Proc. Indian Acad. Sci. (Math. Sci.), Vol. 102, No. 3, December 1992, pp. 225–233.

On the zeros of a class of generalised Dirichlet series—XI

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MS received 17 June 1992

Abstract. A sufficiently large class of generalised Dirichlet series is shown to have lots of zeros in $\sigma > 1/2$. Some examples are (i) $\zeta'(s) - a$ (a any complex constant) (ii) $\alpha - \zeta(s) - \Sigma_{n=0}^{\infty}((n+\sqrt{2})^{-s} - (n+1)^{-s})$ (where α is any positive constant) and (iii) $\alpha + \Sigma_{n=1}^{\infty}(-1)^n(\log n)^{\lambda} n^{-s}$ (where λ is any real constant $> 1/2$ and α any complex constant). Here as is usual we have written $s = \sigma + it$.

Keywords. Zeros; generalised Dirichlet series; Riemann zeta-function.

1. Introduction

In paper [1] of this series we considered zeros of $G(s) = \Sigma_{n=1}^{\infty} a_n n^{-s}$ (under fairly general conditions. We have changed the notation for $F(s)$ to $G(s)$ to avoid a clash of notation later) in the rectangle

$$\{\sigma \geqslant \tfrac{1}{2} - \delta, \quad T \leqslant t \leqslant 2T\}, \tag{1}$$

where $\delta = \delta(T) \to 0$ as $T \to \infty$, and as usual $s = \sigma + it$. The only restrictive condition was something like $\Sigma |a_p|^2 \gg x/\log x$, (the sum being over all primes p subject to $x < p \leqslant 2x$) for all large x and what was irksome was the condition $a_1 \neq 0$. The main object of the present paper is to relax the condition $a_1 \neq 0$ to $a_1 = 0, \ldots, a_{n_0} = 0$ and $a_{n_0+1} \neq 0$ where $n_0 (\geqslant 0)$ is an integer constant. Of course we can (as we do) assume $n_0 \geqslant 1$ since the case $n_0 = 0$ is considered in the paper $X^{[1]}$ of this series. Also the condition involving a_p was designed to include $\zeta(s)$; but if we strengthen the lower bound to say $\Sigma |a_p|^2 \gg x(\log x)^2$ then we can prove that $G(s)$ has at least one zero in

$$\{\sigma > \tfrac{1}{2}, \quad T \leqslant t \leqslant 2T \tag{2}$$

provided only that $|G(s)|$ does not exceed a fixed power of T (assuming T to be sufficiently large). Also by using ideas of this paper and those of [7] it is possible to prove that Riemann hypothesis implies that if $q = [\alpha(\log\log T)^{1/2}]$ (where $\alpha > 0$ is a constant) then

$$\liminf_{T \to \infty} \left\{ \frac{1}{T} \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2/q} \mathrm{d}t \right\} \geqslant \exp(\alpha^{-2}). \tag{3}$$

(We may also formulate a result for $1/H\int_T^{T+H}(\cdots)dt$ where $T \geqslant H \gg \log\log T$). The first of these results follows from a routine application of the method of $X^{[1]}$ (except when $a_1 = 0$ in which case the method of the present paper succeeds) while the second follows from the following observation. Consider $G(s)$ where the a_n are multiplicative over square-free integers n. Then the coefficient of $(p_1 \cdots p_k)^{-s}(p_1, \cdots, p_k$ distinct primes) in $(G(s))^{1/q}$ is the same as in

$$\left(1+\frac{a_{p1}}{p_1^s}\right)^{1/q}\left(1+\frac{a_{p2}}{p_2^s}\right)^{1/q}\cdots\left(1+\frac{a_{pk}}{p_k^s}\right)^{1/q}$$

i.e. $q^{-k}a_{p1}a_{p2}\cdots a_{pk}$. We have then to use the Hardy-Ramanujan theorem as in [7]. We do not give further details of the proof of these results. Instead we define a property P_q of a Dirichlet series $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$ where $\{b_n\}$ is any sequence of complex numbers and $\{\mu_n\}$ is any sequence of real numbers with $b_1 = \mu_1 = 1$, $\mu_1 < \mu_2 < \mu_3 < \cdots$ and $1/C \leqslant \mu_{n+1} - \mu_n \leqslant C$ where $C(\geqslant 1)$ is an integer constant. We assume that the series for $G(s)$ converges absolutely for some complex number s.

