

## A closed clockwork theory: $\mathbb{Z}_2$ parity and more

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**ABSTRACT:** We develop a new class of clockwork theories with an augmented structure of the near-neighbour interactions along a one-dimensional *closed* chain. Such a topology leads to new and attractive features in addition to generating light states with hierarchical couplings via the usual clockwork mechanism. For one, there emerges a  $\mathbb{Z}_2$  symmetry under the exchange of fields resulting in a physical spectrum consisting of states, respectively even and odd under the *exchange* parity with a two-fold degeneracy at each level. The lightest odd particle, being absolutely stable, could be envisaged as a potential dark matter candidate. The theory can also be obtained as a deconstruction of a five-dimensional theory embedded in a geometry generated by a linear dilaton theory on a  $S^1/\mathbb{Z}_2$  orbifold with three equidistant 3-branes. Analogous to the discrete picture, the  $\mathbb{Z}_2$  symmetry in the bulk theory necessitates the existence of a KK spectrum of even and odd states, with doubly degenerate modes at each KK level when subject to certain boundary conditions.

**KEYWORDS:** Effective Field Theories, Extra Dimensions, Hierarchy Problem

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**1 Introduction**

As successful as it might seem, the Standard Model (SM) of particle physics is, by no means, the ultimate theory describing Nature in its totality. The shortcomings, both theoretical and those pertaining to discrepancies with data have led to continuing searches for realistic extensions. Over the years, a plethora of phenomenologically relevant new physics models have been proposed to address the shortcomings of the SM, but only for newer issues to emerge. A recurrent problem pertains to the apparent hierarchy of mass scales and couplings. For instance, models with axions or axion-like particles typically require small axion-SM couplings [1]. The case of neutrino mass models with hierarchically heavy right-handed neutrinos serves as another example [2, 3]. Even within the SM, the fermion masses

display a hierarchy among the different flavors. The issue, therefore, warrants an underlying mechanism to generate such small masses or couplings, the absence of which would require a fine-tuning in the corresponding ultraviolet (UV)-complete theory.

The *clockwork* mechanism proposed in refs. [4–6] apparently offers an interesting solution to the aforementioned conundrum. The original mechanism (applicable to particles of any given spin), generically, entails  $N + 1$  copies of a field theory, each involving a massless field with the masslessness being protected by some global symmetry  $G$ . The entire construction can be pictured as a chain of sites in a one-dimensional lattice in the field space with the full symmetry being  $G^{N+1}$ . Furthermore, introducing near-neighbour interaction terms with a strength parametrised by a dimensionless factor  $q (> 1)$  explicitly breaks the global symmetry to a single factor  $G'$  corresponding to which we obtain a single massless particle. This massless state, a linear combination of the  $N + 1$  fields, interestingly, has a hierarchical distribution along the sites with an exponential localisation towards one of the boundary sites.<sup>1</sup> Consequently, a coupling of the SM sector to the opposite boundary of the *clockwork* space would have a coupling of magnitude  $\sim 1/q^N$  with the massless state. Thus, even without introducing any small coupling, just a moderately large value of  $N$  generates a very sizable suppression. The massive modes (the so-called clockwork “gears”), on the other hand, have nearly-flat distributions along the sites. This is the essence of the *clockwork* mechanism which has found its applications in not only the original motivations around axion physics [4, 5, 9–14] but also in other contexts like flavor hierarchies in masses and mixings [15–18], neutrino masses [19–21], dark matter [22–26], neutron-antineutron oscillation [27] as well as inflationary cosmology [28–30]. The mechanism, in its present form, largely accommodates only Abelian groups, but discussions pertaining to non-Abelian clockwork theories can be found in refs. [31–33].

The formulation of the clockwork theories as described in ref. [6] has another compelling attribute in that the one-dimensional lattice in the theory space can be regarded as the deconstruction of a continuum 5D field theory. The corresponding *warped* geometry turns out to be the one that is generated by a 5D linear dilaton theory of gravity [34, 35] with the fifth-dimension compactified on a  $S^1/\mathbb{Z}_2$  orbifold augmented by a 3-brane at each of its boundaries. This setup, similar to the RS1 model [36], also offers a possible solution to the hierarchy problem along with a rich phenomenology of the dynamical graviton and dilaton modes [37, 38].

Since the mechanism relies on the specific structure of the near-neighbour interactions, it is worth exploring whether there exist extensions of this structure, with different *topologies*, that possess novel phenomenological implications in addition to a similar *clockwork* mechanism. To this effect, we attempt to construct a class of clockwork theories with a modified structure compared to the original such that the resulting hierarchy of the massless particle’s distribution along the sites first increases (decreases) from one end towards an intermediate site  $p$  and then decreases (increases) towards the other end. Further, taking  $p$  to be the centre-most site in the lattice and identifying the two end-sites lead to a theory with a  $\mathbb{Z}_2$  symmetry characterized by an exchange of fields in the two arms about the

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<sup>1</sup>While similar effects had been anticipated in refs. [7, 8] the goals of (and, hence, the field assignments in) the clockwork scenarios are distinctly different.

$p$ -th site. Another distinguishing feature of the construction is that it also generates a spectrum of massive particles (CW gears) which are doubly degenerate at each level with the exception of the heaviest mode.<sup>2</sup> This is but a consequence of the *exchange parity* that emerges naturally and ensures that the lightest odd mode in the spectrum remains absolutely stable and, therefore, could potentially be a dark matter candidate.

We further identify the continuum counterpart of the new theory with the geometry generated by a 5D linear dilaton (LD), albeit augmented by a third 3-brane at the centre of the orbifold, with the  $\mathbb{Z}_2$  symmetry being identified with the coordinate reflection in the fifth dimension about the middle brane. The 4D spectrum, for a theory with a vanishing bulk mass, consists of a purely massless mode and a tower of massive KK excitations with a two-fold degeneracy at each KK level. The latter, automatically, are either odd or even under the continuum version of the *exchange parity* which can be regarded as a form of KK parity in a warped 5D scenario, a feature that is typically absent in two-brane warped scenarios.

The primary goal of this paper is to present the generic structure of the modified scenario and, therefore, we resort to discussing the phenomenological implications only in passing, deferring dedicated studies thereof to future works. The outline of the paper is as follows. In section 2, we review the original *one-sided* clockwork theory and introduce our modified setup in section 3. Here, we limit the discussion to bosonic field theories with Abelian symmetries. Section 4 delineates the continuum limit and the corresponding linear dilaton theory that generates the modified 5D metric. The nature of bulk scalar and Abelian vector fields in the new background geometry is evaluated individually in section 5. In section 6 we comment on the fine tunings in the linear dilaton theory and, finally, we conclude in section 7.

## 2 Prelude

We review here, briskly, the basic elements of the original clockwork mechanism proposed in refs. [4, 6], specifically the case of the scalar clockwork. The idea essentially entails a low energy theory of  $(N + 1)$  Nambu-Goldstone bosons corresponding to the spontaneous breaking of an Abelian theory with a global symmetry  $U(1)^{N+1}$  at some high scale  $f$ . The theory below the scale  $f$ , therefore, has a Goldstone symmetry  $U(1)^{N+1}$  which is broken explicitly to a single (shift) symmetry  $U(1)_{CW}$  by introducing interaction terms among “adjacent” NGB species, namely

$$\mathcal{L}_{CW} = -\frac{f^2}{2} \sum_{j=0}^N \partial_\mu U_j^\dagger \partial^\mu U_j - \frac{m^2 f^2}{2} \left[ \sum_{j=0}^{N-1} U_j^\dagger U_{j+1}^q \right] + \text{h.c.} \quad (2.1)$$

where  $m$  is a mass parameter and  $q$  is a real constant. We adopt the metric signature  $(-, +, +, +, +)$ . Denoting the charge operator corresponding to  $U(1)_j$  by  $Q_j$ , (a simple choice is  $Q_j\{U_j\} = 1, \forall j$ ), the generator corresponding to  $U(1)_{CW}$  is

$$Q_{CW} = \sum_{j=0}^N \frac{Q_j}{q^j}, \quad (2.2)$$

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<sup>2</sup>This is in stark contrast to the *two-sided* theory considered in ref. [20] in the context of neutrino masses.

From a different perspective, each  $U_j$  defines a site on a discrete landscape of scalar field theories and eq. (2.1) simply denotes a chain of sites with near-neighbour interactions. Representing  $U_j = e^{i\phi_j(x)/f}$  ( $j = 0 \dots N$ ) with the  $\phi_j$ 's being the pNGBs, the first term in eq. (2.1) leads to canonical kinetic terms for the  $\phi_j$  while the second yields a  $(N+1) \times (N+1)$  (mass)<sup>2</sup> matrix apart from quartic and higher order terms.

Diagonalizing the free Lagrangian through the unitary transformation

$$\varphi_n = \sum_j a_{nj} \phi_j, \tag{2.3}$$

we expect one exactly massless mode (on account of the residual shift symmetry) and  $N$  massive modes. Indeed, the eigenvalues are

$$m_n^2 = m^2 \begin{cases} 0 & n = 0 \\ 1 + q^2 - 2q \cos \frac{n\pi}{N+1} & n \neq 0 \end{cases}, \tag{2.4}$$

with the corresponding eigenvectors being ( $j = 0 \dots N$ ,  $n = 1 \dots N$ )

$$\begin{aligned} a_{0j} &= \frac{\mathcal{N}_0}{q^j}, & a_{nj} &= \mathcal{N}_n \left[ q \sin \frac{jn\pi}{N+1} - \sin \frac{(j+1)n\pi}{N+1} \right] \\ \mathcal{N}_0^2 &\equiv \frac{q^2 - 1}{q^2 - q^{-2N}} & \mathcal{N}_n^2 &\equiv \frac{2m^2}{(N+1)m_n^2}. \end{aligned} \tag{2.5}$$

The *clockwork*(CW) mechanism can now be illustrated with a simple example. Consider an external operator (i.e., composed of fields other than  $\phi_j$ )  $\mathcal{O}_{\text{ext}}$  which couples to the clockwork sector at the  $N$ -th site as

$$\mathcal{L}_{\text{int}} = y \phi_N \mathcal{O}_{\text{ext}}, \tag{2.6}$$

where  $y$  is a coupling parameter. In the mass basis, this results in

$$\mathcal{L}_{\text{int}} = y \left[ \frac{\mathcal{N}_0}{q^N} \varphi_0 + \sum_{n=1}^N \mathcal{N}_n q \sin \frac{Nn\pi}{N+1} \varphi_n \right] \mathcal{O}_{\text{ext}}. \tag{2.7}$$

Assuming that all the dimensionless parameters in the theory are *a priori*  $\mathcal{O}(1)$ , we see that  $\varphi_0$  couples to the external operator with an exponentially suppressed strength for  $q > 1$  and a sufficiently large  $N$ . On the other hand, the heavy eigenstates couple to the external operator with a nearly flat coupling profile. Thus, the clockwork mechanism generates very small couplings naturally in a theory which does not contain any small parameters whatsoever. This is the crux of the mechanism in a nutshell. In the modified theory that we discuss in the subsequent sections we largely retain this feature, albeit with small modifications. In addition, several new and attractive features emanate automatically.

