Proof of some conjectures on the mean-value of Titchmarsh series – III

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Abstract. With some applications in view, the following problem is solved in some special case which is not too special. Let \( F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) be a generalized Dirichlet series with \( 1 = \lambda_1 < \lambda_2 < \ldots, \lambda_n \leq Dn \), and \( \lambda_{n+1} - \lambda_n \geq D^{-1} \lambda_n^{-1} \), where \( \alpha > 0 \) and \( D(\geq 1) \) are constants. Then subject to analytic continuation and some growth conditions, a lower bound is obtained for \( \int_0^H |F(it)|^2 \, dt \). These results will be applied in other papers to appear later.

Keywords. Titchmarsh series; mean value; lower bounds.

1. Introduction

In the previous papers [1] and [2] with the same title (as the present one) we proved some conjectures made by the second author [4]. In this paper we formulate a new conjecture (which we believe to be true at least in some modified form) and indicate a slight progress towards it.

Conjecture. Let \( 1 = \mu_1 < \mu_2 < \ldots \) be any sequence of real numbers with \( 1/C \leq \mu_{n+1} - \mu_n \leq C \), where \( C(\geq 1) \) is an integer constant and \( n = 1, 2, 3, \ldots \). Let us form the sequence \( 1 = \lambda_1 < \lambda_2 < \ldots \) of all possible (distinct) finite power products of \( 1 = \mu_1, \mu_2, \ldots \) with non-negative integral exponents. Let \( s = \sigma + it, H(\geq 10) \) be a real parameter, and \( \{a_n\} \) \((n = 1, 2, 3, \ldots)\) with \( a_1 = 1 \) be any sequence of complex numbers (possibly depending on \( H \)) such that \( F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) is absolutely convergent at \( s = B \) where \( B \geq 3 \) is an integer constant. Suppose that \( F(s) \) can be continued analytically in \((\sigma \geq 0, 0 \leq t \leq H)\) and that there exist \( T_1, T_2 \) with \( 0 \leq T_1 \leq H^{3/4}, H - H^{3/4} \leq T_2 \leq H \) such that for some \( K(\geq 30) \) there holds

\[
\max_{\sigma \geq 0} \left( |F(\sigma + iT_1)| + |F(\sigma + iT_2)| \right) \leq K.
\]

Finally let \( \sum_{n=1}^{\infty} |a_n| \lambda_n^{-B} \leq H^A \), where \( A(\geq 1) \) is an integer constant. Then there exists a constant \( \delta > 0 \) (depending only on \( A, B \) and \( C \)) such that for all \( H \geq H_0(A, B, C) \) there holds

\[
\frac{1}{H} \int_0^H |F(it)|^2 \, dt \geq \frac{1}{2} \sum_{\lambda_n \leq H} |a_n|^2,
\]

provided that \( H^{-1} \log \log K \) does not exceed a small positive constant.

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Remark 1. We can strengthen the Conjecture (1) by replacing $\frac{1}{2}$ by a more specific function of $H$ which is asymptotic to 1 as $H \to \infty$.

Remark 2. By the method of [1] we can prove that

$$\frac{1}{H} \int_0^H |F(it)| \, dt \geq \frac{1}{2}. \tag{2}$$

The Remark 1 is also applicable.

Remark 3. Under the condition

$$\sum_{\lambda \in \mathcal{X}} |a_n| \leq D_0 (\log X)^K \quad (R = H', \ D_0 \text{ any constant, } X \geq 30), \tag{3}$$

we can prove (2). Remark 1 is also applicable. For both these results the conditions involving $K$ are unnecessary. For the results mentioned in Remarks 2 and 3 we refer the reader to [1] and [5].

Remark 4. Actually the proof of (1) in [1] goes through without serious problems until we come to a lower bound for

$$\frac{1}{H} \int_0^H \left| \sum_{n \in H} a_n \lambda_n^{-\eta} \right|^2 \, dt.$$

To apply Montgomery–Vaughan theorem we need good lower bounds for $\lambda_{n+1} - \lambda_n$. These are not available in general. But we can work with $\mu_n = (n_0 + n - 1)/n_0$ where $n_0 (\geq 2)$ is any integer constant (of course using Montgomery–Vaughan Theorem). Thus in this special case we can prove Conjecture (1). We can also handle $\mu_n = (1 + \beta)^{-1} (n + \beta)$ where $\beta (> 0)$ is any real algebraic constant.

