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QUANTIZATION ERROR IN STEREO IMAGING SYSTEMS

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In this paper a stochastic analysis of the quantization error in a stereo imaging system has been presented. Further the probability density function of the range estimation error and the expected value of the range error magnitude are derived in terms of various design parameters. Further the relative range error is proposed.

Keywords: Quantization error; Stereo images; Range estimation error; Range error magnitude; Relative range error

C.R. Categories: I 2.10, I 3.7

1 INTRODUCTION

The ability to obtain the accurate three-dimensional position information in the presence of limited sensor resolution is a crucial task in computer vision and other triangulation systems. Sensors for computer processing applications produce sampled quantized data, whose spatial resolution is determined by limits in device technology and bandwidth. In computer vision and photogrammetry, normally stereo camera setups as shown in Figure 1 are used for obtaining 3-D data.

In a stereo camera system the two viewing cameras are separated by a distance b along same base line, which is normally taken along the positive x axis. With such a stereo-camera system, two images namely, left image and right image are obtained of any 3-D point. The three-dimensional coordinates of each image point is found by computing the disparities between the corresponding left and right image points. This results in a scanty distribution of reconstructed 3-D points.

Designing of any stereo-system is dependent on various parameters like focal lengths of the viewing cameras, the distance of separation between the two viewing cameras, distance of the cameras from the viewing point and interval of image sampling. The relationship

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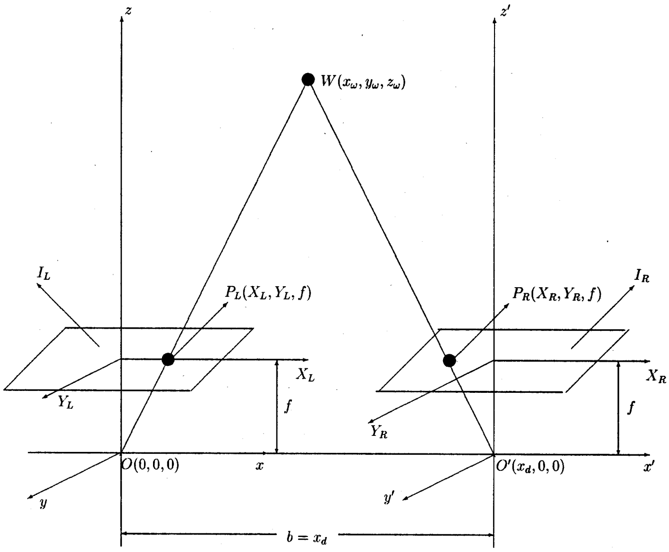

FIGURE 1 Stereo imaging set-up

between the geometry of the stereo-setup and the accuracy in obtaining the actual 3-D position has received scant attention, though it is of great practical importance.

Any stereo set-up need to be designed suitably, to have accurate feature matching and accurate range estimation. Accurate feature matching is possible by choosing the product of focal length and base line distance to be small. On the contrary this product of focal length and base line distance need to be chosen as large as possible for accurate range estimation. Thus these two design criteria of accurate feature matching and accurate range estimation require conflicting requirements. By decreasing the sampling interval, it is also possible to improve the accuracy of range estimation. But it also has its own physical limitations of the imaging set up.

The above constraints necessitated the formulation of acceptable range estimation error of a chosen stereo-imaging set up. Thus it is always possible to choose appropriate design parameters of a stereo-imaging set up, if the acceptable range estimation error is known a proiri. Hence it is necessary to develop a methodology to predict the range estimation error in terms of stereo system parameters. Blostein *et al.* [3] have proposed a mathematical equation in terms of disparity value at a given pixel, so that the probability of the percent range error is less than a given level. But it has been found that use of the relative range error is more advantageous to that of percent range error in formulating relevant mathematical equations for choosing appropriate stereo-imaging set-up. It is also better to derive the expected value of relative range error, which is more useful than deriving the condition for probability of this error being less than a given level. For practical design purposes it is absolutely necessary to formulate the relevant mathematical equations in terms of design parameters of the stereo-system, instead of disparity value at a particular pixel. Though Mcvey *et al.* [4], Verri *et al.* [5] and Matthies *et al.* [6] have given detailed description of stereo quantization error studies, none of them have neither made any stochastic analysis nor derived a closed form expression of the expected range error.

Rodriguez *et al.* [7] have derived the probability density function of the range estimation error and expected value of the range error magnitude in terms of design parameters of the stereo set up. They have also given experimental results to support the accuracy of the theoretical model proposed by them. But Rodriguez *et al.* [7] have assumed that the quantizing errors in x and y coordinate values in the two image planes to be uniformly distributed in order to simplify the calculations. This need not be true in all practical situations. In the work described in this paper the quantizing errors in the respective image planes are assumed to have unimodal distributions, and the depth value z is assumed to be distributed uniformly between $z_{\min}$ and $z_{\max}$. Under the above assumptions the various relevant functions are derived and the marginal density of z coordinate is also calculated.

In order to predict the range estimation error in stereo-imaging to aid the design process, it is useful to derive the expected value of the error. This has necessitated use of a stochastic analysis to derive a closed-form expression for the expected range error. In this paper the probability density function of the range estimation error and the expected value of the range error magnitude are derived in terms of parameters of the stereo system. This paper is an extension of the results of Rodriguez *et al.* [7], for a unimodal distribution of the quantization error in x and y coordinate values, in a stereo imaging system.

2 STEREO IMAGING SETUP

The stereo imaging set-up using two cameras is shown in Figure 1. Let I_L and I_R be the left and right image planes of the pair of cameras C_L and C_R respectively which share a 3-D feature. Let the position and orientation of one camera be known with respect to another and both cameras have a common field of view. Let $OXYZ$ be the rectangular cartesian frame of reference with its origin O at the centre of projection of one of the cameras, say left camera C_L. A point W in 3D-space with its coordinates (x_w, y_w, z_w) with respect to this frame of reference at C_L is viewed by the two cameras C_L and C_R. Let the centre of the right camera C_R be at a point $O'(b, 0, 0)$ with respect to the first camera C_L and is separated by a distance b along a base line which is taken along positive OX axis. Let $O_LX_LY_L$ be the rectangular cartesian system in the left image plane with its origin at $(0, 0, f)$ with respect to $OXYZ$ system. Similarly let $O_RX_RY_R$ be the rectangular cartesian system in the right image plane with its axes parallel to $O_LX_LY_L$ system and its origin at $(b, 0, f)$ with respect $OXYZ$ system. Let the coordinate of the corresponding coordinates of a 3D point $W(x_w, y_w, z_w)$ in the left image plane be $P_L(X_L, Y_L)$ with respect to $O_LX_LY_L$ system and in the right image plane be $P_R(X_R, Y_R)$ with respect to $O_RX_RY_R$ system.

By projecting radially each point in the field of view, through respective focal points on to the left and right image planes, the corresponding left and right image points of any 3D point $W(x_w, y_w, z_w)$ can be obtained. By using collinearity equations [1] and [2],

$$X_L = \frac{fx_w}{z_w} \tag{1}$$

$$X_R = \frac{f(x_w - b)}{z_w} \tag{2}$$

$$Y_L = Y_R = \frac{fy_w}{z_w} \tag{3}$$

These three equations can be inverted to obtain the 3-D coordinate values of x_w, y_w and z_w. Let the disparity d be defined as the difference in coordinate values

$$d = X_L - X_R \tag{4}$$

The corresponding values of x, y and z are given by the inverse perspective projection equations as

$$(x_w, y_w, z = z_w) = \frac{b}{d}(X_L, Y_L, f) \tag{5}$$

Using equation (5) it is possible to compute the three dimensional structure from its stereo images. Throughout the work described in this paper, symbol z_w is replaced by z for convenience in calculations.

3 QUANTIZATION ERROR

Since the imaging set up is discrete in nature, the image coordinates of each pixel can be assumed to suffer from quantization errors of up to $\pm\frac{1}{2}$ pixel. Choosing the image sampling interval as δ, the corresponding quantization error in each of x and y coordinates of the left image (X_L, Y_L) and right image (X_R, Y_R) become $\pm\delta/2$. Hence the error in the disparity $d = X_L - X_R$ reduces to $\pm\delta$. Let the quantized disparity be defined as $\hat{d}$, and disparity error as $\Delta d = \hat{d} - d$, where $-\delta < \Delta d < \delta$, Rodriguez *et al.* [7].