DEFINITION

Let $q(\geqslant 2)$ be an integer. We say that $G(s)$ has the property P_q if there exists a constant $\delta > 0$ and a positive integer $n^* = n^*(\delta)$ (n^* not divisible by q) both depending on $G(s)$ such that $G(s)$ can be continued analytically in

$$\{\sigma \geqslant \tfrac{1}{2} + \delta, \quad T \leqslant t \leqslant 2T\} \tag{4}$$

and has $\gg T$ zeros all of order n^* in this rectangle.

Remarks. Also we consider functions like $\log\zeta(s) - \alpha$ where α is any complex constant. These have singularities but continuable in $\sigma \geqslant 1/2$. We prove that $\log\zeta(s) - \alpha$ has the property P_2 (if we allow analytic continuation except on horizontal lines which contain singularities). *In what follows n^* may depend on T; but n^* will be bounded above by a constant depending only on δ.*

Accordingly our theorems which illustrate our method are

Theorem 1. *The function $\zeta'(s) - \alpha$ has the property P_2 for every complex constant α.*

Theorem 2. *The function log $\zeta(s) - \alpha$ has the property P_2 (in the sense explained in the remark above) for every complex constant α.*

Theorem 3. *The function $G(s) = \alpha - \Sigma_{n=0}^{\infty}(n + \sqrt{2})^{-s}$ has the property P_2 for every positive real constant α.*

Theorem 4. *Let $\lambda(> 1/2)$ be any constant. Then $G(s) = \alpha + \Sigma_{n=1}^{\infty}(-1)^n(\log n)^{\lambda} n^{-s}$ has the property P_2 for every complex constant α.*

Theorem 5. *The function $G(s) = \alpha + \Sigma_{n=1000}^{\infty}(-1)^n(\log\log n)^{3/4} n^{-s}$ has the property P_q (for some integer $q = q(\delta)$) for every complex constant α.*

Remarks. More general results will be found in the later sections of this paper. It is possible to deal with the zeros in $\{\sigma \geqslant 1-\delta, T \leqslant t \leqslant 2T\}$ in a somewhat general setting. These questions will be taken up elsewhere. We would like to remark that our results hold good for zeros of Dirichlet polynomials like $\Sigma_{n \leqslant T} a_n \mu_n^{-s}$ and $\Sigma_{n \leqslant T^{1000}} a_n \mu_n^{-s}$ (with conditions on $\{a_n\}$ of a fairly general nature and somewhat restrictive conditions on $\{\mu_n\}$).

The previous history of Theorems 1 and 2 is well-known and due to many authors. (For references see [8]. Of great relevance here is the work of Bohr and Jessen [4, 5]. But both our methods and results seem to be new).

2. A conjecture and its proof in special cases

We believe that the following conjecture is true (at least in a modified form). In [2] we have proved it in some special cases and these will be used in the present paper. (We stipulate that certain constants shall be integers only for a technical reason which is not serious). We quote from the paper just cited.

Conjecture. Let $1=\mu_1<\mu_2<\cdots$ *be any sequence of real numbers with* $1/C \leqslant \mu_{n+1}-\mu_n \leqslant C$ *where* $C(\geqslant 1)$ *is an integer constant and* $n=1,2,3,\cdots$. *Let us form the sequence* $1=\lambda_1<\lambda_2<\cdots$ *of all possible (distinct) finite power products of* $1=\mu_1,\mu_2,\cdots$ *with non-negative integral exponents. Let* $s=\sigma+it$, $H(\geqslant 10)$ *a real parameter, and* $\{a_n\}(n=1,2,3,\cdots)$ *with* $a_1=1$ *be any sequence of complex numbers (possibly depending on H) such that* $F(s)=\Sigma_{n=1}^{\infty} a_n \lambda_n^{-s}$ *is absolutely convergent at* $s=B$ *where* $B(\geqslant 3)$ *is an integer constant. Suppose that* $F(s)$ *can be continued analytically in* $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$ *and that there exist* T_1, T_2 *with* $0 \leqslant T_1 \leqslant H^{3/4}$, $H-H^{3/4} \leqslant T_2 \leqslant H$ *such that for some* $K(\geqslant 30)$, *there holds*

$$\max_{\sigma \geqslant 0}(|F(\sigma+iT_1)|+|F(\sigma+iT_2)|) \leqslant K. \tag{5}$$