Remark 5. We can formulate Conjecture (1) with no conditions involving $K$, but instead we have to assume condition (3). Remark 1 is also applicable.

Before closing this section we like to make two important remarks. First $\lambda_n \leq \mu_n \leq Cn$ which is obvious because $\{\lambda_n\}$ contains the subsequence $\{\mu_n\}$. Secondly for $x \geq 1$ and $\eta \geq 2C + 1$, we have,

$$\sum_{\lambda \leq x} 1 \leq x^n \sum_{n=1}^\infty \frac{1}{\lambda_n^\eta} \leq x^n \left(1 - \sum_{n=2}^\infty \mu_n^{-\eta}\right)^{-1}
\leq x^n \left(1 - \sum_{n=2}^\infty \left(1 + \frac{n-1}{C}\right)^{-\eta}\right)^{-1}
\leq x^n \left(1 - \left\{\int_0^\infty \left(1 + \frac{u}{C}\right)^{-\eta} \, du\right\}^{-1}
= x^n \left(1 - C \int_0^\infty \frac{du}{(1+u)^\eta}\right)^{-1}
= x^n \left(1 - \frac{C}{\eta - 1}\right)^{-1} \leq 2x^n.$$

Hence in (1) the condition $\lambda_n \leq H^\delta$ is equivalent to a condition of the type $n \leq H^\delta$ with a different constant $\delta > 0$. 
2. Main lemma

Let \( r \) be a positive integer, \( H \geq (r + 5)U \) where \( U \geq 2^{70}(16B)^2 \) and \( M \) and \( N \) are positive integers subject to \( N > M \geq 1 \), and \( B(\geq 3) \) an integer constant. Let \( \{b_m\} (1 \leq m \leq M) \) and \( \{c_n\} (n \geq N) \) be two sequences of complex numbers, \( 1 = \lambda_1 < \lambda_2 < \ldots \) be any increasing sequence of real numbers and let \( A(s) = \sum_{m \leq M} b_m \lambda_m^{-s} \). Let \( B(s) = \sum_{n \geq N} c_n \lambda_n^{-s} \) be absolutely convergent for \( s = B \) and continuous analytically in \( (\sigma \geq 0, 0 \leq t \leq H) \). Write \( g(s) = A(-s)B(s) \),

\[
G(s) = U^{-r} \int_0^U \int_0^U \ldots \int_0^U \prod_{0}^{t} (g(s + i\lambda))
\]

where (here and elsewhere) \( \lambda = u_1 + \ldots + u_r \). Assume that there exist real numbers \( T_1 \) and \( T_2 \) with \( 0 \leq T_1 \leq U, H - U \leq T_2 \leq H \), such that

\[
|g(\sigma + iT_1)| + |g(\sigma + iT_2)| \leq \exp\left(\frac{U}{16B}\right)
\]

uniformly in \( 0 \leq \sigma \leq B \). Let

\[
S_1 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B (U \log \frac{\lambda_n}{\lambda_m})^{-r}
\]

and

\[
S_2 = \sum_{m \leq M, n \geq N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.
\]

Then, we have,

\[
\left|\int_{2U}^{H-(r+3)U} G(it)dt\right| \leq 2B^2 U^{-10} + 54BU^{-1} \int_0^H |g(it)|dt
\]

\[(H + 64B^2)S_1 + 16B^2S_2 \exp\left(-\frac{U}{8B}\right)
\]

Remark. This lemma is borrowed from [1] (see pages 2 to 8).

3. Progress towards the conjecture

From now on we assume that \( 1 = a_1, a_2, a_3, \ldots \) is any sequence of complex numbers. We set \( b_m = a_m \) and \( c_n = a_n \) and assume that \( \sum_{n=1}^{\infty} |a_n|^{-s} \) is convergent.

Lemma 1. We have, with \( \bar{A}(s) = \sum_{m \leq M} a_m \lambda_m^{-s} \),

\[
\int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 dt \geq (H - (r + 5)U - 10\lambda_M \Delta(\lambda_M)) \sum_{m \leq M} |a_m|^2,
\]

where \( \Delta(\lambda_M) = \max_{\mu \neq v} |\lambda_\mu - \lambda_v|^{-1} \).
Proof. Follows from Montgomery-Vaughan Theorem (see [3]).