The error in estimating z dominates over estimated errors in the values of x and y which also suffer quantization errors. In this paper the behaviour of the range estimation error Δz is analysed in terms of system parameters. Let $z_{\min}$ and $z_{\max}$ be the minimum and maximum range values in the field of view. On assuming $0 < z_{\min} \le z \le z_{\max} < bf/\delta$, it follows that $bf + z\Delta d > 0$. Let the quantized disparity in z be denoted by $\hat{z}$, then the range estimation error Δz becomes

$$\begin{aligned}
\Delta z &= \hat{z} - z \\
&= \frac{bf}{\hat{d}} - z \\
&= \frac{bf}{d + \Delta d} - z \\
&= \frac{bf}{(bf/z) + \Delta d} - z \\
&= \frac{-z^2 \Delta d}{bf + z\Delta d}
\end{aligned} \tag{6}$$

If the value of the product bf in the denominator of equation (6) is increased, the value of range estimation error Δz can be reduced. Since $z_{\min} < z < z_{\max}$, using (6), the bound for the range estimation error becomes

$$\begin{gathered}
\frac{-z_{\max}^2 \delta}{bf + z_{\max}\delta} \le \Delta z \le \frac{z_{\max}^2 \delta}{bf - z_{\max}\delta} \\
0 \le |\Delta z| \le \frac{z_{\max}^2 \delta}{bf - z_{\max}\delta}
\end{gathered} \tag{7}$$

Equation (6) implies that the range error increases in magnitude, as the range increases. Let a relative range error be defined as $\Delta z/z$. This relative range error also increases in magnitude as the range increases. Therefore, stereo range estimation is more accurate for nearby objects than for distant ones.

4 PROBABILITY DENSITY FUNCTION OF Δz

It is possible to examine stochastically the range estimation error Δz which is a function of random variables Δd and z. The probability density function of Δz can be formulated by examining the geometrical relationships between the variable Δz and other variables upon which Δz depends. However, such a formulation is a tedious process. To avoid this it is assumed that the quantization errors ΔX_L and ΔX_R in X_L and X_R are independent of each other, and z. Rodriguez *et al.* [7] have assumed the quantization errors in X_L and X_R to be uniformly distributed. These disparities in coordinate values are not usually small. Blostein *et al.* [3] have shown the quantization errors to be very accurate in the case of large disparities. But in the work described in this paper, ΔX_L and ΔX_R are assumed to have unimodal distributions between $-\delta/2$ and $\delta/2$ as shown in Figures 2 and 3, whereas z is assumed to be distributed uniformly between $z_{\min}$ and $z_{\max}$. Under these assumptions the probability density function of Δd can be derived easily as follows.

Let the quantization error in X_L and X_R be ΔX_L and ΔX_R respectively. Since $d = X_L - X_R$ the disparity error becomes

$$\Delta d = \Delta X_L - \Delta X_R \tag{8}$$

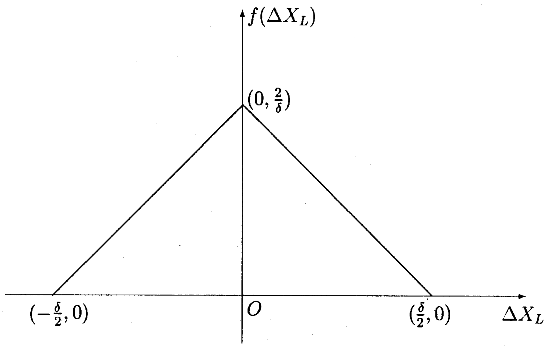

FIGURE 2 Unimodal distribution of error ΔX_L

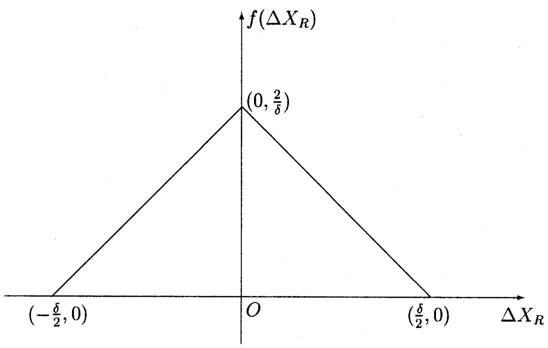

FIGURE 3 Unimodal distribution of error ΔX_R.

Hence the corresponding probability density functions are

$$
\begin{aligned}
f_{\Delta X_L}(\Delta X_L) &= \frac{2(2\Delta X_L + \delta)}{\delta^2}, \quad -\delta/2 < \Delta X_L \le 0 \\
&= \frac{-2(2\Delta X_L - \delta)}{\delta^2}, \quad 0 \le \Delta X_L < \delta/2
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
f_{\Delta X_R}(\Delta X_R) &= \frac{2(2\Delta X_R + \delta)}{\delta^2}, \quad -\delta/2 < \Delta X_R \le 0 \\
&= \frac{-2(2\Delta X_R - \delta)}{\delta^2}, \quad 0 \le \Delta X_R < \delta/2
\end{aligned} \tag{10}
$$

These probability density functions are assumed to be zero outside the specified intervals in the entire work described in this paper. The probability density functions of Δd can be formulated as follows. Let

$$y = \Delta X_R \tag{11}$$

Therefore

$$
\begin{aligned}
\Delta X_L &= y + \Delta d \\
\Delta X_R &= y
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
f(\Delta d, y) &= f(\Delta X_L, \Delta X_R)\left|\frac{\partial(\Delta X_L, \Delta X_R)}{\partial(\Delta d, y)}\right| \\
&= f(\Delta X_L, \Delta X_R)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \\
&= f(\Delta X_L, \Delta X_R) \\
&= f_{\Delta X_L}(\Delta X_L) \cdot f_{\Delta X_R}(\Delta X_R)
\end{aligned}
$$

$$
\begin{aligned}
f(\Delta d, y) &= \frac{4(2\Delta X_L + \delta)(2\Delta X_R + \delta)}{\delta^4}, \quad -\delta/2 < \Delta X_L \le 0, -\delta/2 < \Delta X_R \le 0 \\
&= \frac{-4(2\Delta X_L + \delta)(2\Delta X_R - \delta)}{\delta^4}, \quad -\delta/2 < \Delta X_L \le 0, 0 \le \Delta X_R < \delta/2 \\
&= \frac{-4(2\Delta X_L - \delta)(2\Delta X_R + \delta)}{\delta^4}, \quad 0 \le \Delta X_L < \delta/2, -\delta/2 < \Delta X_R \le 0 \\
&= \frac{4(2\Delta X_L - \delta)(2\Delta X_R - \delta)}{\delta^4}, \quad 0 \le \Delta X_L < \delta/2, 0 \le \Delta X_R < \delta/2
\end{aligned}
$$

Using (11) and (12) the above equations reduce to

$$
\begin{aligned}
f(\Delta d, y) &= \frac{4}{\delta^4}(2y + 2\Delta d + \delta)(2y + \delta), \quad -\delta/2 < y + \Delta d \leq 0, -\delta/2 < y \leq 0 \\
&= \frac{-4}{\delta^4}(2y + 2\Delta d + \delta)(2y - \delta), \quad -\delta/2 < y + \Delta d \leq 0, 0 \leq y < \delta/2 \\
&= \frac{-4}{\delta^4}(2y + 2\Delta d - \delta)(2y + \delta), \quad 0 \leq y + \Delta d < \delta/2, -\delta/2 < y \leq 0 \\
&= \frac{4}{\delta^4}(2y + 2\Delta d\delta)(2y - \delta), \quad 0 \leq y + \Delta d < \delta/2, 0 \leq y < \delta/2
\end{aligned}
$$

The probability density function of Δd is given by,

$$
\begin{aligned}
f_{\Delta d}(\Delta d) &= \frac{4}{\delta^4}\left[\frac{2\delta^3}{3} + 2\delta^2\Delta d + 2\delta\Delta d^2 + \frac{2}{3}\Delta d^3\right], && -\delta \leq \Delta d \leq -\frac{\delta}{2} \\
&= \frac{2}{3\delta^4}(2\delta^3 - 12\delta\Delta d^2 - 12\Delta d^3), && -\frac{\delta}{2} \leq \Delta d \leq 0 \\
&= \frac{2}{3\delta^4}(2\delta^3 - 12\delta\Delta d^2 + 12\Delta d^3), && 0 \leq \Delta d \leq \frac{\delta}{2} \\
&= -\frac{8}{3\delta^4}(\Delta d - \delta)^3, && \frac{\delta}{2} \leq \Delta d \leq \delta
\end{aligned}
\tag{13}
$$

The detailed derivation of the above equation is shown in appendix.