Finally let $\Sigma_{n=1}^{\infty}|a_n|\lambda_n^{-B} \leqslant H^A$ *where* $A(\geqslant 1)$ *is an integer constant. Then there exists a* $\delta_1(>0)$ *(depending only on* A, B, C*) such that for all* $H \geqslant H_0(A,B,C)$ *there holds*

$$\frac{1}{H}\int_0^H |F(it)|^2\,\mathrm{d}t \geqslant \frac{1}{2}\sum_{\lambda_n \leqslant H^{\delta_1}} |a_n|^2, \tag{6}$$

provided that $H^{-1}\operatorname{loglog} K$ *does not exceed a small positive constant.*

Remark. We have used the symbol δ_1 (in place of δ) so that it should not clash with the δ already introduced. Also we recall that 1/2 can be replaced by a quantity ~ 1 (as $H \to \infty$) and whenever we have succeeded in proving this conjecture we have proved it in this stronger form.

We now quote the corollaries to the main theorem of [2].

COROLLARY 1.

Let $\mu_n = n$. *Then the conjecture is true.*

COROLLARY 2.

Let $n_0(\geqslant 2)$ be an integer constant, and $\mu_n = (n_0 + n - 1)/(n_0)$. Then the conjecture is true.

COROLLARY 3.

Let $\beta > 0$ be an algebraic constant, and $\mu_n = ((n+\beta)/(1+\beta))$. Then the conjecture is true. (The conjecture is also true for the choice $\mu_1 = 1$, $\mu_n = n + \beta - 1$ for $n > 1$).

Remark. It is possible to state a slightly more general corollary than Corollary 3. But we do not state it since our ambition is to prove a sufficiently general result.

3. Two important observations

We record the observations as two lemmas.

Lemma 1. *Let $\mu_n = (n_0 + n - 1)/(n_0)$ and $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$ be absolutely convergent for some complex s. Then, we have, for any integer $q > 0$ and σ large enough,*

$$(G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \tag{7}$$

where the λ_n are formed as in the conjecture, $a_1 = 1$, and further whenever $n_0 + n - 1$ is prime $|a_n| = q^{-1}|b_n|$, and so the RHS of (6) is

$$\geqslant \frac{1}{2q^2} \sum_{\mu_n \leqslant H^{\delta_1}} |b_n|^2 \tag{8}$$

where the sum is restricted to those n for which $n = 1$, and also to those n for which $n_0 + n - 1$ is prime.

Proof. It is sufficient to check that if p is a prime $\geqslant n_0 + 1$, the equality

$$\frac{\ell_1 \cdots \ell_k}{n_0^k} = \frac{p}{n_0}$$

where $\ell_1, \cdots, \ell_k$ are integers $\geqslant n_0 + 1$, is not possible except when $k = 1$ and $\ell_1 = p$. This is trivial since p has to divide at least one ℓ_j say ℓ_1. Now

$$n_0^{k-1} = \left(\frac{\ell_1}{p}\right) \quad \ell_2 \cdots \ell_k \geqslant (n_0 + 1)^{k-1}$$

which is impossible unless $k = 1$.

Lemma 2. *Let $G(s) = 1 - \Sigma_{n=2}^{\infty} b_n \mu_n^{-s}$ where b_n are real and non-negative and the series involved converges for some complex s. Then for any integer $q(> 0)$ and σ large enough, we have,*

$$(G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

where the λ_n are as in the conjecture, $a_1 = 1$ and further for $n \geqslant 2$, $a_n \leqslant 0$ and $-a_n \geqslant b_n q^{-1}$ wherever $\lambda_n = \mu_n$.