Lemma 2. We have,

\[ S_2 \leq \lambda_M^2 \left( \sum_{n=1}^{\infty} |a_n| \lambda_n^{-b} \right)^2 \]

and

\[ S_1 \leq 2^r \lambda_M^2 \left( \sum_{n=1}^{\infty} |a_n| \lambda_n^{-b} \right)^{-2} \left( U \lambda_N^{-1} (\lambda_N - \lambda_M) \right)^{-r}. \]

Proof. The first inequality is trivial and the second follows from

\[ \log \frac{\lambda_N}{\lambda_M} = - \log \left( 1 - \frac{\lambda_M}{\lambda_N} \right) > \frac{\lambda_N - \lambda_M}{\lambda_N}. \]

We now make the following.

Hypothesis. \( \{\lambda_n\} \) is any increasing sequence of real numbers satisfying \( \lambda_1 = 1, \lambda_n \leq DN, \lambda_{n+1} - \lambda_n \geq \lambda_n^{-\alpha} D^{-1} \), where \( D(\geq 1) \) is an integer constant and \( \alpha \) a positive constant. Also we assume that

\[ \sum_{n=1}^{\infty} |a_n| \lambda_n^{-B} \leq H^{r/8} \]

where \( 0 < \varepsilon \leq 1/[2(\alpha + 1)] \) and \( r \geq \lceil (200B + 200)\varepsilon^{-1} \rceil \) is any integer. Also \( F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \) shall be as in the introduction except that the \( \{\lambda_n\} \) are not related to the \( \{\mu_n\} \). \( \{\lambda_n\} \) will now be an independent sequence.

From now on we set \( N = M + 1, M = \lceil H^{1/(\alpha + 1) - \varepsilon} \rceil, U = H^{1-(\alpha/2)} + 50B \log \log K_1 \)
where \( K_1 = H^r K \). Note that if \( H \geq (r + 5)U \) is not satisfied, our main theorem (to follow) asserts that a positive quantity is non-negative. Also note that

\[ \min_{0 \leq t \leq H^{3/4}} \max_{\sigma \geq 0} |F(\sigma + it)| \geq \min_{0 \leq t \leq U} \max_{\sigma \geq 0} |F(\sigma + it)| \]

and a similar result holds for the intervals \((H - H^{3/4}, H)\) and \((H - U, H)\).

Lemma 3. We have,

\[ S_2 \leq (DH)^{2B} H^{r/4}, S_1 \leq 2^r (DH)^{2B} H^{r/4} ((2D)^{-\alpha - 2} H^{r/2})^{-r} \]

and

\[ \lambda_M \Delta(\lambda_M) \leq D^{\alpha + 2} H^{1 - \varepsilon}. \]

Proof. We have \( \lambda_M \leq DM \leq DH \) and this proves the first inequality. Also

\[ \begin{align*}
U &\lambda_N^{-1} (\lambda_N - \lambda_M) \geq H^{1 - (\alpha/2)} \lambda_N^{-1} D^{-1} \geq H^{1 - (\alpha/2)} D^{-1} (DN)^{-1 - \alpha} \\
&\geq H^{1 - (\alpha/2)} (2DM)^{-1 - \alpha} D^{-1} \geq H^{1 - (\alpha/2)} (2D)^{-\alpha - 2} H^{-1 + \varepsilon},
\end{align*} \]
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and this proves the second inequality. The third follows from
\[ \lambda_M \Delta(\lambda_M) \leq \lambda_M D \lambda_M^2 \leq D^{s+2} M^{1+\varepsilon} \leq D^{s+2} H^{1-\varepsilon}. \]

The lemma is completely proved.

Now we apply the main lemma (we closely follow the proof of the first main theorem in [1]). Let
\[ A(s) = \sum_{m \leq M} \hat{a}_m \lambda_m^{-s}, \quad \bar{A}(s) = \sum_{m \leq M} a_m \lambda_m^{-s} \]

and
\[ B(s) = \sum_{n \geq N} a_n \lambda_n^{-s}. \]

Then, we have, in \( \sigma \geq B, \) \( F(s) = \bar{A}(s) + B(s) \) and so
\[ |F(it)|^2 = |\bar{A}(it)|^2 + 2 \text{Re} (g(it)) + |B(it)|^2 \]
\[ \geq |\bar{A}(it)|^2 + 2 \text{Re} (g(it)), \]

where \( g(s) = A(-s)B(s) \) Hence
\[ \int_0^H |F(it)|^2 \, dt \]
\[ \geq U^{-r} \int_0^U \cdots \int_0^U \int_2^{U + \lambda} \int_{2U + \lambda}^{H - (r + 3)U + \lambda} (|\bar{A}(it)|^2 + 2 \text{Re} g(it)) \, dt \]
\[ = J_1 + 2J_2 \text{ say.} \]