From equation (6) it is clear that Δz is a monotonic function of Δd. Therefore,

$$
\begin{aligned}
f_{\Delta z}(\Delta z \mid z) &= f_{\Delta d}(\Delta d)\left|\frac{d(\Delta d)}{d(\Delta z)}\right| \\
&= \frac{bf}{(z + \Delta z)^2}\,\frac{8}{3\delta^4}(\delta + \Delta d)^3, \qquad -\delta \leq \Delta d \leq -\frac{\delta}{2} \\
&= \frac{bf}{(z + \Delta z)^2}\,\frac{8}{3\delta^4}\left(3\frac{\delta^4}{4} - 3\delta^2\Delta d - 6\delta\Delta d^2 - \Delta d^3\right), \qquad -\frac{\delta}{2} \leq \Delta d \leq 0 \\
&= \frac{bf}{(z + \Delta z)^2}\,\frac{2}{3\delta^4}(2\delta^3 - 15\delta\Delta d^2 + 9\Delta d^3), \qquad 0 \leq \Delta d \leq \frac{\delta}{2} \\
&= \frac{bf}{(z + \Delta z)^2}\,\frac{-8}{3\delta^4}(\Delta d - \delta)^3, \qquad \frac{\delta}{2} \leq \Delta d \leq \delta
\end{aligned}
\tag{14}
$$

From the equation (6) it is clear that

$$
\Delta d = \frac{-\Delta z b f}{z(z + \Delta z)} \tag{15}
$$

Therefore from the equations (14) the following four cases arise,

Case I:

$$\text{If } \frac{z^2\delta}{(2bf - z\delta)} \leq \Delta z \leq \frac{z^2\delta}{(bf - \delta z)}, \quad \text{then } -\delta \leq \Delta d \leq -\frac{\delta}{2}$$

and

$$\begin{aligned} f_{\Delta z}(\Delta z \mid z) &= f_{\Delta d}(\Delta d)\left|\frac{d(\Delta d)}{d(\Delta z)}\right| \\ &= \frac{bf}{(z+\Delta z)^2}\left\{\frac{8}{3\delta^4}\left(\delta - \frac{\Delta z bf}{z(z+\Delta z)}\right)^3\right\} \\ &= \frac{bf}{(z+\Delta z)^5}\frac{8}{3\delta^4}\left[\frac{\delta z(z+\Delta z) - \Delta z bf}{z}\right]^3 \end{aligned}$$

Case II:

$$\text{If } \frac{z^2\delta}{(2bf - z\delta)} \geq \Delta z \geq 0, \quad \text{then } -\frac{\delta}{2} \leq \Delta d \leq 0$$

and

$$f_{\Delta z}(\Delta z|z) = \frac{bf}{(z+\Delta z)^2}\left\{\frac{2}{3\delta^4}\left(2\delta^3 - 12\delta\frac{b^2f^2(\Delta z)^2}{z^2(z+\Delta z)^2} - 12\frac{b^3f^3(\Delta z)^3}{z^3(z+\Delta z)^3}\right)\right\}$$

Case III:

$$\text{If } -\frac{z^2\delta}{(2bf + z\delta)} \leq \Delta z \leq 0, \quad \text{then } 0 \leq \Delta d \leq \frac{\delta}{2}$$

and

$$f_{\Delta z}(\Delta z|z) = \frac{bf}{(z+\Delta z)^2}\left\{\frac{2}{3\delta^4}\left(2\delta^3 - 12\delta\frac{b^2f^2(\Delta z)^2}{z^2(z+\Delta z)^2} + 12\frac{b^3f^3(\Delta z)^3}{z^3(z+\Delta z)^3}\right)\right\}$$

Case IV:

$$\text{If } -\frac{z^2\delta}{(bf + z\delta)} \leq \Delta z \leq -\frac{z^2\delta}{(2bf + \delta z)}, \quad \text{then } \frac{\delta}{2} \leq \Delta d \leq \delta$$

and

$$\begin{aligned} f_{\Delta z}(\Delta z|z) &= \frac{bf}{(z+\Delta z)^2}\left\{\frac{8}{3\delta^4}\left(\delta + \frac{\Delta z bf}{z(z+\Delta z)}\right)^3\right\} \\ &= \frac{bf}{(z+\Delta z)^5}\frac{8}{3\delta^4}\left[\frac{\delta z(z+\Delta z) + \Delta z bf}{z}\right]^3 \end{aligned}$$

Further the probability density function of Δz is given by,

$$f_{\Delta z}(\Delta z) = \int_{-\infty}^{\infty} f_{\Delta z}(\Delta z|z) f_z(z) dz \tag{16}$$

It is assumed that z is uniformly distributed between $z_{\min}$ and $z_{\max}$. So for $0 < \Delta z \leq z^2\delta/2bf - \delta z$,

$$f_{\Delta z}(\Delta z) = \frac{2bf}{3\delta^4(z_{\max} - z_{\min})} \int_{z_{\min}}^{z_{\max}} \left[\frac{2\delta^3}{(z+\Delta z)^2} - \frac{12\delta b^2 f^2 (\Delta z)^2}{z^2(z+\Delta z)^4} - \frac{12 b^3 f^3 (\Delta z)^3}{z^3 (z+\Delta z)^5} \right] g_1(z) dz$$

where

$$g_1(z) = \begin{cases} 1, & \text{for } \Delta z \leq \dfrac{\delta z^2}{2bf - \delta z} \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

It is easily shown that $g_1(z) = 1$, iff $z \geq z^+$, where

$$z^+ = \frac{(-\delta \Delta z + \sqrt{\delta^2 \Delta z^2 + 8bf\delta\Delta z})}{2\delta} \tag{18}$$

So for, $\Delta z > 0$,

$$f_{\Delta z}(\Delta z) = \frac{2bf}{3\delta^4(z_{\max} - z_{\min})} \int_{z_0^+}^{z_{\max}} \left[\frac{2\delta^3}{(z+\Delta z)^2} - \frac{12\delta b^2 f^2 (\Delta z)^2}{z^2(z+\Delta z)^4} - \frac{12 b^3 f^3 (\Delta z)^3}{z^3 (z+\Delta z)^5} \right] dz$$

$$\begin{aligned} f_{\Delta z}(\Delta z) = \frac{2bf}{3\delta^4(z_{\max} - z_{\min})} \Bigg\{ & -\frac{2\delta^3}{z+\Delta z} - 12\delta b^2 f^2 (\Delta z)^2 \left[\frac{4}{(\Delta z)^5} \log\left(1 + \frac{\Delta z}{z}\right) \right. \\ & \left. - \frac{1}{z(\Delta z)^4} - \frac{3}{(z+\Delta z)(\Delta z)^4} - \frac{1}{(\Delta z)^3 (z+\Delta z)^2} - \frac{1}{3(\Delta z)^2 (z+\Delta z)^3} \right] \\ & - 12 b^3 f^3 (\Delta z)^3 \left[\frac{15}{(\Delta z)^7} \log\left(\frac{z}{z+\Delta z}\right) + \frac{5}{z(\Delta z)^6} - \frac{1}{2(\Delta z)^5 z^2} + \frac{10}{(\Delta z)^6 (z+\Delta z)} \right. \\ & \left. \left. + \frac{3}{(\Delta z)^5 (z+\Delta z)^2} + \frac{1}{(\Delta z)^4 (z+\Delta z)^3} + \frac{1}{4(\Delta z)^3 (z+\Delta z)^4} \right] \Bigg\} \right|_{z_0^+}^{z_{\max}} \end{aligned} \tag{19}$$

where $z_0^+ = \max\{z_{\min}, z^+\}$

Similarly when $\dfrac{z^2\delta}{2bf - z\delta} \leq \Delta z \leq \dfrac{z^2\delta}{bf - \delta z}$,

$$f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max} - z_{\min})} \int_{z_{\min}}^{z_{\max}} \left[\frac{\delta z(z+\Delta z) - \Delta z bf}{z} \right]^3 \frac{g_2(z)}{(z+\Delta z)^5} dz$$

where

$$g_2(z) = \begin{cases} 1, & \text{if } \dfrac{z^2\delta}{2bf - z\delta} \leq \Delta z \leq \dfrac{\delta z^2}{bf - \delta z} \\ 0, & \text{otherwise} \end{cases}$$

It is clear that $g_2(z) = 1$, iff $z \geq z^{++}$, where

$$z^{++} = \frac{(\delta \Delta z + \sqrt{\delta^2 \Delta z^2 + 4bf\delta\Delta z})}{2\delta} \tag{20}$$

Hence in this case,

$$f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max} - z_{\min})} \int_{z_0^{++}}^{z_{\max}} \left[\frac{\delta z(z + \Delta z) - \Delta z b f}{z} \right]^3 \frac{dz}{(z + \Delta z)^5} \tag{21}$$

where $z_0^{++} = \max\{z_{\min}, z^{++}\}$

$$\begin{aligned} f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max} - z_{\min})} \Bigg\{ &- \frac{\delta^3}{z + \Delta z} - 3\delta^2(\Delta z)^2 bf \left[\frac{1}{(\Delta z)^3} \log\left(\frac{z}{z + \Delta z} \right) \right. \\ &\left. + \frac{1}{(z + \Delta z)(\Delta z)^2} + \frac{1}{2\Delta z(z + \Delta z)^2} \right] + 3\delta b^2 f^2 (\Delta z)^2 \left[\frac{4}{(\Delta z)^5} \log\left(1 + \frac{\Delta z}{z} \right) \right. \\ &\left. - \frac{1}{z(\Delta z)^4} - \frac{3}{(z + \Delta z)(\Delta z)^4} - \frac{1}{(\Delta z)^3(z + \Delta z)^2} - \frac{1}{3(\Delta z)^2(z + \Delta z)^3} \right] \\ &- b^3 f^3 (\Delta z)^3 \left[\frac{15}{(\Delta z)^7} \log\left(\frac{z}{z + \Delta z} \right) + \frac{5}{z(\Delta z)^6} - \frac{1}{2(\Delta z)^5 z^2} + \frac{10}{(\Delta z)^6(z + \Delta z)} \right. \\ &\left. \left. + \frac{3}{(\Delta z)^5(z + \Delta z)^2} + \frac{1}{(\Delta z)^4(z + \Delta z)^3} + \frac{1}{4(\Delta z)^3(z + \Delta z)^4} \right] \Bigg\} \right|_{z_0^{++}}^{z_{\max}} \end{aligned} \tag{22}$$