4. Proof of theorems 1, 3, 4 and 5

We sketch the proof in a general setting. Note that after an easy normalisation the functions in question look like $G(s) = \Sigma_{n=1}^{\infty} b_n \mu_n^{-s}$, where $b_1 = 1$, $\{\mu_n\}$ as in any of the Corollaries 1, 2 or 3 (of § 2), which converges absolutely for some complex number s and is analytically continuable in $\sigma > 1/2$. It is easy to see that, for $\sigma = 1/2 + \delta$,

$$\frac{1}{T}\int_T^{2T} |G(\sigma + it)|^2 dt \ll \sum_{n=1}^{\infty} |b_n|^2 n^{-1-2\delta} = V(2\delta), \quad \text{say.} \tag{9}$$

From this and the fact that the absolute value of an analytic function at the centre of a circle is majorised by its mean-value over the disc enclosed by it, it follows that

$$\sum_{|I| = H} \max_{s \in ([(1/2)+\delta, \infty) \times I)} |G(s)|^2 \ll \delta^{-2} V(\delta) T \tag{10}$$

where I runs over all disjoint intervals of length H into which $[T, 2T]$ can be divided with a suitable meaning at the end points. We assume that $H \leqslant T^{1/2}$ and that H is a large enough function of δ. From (10) it follows that

$$\#\{I : |I| = H, \max |G(s)|^2 \geqslant \delta^{-3} V(\delta) H\} \ll \delta T/H. \tag{11}$$

Let $q \geqslant 2$ be an integer. In order to obtain the lower bound

$$\frac{1}{H}\int_I |G(s)|^{2/q} dt \gg \sum_{\lambda_n \leqslant H^{\delta_1}} |a_n|^2 n^{-1-2\delta}, \quad (s = \tfrac{1}{2} + \delta + it), \tag{12}$$

we have to check the condition that $H^{-1} \log\log K$ shall not exceed a small positive constant. In (12) $\{a_n\}$ are defined by

$$F(s) = (G(s))^{1/q} = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}.$$

If we assume that in $[\frac{1}{2} + \delta, \infty] \times I$, $F(s)$ is regular (i.e. $G(s)$ has no zeros of order not divisible by q) then (12) holds if H exceeds a large constant depending on δ since we can take $K = \delta^{-3} V(\delta) H$ provided we omit the intervals counted in (11). Also

$$\#\left\{I : \frac{1}{H}\int_I |G(s)|^2 dt \geqslant \eta^{-1} V(2\delta)\right\} \ll \frac{\eta T}{H}, \quad (s = \tfrac{1}{2} + \delta + it), \tag{13}$$

where $\eta > 0$ is a small constant.

Hence we have $\gg TH^{-1}$ intervals I (with $|I| = H$) for which (12) holds and also

$$\frac{1}{H}\int_I |G(s)|^2 dt \leqslant \eta^{-1} V(2\delta). \tag{14}$$

We now show that each of the rectangles $[\frac{1}{2} + \delta, \infty] \times I$ (for these I) must contain a zero of $G(s)$ of order not divisible by q (if we impose a suitable condition on $V(2\delta)$ and $V(\delta)$). Otherwise from (12) and (14) we must have

$$\left(\frac{D_1}{q^2}\sum_{n\leqslant H^{\delta_1}}' |b_n|^2 n^{-1-2\delta}\right)^q < D_2\eta^{-1}\,V(2\delta) \tag{15}$$

where $D_1 > 0$, $D_2 > 0$, and η are independent of T, $\bar{H}$, q and δ. Also the accent restricts the sum as in (8). If the $\{\mu_n\}$ are as in Corollary 3 we end up with

$$\left(\frac{D_1}{q^2}\sum_{n\leqslant H^{\delta_1}} |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}\,V(2\delta). \tag{16}$$

Since we are interested in finding some $H = H(\delta)$ contradicting (15) and (16) we can as well contradict

$$\left(\frac{D_1}{q^2}\sum_{n=1}^{\infty}' |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}\,V(2\delta) \tag{17}$$

for proving Theorems 1,4 and 5. To prove Theorem 3 we have to contradict

$$\left(\frac{D_1}{q^2}\sum_{n=1}^{\infty} |b_n|^2 n^{-1-2\delta}\right)^q \leqslant D_2\eta^{-1}\,V(2\delta). \tag{18}$$

It is a trivial matter to check that (17) and (18) are false for the particular cases in question. This completes the proofs of Theorems 1, 3, 4 and 5 except for the remark concerning n^* (for this see § 7).

