Fluctuating nonlinear hydrodynamics of flocking

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Abstract

Starting from a microscopic model, the continuum field theoretic description of the dynamics of a system of active ingredients or "particles" is presented. The equations of motion for the respective collective densities of mass and momentum follow exactly from that of a single element in the flock. The single particle dynamics has noise and anomalous momentum dependence in its frictional terms. The equations for the collective densities are averaged over a local equilibrium distribution to obtain the corresponding coarse grained equations of fluctuating nonlinear hydrodynamics (FNH). The latter are the equations used frequently for describing active systems on the basis of intuitive arguments. The transport coefficients which appear in the macroscopic FNH equations are determined in terms of the parameters of the microscopic dynamics.

I. INTRODUCTION

An active system consists of particles or individual elements which have the ability to transduce extra free energy [1, 2]. Generic examples of active matter are, swimming microbes, schools of fish, swarms of birds etc. The active systems considered here are characterized by short ranged interactions, *i.e.*, each element in a flock is influenced by the neighboring elements within a specific distance which is much smaller than the size of the flock. With the interpretation that the velocity vector of the individual units of a flock is like the magnetic spin, the collective system is analogous to the magnetic system studied in theory of phase transitions in statistical mechanics. The formation of the ordered state is like a dynamic analogue of the spontaneous breaking of the continuous symmetry in a spin system. The ordered, coherently moving state of a flock survive the randomness, making a uniformly moving, arbitrarily large flock possible. This is similar to the occurrence of the ferromagnetic phase at finite temperature surviving thermal fluctuations. The occurrence of the collective phenomena of a coherently moving flock is controlled by the inter particle interaction, rather than the aligning effects of an external field causing it. The randomness of the directions of movements of the individual members of the system introduces the stochastic element to the flocking problem in much the same way that thermal fluctuations do at nonzero temperature in an equilibrium ferromagnet. The movement of the individual elements are therefore characterized by some degree of randomness which constitutes a noise. The latter is assumed to have short ranged correlation over space and time.

Statistical mechanics traditionally deals with systems consisting of "passive" particles, which move as result of interactions with their neighbors or with external fields. In an active system on the other hand, the particles or the individual elements are self-driven. For understanding how fluctuations around equilibrium decay in self driven systems, the generalized hydrodynamic approach [3, 4] using coarse grained densities of physical properties constitutes a powerful tool. As a first step, the non-equilibrium steady state of a self-driven system is treated by considering a small driving force acting on a thermal equilibrium state [5]. An effective way of studying the long-distance, long-time properties of a system consisting of a large number of active elements is to use the field theoretic approach. This involves describing the time evolution of a set of coarse grained densities for the system, in terms of stochastic partial differential equations. These equations of generalized hydrody-

namics provide a powerful tool for studying a many particle system in equilibrium [6], as well as in out of equilibrium [7] conditions. The same approach has been used in studying how fluctuations around equilibrium decay in a self driven system like a flocking of active ingredients or "particles" [8]. In the spirit of weakly nonlinear analysis, well-controlled expansions in the vicinity of ordering transitions are performed. Analysis of the corresponding equations of fluctuating nonlinear hydrodynamics (FNH)[9, 10] of the polar active particles predict a non-equilibrium phase transition from a disordered state to a state of long-range order in terms of the particle velocities. The present paper deals with description of the dynamics of active systems in terms of continuum field theoretic models. It is known that long-distance, long-time properties of various many particle systems in equilibrium [6], as well as in out of equilibrium [7] states are correctly reproduced with similar models. The dynamic behavior of a flocking system of active ingredients or "particles" was proposed in Ref. [8] using the same approach. Simulations of automata models of active systems, having short ranged interactions, full rotation invariance, and nonzero fluctuations, show that the possibility of having a coherently moving flock does exist. This is different from the corresponding predictions from statistical mechanics for an equilibrated spin system. For the latter case spontaneous breaking of the continuous symmetry in two or less dimensions is not possible[11]. The nonequilibrium nature of the flock movement and hence dynamics plays a key role for the occurrence of the state with long range order.

Construction of the equations of generalized hydrodynamics starting from a microscopic model of the active constituents is a key step in developing the field theoretic models of such systems. The hydrodynamic equations for an active system have been written in the past purely relying on considerations of the conservation laws and the symmetries of the problem [8]. For an isotropic fluid rotational invariance, space and time translation invariance, and Galilean invariance must be preserved. The corresponding conservation laws are respectively for the number of particles, momentum, and energy of the whole system. The basic conservation law for the active system, used here, is the number conservation signified by the continuity equation for density ρ with momentum current $\mathbf{g}$ as flux. The individual species do not reproduce or die in the flock. They follow Brownian dynamics, and momentum conservation does not hold. For the flocks in which momentum conservation can be justified, hydrodynamics[12, 13] is different. In this paper we construct the equation of motion for the velocity field $\mathbf{v}(\mathbf{x}, t)$ in the flock starting from the appropriate microscopic level description

of the active system in terms of a Langevin equation. The velocity field equation obtained here, using the full symmetry considerations, is similar to the corresponding equation used in the analysis of Ref. [8]. It also has additional nonlinearities involving velocity and density fields. The underlying universal behavior of such systems is controlled by symmetries and conservation laws and not the microscopic details. We show how the various transport coefficients appearing in the FNH equation are linked to respective dissipative constants in the equation for the particle dynamics. For an active system this random noise can arise from driving at widely different time scales or due to statistical fluctuations, rooted in the randomness in the individual motions. The paper is organized as follows: In the next section we introduce the microscopic equations of motion. Section III describes the coarse graining of the microscopic equations and deduction of the continuum equations of fluctuating nonlinear hydrodynamics for flocking system. We end the paper with a brief discussion of the present work with reference to other attempts in the problem.

II. THE MICROSCOPIC EQUATIONS OF MOTION

In a complex fluid the equation of motion of a single particle is often assumed to be a Langevin equation with noise. For describing the active system of polar particles we adopt a similar approach. Using the concept of universality class introduced with respect to critical phenomena and condensed matter physics, models having same general features are expected to behave similarly. To demonstrate the micro-dynamic equations, it is therefore useful to consider the simplest discrete model[14] for the motion of active elements of flocks in two dimensions. Let $\hat{\theta}_i(t)$ be the direction of the i-th element at time t. The equation of motion for this angular variable for an individual element of the flock is determined in terms the $\hat{\theta}$'s for the neighboring elements within a circle of radius R_0 around the i-th element. In addition, the element i is also affected by random component $\eta_i(t)$ which is assumed to have short ranged correlations. The equation of motion of the i-th element of the flock is schematically written as

$$\dot{\hat{\theta}}_i(t) = \Gamma_i[\hat{\theta}] + \eta_i(t) \ . \tag{1}$$

$\Gamma_i[\hat{\theta}]$ is a function describing the characteristic dynamics of the flock. For the two dimensional model represented by Eqn. (1), the hydrodynamic equations have been obtained [15, 16] starting from the corresponding Boltzmann equation description[17].