Similarly for, $-\dfrac{z^2\delta}{2bf + \delta z} \le \Delta z < 0$

$$f_{\Delta z}(\Delta z) = \frac{2bf}{3\delta^4(z_{\max} - z_{\min})} \int_{z_{\min}}^{z_{\max}} \left[\frac{2\delta^3}{(z + \Delta z)^2} - \frac{12\delta b^2 f^2 (\Delta z)^2}{z^2(z + \Delta z)^4} + \frac{12 b^3 f^3 (\Delta z)^3}{z^3(z + \Delta z)^5} \right] g_3(z)\, dz$$

where

$$g_3(z) = \begin{cases} 1, & \text{for } \Delta z \ge -\dfrac{\delta z^2}{2bf + \delta z} \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

Clearly $g_3(z) = 1$, iff $z \ge z^-$, where

$$z^- = \frac{(-\delta\Delta z + \sqrt{\delta^2 \Delta z^2 - 8bf\delta\Delta z})}{2\delta} \tag{24}$$

$$\begin{aligned} f_{\Delta z}(\Delta z) = \frac{2bf}{3\delta^4(z_{\max} - z_{\min})} \Bigg\{ &- \frac{2\delta^3}{z + \Delta z} - 12\delta b^2 f^2 (\Delta z)^2 \left[\frac{4}{(\Delta z)^5} \log\left(1 + \frac{\Delta z}{z} \right) \right. \\ &\left. - \frac{1}{z(\Delta z)^4} - \frac{3}{(z + \Delta z)(\Delta z)^4} - \frac{1}{(\Delta z)^3(z + \Delta z)^2} - \frac{1}{3(\Delta z)^2(z + \Delta z)^3} \right] \\ &+ 12 b^3 f^3 (\Delta z)^3 \left[\frac{15}{(\Delta z)^7} \log\left(\frac{z}{z + \Delta z} \right) + \frac{5}{z(\Delta z)^6} - \frac{1}{2(\Delta z)^5 z^2} + \frac{10}{(\Delta z)^6(z + \Delta z)} \right. \\ &\left. \left. + \frac{3}{(\Delta z)^5(z + \Delta z)^2} + \frac{1}{(\Delta z)^4(z + \Delta z)^3} + \frac{1}{4(\Delta z)^3(z + \Delta z)^4} \right] \Bigg\} \right|_{z_0^{-}}^{z_{\max}} \end{aligned} \tag{25}$$

where $z_0^- = \max\{z_{\min}, z^-\}$

Similarly when $\dfrac{-z^2\delta}{bf+\Delta z\delta} \le \Delta z \le \dfrac{-z^2\delta}{2bf+\delta z}$,

$$f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max}-z_{\min})}\int_{z_{\min}}^{z_{\max}}\left[\frac{\delta z(z+\Delta z)+\Delta zbf}{z}\right]^3 \frac{g_4(z)}{(z+\Delta z)^5}\,dz$$

where

$$g_4(z) = \begin{cases} 1, & \text{if } \dfrac{-z^2\delta}{bf+z\delta} \le \Delta z \le \dfrac{-\delta z^2}{2bf+\delta z} \\ 0, & \text{otherwise} \end{cases}$$

It is clear that $g_4(z) = 1$, iff $z \ge z^{--}$, where

$$z^{--} = \frac{(-\delta\Delta z + \sqrt{\delta^2\Delta z^2 - 4bf\delta\Delta z})}{2\delta} \tag{26}$$

Hence in this case,

$$f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max}-z_{\min})}\int_{z_0^{--}}^{z_{\max}}\left[\frac{\delta z(z+\Delta z)+\Delta zbf}{z}\right]^3 \frac{dz}{(z+\Delta z)^5} \tag{27}$$

where $z_0^{--} = \max\{z_{\min}, z^{--}\}$.
Hence

$$\begin{aligned} f_{\Delta z}(\Delta z) = \frac{8bf}{3\delta^4(z_{\max}-z_{\min})}\Bigg\{&-\frac{\delta^3}{z+\Delta z} + 3\delta^2(\Delta z)^2 bf\left[\frac{1}{(\Delta z)^3}\log\left(\frac{z}{z+\Delta z}\right)\right. \\ &\left.+\frac{1}{(z+\Delta z)(\Delta z)^2} + \frac{1}{2\Delta z(z+\Delta z)^2}\right] + 3\delta b^2 f^2(\Delta z)^2\left[\frac{4}{(\Delta z)^5}\log\left(1+\frac{\Delta z}{z}\right)\right. \\ &\left.-\frac{1}{z(\Delta z)^4} - \frac{3}{(z+\Delta z)(\Delta z)^4} - \frac{1}{(\Delta z)^3(z+\Delta z)^2} - \frac{1}{3(\Delta z)^2(z+\Delta z)^3}\right] \\ &+ b^3f^3(\Delta z)^3\left[\frac{15}{(\Delta z)^7}\log\left(\frac{z}{z+\Delta z}\right) + \frac{5}{z(\Delta z)^6} - \frac{1}{2(\Delta z)^5 z^2} + \frac{10}{(\Delta z)^6(z+\Delta z)}\right. \\ &\left.\left.+\frac{3}{(\Delta z)^5(z+\Delta z)^2} + \frac{1}{(\Delta z)^4(z+\Delta z)^3} + \frac{1}{4(\Delta z)^3(z+\Delta z)^4}\right]\Bigg\}_{z_0^{--}}^{z_{\max}} \end{aligned} \tag{28}$$

where $z_0^{--} = \max_{\min}, z^{--}\}$ and

$$z^{--} = \frac{-\delta\Delta z + \sqrt{\delta^2(\Delta z)^2 - 4\delta bf\Delta z}}{2\delta} \tag{29}$$

Finally, for $\Delta z = 0$,

$$\begin{aligned} f_{\Delta z}(0) &= \frac{2bf}{3\delta^4(z_{\max}-z_{\min})}\int_{z_{\min}}^{z_{\max}}\frac{2\delta^3}{z^2}\,dz \\ &= \frac{4bf}{3\delta(z_{\min}z_{\max})} \end{aligned} \tag{30}$$

5 RELATIVE RANGE ERROR

For practical applications like object recognition, where the objects normally occupy only a small fraction of the total range, the relative range error becomes a more of descriptive quantity. However relative range error $\varepsilon = [|\Delta z|/z_{\max} - z_{\min}]$, is mostly used to measure the accuracy of a stereo imaging system. The range resolution is described better by this relative range error as compared to percent error ($|\Delta z|/z$).

The expected value of the relative range error can be computed only after deriving the expected value of the range error. The derivation is based on equations (6) and (13) as described below.

The Expected value of the range error is given by,

$$\begin{aligned} E(|\Delta z|/z) &= \int_{-\infty}^{\infty} f_{\Delta d}(\Delta d)\, d(\Delta d) \\ &= \int_{0}^{\delta/2} \frac{z^2 \Delta d}{bf + z\Delta d}\left(\frac{2}{3\delta^4}\right)(2\delta^3 - 12\delta(\Delta d)^2 + 12(\Delta d)^3) d(\Delta d) \\ &\quad + \int_{\delta/2}^{\delta} \frac{z^2 \Delta d}{bf + z\Delta d}\left(\frac{-8}{3\delta^4}\right)(\Delta d - d)^3 d(\Delta d) \\ &\quad - \int_{-\delta}^{-\frac{\delta}{2}} \frac{z^2 \Delta d}{bf + z\Delta d}\left(\frac{8}{3\delta^4}\right)(\delta + \Delta d)^3 d(\Delta d) \\ &\quad - \int_{-\delta/2}^{0} \frac{z^2 \Delta d}{bf + z\Delta d}\left(\frac{2}{3\delta^4}\right)(2\delta^3 - 12\delta(\Delta d)^2 - 12(\Delta d)^3) d(\Delta d) \end{aligned} \tag{31}$$

