. Thus the Lorentz transformation is a ‘proper’ one and ensures that the tensor, under Lorentz transformation, remains a symmetric tensor without vectorial components as would be expected from equation(3.15). Rewriting equation (3.13)

$$g_{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{(2k_0)}} \left(a^\dagger_{\mu\nu} \; e^{ikx} + a_{\mu\nu} \; e^{-ikx} \right) \tag{3.21}$$

With operator relations (3.9), we have quantised the tensor operator in flat space time. By using the same flat metric $\eta^{\mu\nu}$, we get

$$g^{\mu\nu}(x) = \int \frac{d^3k}{(2\pi)^3 \sqrt{(2k_0)}} \left(a^{\mu\nu\dagger} \; e^{ikx} + a^{\mu\nu} \; e^{-ikx} \right) \tag{3.22}$$

Let us consider the operator product $:g^{\mu\nu}(x) g_{\nu\lambda}(x):$. The fermions separate out. For the two exponentials contain the factors $|z|$=1 in equation (1.8),

$$\langle \; \sum_{n=1}^\infty \frac{1}{n} \; \alpha_\mu^{-n} \sum_{n=1}^\infty \frac{1}{n} \; \alpha_\nu^n \; \rangle = \eta_{\mu\nu} \; \sum_{n=1}^\infty \frac{1}{n} = \eta_{\mu\nu} \; \zeta(1)$$

where $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ is the zeta function. This will be absorbed in the normalisation factors. It is easier to use string theory notation so that, as in the above,

$$g^{\mu\nu}(x) g_{\nu\lambda}(x) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4k_0 k_0'}} \, |c|^2 \, \langle k, 0|0, k'\rangle \; e^{-k.k'\zeta(1)} \; f^{\mu\nu}{}_{\nu\lambda} = |c|^2 \int \frac{d^3k}{(2\pi)^3} \; e^{-k^2\zeta(1)} \; f^{\mu\nu}{}_{\nu\lambda} \, \langle k, 0|0, k\rangle = \delta^\mu{}_\lambda, \tag{3.23}$$

where the Kronecker delta δ^μ_λ is a mixed tensor. The coefficient $|c|^2$ must be adjusted to have

$$5|c|^2 \int \frac{d^3k}{(2\pi)^3}\; e^{-k^2\zeta(1)}\;\; \langle k,0|0,k\rangle = 1, \tag{3.24}$$

since

$$f^{\mu\nu}{}_{\nu\lambda} = \eta^\mu{}_\nu \eta^\nu{}_\lambda + \eta^\mu{}_\lambda \eta^\nu{}_\nu. \tag{3.25}$$

If $g_{\mu\nu}$ is a covariant tensor, then its inverse $g^{\mu\nu}$, is the contravariant tensor. Other than the scalars and zero, the Kronecker tensor δ^μ_ν is the only tensor where components are same in any coordinate system. Furthermore, the flat space metric $\eta^{\mu\nu}$ also satisfy $\eta^{\mu\nu}\eta_{\nu\lambda} = \delta^\mu_\lambda$. Thus we have found a space dependant metric like the flat space one. This meets the first criterion of Weinberg [25] for the application of the principle of equivalence. In the next section, we shall show that there exists a mapping from any curved space time point x' to the point x where affine is zero so that the equations with derivatives or other equations remain the same for all space time, flat or curved.

The relation (3.23) is 'proper' Lorentz invariant as per earlier extensive derivation of tadpole cancellation. So, by the Principle of Equivalence, $g_{\mu\nu}$ is the true metric tensor operator for all space time points, flat or curved, of the gravitational field. A critical discussion on this matter has been given by Padmanabhan [24].

The commutator of the fields reads

$$\left[g^{\mu\nu}(x), g^{\lambda\sigma}(y)\right] = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k_0}\left(\; e^{ik(x-y)} \;-\; e^{-ik(x-y)}\right) f^{\mu\nu,\lambda\sigma}. \tag{3.26}$$

The propagator is

$$\Delta^{\mu\nu,\lambda\sigma}(x) = \langle 0|T\left(g^{\mu\nu}(x),\; g^{\lambda\sigma}(y)\right)|0\rangle = \frac{1}{(2\pi)^4}\int d^4k\; \Delta_F^{\mu\nu,\lambda\sigma}(k)\; e^{ik(x-y)}, \tag{3.27}$$

where

$$\Delta_F^{\mu\nu,\lambda\sigma}(k) = \frac{1}{2}\; f^{\mu\nu,\lambda\sigma}\; \frac{1}{k^2 - i\epsilon}. \tag{3.28}$$