By Lemmas 1 and 3, we have,
\[ J_1 \geq (H - (r + 5)U - 10D^{s+2} H^{1-\varepsilon}) \sum_{n \leq M} |a_n|^2. \]

Again, we have, for \( 0 \leq \sigma \leq B, \)
\[ |g(s)| = |A(-s)B(s)| = |A(-s)(F(s) - A(s))| \]
\[ \leq \left( \sum_{n \leq M} |a_n|^2 \lambda_n^B \right) K + \left( \sum_{n \leq M} |a_n|^2 \lambda_n^B \right)^2 \]
\[ \leq \lambda_M^B H^{r/8} K + \lambda_M^4 B H^{r/4}. \]

Hence
\[ |g(s)|_{t=T_1} + |g(s)|_{t=T_2} \leq 2K \lambda_M^4 B H^{r/8} (H - ru/8 \lambda_M^{-2B} + K^{-1}) \]
\[ \leq K (DH)^{4B} H^{r/4} \leq H' K = K_1, \]

the last two inequalities being true for instance if \( H \geq 10D. \) Observe that
\[ \exp \left( \frac{U}{16B} \right) \geq \exp \left( \frac{50}{16} \log \log K_1 \right) \geq K_1 \]

and hence the condition on \( g \) and \( U \) required by the main lemma is satisfied. Hence by the main lemma, we have,
\[ |J_2| \leq \frac{2B^2}{U^{10}} + \frac{54B}{U} \int_0^H |g(it)| \, dt + (H + 64B^2)S_1 + 16B^2S_2 \exp\left(-\frac{U}{8B}\right) \]

provided \( H \geq (r + 5)U \) and \( U \geq 2^{10}(16B)^2 \). As remarked already we can ignore the condition \( H \geq (r + 5)U \). Also we will satisfy \( H \geq (50rBD^{\frac{\sigma+2}{\sigma+1}})^{\frac{\sigma}{2}} \) and we will show later that this implies \( U \geq 2^{10}(16B)^2 \). We can assume that \( \int_0^H |F(it)|^2 \, dt \leq H \sum_{\eta \leq M} |a_\eta|^2 \)

(otherwise the result asserted by the main theorem to follow, is trivially true). Hence

\[ \int_0^H |g(it)| \, dt = \int_0^H |A(-it)B(it)| \, dt \]

\[ \leq \int_0^H |A(-it)|^2 \, dt + \int_0^H |B(it)|^2 \, dt \]

\[ \leq 3 \int_0^H |A(-it)|^2 \, dt + 2 \int_0^H |F(it)|^2 \, dt \]

(on noting that \( B(it) = F(it) - \bar{A}(it) \))

\[ \leq (5H + 10D^{2+2}H^{1-\epsilon}) \sum_{\eta \leq M} |a_\eta|^2 \]

by Montgomery–Vaughan Theorem and the third part of Lemma 3. Hence

\[ 2|J_2| \leq \frac{4B^2}{H^5} + \frac{108B}{H^{1-(e/2)}} (5H + 10D^{2+2}H^{1-\epsilon}) \sum_{\eta \leq M} |a_\eta|^2 \]

\[ + (2H + 128B^2)S_1 + 32B^2S_2 \exp\left(-\frac{U}{8B}\right) \]

\[ \leq \left\{ \frac{4B^2}{H^5} + \frac{108B}{H^{1-(e/2)}} (5H + 10D^{2+2}H^{1-\epsilon}) + (2H + 128B^2)S_1 \right. \]

\[ + 32B^2S_2 \exp\left(-\frac{U}{8B}\right) \] \sum_{\eta \leq M} |a_\eta|^2. \]

Thus

\[ \int_0^H |F(it)|^2 \, dt \geq (H - S_3) \sum_{\eta \leq M} |a_\eta|^2, \]

where \( S_3 > 0 \) and

\[ S_3 = (r + 5)U + 10D^{2+2}H^{1-\epsilon} + \frac{4B^2}{H^5} + \frac{108B}{H^{1-e/2}} (5H + 10^{\sigma+2}H^{1-\epsilon}) \]

\[ + (2H + 128B^2)2^\sigma(DH)^{2\beta}(2D)^{\sigma+2}H^{-\sigma/4} \]

\[ + 32B^2(DH)^{2\beta}H^{\sigma/4}\gamma(8B)^\gamma H^-\kappa(2-e)/2. \]