### 3 The closed clockwork

As seen in the preceding section, the original clockwork scenario could be thought of as an open chain of  $U_j$ 's with their charges, under the remnant  $U(1)_{\text{CW}}$ , being given by a

geometric progression. In contrast, we consider a theory with a closed chain, restricting ourselves to a discussion of only scalars and Abelian gauge fields.<sup>3</sup>

### 3.1 Scalar clockwork

A closed clockwork chain implies, of course, that either the charge ratio  $q$  has to be unity or that the geometric progression needs to invert at a point. Since the first alternative does not lead to a hierarchy of couplings, we adopt the second. To be very specific, we consider an even number  $2p$  of pseudo-NGBs and modify the Lagrangian of eq. (2.1) to

$$\begin{aligned} \mathcal{L}_{CW} &= -\frac{f^2}{2} \sum_{j=0}^{2p-1} \partial_\mu U_j^\dagger \partial^\mu U_j - \frac{m^2 f^2}{2} \left[ \sum_{j=0}^{p-1} U_j^\dagger q U_{j+1} + \sum_{j=p}^{2p-1} U_j^\dagger U_{j+1}^q \right] + \text{h.c.} \\ &= -\frac{1}{2} \sum_{j=0}^{2p-1} \partial^\mu \phi_j \partial_\mu \phi_j + \frac{m^2}{2} \left[ \sum_{j=0}^{p-1} (q\phi_j - \phi_{j+1})^2 + \sum_{j=p}^{2p-1} (\phi_j - q\phi_{j+1})^2 \right] + O(\phi^4). \end{aligned} \tag{3.1}$$

Note that we have maintained the same charge ratio  $q$  in both arms which necessitates the pivoting to occur exactly midway, i.e., at the  $p^{\text{th}}$  site. Unequal ratios in the two arms could be chosen too, but only at the cost of losing some very attractive features.

The identification  $U_{2p} \equiv U_0$  (ensuring that  $\phi_0$  couples to both  $\phi_1$  and  $\phi_{2p-1}$ ) renders this a *closed clockwork* scenario and distinguishes it from the two-sided but open-ended scenario presented in ref. [20]. This single difference turns out to have profound implications as we would see below.

Before discussing the mass spectrum, it is worthwhile to enumerate the symmetries of the theory. As before, the Lagrangian in eq. (3.1) has a residual shift symmetry  $U(1)_{CW}$  which is now defined by the generator,

$$\mathcal{Q}_{CW} = \sum_{j=0}^{p-1} \frac{Q_j}{q^{2p-j}} + \sum_{j=p}^{2p-1} \frac{Q_j}{q^j}, \tag{3.2}$$

where, as before, we have assumed that each  $U_j$  has a unit charge under the symmetry  $U(1)_j$ . Additionally, the theory has a  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$  symmetry, where the first  $\mathbb{Z}_2$  corresponds to the rigid exchanges of the fields, namely,  $\phi_j \rightarrow \phi_{2p-j}$  and the second to the exchanges  $\phi_j \rightarrow -\phi_{2p-j}$ , both  $\forall j$ .<sup>4</sup> A consequence of these symmetries is that the eigenmodes must be either even or odd under it. In other words, the eigenstates may be represented as

$$\varphi_n^{(\pm)} = \sum_j a_{nj}^{(\pm)} \phi_j, \tag{3.3}$$

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<sup>3</sup>Incorporating non-Abelian groups in discrete clockwork scenarios introduces certain complications. For example, in the scalar theory, if the fields  $U_j$  were to be bi-fundamental representations of global non-Abelian groups  $G_{j,L} \times G_{j,R}$ , the non-trivial case  $q > 1$  preserves (post explicit breaking of the full symmetry) only a diagonal subgroup  $H$ . Consequently, this leads to a pseudo-Goldstone multiplet susceptible to loop-level corrections to its mass [32]. On the other hand, non-Abelian gauge invariance would typically necessitate the trivial case  $q = 1$  for which the clockwork mechanism does not exist. One may, however, choose to invoke a clockwork hierarchy at the expense of losing gauge invariance [31, 39].

<sup>4</sup>Equivalently, these can be recast as the trivial rigid  $\phi_j \rightarrow -\phi_j$  and  $\phi_j \rightarrow \phi_{2p-j}$ , where the former simply reflects the invariance under charge conjugation (namely,  $U_j \rightarrow U_j^*$ ). The second symmetry, therefore, is the only nontrivial discrete transformation under which the theory is invariant.

where  $(\pm)$  denote the even and odd modes respectively and  $a_{nj}^{(\pm)}$  are the corresponding elements of the transformation matrix, satisfying the eigenvalue equation,

$$\sum_j [M_\phi^2]_{ij} a_{nj} = m^2 \lambda_n a_{ni}, \forall n, i. \quad (3.4)$$

Reverting to the  $(2p \times 2p)$  mass matrix, it is given by

$$M_\phi^2 = m^2 \begin{pmatrix} & \text{(col 0)} & & & \text{(col } p) & & & & \text{(col } 2p-1) \\ \text{(row 0)} & 2q^2 & -q & 0 & 0 & \dots & \dots & \dots & -q \\ & -q & q^2 + 1 & -q & 0 & \dots & \dots & \dots & 0 \\ & 0 & \ddots & \ddots & & \dots & \dots & \dots & \vdots \\ & \vdots & & \ddots & \ddots & \dots & \dots & \dots & \vdots \\ & \dots & -q & q^2 + 1 & -q & 0 & \dots & \dots & \vdots \\ \text{(row } p) & 0 & \dots & -q & 2 & -q & 0 & \dots & \dots \\ & 0 & \dots & \dots & -q & q^2 + 1 & -q & 0 & \vdots \\ & \vdots & \dots & \dots & & \ddots & \ddots & \dots & \vdots \\ \text{(row } 2p-1) & -q & 0 & 0 & 0 & \dots & \dots & -q & q^2 + 1 \end{pmatrix}. \quad (3.5)$$

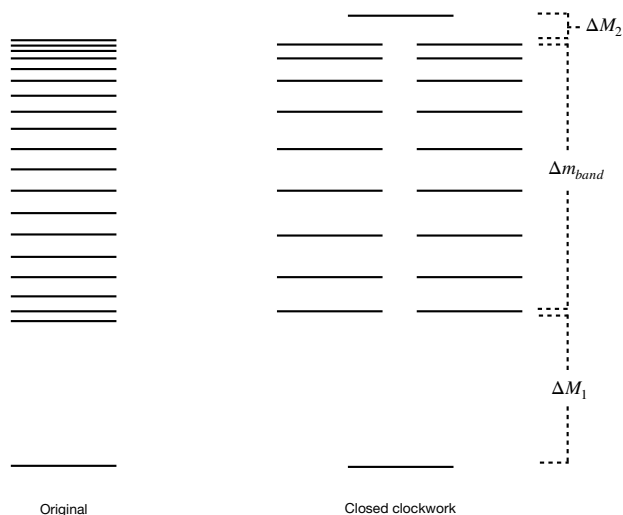
When compared to the mass matrices obtained in ref. [6] from eq. (2.1) or in ref. [20], differences appear only in the  $(0, 0)$ ,  $(p, p)$ ,  $(0, 2p-1)$ ,  $(2p-1, 0)$  and  $(2p-1, 2p-1)$  elements and, together, these would turn out to have very interesting consequences. For one, the eigenvalues are

$$m_n^2 \equiv m^2 \lambda_n = m^2 \begin{cases} 0 & n = 0, \\ 2(1 + q^2) & n = p, \\ 1 + q^2 - 2q \cos \frac{n\pi}{p} & \text{otherwise.} \end{cases} \quad (3.6)$$

A few remarks on the physical spectrum are now in order

- As is evident from eq. (3.6), the spectrum consists of an exactly massless mode  $\varphi_0$  and is bounded from above by the massive mode  $\varphi_p$ .
- In addition to the mass gap between the lightest mode  $\varphi_0$  and the level 1 modes  $\varphi_1/\varphi_{2p-1}$  ( $\Delta M_1$ ) there exists another one ( $\Delta M_2$ ) between the heaviest state  $\varphi_p$  and the penultimate  $\varphi_{p-1}/\varphi_{p+1}$  states.<sup>5</sup> This is in contrast with the case in [4, 6] wherein only the lower gap exists.
- The intermediate states are doubly degenerate at each level, with each degenerate subspace being spanned by the eigenmodes  $\varphi_n$  and  $\varphi_{2p-n}$ , or, equivalently,  $\varphi_n^\pm$ . These fill a band of width  $\Delta m_{\text{band}} \sim 2m$  with mass-splittings  $\delta m_n/m_n$  of  $\mathcal{O}(1/2p)$  for sufficiently large  $p$ . The mass-splitting first increases in the band as we move up the

<sup>5</sup>With  $\Delta M_{1,2}^2 \approx m^2(q-1)^2$ , the second gap ( $\Delta M_2$ ) is noticeable only for large  $q$ .



**Figure 1.** Schematic depiction of the physical states in the *closed* clockwork theory (right) and in the original one-sided theory (left) for  $N = 20$ .

spectrum and then decreases progressively towards the end of the band. Figure 1 shows a schematic of the physical modes emerging in the closed clockwork model along with that of the original model for comparison. Clearly, the inter-level spacings in the closed scenario are larger than those seen in the original one-sided clockwork theories.

- This degeneracy is not simply a consequence of the  $\mathbb{Z}_2$  symmetry alone.<sup>6</sup> Rather, this symmetry itself is just a consequence of the assumptions undertaken in our construction, namely, the universality (i.e., the site-independence) of the  $U(1)_j$  charges  $Q_j$  and the mass parameters  $m^2$  and  $f^2$  as well as the *closed* nature of the interactions in the sense described before. Together, these lead to a pseudo-tridiagonal mass-squared matrix, with a very specified distortion. The corresponding eigenvalue equations are nothing but second-order  $q$ -difference equations, resulting in a degenerate spectrum. Any deformity in the universal nature of the couplings and/or the structure of the interactions near the pivots would lead to non-degeneracies in the masses even if the  $\mathbb{Z}_2$  symmetry exists. The details can be found in appendix A.2.

It should be noted that, for small deviations from uniformity, we would obtain quasi-degenerate states.

The discussion above should convince the reader that this setup is not merely a trivial joining of two open-ended CW chains, but a distinct theory with unique properties.

The aforementioned transformation matrix elements  $a_{nj}^{(\pm)}$  are given by

$$a_{0j}^{(+)} = \mathcal{N}_0^+ \begin{cases} q^{j-2p}, & 0 \leq j \leq p \\ q^{-j}, & j > p \end{cases}, \quad a_{pj}^{(+)} = \mathcal{N}_p^+ \begin{cases} (-q)^{-j}, & 0 \leq j \leq p \\ (-q)^{j-2p}, & j > p \end{cases} \quad (3.7)$$

<sup>6</sup>Were that to be the case, such a degeneracy could as well be seen in a two-sided open chain which, evidently, we do not encounter.



$$a_{nj}^{(+)}(n < p) = \mathcal{N}_n^+ \begin{cases} \sin \frac{jn\pi}{p} + \frac{2q}{q^2-1} \sin \frac{n\pi}{p} \cos \frac{jn\pi}{p}, & 0 \leq j \leq p \\ -\sin \frac{jn\pi}{p} + \frac{2q}{q^2-1} \sin \frac{n\pi}{p} \cos \frac{jn\pi}{p}, & p < j \leq 2p-1 \end{cases}, \quad (3.8)$$

and,

$$a_{nj}^{(-)}(n > p) = \mathcal{N}_n^- \sin \frac{jn\pi}{p}. \quad (3.9)$$

The normalization factors  $\mathcal{N}_{0,p}$  and  $\mathcal{N}_n^\pm$  are given by

$$\begin{aligned} \mathcal{N}_0^2 &= \frac{q^{4p}(q^2-1)}{(q^{2p}-1)(q^2+1)} & \mathcal{N}_p^2 &= \frac{q^{2p}(q^2-1)}{(q^{2p}-1)(q^2+1)} \\ (\mathcal{N}_n^+)^2 &= \frac{(q^2-1)^2}{p \left[ 1 + q^4 - 2q^2 \cos \frac{2n\pi}{p} \right]} & (\mathcal{N}_n^-)^2 &= p^{-1}. \end{aligned} \quad (3.10)$$

Having established the identity of the physical states, let us now consider an external operator  $\mathcal{O}_E$  (obviously transforming trivially under the  $\mathbb{Z}_2$ ) coupling to the CW sector only at the  $j$ -th site, viz.,

$$\mathcal{L}_{\text{ext}} = -\mathcal{O}_E \phi_j = - \left[ \sum_{n=0}^p a_{jn}^+ \varphi_n^+ + \sum_{n=p+1}^{2p-1} a_{jn}^- \varphi_n^- \right] \mathcal{O}_E. \quad (3.11)$$

The  $\mathbb{Z}_2$  symmetry is, thus, explicitly broken<sup>7</sup> as the external operator couples to both even and odd modes. Note, though, that if the coupling site is chosen to be either of  $j = 0, p$ , the operator couples only to the even modes as the odd wavefunctions vanish there. The mixing matrix elements in eq. (3.7)–(3.9) specify that the light mode ( $\varphi_0$ ) is localised towards the  $j = p$  site, away from which its distribution falls exponentially, acquiring a weight  $\sim q^{-p}$  at  $j = 0$ . In contrast, the heaviest mode ( $\varphi_p$ ) is localised near  $j = 0$ , with its weight decreasing exponentially towards  $j = p$  along both arms. Thus, at  $j = 0$ ,  $\mathcal{O}_E$  couples to  $\varphi_0$  with a suppressed coupling ( $\sim q^{-p}$ ) and to  $\varphi_p$  with a strength  $\sim \mathcal{O}(1)$ . This order of strengths reverses for an  $\mathcal{O}_E$  coupling at the  $j = p$  site instead. On the other hand, the rest of the massive modes have a nearly-flat distribution along the sites.