A. Single particle dynamics

In a general context, the flock of N elements is described microscopically by the respective position and momentum coordinates $\{\mathbf{x}_\alpha, \mathbf{p}_\alpha\}$ of the α-th unit for $\alpha = 1, ...N$. We will work here with the time evolution for the momentum $\mathbf{p}_\alpha$ of the α-th element of the flock of N elements as being given by the following Langevin equation:

$$\frac{dp^i_\alpha(t)}{dt} = -\nabla^i_\alpha \sum_\nu U(\mathbf{x}_\alpha(t) - \mathbf{x}_\nu(t)) - \sum_\nu \zeta^{ij}_{\alpha\nu}(\mathbf{p}_\nu) p^j_\nu + \xi^i_\alpha(t) \tag{2}$$

The Greek indices α, ν, etc. denoting the different elements of the flock range from 1 to N. The Latin indices i, j denote the Cartesian coordinates and run over from 1 to the dimensionality d. Thus, ∇^i_α denotes the derivative with respect to i-th component of $\mathbf{x}_\alpha$. In Eqn. (2) and in rest of this paper we follow, the Einstein summation convention for the repeated Latin (Cartesian coordinates) indices. The first two terms on the right hand side of Eqn. (2) represent the deterministic part of the dynamics respectively signifying reversible and irreversible components. A key new feature of this dynamics is that the dissipative coefficient $\zeta^{ij}_{\alpha\nu}(\mathbf{p}_\nu)$ is dependent on $\mathbf{p}_\alpha$ and the frictional drag on the α-th element depends on its own velocity, as well as those of its neighbors. The stochastic or random part of the dynamics of the particle α is the noise ξ^i_α. Its correlation is defined in terms of the symmetric matrix $\Gamma^{ij}_{\alpha\nu}$ as

$$\langle \xi^i_\alpha(t) \xi^j_\nu(t') \rangle_{\rm F} = 2\Gamma^{ij}_{\alpha\nu} \delta(t - t') \quad , \tag{3}$$

where the subscript F on the angular bracket refers to the averaging over the fast degrees of freedom in the dynamics.

B. Dynamics of collective modes

At the collective level, the flock of N elements is described microscopically by the respective position and momentum coordinates $\{\mathbf{x}_\alpha, \mathbf{p}_\alpha\}$ of the α-th unit for $\alpha = 1, ...N$. For the collective modes we focus on the microscopic mass and momentum densities $\hat{\rho}(\mathbf{x}, t)$ and $\hat{\mathbf{g}}(\mathbf{x}, t)$ which are respectively defined as

$$\hat{\rho}(\mathbf{x}, t) \; = \; \sum_{\alpha=1}^{N} m \delta(\mathbf{x} - \mathbf{x}_\alpha(t)) \quad , \tag{4}$$

$$\hat{\mathbf{g}}(\mathbf{x}, t) \; = \; \sum_{\alpha=1}^{N} \mathbf{p}_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha(t)) \quad . \tag{5}$$

The velocity field $\mathbf{v}(\mathbf{x},t)$ is not a statistically defined quantity and as will be shown below, is an auxiliary field in a local equilibrium description of a flock. A microscopic statistical mechanical description of an equilibrium fluid involves a set of collective modes which signify specific conservation laws or broken symmetries of the system. For simplicity, all elements are taken to be equal mass m. The equation of motion for $\hat{\rho}$ follows by taking a time derivative of $\hat{\rho}$ and is the continuity equation,

$$\frac{\partial \hat{\rho}}{\partial t} + \nabla_j \hat{\mathrm{g}}_j = 0, \tag{6}$$

with $\hat{\mathbf{g}}$ as the flux mass for density. Similarly taking a time derivative of $\hat{\mathrm{g}}_i(\mathbf{x},t)$ we obtain the corresponding equation of motion for the momentum density. Using Eqn. (2) for time evolution of $\mathbf{p}_\alpha^i$, we obtain the equation of motion for $\hat{\mathrm{g}}_i(\mathbf{x},t)$ as:

$$\frac{\partial}{\partial t}\hat{\mathrm{g}}_i(\mathbf{x},t) + \hat{\mathrm{V}}_i(\mathbf{x},t) + \hat{\mathcal{F}}_i(\mathbf{x},t) = \hat{\theta}_i(\mathbf{x},t) \quad . \tag{7}$$

The deterministic part of the equation of motion (7) has reversible and irreversible terms, arising from the corresponding contributions in the microscopic equation (2). The time reversible part $\hat{\mathrm{V}}_i(\mathbf{x},t)$ has two terms arising from the time derivatives acting on $\delta(\mathbf{x}-\mathbf{x}_\alpha)$ and $\mathbf{p}_\alpha$. We obtain for the reversible and dissipative contributions of Eqn. (7)

$$\hat{\mathrm{V}}_i \;=\; \nabla_j \Big[\sum_\alpha p_\alpha^i p_\alpha^j \delta(\mathbf{x}-\mathbf{x}_\alpha) \Big] + \hat{\rho}(\mathbf{x},t)\nabla_i \int d\mathbf{x}' U(\mathbf{x}-\mathbf{x}')\hat{\rho}(\mathbf{x}',t) \quad , \tag{8}$$

$$\hat{\mathcal{F}}_i \;=\; \sum_{\alpha,\nu} \zeta_{\alpha\nu}^{ij}(\mathbf{p}_\nu) p_\nu^j \delta(\mathbf{x}-\mathbf{x}_\alpha) \quad . \tag{9}$$

The random part in the right hand side of Eqn. (7) arises from the noise ξ_α of Eqn. (2), and is obtained as

$$\hat{\theta}_i(\mathbf{x},t) = \sum_\alpha \xi_\alpha^i \delta(\mathbf{x}-\mathbf{x}_\alpha) \quad . \tag{10}$$

Correlation of the noise $\hat{\theta}$ is obtained using that for the white noise $\xi_\alpha^i(t)$ and by applying translational invariance.

$$\langle \hat{\theta}_i(\mathbf{x},t)\hat{\theta}_j(\mathbf{x}',t')\rangle_{\mathrm{F}} = \Gamma(\mathbf{x}-\mathbf{x}')\hat{\rho}(\mathbf{x}')\delta_{ij}\delta(t-t') \quad . \tag{11}$$

For the isotropic system, we take $\Gamma_{\alpha\nu}^{ij} \equiv \Gamma_0 \delta_{ij}\delta_{\alpha\nu}$, where Γ_0 is a constant obtaining a local $\Gamma(\mathbf{x}) = \Gamma_0\delta(\mathbf{x})$ [8].

