Proc. Indian Acad. Sci. (Math. Sci.), Vol. 102, No. 2, August 1992, pp. 83–91.

Proof of some conjectures on the mean-value of Titchmarsh series – III

R BALASUBRAMANIAN and K RAMACHANDRA*

The Institute of Mathematical Sciences, Madras 600 113, India

*School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005, India

MS received 17 June 1992

Abstract. With some applications in view, the following problem is solved in some special case which is not too special. Let $F(s)=\Sigma_{n=1}^{\infty} a_n\lambda_n^{-s}$ be a generalized Dirichlet series with $1=\lambda_1<\lambda_2<\ldots,\lambda_n\leqslant Dn$, and $\lambda_{n+1}-\lambda_n\geqslant D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha>0$ and $D(\geqslant 1)$ are constants. Then subject to analytic continuation and some growth conditions, a lower bound is obtained for $(1/H)\int_0^H|F(it)|^2\,dt$. These results will be applied in other papers to appear later.

Keywords. Titchmarsh series; mean value; lower bounds.

1. Introduction

In the previous papers [1] and [2] with the same title (as the present one) we proved some conjectures made by the second author [4]. In this paper we formulate a new conjecture (which we believe to be true at least in some modified form) and indicate a slight progress towards it.

Conjecture. *Let $1=\mu_1<\mu_2<\ldots$ be any sequence of real numbers with $1/C\leqslant\mu_{n+1}-\mu_n\leqslant C$ where $C(\geqslant 1)$ is an integer constant and $n=1,2,3,\ldots$. Let us form the sequence $1=\lambda_1<\lambda_2<\ldots$ of all possible (distinct) finite power products of $1=\mu_1,\mu_2,\ldots$ with non-negative integral exponents. Let $s=\sigma+it$, $H(\geqslant 10)$ be a real parameter, and $\{a_n\}$ $(n=1,2,3,\ldots)$ with $a_1=1$ be any sequence of complex numbers (possibly depending on H) such that $F(s)=\Sigma_{n=1}^{\infty}a_n\lambda_n^{-s}$ is absolutely convergent at $s=B$ where $B\geqslant 3$ is an integer constant. Suppose that $F(s)$ can be continued analytically in $(\sigma\geqslant 0, 0\leqslant t\leqslant H)$ and that there exist T_1, T_2 with $0\leqslant T_1\leqslant H^{3/4}, H-H^{3/4}\leqslant T_2\leqslant H$ such that for some $K(\geqslant 30)$ there holds*

$$\max_{\sigma\geqslant 0}(|F(\sigma+iT_1)|+|F(\sigma+iT_2)|)\leqslant K.$$

Finally let $\Sigma_{n=1}^{\infty}|a_n|\lambda_n^{-B}\leqslant H^A$ where $A(\geqslant 1)$ is an integer constant. Then there exists a constant $\delta>0$ (depending only on A, B and C) such that for all $H\geqslant H_0(A,B,C)$ there holds

$$\frac{1}{H}\int_0^H|F(it)|^2\,dt\geqslant\frac{1}{2}\sum_{\lambda_n\leqslant H^\delta}|a_n|^2, \tag{1}$$

provided that H^{-1} loglog K does not exceed a small positive constant.

* to whom correspondence should be addressed

Remark 1. We can strengthen the Conjecture (1) by replacing $\frac{1}{2}$ by a more specific function of H which is asymptotic to 1 as $H \to \infty$.

Remark 2. By the method of [1] we can prove that

$$\frac{1}{H}\int_0^H |F(it)|\,\mathrm{d}t \geqslant \frac{1}{2}. \tag{2}$$

The Remark 1 is also applicable.

Remark 3. Under the condition

$$\sum_{\lambda_n \leqslant X} |a_n| \leqslant D_0(\log X)^R \ (R = H^\varepsilon,\ D_0 \text{ any constant},\ X \geqslant 30), \tag{3}$$

we can prove (2). Remark 1 is also applicable. For both these results the conditions involving K are unnecessary. For the results mentioned in Remarks 2 and 3 we refer the reader to [1] and [5].