5. Some generalisations

It is plain that we can prove analogues of Theorem 1 (also Theorem 2 as will be seen) to $\zeta''(s)$, $\zeta'''(s), \cdots$, derivatives of L-functions and also to derivatives of the zeta and L-functions of any quadratic field. We can also prove the analogues of Theorems 3, 4 and 5 to more general Dirichlet series. We are particularly interested in (stating the analogue for) a class of functions in which we were interested in [3]. We proceed to recall their definition.

Let $\chi(n)(n = 1, 2, 3, \cdots)$ be a periodic sequence of complex numbers not all zero (if the period is k we require that there is at least one integer n with $(n, k) = 1$ and $\chi(n) \neq 0$) such that the sum $\Sigma\chi(n)$ extended over a period is zero. Let $f(x)$ be a positive real valued function of x defined for $x \geqslant 1$ such that for every fixed $\varepsilon > 0$, $f(x)x^{\varepsilon}$ is increasing and $f(x)x^{-\varepsilon}$ is decreasing for all $x \geqslant x_0(\varepsilon)$. Let $\{d_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of complex numbers satisfying $f(n) \ll |d_n| \ll f(n)$ and for all $X \geqslant 1$ we should have

$$\sum_{X\leqslant n\leqslant 2X} |d_{n+1} - d_n| \ll f(X).$$

The functions that we wish to consider are

$$G(s) = \sum_{n=1}^{\infty} \chi(n) d_n n^{-s}.$$

Let us suppose that the expression

$$E(\delta)=\left(\sum_{n=1}^{\infty}(f(n))^2 n^{-1-2\delta}\right)^{1/2}\left(\sum_{n=1}^{\infty}f(n)(\log(n+1))^{-1}n^{-1-2\delta}\right)^{-1} \tag{19}$$

tends to zero as $\delta\to 0$. Then, we have

Theorem 6. *The function $G(s)-\alpha$ has the property P_2 for every complex constant α.*

Proof. This follows from the arguments of § 5 and § 7. We have only to observe that $f(x)\ll f(2x)\ll f(x)$ and that $\pi(x)\asymp x/\log x$.

Remark. We can also state a similar theorem for the property $P_q(q=q(\delta))$.

6. Proof of theorem 2

The proof is not very much different from the one sketched in § 4. Note that we have the density theorem that $N(\sigma, T)$ defined by

$$\#\{\rho:\zeta(\rho)=0\ \operatorname{Re}\rho\geqslant\sigma, |\operatorname{Im}\rho|\leqslant T\}$$

is $O(T^{\nu(1-\sigma)}(\log T)^5)$ where $\nu=3/(2-\sigma)$ due to Ingham [6] (see also page 236 of [8]). The O-constant is independent of σ and T. In view of this theorem the number of t-intervals I of constant length $H=H(\delta)$, satisfying $T\leqslant t\leqslant 2T$ such that $[\frac{1}{2}+\delta/2,\infty)\times I$ is zero free is $\sim T/H$. This and the remark in § 7 are enough for the proof of Theorem 2.

Remark 1. We may also remark that the analogue of Theorem 2 is true for the logarithm of a finite power product (with complex exponents not all zero) of ordinary L-functions or L-functions of a fixed quadratic field since for these L-functions the function $N(\sigma, T)$ is $O(T^{\nu'(1-\sigma)}(\log T)^{C_0})$ where $\nu'=4/(3-2\sigma)$ and C_0 is an absolute constant. The O-constant depends on the modulii of the characters.

Remark 2. Starting from Theorem 2 one may deduce easily the following.

Theorem 7. *The function $\zeta(s)-e^{\alpha}$ has the property P_2 for every complex constant α.*

7. Completion of proofs

We have proved that for the functions in question the number of distinct zeros in $\{\sigma\geqslant\frac{1}{2}+\delta,\ T\leqslant t\leqslant 2T\}$ whose orders are not divisible by q is $\gg T$. But by a slight variant of the considerations of the proof we can secure that the $\gg TH^{-1}$ intervals I selected for the contradiction have the property that in the rectangles $[\frac{1}{2}+\delta/2,\infty]\times I$ the functions are bounded by a function of δ. By Jensen's theorem it follows that the number of zeros (in these rectangles) counted with multiplicity is bounded. Thus the orders of the $\gg T$ zeros as proved already in § 4, § 5 and § 6 are bounded by a function of δ alone. Hence (by classifying these zeros according to their orders) we see that $\gg T$ zeros (in at least one class) have the same order (a fixed integer not divisible by q). This completes the proof of all our assertions.