III. COARSE GRAINED EQUATIONS

Having identified the dependence of Eqn. (7) on the microscopic set of coordinates $\{\hat{\rho}, \hat{\mathbf{g}}\}$, we now focus on averaging this equation over a suitable non equilibrium ensemble. The nonequilibrium averages of the microscopic densities define the corresponding coarse grained fields, as

$$< \hat{\rho}(\mathbf{x}, t) >_{\rm n.e} \; = \; \rho(\mathbf{x}, t) \tag{12}$$

$$< \hat{\mathbf{g}}(\mathbf{x}, t) >_{\rm n.e} \; = \; \mathbf{g}(\mathbf{x}, t). \tag{13}$$

The averaged equations therefore describe the time evolution of the coarse grained densities $\rho(\mathbf{x}, t)$ and $\mathbf{g}(\mathbf{x}, t)$. These stochastic partial differential equations with smooth spatial and temporal variations form the basis of a macroscopic hydrodynamic description of the flocking system.

A. Local equilibrium distribution

The averaging procedure mentioned above involves integrating over the phase space variables $\{\mathbf{x}_\alpha, \mathbf{p}_\alpha\}$ with respect to a suitable probability distribution for the corresponding ensemble. The latter is chosen here by assuming that system has reached a state of local equilibrium in which root mean square of momentum is constant. To facilitate the analysis, we will consider below the fluid from a co-moving frame (denoted by prime) which has the local velocity $\mathbf{v}(\mathbf{r}, t)$ in a continuum description. The position and momenta coordinates in the co-moving frame are obtained by canonical transformation from the rest frame,

$$\mathbf{x}_\alpha = \mathbf{x}'_\alpha, \quad \text{and} \quad \mathbf{p}_\alpha = \mathbf{p}'_\alpha + m\mathbf{v}(\mathbf{x}'_\alpha) \quad . \tag{14}$$

In the co-moving frame the fluid is locally at rest. Using the concept of Gibbsian ensemble, the distribution function [18] for the local equilibrium state is obtained as

$$f_{\rm le}(\mathit{\Gamma}'_N, t) = Q_l^{-1} \exp\Big[-\beta \Big\{ H' - \int d\mathbf{x} \mu(\mathbf{x}, t) \hat{\rho}'(\mathbf{x}) \Big\} \Big] \equiv Q_l^{-1} \exp\Big(-\beta \tilde{H}' \Big) \quad , \tag{15}$$

where $\mathit{\Gamma}'_N$ symbolizes the phase space coordinates and H' is Hamiltonian in the local rest frame in terms of primed coordinates $\mathbf{p}'_\alpha$. The factor β in the right hand side of Eqn. (15) is related to the average kinetic energy $\epsilon_0 = d/(2\beta)$. The chemical potential in the local

equilibrium ensemble is $\mu(\mathbf{x})$, while Q_l is the normalization constant for the distribution $f_{\rm le}$. Average over the local equilibrium distribution (15) will be denoted with angular brackets $\langle ....\rangle$. Using the transformation rules (14), we relate[3] the local densities $\{\rho, \mathbf{g}\}$ in the two frames as:

$$\hat{\rho}(\mathbf{x}) \;=\; \hat{\rho}'(\mathbf{x}), \tag{16}$$

$$\hat{\mathbf{g}}(\mathbf{x}) \;=\; \hat{\mathbf{g}}'(\mathbf{x}) + \hat{\rho}'(\mathbf{x})\mathbf{v}(\mathbf{x}) \quad . \tag{17}$$

By taking a local equilibrium average of these relations and using the $\mathbf{p}', -\mathbf{p}'$ symmetry of the distribution in the co-moving frame, we obtain

$$\mathbf{g} = \rho\mathbf{v}. \tag{18}$$

The local velocity field $\mathbf{v}(\mathbf{x}, t)$ is thus obtained through a nonlinear relation involving the conserved densities.

B. Time reversible dynamics

The microscopic form of the continuity equation (6) linking $\hat{\rho}, \hat{\mathbf{g}}$ on averaging obtains, in terms of the coarse grained densities the equation

$$\frac{\partial\rho(\mathbf{x}, t)}{\partial t} + \nabla \cdot \mathbf{g}(\mathbf{x}, t) = 0. \tag{19}$$

Next we focus on the equation of motion (7) for the momentum density $\mathbf{g}(\mathbf{x}, t)$. The reversible part of the equation of motion for $g_i(\mathbf{x}, t)$ is obtained by taking an average of $\hat{\rm V}_i$, defined in Eqn. (8), over the non equilibrium ensemble (15). Evaluation of ${\rm V}_i$ is same as was done in the standard case [18] of non-active particles. For clarity we briefly review it in the Appendix IV. The coarse grained expression for the reversible part obtained by averaging over the phase space variables is

$${\rm V}_i \equiv \langle\hat{\rm V}_i\rangle = \nabla_j \left[\frac{g_i(\mathbf{x})g_j(\mathbf{x})}{\rho(\mathbf{x})}\right] + \rho(\mathbf{x})\nabla_i\mu(\mathbf{x}) \quad . \tag{20}$$

The first term on the RHS of the defining expression for ${\rm V}_i$ is the well known convective nonlinear term, essential for maintaining Galilean invariance in hydrodynamic equations.