Remark 4. Actually the proof of (1) in [1] goes through without serious problems until we come to a lower bound for

$$\frac{1}{H}\int_0^H \left|\sum_{n \leqslant H^\delta} a_n \lambda_n^{-it}\right|^2 \mathrm{d}t.$$

To apply Montgomery–Vaughan theorem we need good lower bounds for $\lambda_{n+1} - \lambda_n$. These are not available in general. But we can work with $\mu_n = (n_0 + n - 1)/n_0$ where $n_0(\geqslant 2)$ is any integer constant (of course using Montgomery–Vaughan Theorem). Thus in this special case we can prove Conjecture (1). We can also handle $\mu_n = (1+\beta)^{-1}(n+\beta)$ where $\beta(>0)$ is any real algebraic constant.

Remark 5. We can formulate Conjecture (1) with no conditions involving K, but instead we have to assume condition (3). Remark 1 is also applicable.

Before closing this section we like to make two important remarks. First $\lambda_n \leqslant \mu_n \leqslant Cn$ which is obvious because $\{\lambda_n\}$ contains the subsequence $\{\mu_n\}$. Secondly for $x \geqslant 1$ and $\eta \geqslant 2C+1$, we have,

$$\begin{aligned}
\sum_{\lambda_n \leqslant x} 1 &\leqslant x^\eta \sum_{n=1}^{\infty} \frac{1}{\lambda_n^\eta} \leqslant x^\eta\left(1 - \sum_{n=2}^{\infty} \mu_n^{-\eta}\right)^{-1} \\
&\leqslant x^\eta\left\{1 - \sum_{n=2}^{\infty}\left(1 + \frac{n-1}{C}\right)^{-\eta}\right\}^{-1} \\
&\leqslant x^\eta\left\{1 - \int_0^\infty \left(1 + \frac{u}{C}\right)^{-\eta} \mathrm{d}u\right\}^{-1} = x^\eta\left(1 - C\int_0^\infty \frac{\mathrm{d}u}{(1+u)^\eta}\right)^{-1} \\
&= x^\eta\left(1 - \frac{C}{\eta - 1}\right)^{-1} \leqslant 2x^\eta.
\end{aligned}$$

Hence in (1) the condition $\lambda_n \leqslant H^\delta$ is equivalent to a condition of the type $n \leqslant H^\delta$ with a different constant $\delta > 0$.

2. Main lemma

Let r be a positive integer, $H \geqslant (r+5)U$ where $U \geqslant 2^{70}(16B)^2$ and M and N are positive integers subject to $N > M \geqslant 1$, and $B(\geqslant 3)$ an integer constant. Let $\{b_m\}$ $(1 \leqslant m \leqslant M)$ and $\{c_n\}$ $(n \geqslant N)$ be two sequences of complex numbers, $1 = \lambda_1 < \lambda_2 < \ldots$ be any increasing sequence of real numbers and let $A(s) = \Sigma_{m \leqslant M} b_m \lambda_m^{-s}$. Let $B(s) = \Sigma_{n \geqslant N} c_n \lambda_n^{-s}$ be absolutely convergent for $s = B$ and continuable analytically in $(\sigma \geqslant 0, 0 \leqslant t \leqslant H)$. Write $g(s) = A(-s)B(s)$,

$$G(s) = U^{-r} \int_0^U \mathrm{d}u_r \ldots \int_0^U \mathrm{d}u_1 (g(s+i\lambda))$$

where (here and elsewhere) $\lambda = u_1 + \ldots + u_r$. Assume that there exist real numbers T_1 and T_2 with $0 \leqslant T_1 \leqslant U$, $H - U \leqslant T_2 \leqslant H$, such that

$$|g(\sigma + iT_1)| + |g(\sigma + iT_2)| \leqslant \exp\exp\left(\frac{U}{16B}\right).$$

uniformly in $0 \leqslant \sigma \leqslant B$. Let

$$S_1 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B 2^r \left(U \log \frac{\lambda_n}{\lambda_m}\right)^{-r}$$

and

$$S_2 = \sum_{m \leqslant M, n \geqslant N} |b_m c_n| \left(\frac{\lambda_m}{\lambda_n}\right)^B.$$

Then, we have,

$$\left| \int_{2U}^{H-(r+3)U} G(it)\mathrm{d}t \right| \leqslant 2B^2 U^{-10} + 54BU^{-1} \int_0^H |g(it)|\mathrm{d}t$$
$$+ (H + 64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

Remark. This lemma is borrowed from [1] (see pages 2 to 8).