### 3.1.1 A possible dark matter candidate?

The preceding discussion establishes that, in the event of the SM coupling with the CW sector at either of the sites at  $j = 0$  or  $j = p$ , the theory preserves the exchange parity (i.e., the  $\mathbb{Z}_2$  symmetry). Consequently, the lightest odd CW particle (LOCP) is rendered absolutely stable and, hence, can potentially be a dark matter (DM) candidate (and, by the same token, lead to additional sources of missing-momentum signals at colliders). There is, though, a key difference with most popular DM models. The vanishing of the  $\mathbb{Z}_2$ -odd CW state wavefunctions at the ( $\mathbb{Z}_2$ ) fixed points implies that these do not couple directly to the SM sector. Consequently, the requisite relic abundance would be realised primarily through a pair of DM particles annihilating into the even CW modes which can further

<sup>7</sup>It would have remained unbroken had  $\mathcal{O}_E$  coupled with identical strength to  $\phi_{2p-j}$  as well.

annihilate and/or decay into SM particles. Thus, the odd tower would act as a secluded dark sector. In such models, co-annihilations could play a significant role in enhancing the total cross-sections by virtue of the degeneracy in the physical states. Of course, the extent of such enhancements would depend on the mass-splittings dictated by the parameters of the theory. Another mode of enhancement in the annihilation could be effected by introducing couplings that break  $U(1)_{CW}$  softly, e.g., a trilinear coupling of CW fields that opens a new channel of pair annihilation and co-annihilation via a mediator which could further introduce resonance enhancements in  $s$ -channel processes and/or enhancements from the lightness of the mediator in  $t$ -channel processes. Typically, for a small number of sites ( $p \lesssim 5$ ), the latter channel would prove to be the dominant one. We limit the discussion here to only highlighting the possibility of the aforementioned DM phenomenology and leave the detailed analysis to an accompanying work.

### 3.2 Vector clockwork

Having defined the scalar theory, one for vector fields proceeds quite analogously. We start with a  $U(1)^{2p}$  theory with identical gauge couplings  $g$  alongwith  $2p$  copies of complex scalar fields  $\phi_j$ , such that the Lagrangian is

$$\mathcal{L}_{CW} = \sum_{j=0}^{2p-1} \left\{ -\frac{1}{4} F_{\mu\nu j} F_j^{\mu\nu} - (D_\mu \phi_j)^* (D^\mu \phi_j) - \lambda \left( |\phi_j|^2 - \frac{f^2}{2} \right)^2 \right\}, \quad (3.12)$$

with the identification  $(F_N, \phi_N) \equiv (F_0, \phi_0)$ . Although the gauge symmetry does not require<sup>8</sup> the parameters  $\lambda$  and  $f$  to be site-independent, we, nonetheless, assume universal values. This restriction not only leads to a mass matrix similar to that obtained in the scalar case with the attendant phenomenology (see the discussion in section 3.1 and appendix A.2), but also facilitates an embedding in a higher dimensional theory. The complex scalars  $\phi_j$  have non-zero charges under  $U(1)_j \times U(1)_{j+1}$ , viz.

$$\begin{aligned} (j = 0, \dots, p-1) &: (q, -1) \\ (j = p, \dots, N-1) &: (-1, q) . \end{aligned}$$

With this configuration in place, an inspection of the symmetries in eq. (3.12) shows that each of the complex fields  $\phi_j$  is uncharged under a particular combination of the Abelian symmetry transformations, namely the one corresponding to the generator  $\mathcal{Q}_{CW}$  given in eq. (3.2). This implies that, even on *all* the scalars  $\phi_j$  acquiring non-zero vacuum expectation values, one of the  $2p$   $U(1)$  factors must remain unbroken, and the physical spectrum would include a massless gauge boson and a massless NGB. Following the SSB, the complex scalars can be represented as,

$$\phi_j = \frac{1}{\sqrt{2}} (\chi_j + f) e^{i\pi_j/f}, \quad (3.13)$$

---

<sup>8</sup>Similarly, the terms  $\sum_{i,j} |\phi_i|^2 |\phi_j|^2$  are also allowed by symmetry considerations. However, since their introduction does not bring about a qualitative change, we choose to omit them for the sake of simplicity.

where  $\chi_j$  and  $\pi_j$  are real. Integrating out the heavy radial degrees of freedom  $\chi_j$ , we have a theory of massive U(1) gauge fields below the SSB scale  $f$  defined by

$$\begin{aligned} \mathcal{L}_{\text{eff}} = & - \sum_{j=0}^{2p-1} \frac{1}{4} (F_{\mu j})^2 + \frac{g^2 f^2}{2} \left\{ \sum_{j=0}^{p-1} \left[ \frac{1}{f} \partial_\mu \pi_j + g (q A_{\mu j} - A_{\mu j+1}) \right]^2 \right. \\ & \left. + \sum_{j=p}^{2p-1} \left[ \frac{1}{f} \partial_\mu \pi_j + g (A_{\mu j} - q A_{\mu j+1}) \right]^2 \right\}. \end{aligned} \quad (3.14)$$

apart from a gauge-fixing term, that we specify later. As eq. (3.14) shows, the mass terms for the gauge fields resemble those for the scalar CW. In the physical basis, the full theory is now given by,

$$\mathcal{L}_{\text{eff}}^{(\text{mass})} = \sum_{n=0}^{2p-1} \left\{ -\frac{1}{4} (\mathcal{F}_{\mu\nu}^n)^2 + \frac{1}{2} m_n^2 \left( \mathcal{A}_\mu^n + \frac{1}{m_n} \partial_\mu \tilde{\pi}_n \right)^2 \right\} + \mathcal{L}_{GF}, \quad (3.15)$$

where

$$m_n^2 = \begin{cases} 0 & n = 0 \\ 2g^2 f^2 (1 + q^2) & n = p \\ g^2 f^2 \left( 1 + q^2 - 2q \cos \frac{n\pi}{p} \right) & n \neq 0, p \end{cases} \quad (3.16)$$

and the eigenstates  $\mathcal{A}_\mu^n$  (corresponding to  $\mathcal{F}$ ) and  $\tilde{\pi}_n$  are defined as

$$\begin{aligned} \mathcal{A}_{\mu n} & \equiv \sum_{j=0}^{2p-1} a_{nj} A_{\mu j} \\ \text{and } \tilde{\pi}_n & \equiv \sum_{j=0}^{2p-1} b_{nj} \pi_j \\ & = \begin{cases} \sum_{j=0}^{2p-1} a_{0j} \pi_j & n = 0 \\ \frac{gf}{m_n} \left[ \sum_{j=0}^{p-1} (q a_{nj} - a_{n,j+1}) \pi_j + \sum_{j=p}^{2p-1} (-a_{nj} + q a_{n,j+1}) \pi_j \right] & n > 0 \end{cases} \end{aligned} \quad (3.17)$$

A convenient choice for the gauge-fixing term is given by

$$\mathcal{L}_{GF} = \frac{-1}{2\xi} \sum_{n=1}^{2p-1} (\partial^\mu \mathcal{A}_{\mu n} - \xi m_n \tilde{\pi}_n)^2. \quad (3.18)$$

In the unitary gauge ( $\xi \rightarrow \infty$ ), all of the  $(2p - 1)$  NGBs decouple from the theory.<sup>9</sup> One massless scalar mode  $\tilde{\pi}_0$ , however, remains as a physical state in the spectrum, as expected

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<sup>9</sup>The whole point of choosing the unitary gauge here is to elucidate the existence of a massless scalar mode (a Goldstone boson that is not Higgsed) in the theory. One could, in general, choose a more convenient gauge to facilitate computational simplicity.

from the preceding symmetry argument.<sup>10</sup> In summary, therefore, there are  $2p - 1$  massive photons with a degeneracy identical to that of the CW scalars, a massless photon and a massless scalar. The light scalar may acquire a mass if any of the shift symmetries is broken explicitly.

Similar to the scalar case in section 3.1 the complete spectrum obtained here can be classified as even and odd eigenstates of clockwork parity. Explicitly, the matrix elements  $a_{nj}$  are the same as obtained in eq. (3.7)–(3.10). Consequently, the physical modes have the same nature of localised couplings with an external sector as that of the scalar CW in eq. (3.11). The distinction in this case is that the external operator coupling to the  $j$ -th site is a current charged under  $U(1)_j$  of the CW sector.

#### 4 The continuum perspective

Keeping the discrete theories in perspective, we now explore the possibility of obtaining them as deconstructions of five dimensional bulk theories, an advantage being that in doing this, we would encounter some new possibilities. To identify the appropriate underlying 5D geometry, it is convenient to assume a form of the metric that incorporates at least those symmetries encountered in the clockwork theories along the discrete lattice. Since, in our case, the latter includes a  $\mathbb{Z}_2$  symmetry under field exchanges, a logical ansatz for the 5D metric (with the fifth dimension being compactified to a circle of radius  $R$ ) would be

$$ds^2 = X(|z - z_p|)dx^2 + Z(|z - z_p|)dz^2, \tag{4.1}$$

where  $X$  and  $Z$  are, *a priori*, unknown functions of the extra dimension<sup>11</sup> and, by construction, are  $\mathbb{Z}_2$  symmetric about  $z_p = \pi R/2$ . Here,  $x$  denotes the 4D spacetime coordinates and  $z$  is the fifth coordinate. The action for a massless bulk scalar in this background geometry would, then, be given by

$$\begin{aligned} \mathcal{S} &= \int d^4x \int_0^{\pi R} dz \sqrt{-g} \left[ -\frac{1}{2} g^{MN} \partial_M \phi(x, z) \partial_N \phi(x, z) \right] \\ &= -\frac{1}{2} \int d^4x \int_0^{\pi R} dz X^2 Z^{1/2} \left[ X^{-1} (\partial_\mu \phi)^2 + Z^{-1} (\partial_z \phi)^2 \right]. \end{aligned} \tag{4.2}$$

Re-scaling the scalar field  $\phi \rightarrow X^{-1/2} Z^{-1/4} \phi$  in order to obtain canonical kinetic terms post discretization, we have

$$\mathcal{S} = -\frac{1}{2} \int d^4x \int_0^{\pi R} dz \left\{ (\partial_\mu \phi)^2 + X^2 Z^{-1/2} \left[ \partial_z \left( X^{-1/2} Z^{-1/4} \phi \right) \right]^2 \right\}. \tag{4.3}$$

We, now, discretize the  $z$ -dimension into a lattice of  $2p$  sites with uniform inter-site spacing  $a = \pi R/2p$  and specify a site  $j$  as  $z \rightarrow ja$ . The  $z$ -derivative in the second term of the action

<sup>10</sup>Similar entities have been identified before in the context of periodic moose structures in refs. [40, 41].