C. The dissipative dynamics

Next we focus on averaging the frictional term $\hat{\mathcal{F}}_i$ in the microscopic equation of motion (7) for $\hat{g}_i$. $\hat{\mathcal{F}}_i$ consists of a self or diagonal part $\hat{\mathcal{F}}_i^s$ (for $\alpha = \nu$) and a collective or non-diagonal part $\hat{\mathcal{F}}_c^i$ ($\alpha \neq \nu$). For an isotropic system the dissipative coefficient $\zeta_{\alpha\nu}^{ij}(\mathbf{p}_\nu)$ in Eqn. (9) is split in the schematic form

$$\zeta_{\alpha\nu}^{ij}(\mathbf{p}_\nu) = \delta_{ij}\left[\delta_{\alpha\nu}\zeta_0(\mathbf{p}_\alpha) + (1-\delta_{\alpha\nu})\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)\right] \tag{21}$$

We will assume that the total drag of a specific element α on all of its neighbors vanish $\sum_\nu \tilde{\zeta}_{\nu\alpha} = 0$. To proceed further we need the momentum dependence of the dissipative coefficients ζ_0 and $\tilde{\zeta}_{\alpha\nu}$ introduced in Eqn. (21). As a first step, we confine this momentum dependence to quadratic order in the respective frictional coefficients

$$\zeta_0(\mathbf{p}_\alpha) \;=\; \Delta_0 a_0 - b_{ln} p_\alpha^l p_\alpha^n, \tag{22}$$

$$\frac{1}{2}\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu) \;=\; \Delta_0 A_{\alpha\nu} - B_{\alpha\nu}^{ln} p_\nu^l p_\nu^n \quad . \tag{23}$$

The factor $\Delta_0 =< p_\alpha^i p_\alpha^j >= 2m\epsilon_0/d$. For the isotropic system the parameters $\{a_0, b_{ln}\}$ for the $\zeta_0(\mathbf{p}_\alpha)$ are assumed to be independent of the particle label α. For the off diagonal part $\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)$, the parameters $\{A_{\alpha\nu}, B_{\alpha\nu}^{ln}\}$ are introduced for each pair of particles $\{\alpha, \nu\}$ with $\alpha \neq \nu$. In a system for which translational invariance and isotropy hold, it will be dependent only on the distance r (say) between the two points. In order to preserve the $\mathbf{p}, -\mathbf{p}$ symmetry, the linear terms have been left out in the above expressions. Using the definition (21), together with the forms (22)-(23) in the expression (9) for the dissipative part $\hat{\mathcal{F}}_i$, we obtain respectively $\hat{\mathcal{F}}_s^i$ and $\hat{\mathcal{F}}_c^i$. The resulting expressions for $\hat{\mathcal{F}}_s^i$ is written in terms of the primed variables of the co-moving frame using the transformation rules (14). This is similar to the comparisons made above for averages of $\hat{\rho}$ and $\hat{\mathbf{g}}$ in the lab frame and the comoving frame. On averaging over the distribution (15) and applying the $\{\mathbf{p}', -\mathbf{p}'\}$ symmetry of H', we obtain for the self part $\mathcal{F}_s^i$ the result,

$$\mathcal{F}_s^i \equiv \langle \hat{\mathcal{F}}_s^i \rangle = \Delta_0 \tilde{a}_{ik}\rho(\mathbf{x})v_k(\mathbf{x}) - m^2 b_{ln} v_l v_n \rho(\mathbf{x}) v_i(\mathbf{x}) \tag{24}$$

where $\tilde{a}_{ik} = (a_0 - b_{ll})\delta_{ik} - 2b_{ik}$. Using isotropy to write the matrix $b_{ik} = \delta_{ik} b_0$, the expression for $\mathcal{F}_s^i$ further simplifies and reduces to,

$$\mathcal{F}_s^i = \left\{\alpha_0 - \beta_0 |\mathbf{v}(\mathbf{x},t)|^2\right\} g_i(\mathbf{x},t) \quad . \tag{25}$$

The constants a_0 and b_0 are obtained as,

$$\alpha_0 \;=\; \Delta_0\{a_0 - (d+2)b_0\}\;, \tag{26}$$

$$\beta_0 \;=\; m^2 b_0 \quad , \tag{27}$$

for d dimensions.

The averaging of the off diagonal contribution $\hat{\mathcal{F}}_c^i$ to the dissipative part $\hat{\mathcal{F}}_i$ is somewhat more complex. We express it in terms of a gradient expansion as,

$$\hat{\mathcal{F}}_c^i = \sum_{\alpha\nu}\left[\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)x_{\nu\alpha}^l\nabla_l + \frac{1}{2}\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)x_{\nu\alpha}^k x_{\nu\alpha}^l\nabla_k\nabla_l + ...\right]p_\nu^i\delta(\mathbf{x}-\mathbf{x}_\nu) \quad . \tag{28}$$

The sum over the index "$\alpha\nu$" in Eqn. (28) for all pairs of particles $\{\alpha,\nu\}$ with $\alpha \neq \nu$. The ansatz (23) for $\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)$ is used in Eqn. (28). On averaging $\hat{\mathcal{F}}_i$ over the local equilibrium distribution, respective terms in the right hand side of Eqn. (28) contribute. The double sums over the configurational coordinates $\{\mathbf{x}_\alpha, \mathbf{x}_\nu\}$ are evaluated in two steps. First, for a fixed ν we sum over all values of $\mathbf{x}_{\alpha\nu} = \mathbf{x}_\alpha - \mathbf{x}_\nu$. Using translational invariance this sum is taken to be independent of $\mathbf{x}_\nu$. The terms linear in $x_{\nu\alpha}$ contribute zero on averaging due to mutual cancellations in an isotropic environment. The average of the second order terms in $x_{\nu\alpha}$ is nonzero. Hence the local equilibrium average $\mathcal{F}_c^i$ is expressed in terms of tensors of rank two and four. The latter are defined in terms of parameters $\{A_{\alpha\nu}, B_{\alpha\nu}^{ij}\}$ introduced in Eqn. (23). For an isotropic system, these are respectively expressed in terms of phenomenological constants A_0, B_0, and B_0' as follows:

$$\langle\sum_{\alpha\nu} A_{\alpha\nu}x_{\nu\alpha}^i x_{\nu\alpha}^j\rangle_{\rm c} \;=\; A_0\delta_{ij} \tag{29}$$

$$\langle\sum_{\alpha\nu} B_{\alpha\nu}^{ij}x_{\nu\alpha}^k x_{\nu\alpha}^l\rangle_{\rm c} \;=\; B_0\delta_{ij}\delta_{kl} + \frac{B_0'}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \quad . \tag{30}$$