We note that

$$\begin{aligned} f^{\mu\nu,\lambda\sigma} &= \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} \\ &= \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\sigma} + \eta^{\mu\nu}\eta^{\lambda\sigma} \\ &= f^{(2)\mu\nu,\lambda\sigma} + f^{(0)\mu\nu,\lambda\sigma}, \end{aligned} \tag{3.29}$$

where

$$f^{(2)\mu\nu,\lambda\sigma} = \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\lambda\sigma} \qquad \text{and} \qquad f^{(0)\mu\nu,\lambda\sigma} = \eta^{\mu\nu}\eta^{\lambda\sigma}.$$

The first term of equation (3.29) is the coefficient of the graviton propagator $\Delta_F^{graviton}$ and the second is the coefficient of dilaton propagator $\Delta_F^{dilaton}$ with appropriate factors. The Feynman propagator for the graviton $h_{\mu\nu}(x)$ and the dilaton $D(x)$ are

$$\langle 0|N(h_{\mu\nu}(x)h_{\lambda\sigma}(y))|0\rangle = \Delta^{graviton}_{F\mu\nu,\lambda\sigma}(x-y) = \int \frac{d^4k}{(2\pi)^4}\; \Delta^{graviton}_{F\mu\nu,\lambda\sigma}(k)\; e^{ik(x-y)}, \tag{3.30}$$

and

$$\langle 0|N(D(x)D(y))|0\rangle = \Delta_F^{dilaton}(x-y) = \int \frac{d^4k}{(2\pi)^4}\; \Delta_F^{dilaton}(k)\; e^{ik(x-y)}, \tag{3.31}$$

where

$$\begin{aligned} \Delta^{graviton}_{F\mu\nu,\lambda\sigma}(k) &= \frac{1}{2} f^{(2)}_{\mu\nu,\lambda\sigma}\frac{1}{(k^2 - i\epsilon)}, \\ \text{and} \quad \Delta_F^{dilaton}(k) &= \frac{1}{2}\frac{1}{(k^2 - i\epsilon)}. \end{aligned} \tag{3.32}$$

Various calculations, in quantum gravity, will eventually be reduced to contraction of metrics $g_{\mu\nu}$'s at different space time points. The traceless spin-2 graviton part can be separated out with some caution. It has always been opined that quantisation of metric field strength is essential and contains even more information than the quantisation of gravitational field with gravitons only. For an example, let us construct a spin-2 graviton state, like the string state,

$$\Phi^{\dagger}_{\mu\nu}(x) = \; : g_{\mu\nu}(x) - \frac{1}{2}\eta_{\mu\nu} g^{\kappa}{}_{\kappa}(x) : \tag{3.33}$$

$$= \; : \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}}\left(a^{\dagger}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\; a^{\dagger\kappa}{}_{\kappa}\right)\; e^{ikX} : \tag{3.34}$$

$$\leftrightarrow \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}} \sum_{i,j} c_{ij}\left(b^{\dagger}_{i\mu}\; b^{\dagger}_{j\nu} - \frac{1}{2}\eta_{\mu\nu} b^{\dagger}_{i\kappa}\; b^{\dagger\kappa}_{j}\right)|0,k\rangle. \tag{3.35}$$

This is a very simple way for the string states to create the quanta of general relativity. One can easily check that the graviton propagator comes out correctly. Similarly, we can write the dilaton state. The one dilaton state is

$$D^{\dagger}(x) \leftrightarrow \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2k_0}}\sum_{i,j} c_{ij}\; \frac{1}{2} b^{\dagger}_{i\kappa}\; b^{\dagger\kappa}_{j}\; |0,k\rangle. \tag{3.36}$$

It will be instructive, to find a projection operator which can project out the spin two graviton from $f^{\mu\nu,\lambda\sigma}$ containing both spin-2 graviton and spin-0 dilaton. In fact, such a projection operator turns out to be

$$\mathcal{P}^{\kappa\rho}{}_{\lambda\sigma} = \frac{1}{2}\left(\eta^{\kappa}{}_{\lambda}\eta^{\rho}{}_{\sigma} + \eta^{\kappa}{}_{\sigma}\eta^{\rho}{}_{\lambda} - \eta^{\kappa\rho}\eta_{\lambda\sigma}\right). \tag{3.37}$$