On integration, the final expression reduces to

$$\begin{aligned} E(|\Delta z|/z) &= (\delta^2 z^2 - 6b^2 f^2)\frac{4bf}{\delta^3 z^2}\log\left(\frac{b^2 f^2}{b^2 f^2 - \frac{z^2\delta^2}{4}}\right) + \frac{8}{z^3}\left(\frac{bf}{\delta}\right)^4 \log\left(\frac{bf + \frac{z\delta}{2}}{bf - \frac{z\delta}{2}}\right) \\ &\quad - \frac{8}{z^3}\left(\frac{bf}{\delta}\right)^3 + \frac{6bf}{\delta} + \frac{8bf}{3\delta^4}\left(\delta - \frac{bf}{z}\right)^3 \log\left(\frac{b^2 f^2 - \frac{z^2\delta^2}{4}}{b^2 f^2 - z^2\delta^2}\right) \\ &\quad - \frac{8b^2 f^2}{\delta^2 z} \end{aligned} \tag{32}$$

Assuming again z to be uniformly distributed between $z_{\min}$ and $z_{\max}$, yields

$$\begin{aligned} E(|\Delta z|) &= \int_{-\infty}^{\infty} E(|\Delta z|/z) f_z(z) dz \\ &= \frac{1}{z_{\max} - z_{\min}} \int_{z_{\min}}^{z_{\max}} \Bigg[(\delta^2 z^2 - 6b^2 f^2)\frac{4bf}{\delta^3 z^2}\log\left(\frac{b^2 f^2}{b^2 f^2 - \frac{z^2\delta^2}{4}}\right) \\ &\quad + \frac{8}{z^3}\left(\frac{bf}{\delta}\right)^4 \log\left(\frac{bf + \frac{z\delta}{2}}{bf - \frac{z\delta}{2}}\right) - \frac{8}{z^3}\left(\frac{bf}{\delta}\right)^3 + \frac{6bf}{\delta} \\ &\quad + \frac{8bf}{3\delta^4}\left(\delta - \frac{bf}{z}\right)^3 \log\left(\frac{b^2 f^2 - \frac{z^2\delta^2}{4}}{b^2 f^2 - z^2\delta^2}\right) - \frac{8b^2 f^2}{\delta^2 z}\Bigg] dz \end{aligned} \tag{33}$$

The above integral can be evaluated using the following power series expansion and the condition $z < (bf/\delta) < (2bf/\delta)$,

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad \text{for } -1 < x \leq 1$$

Substituting the above power series expansion in equation (33) and integrating term by term reduces to:

$$E(|\Delta z|) \simeq \frac{1}{z_{\max} - z_{\min}} (I_{E1} + I_{E2} + I_{E3} + I_{E4} + I_{E5} + I_{E6})$$

In the above equation the expressions for I_{Ei}'s, $i = 1, 2, \ldots, 6$ are

$$\begin{aligned}
I_{E1} &= -\frac{\delta^3}{16b^3f^3}(z_{\min} - z_{\max})^4 + \frac{\delta}{4bf}(z_{\min} - z_{\max})^2 + \frac{6bf}{\delta}\log\left(\frac{z_{\min}}{z_{\max}}\right) \\
I_{E2} &= 8\left(\frac{bf}{\delta}\right)^4 \left\{ \frac{\delta}{bf}\left(\frac{z_{\max} - z_{\min}}{z_{\max} z_{\min}}\right) + \frac{\delta^2}{2b^2f^2}\log\left(\frac{z_{\max}(2bf + z_{\min}\delta)}{z_{\min}(2bf + z_{\max}\delta)}\right) \right. \\
&\quad \left. - \frac{4\delta^3}{2bf}\left(\frac{(z_{\max} - z_{\min})}{(2bf + z_{\max}\delta)(2bf + z_{\min}\delta)}\right) \right\} \\
I_{E3} &= 16\left(\frac{bf}{\delta}\right)^3 \left[\frac{1}{z_{\max}^2} - \frac{1}{z_{\min}^2}\right] \\
I_{E4} &= \frac{6bf}{\delta}\int_{z_{\min}}^{z_{\max}} dz \\
&= \frac{6bf}{\delta}(z_{\max} - z_{\min}) \\
I_{E5} &= \frac{-8bf}{3\delta^4}\int_{z_{\min}}^{z_{\max}} \left(\delta - \frac{bf}{z}\right)^3 \log\left(\frac{(b^2f^2 - z^2\delta^2)}{\left(b^2f^2 - \frac{z^2\delta^2}{4}\right)}\right) dz \\
&= (J_{E1} + J_{E2} + J_{E3} + J_{E4} + J_{E5} + J_{E6} + J_{E7} + J_{E8}),
\end{aligned}$$

where

$$\begin{aligned}
J_{E1} &= \frac{8bf}{2\delta^2}\left(\frac{-1}{\delta^2}\right)\left[2bf\log\left[\frac{(2bf - z_{\max}\delta)(2bf + z_{\min}\delta)}{(2bf - z_{\min}\delta)(2bf + z_{\max}\delta)}\right] + 2(z_{\max} - z_{\min})\delta\right] \\
&= -\frac{8bf}{\delta^4}\left[bf\log\left[\frac{(2bfz_{\max}\delta)(2bf + \Delta z_{\min}\delta)}{(2bf - z_{\min}\delta)(2bf + \Delta z_{\max}\delta)}\right] + (z_{\max} - z_{\min})\delta\right] \\
J_{E2} &= \frac{12b^2f^2}{\delta^5}\left[\log\frac{(2bf + z_{\max}\delta)(2bf - z_{\max}\delta)}{(2bf + z_{\min}\delta)(2bf - z_{\min}\delta)}\right] \\
&= \frac{12b^2f^2}{\delta^5}\left[\log\left(\frac{4b^2f^2 - z_{\max}^2\delta^2}{4b^2f^2 - z_{\min}^2\delta^2}\right)\right]
\end{aligned}$$

$$J_{E3} = \frac{6b^2f^2}{\delta^2}\left[\log\frac{(2bf + z_{\max}\delta)(2bf - z_{\min}\delta)}{(2bf + z_{\min}\delta)(2bf - z_{\max}\delta)}\right]$$

$$J_{E4} = -\frac{2b^2f^2}{\delta^2}\left[\log\frac{z_{\max}}{z_{\min}} + \frac{1}{2}\log\left(\frac{4b^2f^2 - z_{\min}^2\delta^2}{4b^2f^2 - z_{\max}^2\delta^2}\right)\right]$$

$$J_{E5} = \frac{24bf}{\delta}\left\{\frac{1}{4}\frac{(z_{\max} - z_{\min})}{(2bf - z_{\max}\delta)(2bf - z_{\min}\delta)} + \frac{1}{4}(z_{\max} - z_{\min}) + \frac{bf}{\delta}\log\left(\frac{2bf - z_{\max}\delta}{2bf - z_{\min}\delta}\right)\right\}$$

$$\begin{aligned} J_{E6} = -36b^2f^2 &\left\{\frac{1}{2\delta^2}\left\{2bf\left[\frac{(z_{\max} - z_{\min})\delta}{(2bf - z_{\max}\delta)(2bf - z_{\min}\delta)}\right] + \log\left(\frac{2bf - z_{\max}\delta}{2bf - z_{\min}\delta}\right)\right\}\right. \\ &+ \frac{1}{2\delta^2}\left\{2bf\left[\frac{(z_{\min} - z_{\max})\delta}{(2bf + z_{\max}\delta)(2bf + z_{\min}\delta)}\right] + \log\left(\frac{2bf + z_{\max}\delta}{2bf + z_{\min}\delta}\right)\right\} \\ &\left. + \left[\log\left(\frac{4b^2f^2 - z_{\min}^2\delta^2}{4b^2f^2 - z_{\max}^2\delta^2}\right)\right]\right\} \end{aligned}$$

$$\begin{aligned} J_{E7} = \frac{18b^3f^3}{\delta} &\left\{\frac{(z_{\max} - z_{\min})}{(2bf - z_{\max}\delta)(2bf - z_{\min}\delta)} + \frac{(z_{\min} - z_{\max})}{(2bf + z_{\max}\delta)(2bf + z_{\min}\delta)}\right. \\ &\left. - \frac{1}{2bf\delta}\left[\log\frac{(2bf - z_{\min}\delta)(2bf + z_{\max}\delta)}{(2bf - z_{\max}\delta)(2bf + z_{\min}\delta)}\right]\right\} \end{aligned}$$

$$\begin{aligned} J_{E8} = -12b^3f^3 &\left\{\frac{3}{2bf}\log\frac{z_{\max}}{z_{\min}} + \frac{3}{2bf}\log\left(\frac{4b^2f^2 - z_{\min}^2\delta}{4b^2f^2 - z_{\max}^2\delta}\right)\right. \\ &\left. + \frac{z_{\max} - z_{\min}}{(2bf - z_{\max}\delta)(2bf - z_{\min}\delta)} + \frac{z_{\max} - z_{\min}}{(2bf + z_{\max}\delta)(2bf + z_{\min}\delta)}\right\} \end{aligned} \tag{34}$$

and

$$\begin{aligned} I_{E6} &= -\frac{8b^2f^2}{\delta^2}\int_{z_{\min}}^{z_{\max}} \frac{1}{z}\,dz \\ &= -\frac{8b^2f^2}{\delta^2}\log(z)|_{z_{\min}}^{z_{\max}} \\ &= -\frac{8b^2f^2}{\delta^2}\log\frac{z_{\max}}{z_{\min}} \\ &= \frac{8b^2f^2}{\delta^2}\log\frac{z_{\min}}{z_{\max}} \end{aligned}$$

In all the above expressions the higher order terms are neglected. Finally the expected value of the relative range error is given by

$$E[\epsilon] = \frac{E(|\Delta z\,|)}{z_{\max} - z_{\min}} \tag{35}$$

It is found that the expected value of the relative range error is a function of only the stereo system design parameters namely b, f, δ, $z_{\max}$ and $z_{\min}$. It is also found that only the relative variations in these parameters will have any affect on the expected relative range error.