The subscript c with the average indicates integration with respect to the spatial coordinates only, the momentum variables being already integrated out. Using arguments similar to given below Eqn. (28), it follows that the averages of the respective double sums in Eqns. (29) and (30) are determined by the likelihood of two elements of the flock to be at a given distance r from each other. Hence $\{A_0, B_0, B_0'\}$ are related to equal time pair correlation function [19] for the system and are calculated using standard procedure[20] for specific model choices. In the continuum field theoretic description, the average $\mathcal{F}_c^i$ is finally obtained as a nonlinear functional of the gradients of coarse grained fields $\mathbf{g}(\mathbf{x},t)$ and $\mathbf{v}(\mathbf{x},t)$. The

details of the calculation are presented in the Appendix IV obtaining the leading order result in gradients of the coarse grained fields as

$$-\mathcal{F}_c^i = D_L\nabla^2 g_i + D_1\nabla_i\nabla.\mathbf{g} + D_2\nabla^2\big[|\mathbf{v}|^2 g_i\big] + D_2'\nabla_k\nabla_l\big[v_k v_l g_i\big] \ . \tag{31}$$

The dissipative coefficients appearing in the right hand side of Eqn. (31) are as given in Eqns. (25) and (26) in the Appendix IV.

The coarse grained noise in the $\mathbf{g}$-equation is $\theta(\mathbf{x},t)\equiv\langle\hat{\theta}(\mathbf{x},t)\rangle$. By averaging Eqn. (11) over the same local equilibrium distribution, we obtain the correlation of $\theta(\mathbf{x},t)$ in the simplest form as

$$\langle\theta_i(\mathbf{x},t)\theta_j(\mathbf{x}',t')\rangle_{\rm F} = \Gamma_0\rho(\mathbf{x}')\delta_{ij}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ \ . \tag{32}$$

D. Hydrodynamic Equations

The equations of hydrodynamics are the equations for the mass and momentum densities with smooth space time variation. The continuity equation (19) is the equation for the density. Thy coarse grained version of equation (7) for momentum density is constructed by adding Eqns. (20), (24), and (31). We obtain the following nonlinear partial differential equation for the momentum current field $\mathbf{g}(\mathbf{x},t)$ with multiplicative noise θ_i.

$$\frac{\partial g_i}{\partial t} + \nabla_j\Big[\frac{g_i g_j}{\rho}\Big] + \rho\nabla_i\mu + \big\{\alpha_0 - \beta_0|\mathbf{v}|^2\big\}\, g_i + L_{ij}g_j + \tilde{L}_{kl}\Big[v_k v_l g_i\Big] = \theta_i \ \ . \tag{33}$$

In the last two terms on the left hand side of Eqn. (33), the operators L_{ij} and $\tilde{L}_{ij}$ are obtained in the common form,

$$-L_1\delta_{ij}\nabla^2 - L_2\nabla_i\nabla_j \ \ . \tag{34}$$

The dissipative coefficients $\{L_1, L_2\}$ in the expression (34) are identified as $\{D_L, D_1\}$ and $\{D_2, D_2'\}$ respectively for L_{ij} and $\tilde{L}_{ij}$. The hydrodynamic equation for flocking is often written [8] in terms of the local velocity field $\mathbf{v}(\mathbf{x},t)$ which is related to $\mathbf{g}$ field through the nonlinear relation (18). From (33) we obtain,

$$\begin{aligned}\frac{\partial v_i}{\partial t} + \mathbf{v}.\nabla v_i \ &= \ -\nabla_i\mu - \big\{\alpha_0 - \beta_0|\mathbf{v}|^2\big\}\, v_i + D_L\nabla^2 v_i + D_1\nabla_i\nabla.\mathbf{v} \\ &+ \ D_2\nabla^2\big[|\mathbf{v}|^2 v_i\big] + D_2'\nabla_k\nabla_l\big[v_k v_l v_i\big] + f_i\end{aligned} \tag{35}$$

In reaching Eqn. (35), we assume that terms containing factors of $(\nabla\rho)/\rho$ [21] are small compared to those proportional to $\delta\rho(\mathbf{x},t)$ or $\mathbf{v}(\mathbf{x},t)$. The chemical potential $\mu(\mathbf{x})$ in the local equilibrium ensemble is expressed as a power series in fluctuation of the density over its mean value ρ_0,

$$\mu(\mathbf{x}) = \sum_{n=0}^{\infty} \sigma_n (\rho - \rho_0)^n \quad , \tag{36}$$

where σ_n are the expansion coefficients. Eqn. (35) has the same form as Eqn. (1) in Ref. [8]. The correlation for the noise f_i in the velocity equation (35), is inversely dependent on the density ρ.

IV. DISCUSSION

There have been previous efforts[22–25] to obtain the equations of generalized hydrodynamics starting from the microscopic description of the flocking system. These works primarily dealt with director type models in two dimensions. Generally kinetic-theory models using the so called Boltzmann-Ginzburg-Landau framework can be pursued to compute hydrodynamic transport coefficients for the system. In particular, the deterministic mean-field theory obtained using Kinetic theory methods has been treated for the case of active nematics. Ref. [22] considered a specific nematic particles model for which the hydrodynamic equations were obtained starting from a Boltzmann equation for the time evolution of the one particle distribution function $f(\mathbf{r},t)$. The nonlinear couplings of hydrodynamic fields in the corresponding dynamic equations for active matter are also calculated using the same approach. With the Langevin approach a multiplicative noise [22] in the FNH equation is included. However the noise was obtained from only the collision-less dynamics.

The present work, on the other hand, is based on a generic polar particle model and is not restricted to two dimensions. In the continuum hydrodynamic description we obtain the equation of motion for the velocity field $\mathbf{v}(\mathbf{x},t)$ in a form similar to the one analyzed in Ref. [8]. The complete field equation for time evolution of $\mathbf{v}(\mathbf{x},t)$ including the multiplicative noise follows here from coarse graining the microscopic level Langevin type equation for the single element of the flock. The velocity field is obtained here as a conjugate thermodynamic field to the total momentum within a local equilibrium description of the system. Thus our analysis is based on in a natural way on the local equilibrium description of the active system. The latter forms the very foundations on which equations of hydrodynamics are

based on. So it is natural that these basic ingredients has to be included in the microscopic approach that we present here to obtain the Toner-Tu equations of fluctuating nonlinear hydrodynamics. The dynamics of the non equilibrium system is described here in terms linear constitutive relations involving gradient of the local velocity field $\mathbf{v}(\mathbf{x}, t)$ and hence our deduction of the Equations of hydrodynamics involve the same gradient expansion of the local thermodynamic variables like current field.

Some comments specific to particular terms in the fluctuating nonlinear hydrodynamic equation for the $\mathbf{v}(\mathbf{x}, t)$ is useful to note here.