3. Progress towards the conjecture

From now on we assume that $1 = a_1, a_2, a_3, \ldots$ is any sequence of complex numbers. We set $b_m = a_m$ and $c_n = a_n$ and assume that $\Sigma_{n=1}^{\infty} |a_n| \lambda_n^{-B}$ is convergent.

Lemma 1. *We have, with $\bar{A}(s) = \Sigma_{m \leqslant M} a_m \lambda_n^{-s}$,*

$$\int_{2U}^{H-(r+3)U} |\bar{A}(it)|^2 \mathrm{d}t \geqslant (H - (r+5)U - 10\lambda_M \Delta(\lambda_M)) \sum_{m \leqslant M} |a_m|^2,$$

where $\Delta(\lambda_M) = \max\limits_{\substack{\mu \neq \nu \\ 1 \leqslant \mu, \nu \leqslant M}} |\lambda_\mu - \lambda_\nu|^{-1}$.

Proof. Follows from Montgomery-Vaughan Theorem (see [3]).

Lemma 2. *We have,*

$$S_2 \leqslant \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^2$$

and

$$S_1 \leqslant 2^r \lambda_M^{2B}\left(\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B}\right)^{-2} (U\lambda_N^{-1}(\lambda_N - \lambda_M))^{-r}.$$

Proof. The first inequality is trivial and the second follows from

$$\log\frac{\lambda_N}{\lambda_M} = -\log\left(1-\left(1-\frac{\lambda_M}{\lambda_N}\right)\right) > \frac{\lambda_N - \lambda_M}{\lambda_N}.$$

We now make the following.

Hypothesis. $\{\lambda_n\}$ is any increasing sequence of real numbers satisfying $\lambda_1 = 1$, $\lambda_n \leqslant Dn, \lambda_{n+1} - \lambda_n \geqslant \lambda_{n+1}^{-\alpha} D^{-1}$, where $D(\geqslant 1)$ is an integer constant and α a positive constant. Also we assume that

$$\sum_{n=1}^{\infty} |a_n|\lambda_n^{-B} \leqslant H^{r\varepsilon/8}$$

where $0 < \varepsilon \leqslant 1/[2(\alpha+1)]$ and $r \geqslant [(200B+200)\varepsilon^{-1}]$ is any integer. Also $F(s) = \Sigma_{n=1}^{\infty} a_n\lambda_n^{-s}$ shall be as in the introduction except that the $\{\lambda_n\}$ are not related to the $\{\mu_n\}$. $\{\lambda_n\}$ will now be an independent sequence.

From now on we set $N = M+1$, $M = [H^{(1/(\alpha+1)-\varepsilon)}]$, $U = H^{1-(\varepsilon/2)} + 50B \operatorname{loglog} K_1$ where $K_1 = H^r K$. Note that if $H \geqslant (r+5)U$ is not satisfied, our main theorem (to follow) asserts that a positive quantity is non-negative. Also note that

$$\min_{0\leqslant t\leqslant H^{3/4}} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

$$\geqslant \min_{0\leqslant t\leqslant U} \max_{\sigma\geqslant 0} |F(\sigma+it)|$$

and a similar result holds for the intervals $(H - H^{3/4}, H)$ and $(H-U, H)$.