<sup>11</sup>The function  $Z$  could, of course, be eliminated by a simple redefinition of the coordinate  $z$ , but it is convenient to retain it.

is to be replaced by the difference<sup>12</sup>

$$\partial_z \left( X^{-1/2} Z^{-1/4} \phi \right) \rightarrow a \left( X_{j+1}^{-1/2} Z_{j+1}^{-1/4} \phi_{j+1} - X_j^{-1/2} Z_j^{-1/4} \phi_j \right).$$

Since the discretized derivative involves fields from adjacent sites, namely,  $\phi_{j+1}$  and  $\phi_j$ , we adopt a particular discretization prescription, wherein the factor coupled to the derivative is replaced by

$$X^2 Z^{-1/2} \rightarrow \left( X_j Z_j^{-1/4} \right) \left( X_{j+1} Z_{j+1}^{-1/4} \right), \quad (4.4)$$

which consists of equal powers of the functions  $X$  and  $Z$  evaluated at adjacent sites  $j$  and  $j + 1$ . With this prescription we have,

$$\mathcal{S} = \frac{-1}{2} \int d^4x \sum_{j=0}^{2p-1} \left\{ (\partial_\mu \phi_j)^2 + \frac{4p^2}{\pi^2 R^2} \left( X_j^{1/2} Z_j^{-1/8} Z_{j+1}^{-3/8} \phi_{j+1} - X_{j+1}^{1/2} Z_{j+1}^{-1/8} Z_j^{-3/8} \phi_j \right)^2 \right\}. \quad (4.5)$$

The second term inside the braces is a simple square of a difference of the form  $(f_1[j+1]f_2[j] - f_1[j]f_2[j+1])^2$ , and the sum is explicitly invariant under  $j \rightarrow 2p - j$ , a property that would be absent without the symmetrization encoded in eq. (4.4). In other words, the (natural) prescription of eq. (4.4) preserves, on discretization, the  $\mathbb{Z}_2$  symmetry of the continuum theory.<sup>13</sup> Specifically, the discrete theory defined in eq. (3.1) is obtained with the assignments

$$\begin{aligned} X_j &\propto Z_j \propto \exp\left(\frac{-4k\pi R}{6p}|j-p|\right) \\ \text{and} \quad q &\equiv \exp\left(\frac{k\pi R}{2p}\right), \quad m^2 \equiv \frac{4p^2}{\pi^2 R^2 q}. \end{aligned} \quad (4.6)$$

Therefore, in the continuum limit,  $p \rightarrow \infty$ , the 5D metric has the form,

$$ds^2 = \exp\left(\frac{-4}{3}k|z-z_p|\right) (dx^2 + dz^2), \quad (4.7)$$

where  $z_p = \pi R/2$ . Furthermore, writing the parameters  $q$  and  $m^2$  defined in eq. (4.6) as functions of the lattice spacing  $a$ , which acts as a natural UV regulator here, we have

$$q(a) = e^{ka} \quad \text{and} \quad m^2(a) = \left[ a^2 q(a) \right]^{-1}. \quad (4.8)$$

---

<sup>12</sup>Clearly, the derivative could, in principle, be mapped into inequivalent differences, and these choices would, potentially, lead to different discrete theories. This ambiguity is not unexpected, for it is well known that several inequivalent discrete theories could flow into the same continuum theory.

<sup>13</sup>Note that this discretization prescription is categorically different from the one adopted in [6] which would break the  $\mathbb{Z}_2$  symmetry if employed in our scenario. This also demonstrates the fact that different prescriptions can lead to different discrete theories which is reminiscent of what is usually encountered in the lattice QCD framework, especially in the context of discretizing fermions on a lattice. The distinct low-energy theories, however, lead to the same bulk 5D theory in the  $N \rightarrow \infty$  limit. It is worth stressing here that this does not necessarily mean that the different discrete theories would also correspond to the same effective 4D theory derived from the 5D action which additionally depends on the particulars of the boundary conditions imposed. Different boundary conditions would typically induce distinct 4D theories as will be seen in a later section.

In the far UV ( $a \rightarrow 0$ ) limit, an expansion in  $a$ , leads to a simple functional form for the mass eigenvalues of the discrete theory, namely,

$$\begin{aligned}
 m^2 \left[ 1 + q^2 - 2q \cos\left(\frac{n\pi}{p}\right) \right] &\rightarrow k^2 + \frac{4n^2}{R^2} + \mathcal{O}(1/2p) \\
 2m^2 (1 + q^2) &\rightarrow \infty.
 \end{aligned}
 \tag{4.9}$$

In other words, these evolve into the KK spectrum of a scalar with an apparent bulk mass  $k^2$ . However, as can be easily ascertained, the scalar is actually massless, with this apparent “bulk mass” being an artefact of the warping. Whether we obtain the exact same KK spectrum as above for the bulk theory defined in eq. (4.2) will be ascertained in a subsequent section.

### 4.1 The extended linear dilaton theory

Although the preceding treatment looks somewhat similar to that in ref. [6], the precise outcomes of eq. (4.7) and eq. (4.9) specify a distinct geometry and a consequent mass spectrum that is distinctively different. However, until now we have just assumed the background geometry and it is contingent on us to obtain the metric in eq. (4.7) as a solution of the five-dimensional Einstein equations. To this end, we consider a 5D linear dilaton scenario [34, 35], with the fifth dimension compactified on a  $S^1/\mathbb{Z}_2$  orbifold of radius  $R$ . Furthermore, three rigid 3-branes are placed at  $z = 0, \pi R/2$  and  $\pi R$  respectively. The bulk and brane actions in this setup are, in the Einstein frame,

$$\begin{aligned}
 \mathcal{S}_{\text{Bulk}} &= 2M_5^3 \int d^4x dz \sqrt{-g} \left( \frac{1}{4} \mathcal{R} - \frac{1}{12} g^{MN} \partial_M S \partial_N S - V(S) \right) \\
 \mathcal{S}_{\text{Brane}} &= -2M_5^3 \int d^4x dz \frac{\sqrt{-g}}{\sqrt{g_{zz}}} \sum_{\alpha=1}^3 \lambda_\alpha(S) \delta(z - z_\alpha),
 \end{aligned}
 \tag{4.10}$$

where  $z_\alpha \equiv \{0, \pi R/2, \pi R\}$ . Here,  $\mathcal{R}$  is the 5D Ricci scalar,  $S(x, z)$  is the dilaton field and  $M_5$  is the 5D fundamental mass scale. The bulk and brane potentials,  $V(S)$  and  $\lambda_\alpha(S)$  are given, respectively, by

$$V(S) = -e^{-2S/3} k^2, \quad \lambda_\alpha(S) = \frac{e^{-S/3}}{2} \frac{\Lambda_\alpha}{M_5^3},
 \tag{4.11}$$

where the  $\Lambda$ 's ( $\Lambda_2 = 4kM_5^3 = -\Lambda_1 = -\Lambda_3$ ) are the vacuum energies on the branes. The dimensionful parameter  $k$  provides a measure of the vacuum energy in the bulk and, thereby, the extent of the warping in the resulting geometry.<sup>14</sup> The most general metric with 4D Poincare invariance corresponding to the action in eq. (4.10), written in terms of the conformal coordinates is [44],

$$ds^2 = e^{2\sigma(z)} \left( \eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right), \quad \eta_{\mu\nu} = \text{diag}\{-1, +1, +1, +1\}.
 \tag{4.12}$$

<sup>14</sup>A similar 5D setup was assumed in [42, 43] in the context of RS-like warped scenarios, albeit without a linear dilaton.

The corresponding equations of motion are given by

$$\begin{aligned} (S'' + 3\sigma'S') e^{-2\sigma(z)} &= 4k^2 e^{-S/3} - M_5^{-3} \sum_{\alpha} e^{-(\sigma+S/3)} \Lambda_{\alpha} \delta(z - z_{\alpha}) \\ [9(\sigma'' - \sigma'^2) + S'^2] e^{-2\sigma(z)} &= -3M_5^{-3} \sum_{\alpha} e^{-(\sigma+S/3)} \Lambda_{\alpha} \delta(z - z_{\alpha}) \\ 36\sigma'^2 - S'^2 &= 12k^2 e^{2(\sigma-S/3)}. \end{aligned} \tag{4.13}$$

These are most easily solved by writing the bulk potential  $V(S)$  in terms of a superpotential  $W(S)$  [44], viz.,

$$V(S) = \frac{3}{4} \left( \frac{\partial W(S)}{\partial S} \right)^2 - \frac{1}{3} W(S)^2, \tag{4.14}$$

such that the solutions to the equations

$$S'(z) = 3e^{\sigma(z)} \frac{\partial W(S)}{\partial S}, \quad \sigma'(z) = -\frac{1}{3} e^{\sigma(z)} W(S) \tag{4.15}$$

subject to the junction conditions

$$W(S) \Big|_{z_{\alpha}-\epsilon}^{z_{\alpha}+\epsilon} = e^{-S/3} \frac{\Lambda_{\alpha}}{M_5^3} \quad \text{and} \quad \frac{\partial W}{\partial S} \Big|_{z_{\alpha}-\epsilon}^{z_{\alpha}+\epsilon} = -\frac{1}{3} e^{-S/3} \frac{\Lambda_{\alpha}}{M_5^3} \tag{4.16}$$

(with  $\epsilon$  being an infinitesimally small positive quantity) are also solutions to eq. (4.13). For our case, this implies

$$W(S) = \begin{cases} e^{-S/3} \sum_{\alpha=1}^3 \frac{\Lambda_{\alpha}}{2M_5^3} [\theta(z - z_{\alpha}) - \theta(z_{\alpha} - z)] & z \in (0, \pi R), \\ e^{-S/3} \left[ -\frac{\Lambda_1}{2M_5^3} \theta(-z) + \frac{\Lambda_3}{2M_5^3} \theta(z - \pi R) \right] & z \notin (0, \pi R). \end{cases} \tag{4.17}$$

With the orbifolding dictating that  $\sigma(z)$  be symmetric about the orbifold fixed points, the second of eqs. (4.15) implies that  $W$  must be an odd function across the fixed points. This, along with the conditions imposed by eq. (4.16), fixes the form of  $W(S)$  in the second line of eq. (4.17). With these ingredients in order, we obtain the following solutions to eq. (4.13),

$$\begin{aligned} \sigma(z) &= -\frac{2}{3} k |z - z_p| e^{(\sigma_o - \frac{1}{3} S_o)} + \sigma_o \\ S(z) &= -2k |z - z_p| e^{(\sigma_o - \frac{1}{3} S_o)} + S_o, \end{aligned} \tag{4.18}$$

where  $\sigma_o$  and  $S_o$  are constants of integration, and we use the notation  $z_p = \pi R/2$ . Choosing, for convenience,  $\sigma_o = S_o/3$ , the solutions reduce to<sup>15</sup>

$$\begin{aligned} \sigma(z) &= -\frac{2}{3} k |z - z_p| + \sigma_o \\ S(z) &= -2k |z - z_p| + S_o, \end{aligned} \tag{4.19}$$

---

<sup>15</sup>This can also be regarded as a rescaling of the parameter  $k$  by defining  $k = \tilde{k} e^{-(\sigma_o - \frac{1}{3} S_o)}$ . Relating  $k$  to the vacuum energy in the bulk demands that  $\tilde{k} e^{-(\sigma_o - \frac{1}{3} S_o)} < M_5$ . While  $S_o$  is related to the VEVs of the dilaton at the branes as we will see in the next subsection,  $\tilde{k}$  and  $\sigma_o$  are free parameters which can be suitably configured to get optimum values for  $k$ .