1. The "drag" between two active elements of the flock is adjusted by their momentum and this effect persists up to a characteristic length around each element and this results in creating the corresponding equations at the hydrodynamic level. The Toner-Tu equation follows on coarse graining the micro-dynamic equation within standard approximations of hydrodynamics

2. The non derivative, dissipative term, *i.e.*, $(\alpha_0 - \beta_0|\mathbf{v}|^2)v_i$ in Eqn. (7) implies a nonzero average velocity in the steady state. We have demonstrated here that the nonlinear coupling of momentum fluctuations in the drag term of Eqn. (2) for the microscopic dynamics, is essential in generating this dissipative term in the hydrodynamic equation. This term is not Galilean invariant and breaks momentum conservation.

3. The standard convective nonlinearity term $\mathbf{v}.\nabla v_i$ in the velocity equation [6] in the present form is kept unchanged. In Ref. [8] active matter hydrodynamics this term is changed with multiplicative factor(s) to break the Galilean invariance. In the present calculation an ad hoc factor can be generated in the advective term to break the Galilean invariance by changing the transformation rules between the co-moving frames. However this will be simply shifting the issue (of breaking Galilean invariance) to the microscopic level.

4. The multiplicative noise f_i in Eqn. (35) naturally follows from the coarse graining process. For studying the effects of large fluctuations, the equations of FNH are the starting point. In the present calculation the noise turns out to be multiplicative as a consequence of coarse graining process. General analysis of these equations has been done in the literature starting from the pioneering work of Ref. [8].

5. The present deduction of the equations of fluctuating nonlinear hydrodynamics shows that additional nonlinearities [26] at the order $O(v^3)$ can occur within allowed symmetries of the problem. Those extra cubic nonlinearity in the velocity field appearing in the hydrodynamic equation can be analyzed using dynamic Renormalization group along the lines of Toner and Tu. This will be an entirely new problem.

We present here new approach to obtaining the hydrodynamic equations for the flocking system. Our calculation shows here how the different transport coefficients and the parameters α_0, β_0 in the macro-dynamic equations link with the phenomenological parameters of the microscopic dynamics. Ref. [8] made the crucial observation that the long range ordering is not destroyed in $d = 2$ due to nonlinearity in the dynamics. New simulations of the microscopic equations can be useful for investigating the roles of model dependent constants on the nature of the phase transition and the dynamics. The present calculation reaches the basic formulas of the transport coefficients in terms of statistical averages over a suitable ensemble. Those can be expressed in terms of pair correlation functions in a similar manner as in the theory of normal fluids [27]. The equal time correlations implied here are the ones averaged over different initial conditions for the flock. Once again simulations with phenomenological models would provide the needed input in this.

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Appendix

The coarse grained momentum density is defined as

$$g_i(\mathbf{x},t) =< \hat{g}_i(\mathbf{x},t) > \quad . \tag{1}$$

Here $\langle ... \rangle$ denotes average over the local equilibrium distribution defined in Eqn. (15). This averaging involves integrating out the microscopic or phase space variables of position and momenta for the N particles. The equation of motion for the coarse grained density $g_i(\mathbf{x},t)$ is obtained as,

$$\frac{\partial}{\partial t} g_i(\mathbf{x},t) + \mathrm{V}_i + \mathcal{F}_i = \theta_i(\mathbf{x},t) \quad , \tag{2}$$

where we have denoted as $\mathrm{V}_i = \langle \hat{\mathrm{V}}_i \rangle$, $\mathcal{F}_i = \langle \hat{\mathcal{F}}_i \rangle$ and $\theta_i = \langle \hat{\theta}_i(\mathbf{x},t) \rangle$ respectively as the local equilibrium averages of the reversible, dissipative and random parts of equation (2). In the following we present how the local equilibrium averages are obtained in each case.

1. Reversible part

The time reversible part V_i of the same equation is obtained as a sum of two terms I_1 and I_2 defined in terms of local equilibrium averages[18] as follows:

$$\mathrm{V}_i = I_1 + I_2 \ , \tag{3}$$

$$I_1 = \nabla_j \left\langle \frac{p_\alpha^i p_\alpha^j}{m} \delta(\mathbf{x} - \mathbf{x}_\alpha) \right\rangle \ , \tag{4}$$

$$I_2 = \int dx' (\nabla_i U(\mathbf{x} - \mathbf{x}')) \langle \hat{\rho}(\mathbf{x},t) \hat{\rho}(\mathbf{x}',t) \rangle \quad . \tag{5}$$

To evaluate the local equilibrium averages on the right hand side of Eqns. (4)-(5), we make use of the transformation rules (14), between the position and momentum coordinates of the laboratory and the co-moving frames. The integral I_1 defined in Eqn. (4) is obtained in terms of the primed variables as,

$$\begin{aligned} I_1 &= m^{-1} \nabla_j \left\langle \{p_\alpha'^{\,i} + m v_i(\mathbf{x}'_\alpha)\} \{p_\alpha'^{\,j} + m v_j(\mathbf{x}'_\alpha)\} \delta(\mathbf{x} - \mathbf{x}'_\alpha) \right\rangle \\ &= m^{-1} \nabla_j \left\langle p_\alpha'^{\,i} p_\alpha'^{\,j} \delta(\mathbf{x} - \mathbf{x}'_\alpha) \right\rangle + \nabla_j [v_i(\mathbf{x}) v_j(\mathbf{x}) \rho(\mathbf{x})] \end{aligned}$$

We use in the above equations the local equilibrium distribution function defined in (15). The phase space variables are collectively denoted as $\{\mathbf{x}'_\mathbf{1}, \mathbf{p}'_\mathbf{1} ... \mathbf{x}'_\mathbf{N}, \mathbf{p}'_\mathbf{N}\} \equiv \Gamma'$. The distribution

$f_{\rm le}$ is symmetric in $\mathbf{p}'_\alpha$. By writing $p'^j_\alpha = \partial\tilde{H}'/\partial p'^j_\alpha$ in the the first integral on the right hand side and integrating by parts w.r.t. p'^j_α, the integral I_1 reduces to $\delta_{ij}m^{-1}\Delta_0\rho(\mathbf{x})$. Hence we obtain

$$I_1 = \frac{\Delta_0}{m}\nabla_i\rho(\mathbf{x}) + \nabla_j\left[\frac{g_i(\mathbf{x})g_j(\mathbf{x})}{\rho(\mathbf{x})}\right] \tag{6}$$