Lemma 3. *We have,*

$$S_2 \leqslant (DH)^{2B} H^{r\varepsilon/4}, S_1 \leqslant 2^r (DH)^{2B} H^{r\varepsilon/4}((2D)^{-\alpha-2}H^{\varepsilon/2})^{-r}$$

and

$$\lambda_M \Delta(\lambda_M) \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

Proof. We have $\lambda_M \leqslant DM \leqslant DH$ and this proves the first inequality. Also

$$U\lambda_N^{-1}(\lambda_N - \lambda_M) \geqslant H^{1-(\varepsilon/2)}\lambda_N^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}D^{-1}(DN)^{-1-\alpha}$$
$$\geqslant H^{1-(\varepsilon/2)}(2DM)^{-1-\alpha}D^{-1} \geqslant H^{1-(\varepsilon/2)}(2D)^{-\alpha-2}H^{-1+\varepsilon},$$

and this proves the second inequality. The third follows from

$$\lambda_M \Delta(\lambda_M) \leqslant \lambda_M D \lambda_M^{\alpha} \leqslant D^{\alpha+2} M^{1+\alpha} \leqslant D^{\alpha+2} H^{1-\varepsilon}.$$

The lemma is completely proved.

Now we apply the main lemma (we closely follow the proof of the first main theorem in [1]). Let

$$A(s) = \sum_{m \leqslant M} \bar{a}_m \lambda_m^{-s}, \quad \bar{A}(s) = \sum_{m \leqslant M} a_m \lambda_m^{-s}$$

and

$$B(s) = \sum_{n \geqslant N} a_n \lambda_n^{-s}.$$

Then, we have, in $\sigma \geqslant B$, $F(s) = \bar{A}(s) + B(s)$ and so

$$\begin{aligned} |F(it)|^2 &= |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)) + |B(it)|^2 \\ &\geqslant |\bar{A}(it)|^2 + 2\,\mathrm{Re}\,(g(it)), \end{aligned}$$

where $g(s) = A(-s)B(s)$. Hence

$$\begin{aligned} &\int_0^H |F(it)|^2 \mathrm{d}t \\ &\geqslant U^{-r} \int_0^U \mathrm{d}u_r \ldots \int_0^U \mathrm{d}u_1 \int_{2U+\lambda}^{H-(r+3)U+\lambda} (|\bar{A}(it)|^2 + 2\,\mathrm{Re}\, g(it)) \mathrm{d}t \\ &= J_1 + 2J_2 \text{ say.} \end{aligned}$$

By Lemmas 1 and 3, we have,

$$J_1 \geqslant (H - (r+5)U - 10D^{\alpha+2} H^{1-\varepsilon}) \sum_{n \leqslant M} |a_n|^2.$$

Again, we have, for $0 \leqslant \sigma \leqslant B$,

$$\begin{aligned} |g(s)| &= |A(-s)B(s)| = |A(-s)(F(s) - A(s))| \\ &\leqslant \left(\sum_{n \leqslant M} |a_n| \lambda_n^B \right) K + \left(\sum_{n \leqslant M} |a_n| \lambda_n^B \right)^2 \\ &\leqslant \lambda_M^{2B} H^{r\varepsilon/8} K + \lambda_M^{4B} H^{r\varepsilon/4}. \end{aligned}$$

Hence

$$\begin{aligned} |g(s)|_{t=T_1} + |g(s)|_{t=T_2} &\leqslant 2K \lambda_M^{4B} H^{r\varepsilon/4} (H^{-r\varepsilon/8} \lambda_M^{-2B} + K^{-1}) \\ &\leqslant K(DH)^{4B} H^{r\varepsilon/4} \leqslant H^r K = K_1, \end{aligned}$$

the last two inequalities being true for instance if $H \geqslant 10D$. Observe that

$$\operatorname{expexp}\left(\frac{U}{16B}\right) \geqslant \operatorname{expexp}\left(\frac{50}{16} \operatorname{loglog} K_1\right) \geqslant K_1$$

and hence the condition on g and U required by the main lemma is satisfied. Hence by the main lemma, we have,

$$|J_2| \leqslant \frac{2B^2}{U^{10}} + \frac{54B}{U}\int_0^H |g(it)|\,dt + (H+64B^2)S_1 + 16B^2 S_2 \exp\left(-\frac{U}{8B}\right)$$