6 CONCLUSIONS

A methodology to calculate the marginal density of Δz has been evolved by evaluating all other relevant probability density functions $f_{\Delta X_L}(\Delta X_L)$, $f_{\Delta X_R}(\Delta X_R)$, $f_{\Delta d}(\Delta d)$, $f_{\Delta z}(\Delta z \mid z)$ and $f_{\Delta z}(\Delta z)$. As the relative range error is most useful in assessing the accuracy of a stereo-imaging system, the expected value of the range error magnitude also has been derived. By deriving the expected value of the relative range error which is expressed as a function of stereo system design parameters, namely distance of separation between the centers of the two cameras, focal lengths and image sampling interval, it is possible to study the effect of variation of these design parameters on the expected value of the relative range error. Thus the evaluation of the expected value of the relative range error provides an useful tool in the design of a stereo imaging system which is established by the study in this paper.

7 APPENDIX

The Probability density function of Δd as shown in (13) is derived and its values are shown in four different cases.

7.1 CASE I: $-\delta \leq \Delta d \leq -\delta/2$

$$
\begin{aligned}
f(\Delta d) &= \int_{-\delta/2-\Delta d}^{\delta/2} \frac{-4}{\delta^4}(2y + 2\Delta d + \delta)(2y - \delta)dy \\
&= \frac{-4}{\delta^4}\int_{-\delta/2-\Delta d}^{\delta/2} [4y^2 - 2y\delta + 4y\Delta d - 2\delta\Delta d + 2y\delta - \delta^2]dy \\
&= \frac{-4}{\delta^4}[(4y^3/3) + 2y^2\Delta d - 2y\delta\Delta d - y\delta^2]_{-(\delta/2)-\Delta d}^{\delta/2} \\
&= \frac{-4}{\delta^4}\left[\frac{4\delta^3/8}{3} + 2\frac{\delta^2}{4}\Delta d - 2\frac{\delta^2}{2}\delta\Delta d - \frac{\delta}{2}\delta^2 \right. \\
&\quad - \left\{\frac{4}{3}\left(\frac{-\delta}{2} - \Delta d\right)^3 + 2\left(\frac{-\delta}{2} - \Delta d\right)^2 \Delta d - 2\delta\Delta d\left(\frac{-\delta}{2} - \Delta d\right)\right. \\
&\quad \left.\left. - \left(\frac{-\delta}{2} - \Delta d\right)\delta^2\right\}\right] \\
&= \frac{-4}{\delta^4}\left[\frac{\delta^3}{6} + \frac{\delta^2}{2}\Delta d - \delta^2\Delta d - \frac{\delta^3}{2} + \frac{4}{3}\left(\frac{\delta}{2} + \Delta d\right)^3 - 2\left(\frac{\delta}{2} + \Delta d\right)^2 \Delta d \right. \\
&\quad \left. - 2\delta\Delta d\left(\frac{\delta}{2} + \Delta d\right) - \left(\frac{\delta}{2} + \Delta d\right)\delta^2\right] \\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3} - \frac{\delta^2\Delta d}{2} + \left(\frac{\delta}{2} + \Delta d\right)^2 \left\{\frac{4}{3}\left(\frac{\delta}{2} + \Delta d\right) - 2\Delta d\right\}\right. \\
&\quad \left. - \left(\frac{\delta}{2} + \Delta d\right)\delta(\delta + 2\Delta d)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}+\left(\frac{\delta}{2}+\Delta d\right)^2\left\{\frac{4\delta}{6}+\frac{4}{3}\Delta d-2\Delta d\right\}-2\delta\left(\frac{\delta}{2}+\Delta d\right)^2\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}+\left(\frac{\delta}{2}+\Delta d\right)^2\left\{\frac{2\delta}{3}+\frac{4}{3}\Delta d-2\Delta d-2\delta\right\}\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}+\left(\frac{\delta}{2}+\Delta d\right)^2\left\{-\frac{4\delta}{3}-\frac{2\Delta d}{3}\right\}\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}-\frac{2}{3}(\Delta d+2\delta)\left(\frac{\delta}{2}+\Delta d\right)^2\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}-\frac{2}{3}(\Delta d+2\delta)\left(\frac{\delta^2}{4}+\Delta d^2+\delta\Delta d\right)\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}-\frac{2}{3}\left(\frac{\delta^2}{4}\Delta d+\Delta d^3+\delta\Delta d^2+\frac{\delta^3}{2}+2\delta\Delta d^2+2\delta^2\Delta d\right)\right]\\
&= \frac{-4}{\delta^4}\left[-\frac{\delta^3}{3}-\frac{\delta^2\Delta d}{2}-\frac{\delta^2}{6}\Delta d-\frac{2\Delta d^3}{3}+\frac{2}{3}\delta\Delta d^2-\frac{\delta^3}{2}-\frac{4}{3}\delta\Delta d^2-\frac{4}{3}\delta^2\Delta d\right]\\
&= \frac{4}{\delta^4}\left[\frac{\delta^3}{3}+\delta^2\Delta d\left(\frac{1}{2}+\frac{1}{6}+\frac{4}{3}\right)+\delta\Delta d^2\left(\frac{2}{3}+\frac{4}{3}\right)+\frac{2}{3}\Delta d^3\right]\\
&= \frac{4}{\delta^4}\left[\frac{\delta^3}{3}+\frac{(3+1+8)}{6}\delta^2\Delta d+\frac{6}{3}\delta\Delta d^2+\frac{2}{3}\Delta d^3\right]\\
&= \frac{4}{\delta^4}\left[\frac{2}{3}\delta^3+2\delta^2\Delta d+2\delta\Delta d^2+\frac{2}{3}\Delta d^3\right]\\
&= \frac{8}{3\delta^4}[\delta^3+3\delta^2\Delta d+3\delta\Delta d^2+\Delta d^3]\\
&= \frac{8}{3\delta^4}(\delta+\Delta d)^3
\end{aligned}
\tag{36}
$$

7.2 CASE II: $-\delta/2 \le \Delta d \le 0$

$$
\begin{aligned}
f(\Delta d) &= \int_{-\delta/2-\Delta d}^{0}\frac{4}{\delta^4}(2y+2\Delta d+\delta)(2y+\delta)dy\\
&\quad+\int_{0}^{-\Delta d}\frac{-4}{\delta^4}(2y+2\Delta d+\delta)(2y-\delta)dy\\
&\quad+\int_{-\Delta d}^{\delta/2}\frac{4}{\delta^4}(2y+2\Delta d-\delta)(2y-\delta)dy\\
&= I_1+I_2+I_3
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \frac{4}{\delta^4}\int_{-(\delta/2)-\Delta d}^{0}[4y^2+2y\delta+4y\Delta d+2\delta\Delta d+2y\delta+\delta^2]dy\\
&= \frac{4}{\delta^4}\int_{-(\delta/2)-\Delta d}^{0}[4y^2+4y\delta+4y\Delta d+2\delta\Delta d+\delta^2]dy
\end{aligned}
$$

$$= \frac{4}{\delta^4}\left[\frac{4y3}{3} + 2y^2\delta + 2y^2\Delta d + 2y\delta\Delta d + y\delta^2\right]_{-(\delta/2)-\Delta d}^{0}$$

$$= -\frac{4}{\delta^4}\left[\frac{4}{3}\left(-\frac{\delta}{2} - \Delta d\right)^3 + 2\delta\left(-\frac{\delta}{2} - \Delta d\right)^2 + 2\Delta d\left(-\frac{\delta}{2} - \Delta d\right)^2\right.$$

$$\left. + 2\delta\Delta d\left(-\frac{\delta}{2} - \Delta d\right) + \delta^2\left(-\frac{\delta}{2} - \Delta d\right)\right]$$

$$= \frac{4}{\delta^4}\left[\frac{4}{3}\left(\frac{\delta}{2} + \Delta d\right)^3 - 2\delta\left(\frac{\delta}{2} + \Delta d\right)^2 - 2\Delta d\left(\frac{\delta}{2} + \Delta d\right)^2\right.$$