One can verify that

$$\mathcal{P}^{\kappa\rho}{}_{\lambda\sigma} f^{\mu\nu,\lambda\sigma} = f^{(2)\kappa\rho,\mu\nu}, \tag{3.38}$$

and

$$\mathcal{P}^{\kappa\rho}{}_{\lambda\sigma}\mathcal{P}^{\lambda\sigma}_{\mu\nu} = \mathcal{P}^{\kappa\rho}{}_{\mu\nu}. \tag{3.39}$$

Further, in order to show the graviton quanta in general relativity,let us apply the projection operator (3.37) on the Ricci tensor $R_{\mu\nu}$

$$\mathcal{P}^{\mu\nu}{}_{\lambda\sigma}\; R_{\mu\nu} = R_{\lambda\sigma} - \frac{1}{2} g_{\lambda\sigma}\; R = G_{\lambda\sigma}, \tag{3.40}$$

where

$$R_{\mu\nu} = \partial^{\lambda}\partial_{\lambda} g_{\mu\nu}(x) + \partial_{\nu}\partial_{\mu} g^{\lambda}_{\lambda}(x) - \partial_{\nu}\partial^{\lambda} g_{\mu\lambda}(x) - \partial_{\mu}\partial^{\lambda} g_{\nu\lambda}(x) + g_{\eta\sigma}(x)\left(\Gamma^{\eta}_{\lambda\lambda}\;\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\eta}_{\nu\lambda}\;\Gamma^{\sigma}_{\mu\lambda}\right). \tag{3.41}$$

and $G_{\lambda\sigma}$ is the Einstein's tensor of general relativity. Thus, for the gravitons of gravitational field of general relativity of Einstein, $G_{\lambda\sigma}$=0. It may be noted that we shall retain $|0, k >$ and $< k, 0|$ to indicate that the vacuum of the superstring states is tachyonic.

4. THE AFFINE CONNECTION

There have been many successful ways of deriving Einstein's equation using the principle of equivalence. But the study of quantum gravity implies that this principle should be replaceable by quantum mechanical methods which includes perturbative and nonperturbative expansions. The effect of full gravitational interactions can be obtained by replacing the ordinary derivatives by covariant derivatives or coderivatives. First we wish to justify the quantisation scheme with plane waves for the gravitational field. We shall use classical recipes.

The coderivatives of $A^{\mu}(x)$ is given by

$$D_{\nu}A^{\mu}(x) = A^{\mu}_{;\nu}(x) = \partial_{\nu}A^{\mu}(x) + \Gamma^{\mu}_{\rho\nu}A^{\rho}(x). \tag{4.1}$$

where $\Gamma^{\mu}_{\rho\nu}$ are 64 component affine connections which for Riemann-Christoffel tensors [27]. The transformation property, when mapped from x to x', is

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\sigma\tau} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} \partial x'^{\nu}}. \tag{4.2}$$

The covariant derivative of the symmetric tensor $g_{\mu\nu}$ is

$$D_{\lambda} g_{\mu\nu} = g_{\mu\nu;\lambda} = \partial_{\lambda} g_{\mu\nu} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}. \tag{4.3}$$

But the affines, Γ's and $\partial_{\lambda} g_{\mu\nu}$ vanish in local inertial coordinate system. Once a proper tensor is zero in one coordinate system, it remains zero in all other coordinate systems. Therefore the covariant derivative of $g_{\mu\nu}$ is zero, so we get

$$\partial_{\lambda} g_{\mu\nu} = \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu}. \tag{4.4}$$

Thus all the elements of $g_{\mu\nu}(x)$ are completely determined for all points in space time. By suitably adding three such g's and using equation (3.23)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_{\mu} g_{\kappa\nu} + \partial_{\nu} g_{\kappa\mu} - \partial_{\kappa} g_{\mu\nu} \right). \tag{4.5}$$