provided $H \geqslant (r+5)U$ and $U \geqslant 2^{70}(16B)^2$. As remarked already we can ignore the condition $H \geqslant (r+5)U$. Also we will satisfy $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$ and we will show later that this implies $U \geqslant 2^{70}(16B)^2$. We can assume that $\int_0^H |F(it)|^2\,dt \leqslant H\Sigma_{n\leqslant M}|a_n|^2$ (otherwise the result asserted by the main theorem to follow, is trivially true). Hence

$$\begin{aligned}
\int_0^H |g(it)|\,dt &= \int_0^H |A(-it)B(it)|\,dt \\
&\leqslant \int_0^H |A(-it)|^2\,dt + \int_0^H |B(it)|^2\,dt \\
&\leqslant 3\int_0^H |A(-it)|^2\,dt + 2\int_0^H |F(it)|^2\,dt \\
&\text{(on noting that } B(it) = F(it) - \bar{A}(it)) \\
&\leqslant (5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M} |a_n|^2
\end{aligned}$$

by Montgomery–Vaughan Theorem and the third part of Lemma 3. Hence

$$\begin{aligned}
2|J_2| &\leqslant \frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) \sum_{n\leqslant M}|a_n|^2 \\
&\quad + (2H + 128B^2)S_1 + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right) \\
&\leqslant \left\{\frac{4B^2}{H^5} + \frac{108B}{H^{1-(\varepsilon/2)}}(5H + 10D^{\alpha+2}H^{1-\varepsilon}) + (2H+128B^2)S_1\right. \\
&\quad \left. + 32B^2 S_2 \exp\left(-\frac{U}{8B}\right)\right\} \sum_{n\leqslant M}|a_n|^2.
\end{aligned}$$

Thus

$$\int_0^H |F(it)|^2\,dt \geqslant (H - S_3)\sum_{n\leqslant M}|a_n|^2,$$

where $S_3 > 0$ and

$$\begin{aligned}
S_3 &= (r+5)U + 10D^{\alpha+2}H^{1-\varepsilon} + \frac{4B^2}{H^5} + \frac{108B}{H^{1-\varepsilon/2}}(5H + 10^{\alpha+2}H^{1-\varepsilon}) \\
&\quad + (2H + 128B^2)2^r(DH)^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4} \\
&\quad + 32B^2(DH)^{2B}H^{r\varepsilon/4}r^r(8B)^r H^{-r(2-\varepsilon)/2}.
\end{aligned}$$

Here we have used $\exp\left(-\frac{U}{8B}\right) \leqslant \frac{r^r(8B)^r}{U^r}$. Note that

$$\log\log K_1 \leqslant \log\log(H^r K) \leqslant \log\log K + \log(r\log H)$$
$$\leqslant \log\log K + \log r + \log H$$

and that $\frac{1}{2}(\log H)^2 \leqslant H$ and so $\log H \leqslant 2H^{1/2}$ and so

$$\frac{\log H}{H^{1-\varepsilon/4}} \leqslant \frac{2}{H^{(1/2)-(\varepsilon/4)}} \leqslant 2H^{-1/4}.$$

Hence

$$(r+5)U \leqslant 100Br(\log\log K + \log r + \log H) + (r+5)H^{1-\varepsilon/2}.$$

Thus

$$S_3 \leqslant 100Br\log\log K + r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}\left\{\frac{100B\log H}{H^{1-(\varepsilon/4)}}\right.$$
$$+\frac{(r+5)}{r(\log r)D^{\alpha+2}H^{\varepsilon/4}} + \frac{10}{r(\log r)H^{3\varepsilon/4}} + \frac{4B^2}{H^5} + \frac{108B(15D^{\alpha+2})H}{r(\log r)D^{\alpha+2}H^{2-(3\varepsilon/4)}}$$
$$+\frac{130B^2H2^rD^BH^{2B}(2D)^{r(\alpha+2)}H^{-r\varepsilon/4}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}}$$
$$\left.+\frac{32B^2(DH)^{2B}r^r(8B)^rH^{-r(1-\varepsilon/4)}}{r(\log r)D^{\alpha+2}H^{1-(\varepsilon/4)}}\right\}.$$