The metric in eq. (4.12), therefore, has the explicit form<sup>16</sup>

$$ds^2 = \exp\left(\frac{-4}{3}k|z - z_p|\right) (dx^2 + dz^2). \quad (4.20)$$

So, the extended linear dilaton (LD) theory defined in eq. (4.10) successfully generates the required *clockwork* geometry of eq. (4.7). Evidently, the symmetry of the triple brane setup about  $z = z_p$  in terms of the physical parameters in the theory has induced  $\mathbb{Z}_2$  symmetric solutions for  $\sigma(z)$  and  $S(z)$  and, hence, for the resulting metric, under the transformation  $z \rightarrow \pi R - z$ .

## 4.2 Addressing the hierarchy problem

The role of the linear dilaton model as a solution to the hierarchy problem has been elucidated in detail in refs. [6, 34] and analogous arguments readily follow for our three 3-brane setup. For brevity's sake, we refrain from discussing this in detail and, instead, highlight only the key differences. To begin with, starting with the fundamental scale  $M_5$ , we may obtain, on dimensional reduction, the Planck mass ( $M_P \sim 10^{19}$ ) as the effective mass scale in the 4D Einstein-Hilbert action, viz.,

$$M_P^2 = 2M_5^3 \int_0^{\pi R} dz e^{3\sigma(z)} = \frac{2M_5^3}{k} (1 - e^{-k\pi R}). \quad (4.21)$$

The above relation resembles the one obtained in the RS model [36].

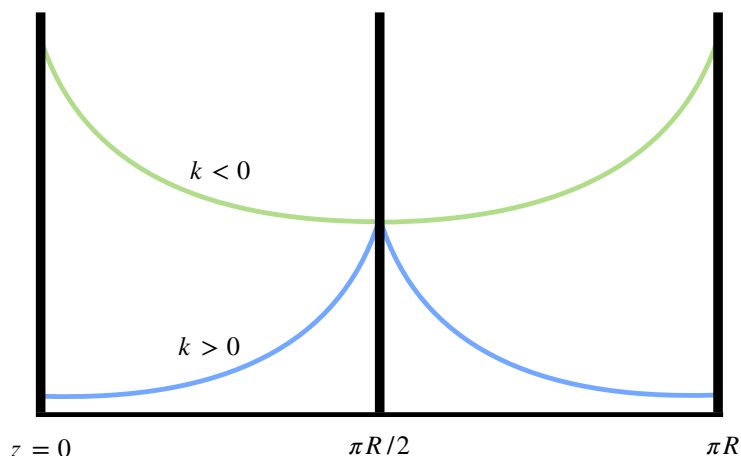
The major difference from the two-brane theory lies in the emergence of two distinct physical scenerios that ameliorate the hierarchy issue. For  $k > 0$ , we have two negative tension (IR) branes at the orbifold boundaries and a positive tension (UV) brane at the central  $\mathbb{Z}_2$  fixed point. We shall call it the IR-UV-IR setup. In this scenario, we have  $M_P \sim M_5$  for a nominally large value of  $k\pi R \sim \mathcal{O}(10)$ . For a given value of  $k$ , though, the required magnitude of  $k\pi R$  is roughly twice of that necessary in two-brane models. This difference stems from the fact that in our three brane model the hierarchy is explained by the warping profile operative between the boundary branes and the brane present in the middle of the orbifold. A standard model (SM) sector localised on the IR brane would have its effective mass parameter<sup>17</sup> ( $m_{IR}$ ) derived from the parameters in the fundamental 5D theory via relations like  $m_{IR} = e^{-ak\pi R} m_0$ , thereby lowering it to the TeV scale. Here,  $a$  is an  $\mathcal{O}(1)$  numerical constant arising from a combination of the exponential in the (induced) 4D metric determinant and that emanating from field redefinitions.  $m_0$  denotes a parameter of the brane localised sector in the full 5D theory, presumably near the Planck scale. Note that this argument is similar to the one presented in the original RS scenario [36].

On the other hand, for  $k < 0$ , the negative tension brane is located at the central fixed point and the positive tension branes at the orbifold boundaries. Hence, we refer to this as the UV-IR-UV setup. The hierarchy is addressed in a similar manner as before for  $|k\pi R| \sim \mathcal{O}(10)$ , only that in this case the fundamental scale,  $M_5$ , is near the TeV scale.

<sup>16</sup>Note that we drop the overall constant factor  $e^{2\sigma_0}$  here as it does not affect the form of the geometry.

<sup>17</sup>For the SM this means the independent mass parameter  $\mu$  in the Higgs sector or, equivalently, the symmetry breaking scale  $v$ .





**Figure 2.** Graphical depiction of the warp factor  $e^{2\sigma(z)}$  in the IR-UV-IR (blue) and the UV-IR-UV (green) scenario.

The triple brane theory, therefore, proposes the plausibility of two distinct scenarios in terms of their role in explaining the Planck-EW hierarchy (figure 2). However, one needs to check for the stability of both the scenarios in terms of the physicality of the emerging dilaton modes. This is necessary as the coefficients of their kinetic terms depend on the parameters of the theory and, hence, may lead to unphysical degrees of freedom for certain values of the parameter space. For a related discussion see appendix B.

Phenomenologically, the two scenarios could have distinct, albeit rich, consequences. For instance, the dynamical dilaton (see appendix B) and graviton modes emerging in the two setups would lead to their different couplings with the SM confined to the IR brane. A more detailed inspection of these aspects, though, is beyond the scope of the current exercise and would be pursued later.

### 4.3 Radius stabilisation

Modulus stabilisation in warped scenarios like the RS model typically involves the addition of a bulk scalar field (the modulus field) whose VEVs at the branes stabilise the size of the extra dimension *à la* the Goldberger-Wise mechanism [45]. In a linear dilaton model the dilaton field itself can accomplish the role of a modulus field<sup>18</sup> [35]. In this section we extend those arguments to our triple brane model<sup>19</sup> by invoking additional potentials on the branes. These potentials could, in principle, arise even as quantum corrections on the brane resulting from the dilaton’s interactions with brane localised matter. Rather than posit a complicated potential, we limit ourselves to the simplest choice, viz.

$$\lambda_\alpha(S) = \frac{e^{-S/3}}{2} \left[ \frac{\Lambda_\alpha}{M_5^3} + \mu_\alpha (S - \mathcal{V}_\alpha)^2 \right], \tag{4.22}$$

<sup>18</sup>Even if the dilaton field is not considered to be responsible for the stabilisation, a GW-like mechanism can be consistently adopted for the purpose. See [6] for a discussion.

<sup>19</sup>Stabilization of triple-brane systems have been studied before in [46] in the context of the RS scenario.

where  $\mu_\alpha$  and  $\mathcal{V}_\alpha$  are constants. For this to be consistent with the junction conditions in eq. (4.16) we must have

$$S(z_\alpha) = \mathcal{V}_\alpha. \tag{4.23}$$

In other words, the junction conditions impose boundary conditions on the dilaton profile such that the dilaton field acquires nonzero vacuum expectation values at the branes. This further ensures that the augmented brane potentials do not generate any backreaction in the system.

Now, taking the general form of the solutions in eq. (4.19) and imposing the appropriate boundary conditions at the end-of-the-world branes, viz.,

$$S(z_\alpha) = -k\pi R + S_o = \mathcal{V}_\alpha, \tag{4.24}$$

(where  $S_o = \mathcal{V}_2$  from eq. (4.23)) we obtain the size of the extra dimension in terms of the vacuum expectation values of the dilaton field at the branes

$$\pi R = \frac{(\mathcal{V}_2 - \mathcal{V}_1)}{k} \quad \text{and} \quad \pi R = \frac{(\mathcal{V}_2 - \mathcal{V}_3)}{k}. \tag{4.25}$$

However, note that, together, the two conditions stipulate that the mechanism is consistent only when the VEVs on the branes located at the orbifold boundaries are equal. While this can be ensured by imposing an exact  $\mathbb{Z}_2$  symmetry (about  $z = z_p$ ) in the full (bulk-brane) theory, it would be interesting to find a mechanism or a more fundamental theory in which such a symmetry would emerge naturally. In the purview of our current discussion, though, the imposition  $\mathcal{V}_1 = \mathcal{V}_3$  might (naively) suggest an additional fine-tuning in the theory, an aspect we comment upon in section 6.

## 5 Bulk fields in clockwork/linear dilaton geometry

Having determined the configuration of the 5D theory which generates the modified *clockwork* geometry, it is imperative to study the nature of bulk field theories in the background specified by the metric in eq. (4.20), so as to ascertain and verify the correspondence drawn earlier between the discrete and the 5D continuum theories. Although it is straightforward to generalise the ensuing discussion to non-Abelian theories we choose to study only bulk Abelian theories here in order to directly compare with the discrete clockwork theories. As before, we limit the discussion to only bosonic fields. We note that, in eq. (4.10), the terms proportional to  $M_5^3 k^2$  are the ones that primarily influence the geometry. In the following bulk field actions neither do we assume any large mass parameters nor do we assume vacuum expectation values for the scalar fields. Consequently, the backreaction due to these fields can be entirely neglected.

### 5.1 Bulk scalar in LDG

Assuming the metric of eq. (4.12)—a valid assumption as long as the backreaction due to the new field is negligible — the action for a scalar field  $\phi$  is given by

$$\mathcal{S} = \frac{-1}{2} \int d^4x \int_0^{\pi R} dz e^{3\sigma(z)} \left[ (\partial_\mu \phi)^2 + (\partial_z \phi)^2 \right] \tag{5.1}$$

which is invariant under the  $\mathbb{Z}_2$  transformation ( $z \rightarrow \pi R - z$ ) with  $z = z_p$  being a fixed point under this transformation.<sup>20</sup> The existence of this symmetry stipulates that the solutions of the equation of motion (EOM) should be classifiable as representations of the corresponding parity operator  $\mathcal{Z}$ , namely<sup>21</sup>

$$\mathcal{Z}\phi^{(\pm)}(x, z) \equiv \phi^{(\pm)}(x, \pi R - z) = \pm\phi^{(\pm)}(x, z). \quad (5.2)$$

Clearly, this is analogous to the  $\mathbb{Z}_2$  parity in the discrete case. The  $U(1)_{CW}$  symmetry in the discrete theory — see eq. (3.2)— is reflected by the invariance of the action under a constant shift of the bulk fields,  $\phi^{(\pm)}(x, z) \rightarrow \phi^{(\pm)}(x, z) + c$ . Effecting the KK decompositions

$$\phi^{(\pm)}(x, z) = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \varphi_n^{(\pm)}(x) f_n^{(\pm)}(z), \quad (5.3)$$

with the obvious normalisation

$$\frac{1}{\pi R} \int_0^{\pi R} dz e^{3\sigma(z)} f^{(\pm)}(n) f^{(\pm)}(n') = \delta^{nn'}, \quad (5.4)$$

the EOMs for  $\varphi^{(\pm)}(x)$  and  $f^{(\pm)}(z)$  are

$$\begin{aligned} (\partial_x^2 - m_n^2) \varphi_n^{(\pm)}(x) &= 0 \\ (\partial_z^2 - k^2 + m_n^2) (e^{3\sigma(z)/2} f_n^{(\pm)}(z)) &= 0. \end{aligned} \quad (5.5)$$

At this stage, the masses  $m_n$  for the KK modes  $\phi_n$  are arbitrary, and are to be determined only by solving for the eigenvalues of the operator in the second equation. This, in turn, needs the boundary conditions (BC) on the bulk field to be specified. With the  $\mathbb{Z}_2$  stipulating that the BC at  $z = 2z_p$  must be given by that at the origin, we choose to impose the BCs at  $z = 0$  and  $z = z_p$ . Given that an individual BC could either be Dirichlet-like ( $\phi(x, z) = 0$ ) or Neumann-like ( $\partial_z \phi(x, z) = 0$ ), various combinations are possible, each with its particular set of consequences, and we examine below the two broad classes:

- **Case I:** Imposing *identical* BCs at  $z = 0, z_p$ , we have two further choices, namely both Dirichlet-like (DD) or both Neumann-like (NN) Explicitly, we obtain the following solutions:

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<sup>20</sup>This, of course, is in addition to the orbifold fixed points.