Here we have used \( \exp\left(-\frac{U}{8B}\right) \leq \frac{f(8B)}{U^{\epsilon}} \). Note that
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\[ \log \log K_1 \leq \log \log (H' K) \leq \log \log K + \log(r \log H) \]
\[ \leq \log \log K + \log r + \log H \]

and that \( \frac{1}{2} (\log H)^2 \leq H \) and so \( \log H \leq 2H^{1/2} \) and so

\[ \frac{\log H}{H^{1 - \epsilon/4}} \leq \frac{2}{H^{(1/2) - (\epsilon/4)}} \leq 2H^{-1/4}. \]

Hence

\[ (r + 5) U \leq 100 Br (\log \log K + \log r + \log H) + (r + 5) H^{1 - \epsilon/2}. \]

Thus

\[ S_3 \leq 100Br \log \log K + r(\log r) D^{x+2} H^{1 - (\epsilon/4)} \left\{ \frac{100B \log H}{H^{1 - (\epsilon/4)}} \right. \]
\[ + \frac{(r + 5)}{r(\log r) D^{x+2} H^{1 - (\epsilon/4)}} + \frac{10}{r(\log r) H^{3 \epsilon/4}} + \frac{4B^2}{H^5} + \frac{108B(15D^{x+2})H}{r(\log r) D^{x+2} H^{2 - (3\epsilon/4)}} \]
\[ + \frac{130B^2 H^2 D H B^2 (2D')^{(x+2)} H^{-r(\epsilon/4)}}{r(\log r) D^{x+2} H^{1 - (\epsilon/4)}} \]
\[ + \frac{32B^2 (DH)^2 D' r(8B) H^{-r(1 - \epsilon/4)}}{r(\log r) D^{x+2} H^{1 - (\epsilon/4)}} \right\} \]

Denote the expression in the last curly bracket by \( S_4 \). Then we have

\[ S_4 \leq \frac{200B}{H^{1/4}} + \frac{2r}{H^{1/4}} + \frac{10}{H^{1/4}} + \frac{4B^2}{H^{1/4}} + \frac{1620B}{H^{1/4}} + \frac{130B^2 D B^2 H^{2B+1 + 1}}{H^{1/4}} \left( \frac{4D^{x+2}}{H^{1/4}} \right)^r \]
\[ + \frac{32B^2 D^{2B} H^{2B+1}}{H^{1/4}} \left( \frac{8Br}{H^{1/2}} \right)^r. \]

Let \( H^{\epsilon/8} \geq 4D^{x+2} \). We have \( H^{\epsilon/8} \geq H^{(200B + 200)e^{-1} - 1)\epsilon/8} \geq H^{2B + 2} \). Let \( H^{1/4} \geq 8Br \). We have \( H^{1/4} \geq H^{2B+1} \). Now both \( H^{\epsilon/8} \geq 4D^{x+2} \) and \( H^{1/4} \geq 8Br \) are satisfied if

\[ H \geq (32BrD^{x+2})^{8/\epsilon}. \]

Hence under this only condition, we have,

\[ S_4 \leq (200B + 2r + 10 + 4B^2 + 1620B + 130B^2 D B + 32B^2 D^{2B}) H^{-\epsilon/4} \]
\[ \leq rB^2 D^{2B} H^{-\epsilon/4} (200 + 2 + 10 + 4 + 1620 + 130 + 32) \]
\[ \leq 2000rB^2 D^{2B} H^{-\epsilon/4} \leq 1 \]

provided \( H \geq (2000rB^2 D^{2B})^{4/\epsilon} \). Now this last condition and \( H \geq (32BrD^{x+2})^{8/\epsilon} \) are both satisfied if \( H \geq (50rBD^{x+2})^{8/\epsilon} \). Finally \( U \geq H^{1 - \epsilon/2} \geq H^{3/4} \geq (50 \times 200B)^{(8/\epsilon)(3/4)} = (10,000B\cdot B)^{6/\epsilon} \geq 2^{13}(6/\epsilon)(16B)^2 (B^{24}/(16B)^2) \geq 2^{70}(16B)^2 \) since \( B \geq 3 \). Collecting, we have proved the following