$$\left. + 2\delta\Delta d\left(\frac{\delta}{2} + \Delta d\right) + \delta^2\left(\frac{\delta}{2} + \Delta d\right)\right]$$

$$= \frac{4}{\delta^4}\left[\frac{4}{3}\left(\frac{\delta}{2} + \Delta d\right)^3 - 2(\delta + \Delta d)\left(\frac{\delta}{2} + \Delta d\right)^2 + \delta\left(\frac{\delta}{2} + \Delta d\right)(\delta + 2\Delta d)\right]$$

$$= \frac{4}{\delta^4}\left[\frac{4}{3}\left(\frac{\delta}{2} + \Delta d\right)^3 - 2\Delta d\left(\frac{\delta}{2} + \Delta d\right)^2\right]$$

$$= \frac{4}{\delta^4}\left[2\left(\frac{\delta}{2} + \Delta d\right)^2\left\{\frac{2}{3}\left(\frac{\delta}{2} + \Delta d\right) - \Delta d\right\}\right] = \frac{8}{3\delta^4}\left(\frac{\delta}{2} + \Delta d\right)^2(\delta - \Delta d)$$

$$I_2 = \int_0^{-\Delta d}[4y^2 + 4y\Delta d - 2\delta\Delta d - \delta^2]dy$$

$$= -\frac{4}{\delta^4}\left[\frac{4y^3}{3} + 2y^2\Delta d - 2y\delta\Delta d - y\delta^2\right]_0^{-\Delta d}$$

$$= -\frac{4}{\delta^4}\left[\frac{-4\Delta d^3}{3} + 2\Delta d^3 + 2\delta\Delta d^2 + \Delta d\delta^2\right]$$

$$= -\frac{4}{\delta^4}\left[\frac{2\Delta d^3}{3} + 2\delta\Delta d^2 + \Delta d\delta^2\right]$$

$$= -\frac{8}{3\delta^4}\left[\Delta d^3 + \frac{3}{2}\delta^2\Delta d + 3\delta\Delta d^2\right]$$

$$I_3 = \frac{4}{\delta^4}\int_{-\Delta d}^{\delta/2}[4y^2 - 2y\delta + 4y\Delta d - 2\delta\Delta d - 2y\delta + \delta^2]dy$$

$$= \frac{4}{\delta^4}\left[\frac{4y^3}{3} - y^2\delta + 2y^2\Delta d - 2y\delta\Delta d - y^2\delta + y\delta^2\right]_{-\Delta d}^{\delta/2}$$

$$= \frac{4}{\delta^4}\left[\frac{4}{3}\frac{\delta^3}{8} - \frac{\delta^2}{4}\delta + 2\frac{\delta^2}{4}\Delta d - 2\frac{\delta}{2}\delta\Delta d - \frac{\delta^2}{4}\delta + \frac{\delta}{2}\delta^2\right.$$

$$
\begin{aligned}
&-\left\{\frac{4}{3}(-\Delta d)^3 - (-\Delta d)^2\delta + 2(-\Delta d)^2\Delta d \right.\\
&\left.-2(-\Delta d)\delta\Delta d - (-\Delta d)^2\delta + (-\Delta d)\delta^2\right\}\Bigg]\\
&= \frac{4}{\delta^4}\left[\delta^3\left(\frac{1}{6}-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}\right)+\frac{\delta^2}{2}\Delta d + (\Delta d)^3\left\{\frac{4}{3}-2\right\}\right]\\
&= \frac{4}{\delta^4}\left[\frac{\delta^3}{6}+\frac{\delta^2\Delta d}{2}-\frac{2}{3}(\Delta d)^3\right]\\
&= \frac{8}{3\delta^4}\left[\frac{\delta^3}{4}+\frac{3}{4}\delta^2\Delta d - \Delta d^3\right]
\end{aligned}
$$

Since

$$
\begin{aligned}
f(\Delta d) &= I_1 + I_2 + I_3\\
f(\Delta d) &= \frac{2}{3\delta^4}[2\delta^3 - 12\delta\Delta d^2 - 12\Delta d^3] \qquad (37)
\end{aligned}
$$

7.3 CASE III: $\delta/2 \le \Delta d \le \delta$

$$
\begin{aligned}
f(\Delta d) &= -\frac{4}{\delta^4}\int_{-(\delta/2)}^{(\delta/2)-\Delta d}[4y^2 + 2y\delta + 4y\Delta d - 2\delta\Delta d - 2y\delta - \delta^2]dy\\
&= -\frac{4}{\delta^4}\left[\frac{4y^3}{3} + y^2\delta + 2y^2\Delta d - 2y\delta\Delta d - y^2\delta + y\delta^2\right]_{-(\delta/2)}^{(\delta/2)-\Delta d}\\
&= -\frac{4}{\delta^4}\left[\frac{4}{3}\left(\frac{\delta}{2}-\Delta d\right)^3 + 2\Delta d\left(\frac{\delta}{2}-\Delta d\right)^2 - 2\delta\Delta d\left(\frac{\delta}{2}-\Delta d\right) - \delta^2\left(\frac{\delta}{2}-\Delta d\right)\right.\\
&\quad \left.-\left\{\frac{4}{3}\left(\frac{-\delta^3}{8}\right) + 2\frac{\delta^2}{4}\Delta d + 2\delta\frac{\delta}{2}\Delta d + \frac{\delta}{2}\delta^2\right\}\right]\\
&= -\frac{4}{\delta^4}\left[2\left(\frac{\delta}{2}-\Delta d\right)^2\left\{\Delta(d + \frac{2}{3} - \left(\frac{\delta}{2}-\Delta d\right)\right\} - \delta\left(\frac{\delta}{2}-\Delta d\right)(2\Delta d + \delta)\right.\\
&\quad \left.+\frac{\delta^3}{6} - \frac{\delta^2}{2}\Delta d - \delta^2\Delta d - \frac{\delta^3}{2}\right]\\
&= -\frac{4}{\delta^4}\left[2\left(\frac{\delta}{2}-\Delta d\right)^2\left(\frac{\delta+\Delta d}{3}\right) - 2\delta\left(\frac{\delta}{2}-\Delta d\right)\left(\frac{\delta}{2}+\Delta d\right) - \frac{\delta^3}{3} - \frac{3\delta^2}{2}\Delta d\right]\\
&= -\frac{4}{\delta^4}\left[\frac{2}{3}\left(\frac{\delta}{2}-\Delta d\right)^2(\delta+\Delta d) - 2\delta\left(\frac{\delta^2}{4}-\Delta d^2\right) - \frac{\delta^3}{3} - \frac{3\delta^2}{2}\Delta d\right]\\
&= -\frac{4}{\delta^4}\left[\frac{2}{3}\left(\frac{\delta}{2}-\Delta d\right)^2(\delta+\Delta d) - \frac{\delta^3}{2} + 2\delta\Delta d^2 - \frac{\delta^3}{3} - \frac{3\delta^2}{2}\Delta d\right]
\end{aligned}
$$

$$= -\frac{4}{\delta^4}\left[\frac{2}{3}\left\{\frac{\delta^3}{4} + \frac{\delta^2}{4}\Delta d - \delta^2\Delta d - \delta\Delta d^2 + \delta\Delta d^2 + \Delta d^3\right\}\right.$$

$$\left. -\frac{5\delta^3}{6} + 2\delta\Delta d^2 - \frac{3\delta^2}{2}\Delta d\right]$$

$$= -\frac{4}{\delta^4}\left[\frac{\delta^3}{6} + \frac{\delta^2}{6}\Delta d - \frac{2}{3}\delta^2\Delta d + \frac{2}{3}\Delta d^3 - \frac{5\delta^3}{6} + 2\delta\Delta d^2 - \frac{3\delta^2}{2}\Delta d\right]$$

$$= -\frac{4}{\delta^4}\left[-\frac{2\delta^3}{3} + \delta^2\Delta d\left(\frac{1}{6} - \frac{2}{3} - \frac{3}{2}\right) + 2\delta\Delta d^2 + \frac{2}{3}\Delta d^3\right]$$

$$= -\frac{4}{\delta^4}\left[-\frac{2\delta^3}{3} - 2\delta^2\Delta d + 2\delta\Delta d^2 + \frac{2}{3}\Delta d^3\right]$$

$$= -\frac{8}{3\delta^4}[-\delta^3 - 3\delta^2\Delta d + 3\delta\Delta d^2 + \Delta d^3]$$

$$= -\frac{8}{3\delta^4}(\Delta d - \delta)^3 \tag{38}$$

7.4 CASE IV: $0 \le \Delta d \le \delta/2$

$$f(\Delta d) = \frac{4}{\delta^4}\int_{-(\delta/2)}^{-\Delta d}(2y + 2\Delta d + \delta)(2y + \delta)dy$$

$$- \frac{4}{\delta^4}\int_{-\Delta d}^{0}(2y + 2\Delta d - \delta)(2y + \delta)dy$$