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POST-SCRIPT. The condition $E(\delta)\to 0$ as $\delta\to 0$ see ((9)) can be proved under various choices of $f(n)$. For example let $(\log n)^2 \leqslant f(n) \leqslant \exp((\log n)^{0\cdot 1})$. Then $E(\delta)\to 0$ as $\delta\to 0$. To see this we begin with a

Lemma. Let $f(n)(n=1,2,3,\cdots)$ *be any sequence of positive real numbers such that* $(\log f(n))(\log n)^{-1}\to 0$ *as* $n\to\infty$. *For any* $\delta>0$ *put*

$$Q_1=\sum_{n=1}^{\infty}(f(n))^2 n^{-1-2\delta},\quad Q_2=\sum_{n=1}^{\infty}(f(n))^2(\log(n+1))^{-1}n^{-1-2\delta},$$

and

$$Q_3=\sum_{1\leqslant \exp(Q_1^{1/4})}(f(n))^2 n^{-1-2\delta}.$$

If $Q_1-Q_3\leqslant\frac{1}{2}Q_1$ *and* $Q_1\geqslant(1/\varepsilon)^2$, $(0<\varepsilon<\frac{1}{2})$, *then* $Q_1\ll\varepsilon Q_2^2$.

Proof. We have

$$\begin{aligned}Q_2 &\geqslant \sum_{1\leqslant\exp(Q_1^{1/4})}(f(n))^2(\log(n+1))^{-1}n^{-1-2\delta}\\ &\gg Q_1^{-1/4}Q_1(\text{with an implied absolute constant, since } Q_3\geqslant\tfrac{1}{2}Q_1)\end{aligned}$$

i.e. $Q_2^2\gg Q_1^{3/2}\gg(1/\varepsilon)Q_1$ since $Q_1\geqslant(1/\varepsilon)^2$.

This completes the proof of the lemma.

COROLLARY.

Let $(\log n)^2\leqslant f(n)\leqslant\exp((\log n)^{0\cdot 1})$. *Then* $E(\delta)\to 0$ *as* $\delta\to 0$.

Proof. In this case $Q_1 \geqslant \Sigma_{n=1}^{\infty}(\log n)^4 n^{-1-2\delta} \asymp \delta^{-5} \geqslant (1/\varepsilon)^2$ if δ is sufficiently small.

We have only to prove that $Q_1 - Q_3 \leqslant \frac{1}{2}Q_1$. Let d be any positive constant. We will show that $Q_4 = \Sigma_{n \geqslant \exp(d\delta^{-1\cdot 25})}(f(n))^2 n^{-1-2\delta}$ tends to zero as $\delta \to 0$. For $n \geqslant \exp(d\delta^{-1\cdot 25})$, we have

$$\frac{n^{2\delta}}{(f(n))^2} \geqslant \frac{n^{2\delta}}{\exp(2(\log n)^{0\cdot 1})} \geqslant \exp\{2\delta \log n - 2(\log n)^{0\cdot 1}\}$$

$$\geqslant \exp\{(\log n)^{0\cdot 1}(2\delta(\log n)^{0\cdot 9} - 2)\}$$

$$\geqslant \exp\{(\log n)^{0\cdot 1}(2\delta(d^{0\cdot 9})(1/\delta)^{1\cdot 125} - 2\}$$

$\geqslant (\log n)^2$ (for all n exceeding an absolute constant if δ is small enough).

Thus $Q_4 \to 0$ as $\delta \to 0$. This proves the corollary completely since $\Sigma_{n=2}^{\infty} n(\log n)^{-2}$ is convergent. (For the validity of $E(\delta) \to 0$ clearly we can impose $(\log n)^{R_1} \leqslant f(n) \leqslant \exp((\log n)^{R_2})$ where $R_1 (> 3/2)$ and $R_2 (< 1 - 4(2R_1 + 1)^{-1})$ are constants).