Denote the expression in the last curly bracket by S_4. Then we have

$$S_4 \leqslant \frac{200B}{H^{\varepsilon/4}} + \frac{2r}{H^{\varepsilon/4}} + \frac{10}{H^{\varepsilon/4}} + \frac{4B^2}{H^{\varepsilon/4}} + \frac{1620B}{H^{\varepsilon/4}} + \frac{130B^2D^BH^{2B+1+1}}{H^{\varepsilon/4}}\left(\frac{4D^{\alpha+2}}{H^{\varepsilon/4}}\right)^r$$
$$+\frac{32B^2D^{2B}}{H^{\varepsilon/4}}H^{2B+1}\left(\frac{8Br}{H^{1/2}}\right)^r.$$

Let $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$. We have $H^{r\varepsilon/8} \geqslant H^{((200B+200)\varepsilon^{-1}-1)\varepsilon/8} \geqslant H^{2B+2}$. Let $H^{1/4} \geqslant 8Br$. We have $H^{r/4} \geqslant H^{2B+1}$. Now both $H^{\varepsilon/8} \geqslant 4D^{\alpha+2}$ and $H^{1/4} \geqslant 8Br$ are satisfied if

$$H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}.$$

Hence under this only condition, we have,

$$S_4 \leqslant (200B + 2r + 10 + 4B^2 + 1620B + 130B^2D^B + 32B^2D^{2B})H^{-\varepsilon/4}$$
$$\leqslant rB^2D^{2B}H^{-\varepsilon/4}(200 + 2 + 10 + 4 + 1620 + 130 + 32)$$
$$\leqslant 2000rB^2D^{2B}H^{-\varepsilon/4} \leqslant 1$$

provided $H \geqslant (2000rB^2D^{2B})^{4/\varepsilon}$. Now this last condition and $H \geqslant (32BrD^{\alpha+2})^{8/\varepsilon}$ are both satisfied if $H \geqslant (50rBD^{B+\alpha+2})^{8/\varepsilon}$. Finally $U > H^{1-\varepsilon/2} \geqslant H^{3/4} \geqslant (50 \times 200B)^{(8/\varepsilon)(3/4)} = (10{,}000B\cdot B)^{6/\varepsilon} \geqslant 2^{13(6/\varepsilon)}(16B)^2\ (B^{24}/(16B)^2) \geqslant 2^{70}(16B)^2$ since $B \geqslant 3$. Collecting, we have proved the following

Main Theorem. *Let $\{\lambda_n\}(n = 1, 2, 3, \ldots)$ with $\lambda_1 = 1$ be any increasing sequence of real numbers with the properties $\lambda_n \leqslant Dn$ and $\lambda_{n+1} - \lambda_n > D^{-1}\lambda_{n+1}^{-\alpha}$ where $\alpha(>0)$ is a constant and $D(\geqslant 1)$ is an integer constant. Let $\{a_n\}(n = 1, 2, 3, \ldots)$ with $a_1 = 1$ be any sequence*

of complex numbers such that $F(s)=\sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ *is absolutely convergent at* $s=B$, *where* $B(\geqslant 3)$ *is an integer constant. Let* $0<\varepsilon<(2(1+\alpha))^{-1}$ *and let* $r(\geqslant[(200B+200)\varepsilon^{-1}])$ *be any integer constant. Let*

$$\sum_{n=1}^{\infty}|a_n|\lambda_n^{-B}\leqslant H^{(r\varepsilon/8)}.$$

Assume that $F(s)$ *possesses an analytic continuation in* $(\sigma\geqslant 0, 0\leqslant t\leqslant H)$ *and that there exist* T_1, T_2 *with* $0\leqslant T_1\leqslant H^{3/4}$, $H-H^{3/4}\leqslant T_2\leqslant H$ *such that for some* $K(\geqslant 30)$ *there holds*

$$|F(\sigma+iT_1)|+|F(\sigma+iT_2)|\leqslant K$$

uniformly in $0\leqslant\sigma\leqslant B$. *Let*

$$H\geqslant(50rBD^{B+\alpha+2})^{8/\varepsilon}.$$

Then, there holds,

$$\frac{1}{H}\int_0^H|F(it)|^2\,dt\geqslant(1-\phi)\sum_{n\leqslant M}|a_n|^2,$$

where

$$M=H^{\theta},\quad \theta=\frac{1}{1+\alpha}-\varepsilon,$$

and

$$\phi=r(\log r)D^{\alpha+2}H^{-\varepsilon/4}+100H^{-1}Br\log\log K.$$

In view of the two closing remarks at the end of § 1 we can be now deduce some corollaries.