$$+ \frac{4}{\delta^4}\int_{0}^{(\delta/2)-\Delta d}(2y + 2\Delta d - \delta)(2y - \delta)dy$$

$$= J_1 + J_2 + J_3$$

$$J_1 = \frac{4}{\delta^4}\int_{-(\delta/2)}^{-\Delta d}[(2y + \delta)^2 + 2\Delta d(2y + \delta)]dy$$

$$= \frac{4}{\delta^4}\left[\frac{(2y + \delta)^3}{3}\frac{1}{2} + 2\Delta d\frac{(2y + \delta)^2}{2}\frac{1}{2}\right]_{-(\delta/2)}^{-\Delta d}$$

$$= \frac{4}{\delta^4}\left[\frac{(2y + \delta)^3}{6} + \Delta d\frac{(2y + \delta)^2}{2}\right]_{-(\delta/2)}^{-\Delta d}$$

$$= \frac{2}{3\delta^4}[(2y + \delta)^3 + 3\Delta d(2y + \delta)^2]_{-(\delta/2)}^{-\Delta d}$$

$$= \frac{2}{3\delta^4}[(-2\Delta d + \delta)^3 + 3\Delta d(-2\Delta d + \delta)^2 - \{0 + 0\}]$$

$$= \frac{2}{3\delta^4}(-2\Delta d + \delta)^2[\delta - 2\Delta d + 3\Delta d]$$

$$= \frac{2}{3\delta^4}(-2\Delta d + \delta)^2[\delta + \Delta d]$$

$$
\begin{aligned}
J_2 &= -\frac{4}{\delta^4}\int_{-\Delta d}^{0}[(2y+\delta)^2 + 2(\Delta d-\delta)(2y+\delta)]dy \\
&= -\frac{4}{\delta^4}\left[\frac{(2y+\delta)^3}{3}\frac{1}{2} + 2(\Delta d-\delta)\frac{(2y+\delta)^2}{2}\frac{1}{2}\right]_{-\Delta d}^{0} \\
&= -\frac{4}{\delta^4}\left[\frac{(2y+\delta)^3}{6} + (\Delta d-\delta)\frac{(2y+\delta)^2}{2}\right]_{-\Delta d}^{0} \\
&= -\frac{2}{3\delta^4}[(2y+\delta)^3 + 3(\Delta d-\delta)(2y+\delta)^2]_{-\Delta d}^{0} \\
&= -\frac{2}{3\delta^4}[\delta^3 + 3(\Delta d-\delta)\delta^2 - \{(\delta-2\Delta d)^3 + 3(\Delta d-\delta)(\delta-2\Delta d)^2\}] \\
&= -\frac{2}{3\delta^4}[\delta^3 + 3\delta^2(\Delta d-\delta) - (\delta-2\Delta d)^2\{\delta-2\Delta d+3\Delta d-3\delta\}] \\
&= -\frac{2}{3\delta^4}[\delta^3 + 3\delta^2(\Delta d-\delta) - (-\delta-2\Delta d)^2\{\Delta d-2\delta\}] \\
&= -\frac{2}{3\delta^4}[\delta^2(\delta+3\Delta d-3\delta) - (\delta-2\Delta d)^2(\Delta d-2\delta)] \\
&= -\frac{2}{3\delta^4}[\delta^2(3\Delta d-2\delta) - (\delta-2\Delta d)^2(\Delta d-2\delta)] \\
&= -\frac{2}{3\delta^4}[3\delta^2\Delta d - 2\delta^3 - \delta^2\Delta d + \delta^2(2\delta) + 4\delta\Delta d^2 - 4\delta\Delta d(2\delta) + 4\Delta d^2(2\delta) - 4\Delta d^2\Delta d] \\
&= -\frac{2}{3\delta^4}[\delta^2\Delta d(3-1-8) + \delta\Delta d^2(4+8) - 4\Delta d^3]
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \frac{4}{\delta^4}\int_{0}^{(\delta/2)-\Delta d}[(2y-\delta)^2 + 2\Delta d(2y-\delta)]dy \\
&= \frac{4}{\delta^4}\left[\frac{(2y-\delta)^3}{3}\frac{1}{2} + 2\Delta d\frac{(2y-\delta)^2}{2}\frac{1}{2}\right]_{0}^{(\delta/2)-\Delta d} \\
&= \frac{4}{\delta^4}\left[\frac{(2y-\delta)^3}{6} + \Delta d\frac{(2y-\delta)^2}{2}\right]_{0}^{(\delta/2)-\Delta d} \\
&= \frac{2}{3\delta^4}[(2y-\delta)^3 + 3\Delta d(2y-\delta)^2]_{0}^{(\delta/2)-\Delta d} \\
&= \frac{2}{3\delta^4}[(\delta-2\Delta d-\delta)^3 + 3\Delta d(\delta-2\Delta d-\delta)^2 - (-\delta^3 + 3\Delta d\delta^2)] \\
&= \frac{2}{3\delta^4}[-8\Delta d^3 + 3\Delta d4\Delta d^2 + \delta^3 - 3\Delta d\delta^2)] \\
&= \frac{2}{3\delta^4}[4\Delta d^3 + \delta^3 - 3\Delta d\delta^2)] \\
J_2 + J_3 &= \frac{2}{3\delta^4}[6\delta^2\Delta d - 12\delta\Delta d^2 + 4\Delta d^3 + 4\Delta d^3 + \delta^3 - 3\Delta d\delta^2] \\
&= \frac{2}{3\delta^4}[3\delta^2\Delta d - 12\delta\Delta d^2 + 8\Delta d^3 + \delta^3] \\
J_1 + J_2 + J_3 &= \frac{2}{3\delta^4}[\delta(\delta^2 - 4\delta\Delta d + 4d^2) + \Delta d(\delta^2 - 4\delta\Delta d + 4\Delta d^2) \\
&\quad + 3\delta^2\Delta d - 12\delta\Delta d^2 + 8\Delta d^3 + \delta^3]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2}{3\delta^4}[\delta^3 - 4\delta^2\Delta d + 4\delta\Delta d^2 + \delta^2\Delta d - 4\delta\Delta d^2 + 4\Delta d^3 \\
&\quad + 3\delta^2\Delta d - 12\delta\Delta d^2 + 8\Delta d^3 + \delta^3] \\
&= \frac{2}{3\delta^4}[2\delta^3 + \delta^2\Delta d(-4+1+3) + \delta\Delta d^2(-12) + 12\Delta d^3] \\
&= \frac{2}{3\delta^4}[2\delta^3 - 12\delta\Delta d^2 + 12\Delta d^3] \qquad (39)
\end{aligned}
$$

Thus the above four cases finally can be written as:

$$
\begin{aligned}
f_{\Delta d}(\Delta d) &= \frac{4}{\delta^4}\left[\frac{2\delta^3}{3} + 2\delta^2\Delta d + 2\delta\Delta d^2 + \frac{2}{3}\Delta d^3\right], && -\delta \le \Delta d \le -\frac{\delta}{2} \\
&= \frac{2}{3\delta^4}(2\delta^3 - 12\delta\Delta d^2 - 12\Delta d^3), && -\frac{\delta}{2} \le \Delta d \le 0 \\
&= \frac{2}{3\delta^4}(2\delta^3 - 12\delta\Delta d^2 + 12\Delta d^3), && 0 \le \Delta d \le \frac{\delta}{2} \\
&= -\frac{8}{3\delta^4}(\Delta d - \delta)^3, && \frac{\delta}{2} \le \Delta d \le \delta
\end{aligned}
$$

References

[1] R. Balasubramanian, Sukhendu Das and K. Swaminathan, Simulation studies for the performance analysis of the reconstruction of a line using stereoscopic projections, *Proc. of Indian Conference on Computer Vision, Graphics and Image Processing*, pp. 338–344, IISc, Bangalore, India, Dec. 20–22, 2000.

[2] R. Balasubramanian, Sukhendu Das and K. Swaminathan, Error analysis in reconstruction of a line in 3-D from two arbitrary perspective views, *International Journal of Computer Mathematics*, **78**, 191–212, 2001. Gordon and Breach Publishing Group, U.K.

[3] S.D. Blostein and T.S. Huang, Error analysis in stereo determination of 3-D point positions, *IEEE Trans. Pattern Analysis and Machine Intelligence*, **6**, 752–765, 1987.

[4] E.S. McVey and J.W. Lee, Some accuracy and resolution aspects of computer vision distance measurements, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, **4**, 646–649, 1982.

[5] A. Verri and V. Torre, Absolute depth estimates in stereopsis, *Journal of Optical Society of America*, **3**, 297–299, 1986.

[6] L. Matthies and S.A. Shafer, Error modeling in stereo navigation, *IEEE Journal of Robatics and Automation*, **3**, 239–248, 1987.

[7] J.J. Rodríguez and J.K. Aggarwal, Quantization error in stereo imaging, *Proceedings of Computer Vision and Pattern Recognition*, 153–158, 1988.