It must be remembered that the field strength has been quantised in terms of the sum of the plane waves only. The result from such quantisation may not be valid in all points in the curved or flat space time. However we can argue as follows. Since both the $g_{\mu\nu}$ and affine are symmetric, for distant parallelism for which there is well laid down proceedure in affine geometry [27], any of the terms of $\Gamma^{\lambda}_{\mu\nu}$ is of the form $\Gamma^{\lambda}_{\mu\nu} \sim -g^{\lambda\kappa} \partial_{\mu} g_{\kappa\nu}$ and a typical derivative is $g^{\mu}_{\alpha,\sigma} \sim -\Gamma^{\mu}_{\rho\sigma}\ g^{\rho}_{\alpha}$. Further, if the affine is symmetric, it can be shown that [27]

$$g^{\alpha\dagger}_{\mu,\nu} = g^{\alpha\dagger}_{\nu,\mu}, \tag{4.6}$$

and can be expressed as the gradient of four scalar $\phi_{\mu,\alpha} = \partial_{\mu} \phi_{\alpha}(x)$, so that for another point in the space time $x'(\mu) = \phi^{\mu}(x)$,

$$g^{\dagger}_{\mu\nu} = \frac{\partial x'_{\mu}}{\partial x_{\nu}}, \quad g_{\nu\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} \tag{4.7}$$

If we substitute this to calculate Γ's of equation (4.2), we obtain

$$\Gamma'^{\lambda}_{\mu\nu} = 0. \tag{4.8}$$

Thus there always exits a mapping from x to x', so that affinity is flat and vice versa, i.e. if there is flat affinity, a particular mapping will make it a symmetric nonzero affinity. This justifies our use of plane wave in the Fourier transform for quantisation of gravitation and the result will be true for all space time points and for other types of expansion of the field. Most importantly $g_{\mu\nu}$ is the covariant and $g^{\mu\nu}$ is contravariant metric tensor of general relativity. They are quantised in the interaction picture as given by equation (3.21).

5. CALCULATION OF AFFINES, RIEMANN AND RICCI TENSORS

Weinberg [25] has given a very convenient form for the Riemann- Christoffel curvature which is useful to calculate the gravitational effects by replacing the ordinary derivatives. The quantum version has the same form

$$R_{\lambda\mu\nu\kappa} = \ : \partial_{\kappa}\partial_{\mu} g_{\lambda\nu}(x) - \partial_{\kappa}\partial_{\lambda} g_{\mu\nu}(x) - \partial_{\nu}\partial_{\mu} g_{\lambda\kappa}(x) - \partial_{\nu}\partial_{\lambda} g_{\mu\kappa}(x) + g_{\eta\sigma} \left(\Gamma^{\eta}_{\nu\lambda}\ \Gamma^{\sigma}_{\mu\kappa} - \Gamma^{\eta}_{\kappa\lambda}\ \Gamma^{\sigma}_{\mu\nu} \right) : \tag{5.1}$$

The Ricci tensor $R_{\lambda\mu\lambda\nu}$ is conveniently written as

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{int(string)}_{\mu\nu} \tag{5.2}$$

where

$$R^{(0)}_{\mu\nu} = \partial^{\lambda}\partial_{\lambda} g_{\mu\nu}(x) + \partial_{\nu}\partial_{\mu} g^{\lambda}_{\lambda}(x) - \partial_{\nu}\partial^{\lambda} g_{\mu\lambda}(x) - \partial_{\mu}\partial^{\lambda} g_{\nu\lambda}(x), \tag{5.3}$$

is without gravitational interaction and

$$R_{\mu\nu}^{int(string)} = g_{\eta\sigma}\left(\Gamma^{\eta}_{\lambda\lambda}\ \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\eta}_{\nu\lambda}\ \Gamma^{\sigma}_{\mu\lambda}\right) \tag{5.4}$$

contains the interaction part.

We now proceed to find expressions for the affines. Affines as quantum operators should be expressions with normal ordering. The product of $g_{\mu\nu}$'s and their derivative contain product of two distinct pieces. All the fermion pairs of the affines separate out from the factors $e^{ik.X}$ as the bosonic α_μ's commute with the fermionic pairs $b_{\mu,j},\ b_{\nu,k}$. The bosonic part again consists of a zero mode piece and nonzero mode piece. We shall use the product form which appears to be different for normal vertex forms. The zero part of each $g_{\mu\nu}(k)$'s tachyonic part piece is

$$W_0(k,z) = exp\left(k.\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}z^n\right)\cdot exp\left(-k.\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n z^{-n}\right). \tag{5.5}$$