<sup>21</sup>Note that we treat the fields  $\phi^{(\pm)}(x, z)$  as two independent degrees of freedom which enables us to simultaneously impose different sets of boundary conditions for the even and odd fields. This follows from the fact that, in the two patches of the orbifold, the fields can be deemed to be *a priori* disconnected (and, hence, independent) with support only on the individual patches. Demanding continuous solutions to the EOM, the  $\mathbb{Z}_2$  symmetry of the system dictates the existence of two (even and odd) fields  $\phi^{(\pm)}(x, z)$  as linear combinations of the piecewise fields. Hereinafter, we resort to discussing the physical properties of the theory directly in terms of the even and odd bulk fields.

**Neumann BC at both  $z = 0$  and  $z_p$  (NN).** This admits only the even solutions, namely (here,  $n \geq 1$ )

$$\begin{aligned}
 f_0^+(z) &= \mathcal{N}_0 \\
 m_0 &= 0 \\
 f_n^+(z) &= \mathcal{N}_n^+ e^{-\frac{3}{2}\sigma(z)} \left\{ -\frac{kR}{2n} \sin \left[ \frac{2n}{R} |z - z_p| \right] + \cos \left[ \frac{2n}{R} (z - z_p) \right] \right\} \\
 m_n^2 &= k^2 + \frac{4n^2}{R^2}.
 \end{aligned} \tag{5.6}$$

**Dirichlet BC at  $z = 0$  and  $z_p$  (DD).** Understandably, these lead only to the odd solutions, namely

$$f_n^-(z) = \mathcal{N}_n^- e^{-\frac{3}{2}\sigma(z)} \sin \left[ \frac{2n}{R} (z - z_p) \right], \quad m_n^2 = k^2 + \frac{4n^2}{R^2}. \tag{5.7}$$

The normalisation factors are given by

$$\mathcal{N}_0 = \sqrt{\frac{k\pi R}{1 - e^{-k\pi R}}}, \quad \mathcal{N}_n^+ = 2\sqrt{2} \frac{n}{m_n R}, \quad \mathcal{N}_n^- = \sqrt{2} \tag{5.8}$$

Thus, the physical spectrum consists of an exactly massless mode ( $\varphi_0$ ) and, beyond a mass gap of magnitude  $\sim k$ , two mass-degenerate KK-towers, with one each of  $\mathbb{Z}_2$ -even ( $\varphi_n^{(+)}$ ) and odd ( $\varphi_n^{(-)}$ ) mode at each level. In a limited sense, this mimics the spectrum of the discrete theory (see eq. (4.9)). Just as in the discrete case, the occurrence of a mass degeneracy can be attributed to the  $\mathbb{Z}_2$  symmetry of the action as well as to the boundary conditions assumed. (For further elucidation of the mass degeneracy, see appendix A.1.) Note that, given the BCs, the EOM has no non-trivial solution for  $m^2 < k^2$ .

The mass eigenvalues are graphically shown in figure 3 for a benchmark set of values for the parameters  $k$  and  $R$ . Evidently, the inter-level spacing increases progressively with  $n$  and then tends to saturate to  $\Delta m_n \sim 2\pi/R$  for large  $n$ . While this might seem at variance with the discrete case, this was only to be expected. If the gravitational interaction could be neglected, the five-dimensional theory could, notionally, be treated as a free and, hence, UV-complete one with the masses defined by  $k$  and  $R^{-1}$ . The spectrum of the discrete theory, on the other hand, must depend on both the lattice spacing  $a$  and the UV-scale  $f$ . Yet, in the event of  $p$  being large, an analogous behaviour is seen for  $n \ll p$  with an opposite behaviour (namely, the subsequent spacings decreasing) being seen for  $n > p$  (see eq. (4.9)). In other words, the mass eigenvalues therein assume the form of the KK spectrum when the RG evolution of the parameters, with respect to the lattice spacing  $a$ , is taken into account in the sense indicated in eq. (4.9).

Now that we have obtained the solutions, the theory in eq. (5.1) can be rewritten in terms of the even and odd bulk fields. However, this would tend to make the action look cumbersome as one adds higher order terms. Since, all that we expect after

dimensional reduction is a 4D theory with the  $\mathbb{Z}_2$  parity intact, it is simpler to consider the equivalent, namely a single bulk field  $\phi(x, z)$  and ascribe it with the expansion

$$\phi(x, z) \equiv \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \left\{ \varphi_n^{(+)}(x) f_n^{(+)}(z) + \varphi_n^{(-)}(x) f_n^{(-)}(z) \right\} \quad (5.9)$$

when the 4D theory is to be determined.

With this definition, we examine the coupling profile of the bulk sector with an operator  $\mathcal{O}_E$  confined to *one* of the branes, i.e., at  $z = z_\alpha \equiv \{0, \pi R/2, \pi R\}$ . To this end, we introduce the interaction action

$$\mathcal{S}_{ext} = - \int d^4x dz \frac{\sqrt{-g}}{\sqrt{g_{zz}}} \exp\left(\frac{-5}{6} S(z)\right) \mathcal{O}_E \phi(x, z) \delta(z - z_\alpha). \quad (5.10)$$

The dilaton factor  $e^{-5S(z)/6}$  has been introduced to induce localisation of the 0-th mode towards one of the fixed points. The numerical constant  $(-5/6)$  is fixed by demanding that the heavy KK modes are de-localised, i.e., the exponential factor in their wavefunctions is eliminated. This choice, although not strictly necessary in the context of the 5D theory alone, is adopted to mimic the coupling profile typical of the clockwork mechanism in the corresponding discrete theory.<sup>22</sup> Using eq. (5.6)–(5.7), the above action simplifies to

$$\begin{aligned} \mathcal{S}_{ext} &= \frac{-1}{\sqrt{\pi R}} \int d^4x dz \frac{\sqrt{-g}}{\sqrt{g_{zz}}} \exp\left(\frac{-5}{6} S(z)\right) \delta(z - z_\alpha) \\ &\quad \left\{ \sum_{n=0}^{\infty} f^{+(n)} \mathcal{O}_E \varphi^{+(n)}(x) + \sum_{n=0}^{\infty} f^{-(n)} \mathcal{O}_E \varphi^{-(n)}(x) \right\} \\ &= \frac{-1}{\sqrt{\pi R}} \int d^4x \left\{ \mathcal{N}_0 e^{-k|z_\alpha - z_p|} \mathcal{O}_E \varphi^{+(0)}(x) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \mathcal{N}_{(n)}^+ \mathcal{O}_E \varphi^{+(n)}(x) \right. \\ &\quad \left. \left[ -\frac{kR}{2n} \sin\left(\frac{2n}{R}|z_\alpha - z_p|\right) + \cos\left(\frac{2n}{R}(z_\alpha - z_p)\right) \right] \right\}. \end{aligned} \quad (5.11)$$

The fact that the odd wavefunctions vanish at the fixed points and that the external operator is trivially invariant under the bulk  $\mathbb{Z}_2$  transformation ensure that the brane localised interactions preserve the  $\mathbb{Z}_2$  parity. Evidently, the coupling of the zero mode at the boundary branes ( $z_\alpha = 0, \pi R$ ) is exponentially suppressed, whereas, that at the central brane ( $z_\alpha = z_p$ ) is  $\sim \mathcal{O}(1)$ . In contrast, the massive modes have nearly similar couplings, of magnitude  $\sim \mathcal{O}(1)$ , at every brane. This, clearly, is analogous to what we obtain in the discrete case discussed in section 3.1.

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<sup>22</sup>In general we can also introduce a dilaton factor  $e^{cS(z)}$  coupling to the overall Lagrangian in the bulk action of eq. (5.1), where  $c$  is a constant which determines the coupling strength. Such a construction has been studied in [33] in the context of the original CW/LD geometry. We assume  $c = 0$  in our modified theory only to simplify matters in hand and generalisation with respect to dilaton couplings would be an interesting aspect to explore.

- **Case II:** an interesting possibility is afforded by the imposition of an *asymmetric* set of BCs, e.g., Neumann BC at  $z = 0$  and Dirichlet BC at  $z = z_p$  or vice-versa. These BCs do not admit an exactly massless mode which is in stark contrast with what we encounter in case **I**. Explicitly,

**Neumann BC at  $z = 0$  and Dirichlet BC at  $z = z_p$  (ND)** leads to the odd solutions

$$f_n^-(z) = \tilde{\mathcal{N}}_n^{(-)} e^{-\frac{3}{2}\sigma(z)} \sin[\beta_n(z - z_p)] \quad (5.12)$$

where  $\beta_n$  are the solutions to the transcendental equation

$$\tan \frac{\beta_n \pi R}{2} = -\frac{\beta_n}{k}, \quad (5.13)$$

and  $m_n^2 = k^2 + \beta_n^2$ . On the other hand,

**Dirichlet BC at  $z = 0$  and Neumann BC at  $z = z_p$  (DN)**, gives the even solutions

$$f_n^+(z) = \tilde{\mathcal{N}}_n^{(+)} e^{-\frac{3}{2}\sigma(z)} \left\{ -\frac{k}{\beta_n} \sin[\beta_n|z - z_p|] + \cos[\beta_n(z - z_p)] \right\} \quad (5.14)$$

Once again,  $\beta_n$  are given by a very similar equation, namely,

$$\tan \frac{\beta_n \pi R}{2} = \frac{\beta_n}{k}. \quad (5.15)$$

with  $m_n^2 = k^2 + \beta_n^2$ . That the (ND) and (DN) cases admit odd and even solutions, respectively, is dictated by the boundary conditions and the continuity conditions at  $z = z_p$  in each case, along with the fact that the theory is  $\mathbb{Z}_2$  symmetric. Clearly, the KK modes are non-degenerate with the even modes being heavier than the odd ones for a particular KK level, and, hence, incommensurable with the discrete CW mass spectrum in the sense shown<sup>23</sup> in eq. (4.9). Similar to case **I**, the inter-level spacings in the individual towers increase as we go higher in the spectrum and tend to saturate for large values of  $n$  for which  $\beta_n \rightarrow (2n - 1)/R \gg k$ . Therefore, any quasi-degeneracy seemingly present in the low lying KK states gets disturbed in the heavier levels.