**Main Theorem.** Let \( \{ \lambda_n \} (n = 1, 2, 3, \ldots) \) with \( \lambda_1 = 1 \) be any increasing sequence of real numbers with the properties \( \lambda_n \leq Dn \) and \( \lambda_{n+1} - \lambda_n > D^{-1} \lambda_n^{a+1} \) where \( a(> 0) \) is a constant and \( D(\geq 1) \) is an integer constant. Let \( \{ a_n \} (n = 1, 2, 3, \ldots) \) with \( a_1 = 1 \) be any sequence
of complex numbers such that $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is absolutely convergent at $s = B$, where $B(\geq 3)$ is an integer constant. Let $0 < \varepsilon < (2(1 + \alpha))^{-1}$ and let $r(\geq [(200B + 200)\varepsilon^{-1}])$ be any integer constant. Let

$$\sum_{n=1}^{\infty} |a_n| \lambda_n^{-B} \leq H^{(r+2)/\varepsilon}.$$ 

Assume that $F(s)$ possesses an analytic continuation in $(\sigma \geq 0, 0 \leq t \leq H)$ and that there exist $T_1, T_2$ with $0 \leq T_1 \leq H^{3/4}$, $H - H^{3/4} \leq T_2 \leq H$ such that for some $K(\geq 30)$ there holds

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K$$

uniformly in $0 \leq \sigma \leq B$. Let

$$H \geq (50rBD^{B+\varepsilon+2})^{B/\varepsilon}.$$

Then, there holds,

$$\frac{1}{H} \int_{0}^{H} |F(it)|^2 dt \geq (1 - \phi) \sum_{n<M} |a_n|^2,$$

where

$$M = H^\theta, \quad \theta = \frac{1}{1 + \alpha} - \varepsilon,$$

and

$$\phi = r(\log r)D^{\varepsilon+2}H^{-\varepsilon/4} + 100H^{-1}Br \log \log K.$$

In view of the two closing remarks at the end of § 1 we can be now deduce some corollaries.

**COROLLARY 1.**

Let $\mu_n = n$. Then the conjecture is true.

**Proof.** We can take $C = 1$, $\alpha = \varepsilon$ and $D = 1$.

**COROLLARY 2.**

Let $n_0(\geq 2)$ be an integer constant and $\mu_n = (n_0 + n - 1)/n_0$. Then the conjecture is true.

**Proof.** First, since $\{\mu_n\}$ is a subsequence of $\{\lambda_n\}$ it follows that $\lambda_n \leq \mu_n \leq n$. To apply the main theorem we have to verify that $\lambda_{n+1} - \lambda_n \geq D^{-1} \lambda_n^{-\varepsilon/2}$ holds with some constant $\alpha > 0$ and $D(\geq 1)$ an integer constant. To prove this we observe that we can assume that $\lambda_{n+1} - \lambda_n \leq 1$. In this case

$$\lambda_{n+1} - \lambda_n = \frac{m_1 \cdots m_k}{n_0} - \frac{n_1 \cdots n_l}{n_0} \geq n_0^{-j},$$

where $j = \max(k, l)$. Now $(1 + (1/n_0))^k \leq \lambda_{n+1}$ and $(1 + (1/n_0))^l \leq \lambda_n$ and so $j = \max(k, l) \leq (\log \lambda_{n+1}) (\log ((n_0 + 1)/n_0))^{-1}$. But $\log(n_0 + 1)/n_0 = - \log(1 - (1/n_0 + 1)) > 1/(n_0 + 1)$. 

Thus \( j \leq (n_0 + 1)(\log \lambda_{n+1}) \) and so
\[
n_0^{-j} \geq \lambda_{n+1}^{-\alpha} \quad \text{where} \quad \alpha = (n_0 + 1)\log n_0.
\]
Plainly we can take \( D = 1 \).

**COROLLARY 3.**

Let \( \beta > 0 \) be an algebraic constant and \( \mu_n = (n + \beta)/(1 + \beta) \). Then the conjecture is true. (The conjecture is also true for the choice \( \mu_1 = 1, \mu_n = n + \beta - 1 \) for \( n > 1 \)).

**Proof.** As before \( \lambda_n \leq \mu_n \leq (n\beta + 2n)(\beta + 1)^{-1} \). Also considering the norm of \( \lambda_{n+1} - \lambda_n \) (in case it is \( \neq 0 \)) we can prove that \( \lambda_{n+1} - \lambda_n \geq D\lambda_n^{-\alpha} \). The latter assertion follows similarly.

**Post-script.** The results of this paper were necessitated by a lot of applications to the zeros of generalized Dirichlet series. All these applications will form the subject matter of our forthcoming paper "On the zeros of a class of generalized Dirichlet series-XI".

**References**