The correlation function is found by using

$$\langle\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_{-n}z_2^n\right)\left(\sum_{n=1}^{\infty}\frac{1}{n}\alpha_n z_1^{-n}\right)\rangle = \eta^{\mu\nu}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z_2}{z_1}\right)^n. \tag{5.6}$$

We have specified earlier that $|z_1|$=1 and $|z_2|$=1 to have the state $|0,k\rangle$,

$$\langle W_0(k_1)W_0(k_2)\rangle = e^{k_1\cdot k_2\zeta(1)}, \tag{5.7}$$

Sometimes this number is left as a parameter λ. We have already adjusted this in equation (3.23). Similarly

$$\langle W_0(k_1)W_0(k_2)W_0(k_3)\rangle = e^{(k_1.k_2+k_3.k_2+k_1.k_3)\zeta(1)}, \tag{5.8}$$

The zero mode parts are given by [12],

$$\langle Z_0(k_1)Z_0(k_2)\rangle = e^{ik_1.x}\ e^{ik_2.x}, \tag{5.9}$$

and

$$\langle Z_0(k_1)Z_0(k_2)Z_0(k_3)\rangle = e^{ik_1.x}\ e^{ik_2.x}\ e^{ik_3.x}. \tag{5.10}$$

The product $V_0 = W_0\ Z_0$ is then

$$\langle V_0(k_1)V_0(k_2)\rangle = e^{ik_1.x}\ e^{ik_2.x}\ e^{(k_2.k_1)\zeta(1)}, \tag{5.11}$$

$$\langle V_0(k_1)V_0(k_2)V_0(k_3)\rangle = e^{ik_1.x}\ e^{ik_2.x}\ e^{ik_3.x}\ e^{(k_2.k_1+k_2.k_3+k_1.k_3)\zeta(1)}. \tag{5.12}$$

and so on, as given in Ref.[12]. The products and derivatives of the string model metrices $g_{\mu\nu}$ satisfy simple commutation relations as given in equation (3.9).

$$\partial^\kappa g^{\mu\nu}(x)\ \partial_\rho g_{\lambda\sigma}(x) = \int \frac{k^\kappa d^3\ k}{\sqrt{2k_0}(2\pi)^3}\frac{k'_\rho d^3\ k'}{\sqrt{2k'_0}(2\pi)^3}\left(a^{\mu\nu\dagger}\ e^{-ikx} - a^{\mu\nu}\ e^{ikx}\right)\left(a^\dagger_{\lambda\sigma}\ e^{-ik'x} - a_{\lambda\sigma}\ e^{ik'x}\right). \tag{5.13}$$

Because of the vacuum $|0,k\rangle$ and $\langle k',0|$,

$$:\partial^\kappa g^{\mu\nu}(x)\ \partial_\rho g^{\lambda\sigma}(x): \ = \int \frac{d^3\ k}{\sqrt{2k_0}(2\pi)^3}\frac{d^3\ k'}{\sqrt{2k'_0}(2\pi)^3}\ \langle k,0|\left[a^{\mu\nu},\ a^{\lambda\sigma\dagger}\right]|0,k'\rangle\ \ e^{-k.k'\zeta(1)}\ k^\kappa k'_\rho \tag{5.14}$$

$$= \int \frac{d^3\ k}{\sqrt{2k_0}(2\pi)^3}\frac{d^3\ k'}{\sqrt{2k'_0}(2\pi)^3}(2\pi)^3\ |c|^2\ f^{\mu\nu,\lambda\sigma}\ \langle k,0|0,k'\rangle e^{-k.k'\zeta(1)}\ k^\kappa k'_\rho$$

$$= \int \frac{d^3\ k}{(2\pi)^3}\ |c|^2\ e^{-k^2\zeta(1)}\ f^{\mu\nu,\lambda\sigma}\ k^\kappa k_\rho\langle k,0|0,k\rangle \tag{5.15}$$

$$= \int \frac{d^3\ k}{(2\pi)^3}\left(\eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda}\right)\ |c|^2\ e^{-k^2\zeta(1)}\ k^\kappa k_\rho\langle k,0|0,k\rangle. \tag{5.16}$$