To calculate the integral I_2, defined in the right hand side of Eqn. (5), we use for the N particle system the operator $\hat{O}$ defined as,

$$i\hat{O} = \Big[\frac{\partial H}{\partial \mathbf{p}_\alpha}\cdot\frac{\partial}{\partial \mathbf{x}_\alpha} - \frac{\partial H}{\partial \mathbf{x}_\alpha}\cdot\frac{\partial}{\partial \mathbf{p}_\alpha}\Big] \quad , \tag{7}$$

where the H is the Hamiltonian. Note that for a Newtonian system, $\hat{O}$ is the Liouville operator. Using the above definition of $\hat{O}$ it is straightforward to show[18] by acting the latter on $\hat{\rho}$ that

$$i\hat{O}\hat{\rho} = -\nabla\cdot\hat{\mathbf{g}} \quad . \tag{8}$$

The local equilibrium average of the quantity $i\hat{O}\hat{g}'_i(\mathbf{x})$ is obtained as,

$$\langle i\hat{O}\hat{g}'_i(\mathbf{x})\rangle \;=\; Q_l^{-1}\int d\Gamma'\Big[\frac{\partial H'}{\partial p'^j_\alpha}\frac{\partial \hat{g}'_i(\mathbf{x})}{\partial x'^j_\alpha} - \frac{\partial H'}{\partial x'^j_\alpha}\frac{\partial \hat{g}'_i(\mathbf{x})}{\partial p'^j_\alpha}\Big]e^{-\beta\tilde{H}'} \quad . \tag{9}$$

The averaging in the grand canonical ensemble is done with respect to the $exp[-\beta\tilde{H}']$ defined in Eqn. (15). Using the standard form for Hamiltonian $H'(\mathbf{x}',\mathbf{p}')$, the right hand side of Eqn. (9) reduces to,

$$\langle i\hat{O}\hat{g}'_i(\mathbf{x})\rangle = \frac{\Delta_0}{m}\nabla_i\rho(\mathbf{x}) + \int d\mathbf{x}'\{\nabla_i U(\mathbf{x}-\mathbf{x}')\}\langle\hat{\rho}(\mathbf{x},t)\hat{\rho}(\mathbf{x}',t)\rangle. \tag{10}$$

Making use of the derivative form of the operator $\hat{O}$, defined in Eqn. (7), it is straightforward to obtain $< i\hat{O}\hat{g}'_i(\mathbf{x}) >$ as,

$$\langle -i\hat{O}\hat{g}'_i(\mathbf{x})\rangle = -\frac{m}{\Delta_0}\int d\mathbf{x}'\mu(\mathbf{x}')\sum_j\nabla'_j\langle\hat{g}'_i(\mathbf{x})\hat{g}'_j(\mathbf{x}')\rangle = -\rho(\mathbf{x})\nabla_i\mu(\mathbf{x}) \quad , \tag{11}$$

where $\mu(\mathbf{x},t)$ is the chemical potential in the local equilibrium ensemble. In reaching the first equality in (11) we have used the relation (8). For the last equality, the correlation of momentum density fields $\langle\hat{g}'_i(\mathbf{x})\hat{g}'_j(\mathbf{x}')\rangle$ is obtained as $(\Delta_0/m)\delta_{ij}\rho(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')$. The last result follows using the average momentum correlation $< p^i_\alpha p^j_\alpha >=\Delta_0\delta_{ij}$. Combining the results (10) and (11), we obtain

$$I_2 = -\frac{\Delta_0}{m}\nabla_i\rho(\mathbf{x}) + \rho(\mathbf{x})\nabla_i\mu(\mathbf{x}) \quad . \tag{12}$$

By adding the results from (6) and (12) we obtain for the reversible part of the flux in the momentum density equation as,

$$\langle \hat{V}_i \rangle = \nabla_j \left[\rho v_i(\mathbf{x}) v_j(\mathbf{x}) \right] + \rho(\mathbf{x}) \nabla_i \mu(\mathbf{x}) \quad . \tag{13}$$

2. Dissipative part

We now focus on coarse graining of the dissipative term $\hat{\mathcal{F}}_i$ on the left hand side of Eqn. (2). This term involves the dissipative coefficient $\zeta^{ij}_{\alpha\nu}$ introduced in Eqns. (21)-(23). The contributions are respectively coming from the diagonal or self component ζ_0 and the off diagonal part $\tilde{\zeta}_{\alpha\nu}$. The momentum dependence of these dissipative constants are assumed as stated in Eqns. (22)-(23). It may be noted in our notation the repeated Greek indices α and ν in defining Eqns. for the friction coefficients $\zeta_0(\mathbf{p}_\alpha)$ and $\tilde{\zeta}_{\alpha\nu}(\mathbf{p}_\nu)$ are not summed over. The constant Δ_0 is related to the average energy ϵ_0 of a particle as: $\Delta_0 = 2m\epsilon_0/d$ in d dimensions. As stated in the main text, we made by choice all the phenomenological coefficients $\{a_0, b_{ln}\}$, and $\{A_{\alpha\nu}, B^{ln}_{\alpha\nu}\}$ of same dimension. Substituting the above forms, $\hat{\mathcal{F}}_i$ reduces to

$$\begin{aligned} \hat{\mathcal{F}}_i &= \left[\Delta_0 a_0 - b_{mn} p^m_\alpha p^n_\alpha \right] p^i_\alpha \delta(\mathbf{x} - \mathbf{x}_\alpha) \\ &+ \nabla_k \nabla_l \Big\{ \Delta_0 A_{\alpha\nu} - B^{mn}_{\alpha\nu} p^m_\nu p^n_\nu \Big\} x^k_{\nu\alpha} x^l_{\nu\alpha} p^i_\nu \delta(\mathbf{x} - \mathbf{x}_\alpha) \\ &\equiv \mathcal{F}^i_s + \mathcal{F}^i_c \end{aligned} \tag{14}$$

First we consider the self contribution by averaging over the local equilibrium ensemble the part $\hat{\mathcal{F}}^i_s$ which is the contribution of the first term on the right hand side of Eqn. (14) above. To compute this term we transform to the co-moving frame and obtain

$$\hat{\mathcal{F}}^i_s = \Big[a_0 \Delta_0 - b_{ln} (p'^l_\alpha + m \mathbf{v}_l(\mathbf{x}'_\alpha))(p'^n_\alpha + m \mathbf{v}_n(\mathbf{x}'_\alpha)) \Big] (p'^i_\alpha + m v_i(\mathbf{x}'_\alpha)) \delta(\mathbf{x} - \mathbf{x}_\alpha) \quad . \tag{15}$$