In addition to the KK spectrum, there exists a single non-trivial  $\mathbb{Z}_2$  even solution (satisfying the **DN** BCs) for  $m^2 < k^2$  given by

$$f_0^+(z) = \tilde{\mathcal{N}}_0^{(+)} e^{-\frac{3}{2}\sigma(z)} \left\{ -\frac{k}{\tilde{\beta}} \sinh[\tilde{\beta}|z - z_p|] + \cosh[\tilde{\beta}(z - z_p)] \right\}, \quad (5.16)$$

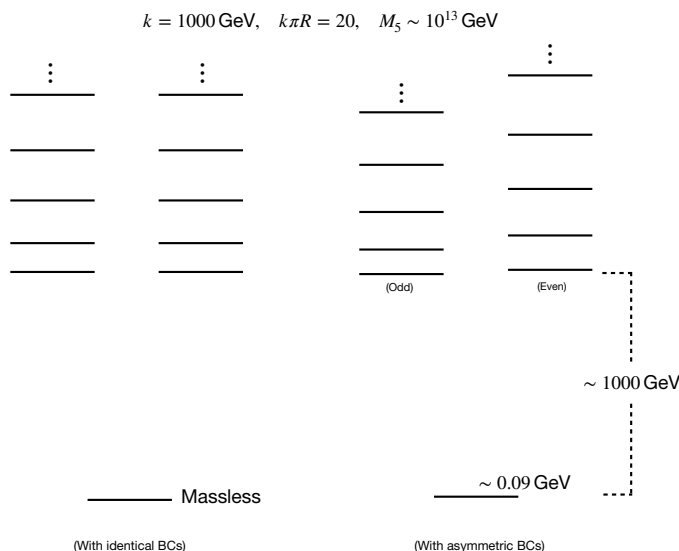
where  $\tilde{\beta} \equiv \sqrt{k^2 - m_0^2}$  is obtained as a solution of the equation

$$\tan \frac{\tilde{\beta} \pi R}{2} - \frac{\tilde{\beta}}{k} = 0. \quad (5.17)$$

This is unlike case **I** which admits a trivial massless mode. For values of  $|k\pi R|$  that potentially address the EW-Planck hierarchy, the preceding equation has the solution

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<sup>23</sup>It remains to be examined whether this scenario has a different (non-CW) discrete analogy that could be derived as a deconstruction of the 5D theory, presumably by invoking a different discrete form for the  $z$ -derivatives than what we have adopted here.



**Figure 3.** Schematic representation of the first few KK levels in the case of identical (left) and asymmetric (right) boundary conditions for a benchmark parameter set.

$\tilde{\beta} \sim k$ , with  $f_0^+(z) \rightarrow \tilde{\mathcal{N}}_0^{(+)}$ , i.e., an *almost* flat profile along the fifth dimension similar to what was obtained for the exactly massless state in case **I**. In other words, the erstwhile flat (and, hence, massless) solution acquires a slight non-triviality in its wavefunction and, thereby, is very slightly lifted in mass. A part of the full spectrum is shown schematically in figure 3 for a specific set of values for  $k$  and  $R$ .

We now turn to the question of the coupling with a brane localised operator. If an interaction of the form shown in eq. (5.10) is assumed, the nature of coupling with the light mode remains quite similar to what we encountered in case **I** for  $|k\pi R| \gg 1$ . The heavier KK modes, too, have similar ( $\mathcal{O}(1)$ ) couplings at the branes as previously seen. The major distinction, however, appears in the case of the odd modes in that they now have vanishing couplings only at the central fixed point. Couplings with a sector confined to one of the boundary branes, therefore, would break the  $\mathbb{Z}_2$  parity explicitly unless there exist couplings with a mirror sector in the opposite brane as well.

At this juncture, an interesting distinction between the two sets of BCs is worth mentioning. Note that the **(NN)** BC preserves the symmetry under a constant shift of the 4D field  $\varphi_0$ , i.e., under  $\varphi_0(x) + c$ , due to the  $z$ -independence of its wavefunction. This, in turn, justifies its masslessness. The **(DD)** BC clearly breaks this symmetry which is why we do not obtain an odd massless mode. The **(DN)** BC, too, breaks this symmetry for the zeroth mode, thereby, leading to it being massive.

## 5.2 Bulk vector in LDG

The discussion of the bulk theory for a massless vector boson runs, to a large extent, parallel to that for the scalar theory. The free field action is, of course, given by

$$\mathcal{S} = \int d^4x dz \sqrt{-g} \left\{ -\frac{1}{4} g^{MM'} g^{NN'} F_{MN} F_{M'N'} \right\} \quad (5.18)$$

where  $M = \{\mu, z\}$  etc. Choosing the generalized  $R_\xi$  gauge-fixing term

$$S_{GF} = \int d^4x dz \sqrt{-g} e^{-4\sigma(z)} \left\{ -\frac{1}{2\xi} \left[ \partial^\mu A_\mu + \xi e^{-\sigma(z)} \partial_z \left( e^{\sigma(z)} A_z \right) \right]^2 \right\} \quad (5.19)$$

allows us to eliminate all the  $A_\mu - A_z$  mixing terms upto total derivatives along the  $z$  dimension. We define the field decomposition to be

$$A_\mu^{(\pm)} = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \mathcal{A}_{\mu n}^{(\pm)}(x) f_n^{(\pm)}(z), \quad A_z^{(\pm)} = \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \mathcal{A}_{z n}^{(\pm)}(x) h_n^{(\pm)}(z) \quad (5.20)$$

with the normalization conditions being

$$\frac{1}{\pi R} \int dz e^{\sigma(z)} f_n^{(\pm)} f_{n'}^{(\pm)} = \delta_{nn'}, \quad (5.21)$$

where the choice  $h_n \equiv (1/m_n) \partial_z f_n$  for  $n > 0$  is dictated by the  $\mathbb{Z}_2$  symmetry in the  $A_\mu - A_z$  mixing terms and  $h_0 \equiv f_0$  is motivated by the fact that the zero mode solutions of the respective EOMs are identical. Dropping the  $(\pm)$  superscripts for brevity, we obtain the following EOMs for  $\mathcal{A}_\mu^{(n)}(x)$ ,  $\mathcal{A}_z^{(n)}$  and  $f^{(n)}(z)$  (for both even and odd modes),

$$\begin{aligned} \partial_\mu \mathcal{F}^{\mu\nu(n)} + m_n^2 \mathcal{A}^{\nu(n)} + \frac{1}{\xi} \partial^\nu \left( \partial^\mu \mathcal{A}_\mu^{(n)} \right) &= 0 \\ \left( \partial_x^2 - \xi m_n^2 \right) \mathcal{A}_z^{(n)}(x) &= 0 \\ \left( \partial_z^2 - \frac{1}{9} k^2 + m_n^2 \right) \left( e^{\sigma(z)/2} f^{(n)}(z) \right) &= 0, \end{aligned} \quad (5.22)$$

where the field strength tensors  $\mathcal{F}^{(n)}$  correspond to the 4D fields  $\mathcal{A}_\mu^{(n)}$ . The equation for  $f(z)$  is similar to the one for the bulk scalar with the only difference being in the coefficient of the  $k^2$  term. Following the terminology therein, the solutions for the *identical* BC case are:

**NN boundary conditions.**

$$\begin{aligned} f^{+(0)}(z) &= \mathcal{N}_0^{(+)} \\ f^{+(n)}(z) &= \mathcal{N}_n^{(+)} e^{-\sigma(z)/2} \left\{ -\frac{kR}{6n} \sin \left[ \frac{2n}{R} |z - z_p| \right] + \cos \left[ \frac{2n}{R} (z - z_p) \right] \right\} \\ m_0^2 &= 0 \\ m_n^2 &= \frac{1}{9} k^2 + \frac{4n^2}{R^2} \end{aligned} \quad (5.23)$$

**DD boundary conditions.**

$$f^{-(n)}(z) = \mathcal{N}_n^{(-)} e^{-\sigma(z)/2} \sin \left[ \frac{2n}{R} (z - z_p) \right], \quad m_n^2 = \frac{1}{9} k^2 + \frac{4n^2}{R^2}, \quad (5.24)$$

where, as before,  $n \geq 1$  and  $(\pm)$  denote  $\mathbb{Z}_2$  even and odd wavefunctions respectively. The normalisation factors,  $\mathcal{N}$ , are given by

$$\mathcal{N}_0 = \sqrt{\frac{k\pi R}{3 \left( 1 - e^{-\frac{1}{3} k\pi R} \right)}}, \quad \mathcal{N}_n^+ = 2\sqrt{2} \frac{n}{m_n R}, \quad \mathcal{N}_n^- = \sqrt{2}. \quad (5.25)$$



As in the scalar case, and as expected, we have exactly one massless mode  $A_{0\mu}^{(+)}$  which is to be identified with the four-dimensional gauge field. The 4D masses of the vector bosons can be attributed to the Goldstone-like degrees of freedom  $\mathcal{A}_z^{(n)}$  ( $n \geq 1$ ) which, in the unitary gauge ( $\xi \rightarrow \infty$ ), vanish from the effective 4D theory. In addition, there remains a massless scalar mode in the spectrum. Therefore, following the treatment in the scalar theory, the spectrum of even and odd KK modes can be classified in terms of the  $\mathbb{Z}_2$  parity,

$$\mathcal{Z}\mathcal{A}_\mu^{(\pm)(n)}(x) = \pm\mathcal{A}_\mu^{(\pm)(n)}(x), \quad \mathcal{Z}\mathcal{A}_z^{(+)(0)}(x) = \mathcal{A}_z^{(+)(0)}(x). \quad (5.26)$$

In other words, the  $\mathbb{Z}_2$  assignments as well as the degeneracy within a level is exactly the same as that for the spectrum obtained in the discrete scenario in section 3.2. The coupling profile of the KK modes with a brane localised operator can be seen by assuming that the operator is a current ( $\mathcal{J}_M = \{\mathcal{J}_\mu, \mathcal{O}_S\}$ ) composed of fields charged under the bulk U(1) gauge group and, therefore, couples to the bulk gauge field. The appropriate interaction is defined by

$$\mathcal{S}_{ext} = - \int d^4x dz \frac{\sqrt{-g}}{\sqrt{g_{zz}}} e^{-\frac{7}{6}S(z)} \mathcal{J}_M A^M(x, z) \delta(z - z_\alpha), \quad (5.27)$$

where  $\mathcal{J}_\mu$  is a vector current,<sup>24</sup>  $\mathcal{O}_S$  is a scalar operator, and, similar to the scalar theory, we assume

$$\begin{aligned} A_\mu(x, z) &\equiv \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \left\{ \mathcal{A}_{\mu n}^{(+)}(x) f_n^{(+)}(z) + \mathcal{A}_{\mu n}^{(-)}(x) f_n^{(-)}(z) \right\} \\ A_z(x, z) &\equiv \frac{1}{\sqrt{\pi R}} \sum_{n=0}^{\infty} \left\{ \mathcal{A}_{z n}^{(+)}(x) h_n^{(+)}(z) + \mathcal{A}_{z n}^{(-)}(x) h_n^{(-)}(z) \right\}. \end{aligned} \quad (5.28)$$

With this expansion, the action simplifies to (in the unitary gauge)

$$\begin{aligned} \mathcal{S}_{ext} &= - \frac{1}{\sqrt{\pi R}} \int d^4x dz \frac{\sqrt{-g}}{\sqrt{g_{zz}}} e^{-\frac{7}{6}S(z)} \left\{ \sum_{n=0}^{\infty} f^{+(n)} \mathcal{J}_\mu \mathcal{A}_{(n)}^{\mu+}(x) + \sum_{n=0}^{\infty} f^{-(n)} \mathcal{J}_\mu \mathcal{A}_{(n)}^{\mu-}(x) \right. \\ &\quad \left. + f^{+(0)} \mathcal{O}_S \mathcal{A}_{z(0)}^{(+)}(x) \right\} \delta(z - z_\alpha) \\ &= - \frac{1}{\sqrt{\pi R}} \int d^4x \left\{ \mathcal{N}_0^{(+)} e^{-\frac{1}{3}k|z_\alpha - z_p|} \mathcal{J}_\mu \mathcal{A}_{(0)}^{\mu+}(x) + \mathcal{N}_0^{(+)} e^{-\frac{1}{3}k|z_\alpha - z_p|} \mathcal{O}_S \mathcal{A}_{z(0)}^{(+)}(x) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} \mathcal{N}_n^{(+)} \left[ - \frac{kR}{6n} \sin \left[ \frac{2n}{R} |z_\alpha - z_p| \right] + \cos \left[ \frac{2n}{R} (z_\alpha - z_p) \right] \right] \mathcal{J}_\mu \mathcal{A}_{(n)}^{\mu+}(x) \right\}, \end{aligned} \quad (5.29)$$

Analogous to the scalar case, and going by the argument mentioned therein, the dilaton factor  $e^{-7S(z)/6}$  has been introduced to induce localisation of the massless modes (both the

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<sup>24</sup>Identifying the external sector with the SM, this can be envisaged as the current associated with some of the popular and well-motivated scenarios with an Abelian gauge group like U(1)<sub>B-L</sub>, U(1)<sub>L<sub>μ</sub>-L<sub>τ</sub></sub>, etc., or with a dark sector group U(1)<sub>D</sub> with the light vector being a Z' or a dark photon-like particle. The light scalar, on the other hand, can be identified as an axion-like particle. Prospective models based on such bulk scenarios, therefore, would simultaneously induce couplings of a Z' (or A') and an ALP with the SM sector.

vector and the scalar) alone towards one of the branes. Further, from eq. (5.23)–(5.24), we see that the coupling profiles of the KK modes at the branes is of the same nature as obtained for the scalar theory and reflects the *clockworking* attribute of the corresponding discrete theory.