COROLLARY 1.

Let $\mu_n=n$. *Then the conjecture is true.*

Proof. We can take $C=1$, $\alpha=\varepsilon$ and $D=1$.

COROLLARY 2.

Let $n_0(\geqslant 2)$ *be an integer constant and* $\mu_n=(n_0+n-1)/n_0$. *Then the conjecture is true.*

Proof. First, since $\{\mu_n\}$ is a subsequence of $\{\lambda_n\}$ is follows that $\lambda_n\leqslant\mu_n\leqslant n$. To apply the main theorem we have to verify that $\lambda_{n+1}-\lambda_n\geqslant D^{-1}\lambda_{n+1}^{-\alpha}$ holds with some constant $\alpha>0$ and $D(\geqslant 1)$ an integer constant. To prove this we observe that we can assume that $\lambda_{n+1}-\lambda_n\leqslant 1$. In this case

$$\lambda_{n+1}-\lambda_n=\frac{m_1\ldots m_k}{n_0^k}-\frac{n_1\ldots n_l}{n_0^l}\geqslant n_0^{-j}$$

where $j=\max(k,l)$. Now $(1+(1/n_0))^k\leqslant\lambda_{n+1}$ and $(1+(1/n_0))^l\leqslant\lambda_n$ and so $j=\max(k,l)\leqslant(\log\lambda_{n+1})(\log((n_0+1)/n_0)^{-1}$. But $\log(n_0+1)/n_0=-\log(1-(1/n_0+1))>1/(n_0+1)$.

Thus $j \leqslant (n_0 + 1)(\log \lambda_{n+1})$ and so

$$n_0^{-j} \geqslant \lambda_{n+1}^{-\alpha} \text{ where } \alpha = (n_0 + 1)\log n_0.$$

Plainly we can take $D = 1$.

COROLLARY 3.

Let $\beta > 0$ be an algebraic constant and $\mu_n = (n + \beta)/(1 + \beta)$. Then the conjecture is true. (The conjecture is also true for the choice $\mu_1 = 1$, $\mu_n = n + \beta - 1$ for $n > 1$).

Proof. As before $\lambda_n \leqslant \mu_n \leqslant (n\beta + 2n)(\beta + 1)^{-1}$. Also considering the norm of $\lambda_{n+1} - \lambda_n$ (in case it is $\neq 0$) we can prove that $\lambda_{n+1} - \lambda_n \geqslant D\lambda_{n+1}^{-\alpha}$. The latter assertion follows similarly.

Post-script. The results of this paper were necessitated by a lot of applications to the zeros of generalized Dirichlet series. All these applications will form the subject matter of our forthcoming paper "On the zeros of a class of generalized Dirichlet series-XI".

References

[1] Balasubramanian R and Ramachandra K, Proof of some conjectures on the mean-value of Titchmarsh series-I, *Hardy–Ramanujan J.* **13** (1990) 1–20

[2] Balasubramanian R and Ramachandra K, Proof of some conjectures on the mean-value of Titchmarsh series-II, *Hardy–Ramanujan J.* **14** (1991) 1–20

[3] Montgomery H L and Vaughan R C, Hilbert's inequality, *J. London Math. Soc.* (2) **8** (1974) 73–82

[4] Ramachandra K, Progress towards a conjecture on the mean-value of Titchmarsh series-I, *Recent Progress in Analytic Number Theory* (eds) H Halberstam and C Hooley (Academic Press) (1981) 303–318

[5] Ramachandra K, Proof of some conjectures on the mean-value of Titchmarsh series with applications to Titchmarsh's phenomenon, *Hardy–Ramanujan J.* **13** (1990) 21–27