By averaging over the distribution (15) and using the $\{\mathbf{p}', -\mathbf{p}'\}$ symmetry of the corresponding Hamiltonian, we obtain

$$\mathcal{F}^i_s = \Delta_0 \left[(a_0 - b_{ll}) \delta_{ik} - 2 b_{ik} \right] \rho(\mathbf{x}) v_k(\mathbf{x}) - m^2 b_{ln} v_l v_n \rho(\mathbf{x}) v_i(\mathbf{x}) \quad . \tag{16}$$

The above expression is further simplified using isotropy to write the matrix $b_{ij} = \delta_{ij} b_0$. Thus the self part $\mathcal{F}^i_s$ of the dissipative term $\mathcal{F}_i$, as defined in Eqn. (14), is obtained as

$$\mathcal{F}^i_s = \{ \alpha_0 - \beta_0 |\mathbf{v}(\mathbf{x}, t)|^2 \} g_i(\mathbf{x}, t) \quad . \tag{17}$$

In d dimensions the constants α_0 and β_0 are respectively obtained as

$$\alpha_0 \;=\; \Delta_0(a_0 - (d+2)b_0), \tag{18}$$

$$\beta_0 \;=\; m^2 b_0 \quad . \tag{19}$$

Next, we consider the contribution $\hat{\mathcal{F}}_c^i$ of the dissipative part coming from the non-diagonal part of $\zeta_{\alpha\nu}$ and is defined in Eqn. (14). Similar to the case of the self component, we transform the corresponding microscopic quantity $\hat{\mathcal{F}}_c^i$ in this case to the co-moving frame and take an average over only the momentum coordinates of the phase space variables Γ' to obtain

$$\begin{aligned}\mathcal{F}_c^i \;=\; \nabla_k\nabla_l\Big\langle \Big[\Delta_0 A_{\alpha\nu} v_i(\mathbf{x}_\nu) - B^{mn}_{\alpha\nu}\Big\{\Delta_0(\delta_{mn}v_i(\mathbf{x}_\nu) + \delta_{im}v_n(\mathbf{x}_\nu) \\ +\; \delta_{in}v_m(\mathbf{x}_\nu)) + m^2 v_m(\mathbf{x}_\nu)v_n(\mathbf{x}_\nu)v_i(\mathbf{x}_\nu)\Big\}\Big] m\delta(\mathbf{x}-\mathbf{x}_\nu)x^k_{\nu\alpha}x^l_{\nu\alpha}\Big\rangle_{\rm C}\end{aligned} \tag{20}$$

The subscript C with the average indicates integration with respect to the spatial coordinates only, the momentum variables being already integrated out. To evaluate the double sums over the configurational coordinates $\{\mathbf{x}_\alpha, \mathbf{x}_\nu\}$ in the right hand side of Eqn. (20) we use translational invariance for the averaged quantity. Thus the summation over two particle indices $\{\alpha, \nu\}$, is calculated in two steps. First we sum over all values of $\mathbf{r}_{\alpha\nu} = \mathbf{x}_\alpha - \mathbf{x}_\nu$, for a fixed ν and assume that the sum is independent of the origin ν.

$$\begin{aligned}\mathcal{F}_c^i \;=\; \nabla_k\nabla_l\Big\langle \Delta_0\Big\{A_{\alpha\nu}x^k_{\nu\alpha}x^l_{\nu\alpha}\Big\}\hat{\rho}(\mathbf{x})v_i(\mathbf{x}) - \Big\{B^{mn}_{\alpha\nu}x^k_{\nu\alpha}x^l_{\nu\alpha}\Big\} \\ \times\; \Big[\Delta_0\big(\delta_{mn}v_i(\mathbf{x}) + \delta_{im}v_n(\mathbf{x}) + \delta_{in}v_m(\mathbf{x})\big) + m^2 v_m(\mathbf{x})v_n(\mathbf{x})v_i(\mathbf{x})\Big]\hat{\rho}(\mathbf{x})\Big\rangle_{\rm C}\end{aligned} \tag{21}$$

The quantities within the big curly brackets on the right hand side of Eqn. (21) are summations of functions which depend on the separation between the particles α and ν, being summed for all pairs. We approximate the local equilibrium averages of the corresponding quantities for an isotropic system in terms of expressions for the dissipative coefficients involving the pair correlation function of the system. For an isotropic system, the two tensors of rank 2 and 4 are respectively described in terms of parameters A_0 and $\{B_0, B_0'\}$:

$$\Big\langle A_{\alpha\nu}x^k_{\nu\alpha}x^l_{\nu\alpha}\Big\rangle_{\rm C} \;=\; A_0\delta_{kl} \tag{22}$$

$$\Big\langle B^{pn}_{\alpha\nu}x^k_{\nu\alpha}x^l_{\nu\alpha}\Big\rangle_{\rm C} \;=\; B_0\delta_{pn}\delta_{kl} + \frac{1}{2}B_0'(\delta_{pk}\delta_{nl} + \delta_{pl}\delta_{nk}) \quad . \tag{23}$$

Using the definitions (22)-(23), and some trivial algebra, the local equilibrium average of $\hat{\mathcal{F}}_c^i$ is approximated in terms of the coarse grained density $\rho = \langle\hat{\rho}\rangle$ and the coupled velocity fields $\mathbf{v}(\mathbf{x}, t)$. We obtain the result

$$-\mathcal{F}_c^i = D_L\nabla^2 g_i + D_1\nabla_i\nabla.\mathbf{g} + D_2\nabla^2\big[|\mathbf{v}|^2 g_i\big] + D_2'\nabla_k\nabla_l\big[v_k v_l g_i\big] \ . \tag{24}$$

We have defined the dissipative constants in terms of three phenomenological constants, A_0, B_0, and B_0' the following coefficients,

$$D_L \ = \ \Delta_0\{(d+2)B_0 + B_0' - A_0\} \tag{25}$$

$$D_1 \ = \ 2\Delta_0 B_0', \qquad D_2 = B_0 m^2, \qquad D_2' = B_0' m^2 \ \ . \tag{26}$$

Computation of the expressions for the transport coefficient will require the knowledge or construction of the model for the microscopic level dissipation coefficients for the system as stated in Eqn. (22)-(23) in terms of various phenomenological constants.