We use the above equation to obtain,

$$(\partial^\kappa g^{\mu\nu}(x))\ \ (\partial_\rho g_{\lambda\sigma}(x)) = \int \frac{d^3\ k}{(2\pi)^3} k^\kappa k_\rho f^{\mu\nu}{}_{\lambda\sigma} \langle k,0|0,k\rangle. \tag{5.17}$$

So, the general form of an affine given by

$$\Gamma^\lambda_{\mu\nu}(x) = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}\right), \tag{5.18}$$

is in operator form reduces to

$$:\Gamma^\lambda_{\mu\nu}(x): \ = i \int \frac{d^3k}{(2\pi)^3} (\eta^\lambda_\nu k_\mu \ + \ \eta^\lambda_\mu k_\nu\ )\langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}. \tag{5.19}$$

Here we have used the relation

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} k^\mu k'^\nu \langle k,0|0,k\rangle\langle k',0|0,k'\rangle &= \lim_{x\to 0} \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3} \partial^\mu\partial^\nu e^{i(k-k')x} \langle k,0|0,k\rangle\langle k',0|0,k'\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} k^\mu k^\nu \langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}. \end{aligned} \tag{5.20}$$

Thus we get the affines occurring in equation(5.4) as follows.

$$:\Gamma^\eta_{\lambda\lambda}(x): \ = \ i \int \frac{d^3k}{(2\pi)^3} (2k^\eta)\langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}, \tag{5.21}$$

$$:\Gamma^\sigma_{\mu\nu}(x): \ = \ i \int \frac{d^3k'}{(2\pi)^3} (\eta^\sigma_\nu k'_\mu + \eta^\sigma_\mu k'_\nu)\langle k',0|0,k'\rangle |c|^2\ e^{-k'^2\zeta(1)}, \tag{5.22}$$

$$:\Gamma^\eta_{\nu\lambda}(x): \ = \ i \int \frac{d^3k}{(2\pi)^3} (\eta^\eta_\nu k_\lambda + \eta^\eta_\lambda k_\nu)\langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}, \tag{5.23}$$

and

$$:\Gamma^\sigma_{\mu\lambda}(x): \ = \ i \int \frac{d^3k'}{(2\pi)^3} (\eta^\sigma_\mu k'_\lambda + \eta^\sigma_\lambda k'_\mu)\langle k',0|0,k'\rangle |c|^2\ e^{-k'^2\zeta(1)}. \tag{5.24}$$

Taking the product and difference of the affines at the same point 'x', and using equation(5.20) we obtain the interaction part of the Riemann tensor from equation(5.4),

$$R^{int(string)}_{\mu\nu} = \int \frac{d^3k}{(2\pi)^3} \left(k^2\eta_{\mu\nu} - k_\mu k_\nu\right) \langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}. \tag{5.25}$$

Let us now consider the expression, which is the remaining exact part of the Ricci tensor

$$R^{int}_{\mu\nu} = \ :\frac{1}{2} g_{\mu\nu} \left(\partial_\lambda\partial_\sigma g^{\lambda\sigma} - \Box g^\gamma_\gamma\right):. \tag{5.26}$$

Using the equations(5.16) and (5.17), equation (5.26) can be written as

$$R^{int}_{\mu\nu} = \int \frac{d^3k}{(2\pi^3)} \left(k^2\eta_{\mu\nu} - k_\mu k_\nu\right) \langle k,0|0,k\rangle |c|^2\ e^{-k^2\zeta(1)}. \tag{5.27}$$

So,

$$R^{int(string)}_{\mu\nu} = R^{int}_{\mu\nu}. \tag{5.28}$$

Thus the Ricci tensor of general relativity,

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{int}_{\mu\nu}. \tag{5.29}$$

is exactly reproduced in quantised form. It is perhaps one of the rare instances where the first order perturbation, as suggested, gives the exact result. The product of operators in the expression for the affains and Ricci tensor appear to be independent of x. This is usually the case of operator product expansions [21], or the correlation function in Green function method of calculations in quantum field theory.

6. CONCLUSION

We have constructed a symmetric second rank tensor which represents the gravitational metric field strength. Using the equivalence of string states and field operators, this has been quantised in terms of the creation and annihilation operators given in equation (3.21). The metric quantum operator has all the properties of the metric of general relativity. In the interaction picture, it satisfies the weak field Einstein equation $R^{(0)}_{\mu\nu} = 0$ for Ricci tensor, without the affines. The Riemann-Christoffel affines which occur by replacing the ordinary derivatives by coderivatives are evaluated in detail. It is found that, as another rare case, the first order calculation of $R^{int}_{\mu\nu}$ gives the exact $R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{int}_{\mu\nu} = 0$ of general relativity in vacuum. The gravitational self stress energy in vacuum also vanishes, as shown below. The exact Einstein equation can be written, following Weinberg [25],

$$R^{(0)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(0)\lambda}{}_{\lambda} = -8\pi G(T_{\mu\nu} + t_{\mu\nu}), \tag{6.1}$$

where $t_{\mu\nu}$ is the exact energy momentum tensor of the gravitational field itself and is given by

$$t_{\mu\nu} = \frac{1}{8\pi G}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\lambda}{}_{\lambda} - R^{(0)}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}R^{(0)\lambda}{}_{\lambda}\right]. \tag{6.2}$$

This is seen to vanish due to equations (5.3) and (5.29).

So far, quantum gravity in vacuum has appeared non-renormalisable. Here we have attempted to show a new way for further investigation to renormalise quantum gravity using superstring states. Unlike most of the previous works on quantum gravity, we have not used a power series expansion of the metric tensor $g_{\mu\nu}(x)$ around a flat metric $\eta_{\mu\nu} = (- \ + \ + \ +)$ with a coupling constant κ having the dimension of length as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\phi_{\mu\nu}. \tag{6.3}$$

Since the Riemann tensor and as a consequence, each term of the interaction Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{\sqrt{-g}}{16\pi G}g^{\dagger}_{\mu\nu}R^{(0)}_{\mu\nu} \\ &= \frac{\sqrt{-g}}{16\pi G}\left[-\partial^{\lambda}g^{\dagger}_{\mu\nu}\partial_{\lambda}g_{\mu\nu} - \partial_{\nu}g^{\dagger}_{\mu\nu}\partial_{\mu}g^{\lambda}_{\lambda} + \partial_{\nu}g^{\dagger}_{\mu\nu}\partial_{\lambda}g_{\mu\lambda} + \partial^{\mu}g^{\dagger}_{\mu\nu}\partial_{\lambda}g_{\nu\lambda}.\right] + \text{total derivatives} \end{aligned} \tag{6.4}$$

contains the product of two affines, each with one derivative and two $\phi_{\mu\nu}$'s of equation(6.3), the power counting for renormalisability comes out to be $N = 6$. As a result, the theory has very little chance of making quantum gravity finite by the usual renormalisation procedure. When $\eta_{\mu\nu}$ is the zeroth order of $g_{\mu\nu}$ in the conventional perturbation calculation, one would get the graviton denoted by the Ricci tensor $R^{(1)}_{\mu\nu}$ as given by Weinberg [25]. The complication in writing the equations for $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ increases enormously; it will also be complicated , if not worse, in quantum gravity. Instead, by treating the metric tensor as operator function in the gravitational field in our case, the exact Ricci tensor is obtained in the first order itself.

In summary,we construct a four dimensional superstring physical state as a second rank Lorentz tensor equivalent operator with zero mass. These states are quantised as exact equivalent operators like the Gupta-Bleuler formalism. The metric tensor, which is not traceless, is found to contain both the graviton and the dilaton. The interaction Lagrangian, as usual, is the difference of the product of two affines. The product of two field strengths like $g^{\mu\nu}(x)g_{\nu\lambda}(x)$ which, when treated as quantum operators, turns out to be δ^{μ}_{λ}. In actual calculation, the two metrics at different space time are usually contracted but not the traceless field quanta of graviton alone. As a result, the number for renormalisability of these becomes $N = 4$. So the interaction Lagrangian theory, as formulated here, is possibly renormalisable. Perhaps the zeroth and first order are enough for obtaining correct results and may be a gift of the principle of equivalence.

It will be interesting and very much necessary to study the interaction of gravity with matter and radiation not only for self vacuum as we have done , and then establish our procedure for renormalisation of the quantum theory of gravity with external energy-momentum tensor. One must also calculate graviton-graviton scattering to make definite statement about the renormalisability of quantum gravity interacting with matter. It is worthwhile to point out that Mandelstam [5] has noted that it is only necessary 'to treat a gravitational field within itself since such a system possesses all the essential complications of the problem of gravity'.

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