By analogy with the scalar case, again, it is straightforward to see that the EOMs in eq. (5.22) have solutions corresponding to asymmetric BCs as well with no mass degeneracy. Of course, in this case, there exists a light vector mode with a nonzero mass, with the corresponding light scalar acting as a Goldstone boson which vanishes altogether in the unitary gauge. This can also be understood by noting that the (DN) BC does not preserve the 4D gauge invariance of the zeroth mode as opposed to that in the (NN) case.

## 6 Perturbation and fine tunings in the LD theory: an aside

The degeneracy between the odd- and even-modes could be considered a surprise, and it is worth pondering if it is the result of a fine-tuning of parameters. Given this, it is worth reexamining, at this stage, the linear dilaton theory of section 4 for the extent of any extra fine-tuning beyond the usual one encountered in generic warped theories associated with the vanishing of the 4D cosmological constant (CC). Apart from the equality between the brane tensions and the bulk curvature mentioned in section 4.1, viz.

$$\Lambda_2 = 4kM_5^3 = -\Lambda_1 = -\Lambda_3, \tag{6.1}$$

the present theory seemingly demands an additional tuning in order to arrange for the relation  $\mathcal{V}_1 = \mathcal{V}_3$  of the VEVs required to stabilize the fifth dimension. However, note that, barring the last equality in eq. (6.1), the rest is exactly the same as in the usual CW (or its linear-dilaton cousin) and does not introduce any fine-tuning beyond the usual one. As for the last equality, clearly  $\Lambda_1 = \Lambda_3$  is mandated by the  $\mathbb{Z}_2$  inherent to the formulation and hence, unless the  $\mathbb{Z}_2$  is explicitly broken, this does not represent an extra fine-tuning either. Exactly same is the case for  $\mathcal{V}_1 = \mathcal{V}_3$ . At this level, then, the degeneracy between the even and odd states of a bulk field in the LD background seems a natural outcome.

The situation, however, could change drastically if we relax either of the two conditions inherent to the said degeneracy, namely the  $\mathbb{Z}_2$  and the uniformity of the CW coupling (reflected, in the continuum, by a uniform warping). We now examine each in turn, starting with the second. While our treatment might (and justifiably) be termed an ad hoc one, in the absence of any knowledge of the fundamental theory that stipulates either the  $\mathbb{Z}_2$  or the properties of the dilaton, possible deviations from our assumptions can hardly be dismissed, and, indeed, could very well be caused by quantum corrections, if nothing else. We maintain that the discussion below, while only an illustrative one, yet captures the qualitative features.

### 6.1 Non-uniform warping

We begin by examining the case where the  $\mathbb{Z}_2$  is intact, but the warping is no longer uniform (linear). In other words, we are interested in examining the consequences of replacing

$$\sigma(z) = \frac{-2}{3}k|z - z_p| \rightarrow \frac{-2}{3}K(z)|z - z_p|$$

where  $K(z)$  varies from a constant value by only a small amount, viz.

$$K(z) = k + \delta k(z), \tag{6.2}$$

so that a perturbative treatment may be permissible. As argued above, such a change may be brought about by, for example, altering the dilaton potential (presumably away from the linear dilaton paradigm) and adjusting the brane potentials adequately so as to have a consistent solution. However, rather than delve into the origin of such a change, we treat eq. (6.2) as determining an effective background for the bulk scalar field  $\phi$ .

Such a change would, naturally, be expected to break the degeneracy between the even and odd modes by an extent determined by the magnitude of the shift in  $k$ . With  $\delta k(z)$  being an unknown function, admitting even oscillatory behaviour, a robust measure of this shift is necessary and, to this end, we define a relative shift magnitude by

$$\frac{\delta_K}{k} \equiv \left( \frac{1}{\pi R} \int_0^{\pi R} dz \left[ \frac{\delta k(z)}{k} \right]^2 \right)^{1/2}. \tag{6.3}$$

A nonzero  $\delta k(z)$  would lead to a shift in the hitherto degenerate (for each  $n$ ) mass-squared matrices, not only in the diagonal terms, but also in the off-diagonal ones. For small  $\delta_K$ , these shifts can be calculated perturbatively, leading to

$$\begin{aligned} \Delta m_n^{(++)2} &= \frac{2k}{\pi R} \int_0^{\pi R} dz e^{3\sigma(z)} \left[ f_n^+(z) \right]^2 \delta k(z) \\ \Delta m_n^{(--)^2} &= \frac{2k}{\pi R} \int_0^{\pi R} dz e^{3\sigma(z)} \left[ f_n^-(z) \right]^2 \delta k(z) \quad . \\ \Delta m_n^{(+-)^2} &= \frac{2k}{\pi R} \int_0^{\pi R} dz e^{3\sigma(z)} f_n^+(z) f_n^-(z) \delta k(z) \end{aligned} \tag{6.4}$$

(Of course, if  $\mathbb{Z}_2$  is maintained by  $\delta k(z)$ , then  $\Delta m_n^{(+-)^2} = 0$ ). Given the above, the mass-splitting introduced at each KK level is given by

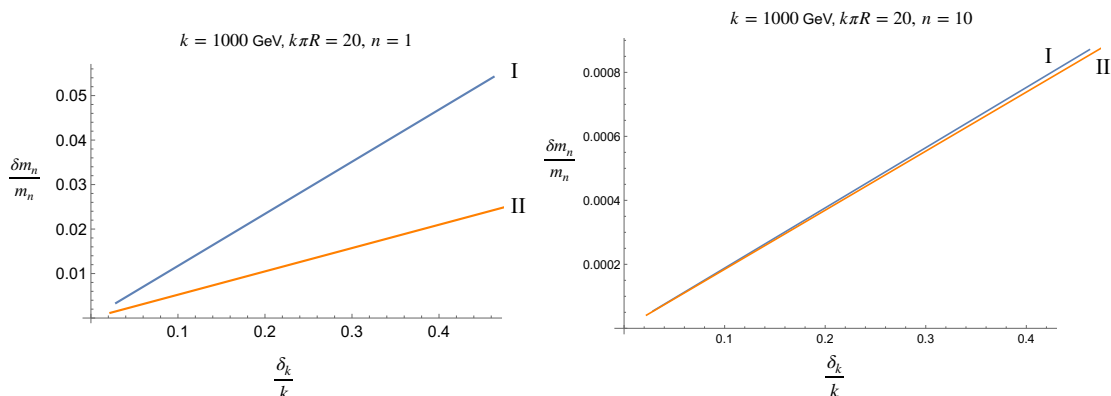
$$\Delta m_n^2 = \sqrt{\left( \Delta m_n^{(++)2} - \Delta m_n^{(--)^2} \right)^2 + 4 \left( \Delta m_n^{(+-)^2} \right)^2}, \tag{6.5}$$

With the mass splitting being an observable, the functional dependence of the relative splitting  $\Delta m_n/m_n$  on the quantity  $\delta_K/k$  can be used as a diagnostic for the extent of fine tuning in the theory [47]. To quantify this, we need to specify the form of  $\delta k(z)$ . As illustrative examples, we consider two forms, namely

$$\begin{aligned} \text{Case I :} \quad \delta k(z) &= \epsilon k \cos \left[ k \left( z - \frac{\pi R}{2} \right) \right] \\ \text{Case II :} \quad \delta k(z) &= \epsilon k \exp \left[ -k \left| z - \frac{\pi R}{2} \right| \right], \end{aligned} \tag{6.6}$$

where  $\epsilon$  is a small constant. Figure 4 shows the consequent variation of the relative splittings with the fractional shift in  $k$ . A few features are immediately clear:

- With the products  $e^{3\sigma(z)} [f_n^\pm(z)]^2$  being simple oscillatory functions for  $n > 0$ , the extent of the correction is governed, to a very large extent, by the profile of  $\delta k(z)$ .



**Figure 4.** Relative mass splitting as a function of the fractional change  $\delta_K/k$  for the KK levels (a)  $n = 1$  and (b)  $n = 10$ .

- Typically,  $\Delta m_n/m_n \ll \delta_K/k$ . This is not difficult to understand, especially for the two profiles that is presented. Although, in general, the contribution to  $\Delta m_n^{(++)^2}$  and  $\Delta m_n^{(--)^2}$ , individually, would tend to be maximized when the peaks of  $\delta k(z)$  coincide with the peaks of  $\exp[3\sigma(z)]f^2$ , the oscillatory behaviour of the latter would tend to average out the two contributions to comparable values. Thus, the consequent splitting would not change significantly for other profiles as well.
- In particular, for Case I, the oscillator profile leads to cancellations between corrections from different  $z$ -regions. Even then, the corrections are still much larger than those for Case II, wherein  $\delta k(z)$  quickly falls off away from the central brane leading to sizable contributions only from a narrow sliver.
- The absolute splittings tend to be smaller for higher KK-levels as the weight function  $\exp[3\sigma(z)]f^2$  becomes rapidly oscillating as  $n$  increases, leading to  $\Delta m_n^{(++)^2}$  and  $\Delta m_n^{(--)^2}$  being similar in magnitude. The relative splitting is further suppressed as the (unaltered) contribution from the momentum in the fifth direction grows quickly.

In summary, the relative smallness of  $\Delta m_n/m_n$  *vis a vis* the ‘perturbation’  $\delta_K/k$  implies that the Barbieri-Giudice [47] measure of fine-tuning is quite small in the present context.

## 6.2 A broken $\mathbb{Z}_2$

We, now, turn to the second possibility mentioned above, namely maintain a linear  $\sigma(z)$ , but break the discrete symmetry. Naively, the simplest way to achieve this would be to either shift the intermediate brane from the geometrical centre or assume the mass parameter  $k$  to be asymmetric around the central fixed point in the bulk, or both.

It should be realized, though, that consistency would, generically demand that unequal  $k$  in the two halves should be accompanied by a shift in the position of the central brane. The easiest way to understand this is to fall back on the discrete case, where  $q$  — the charge scaling factor between consecutive sites — had taken the place of  $k$  in the continuum and  $p$  — the number of steps between the two pivots — was analogous to the radius  $r \equiv R/2$ .

If the pair  $(q, p)$  were different in the two halves, clearly, consistency at the two pivots demands<sup>25</sup>  $q_1^{p_1} = q_2^{p_2}$ . In the continuum language, this translates to

$$r_1 = R - r_2 = \frac{k_2}{k_1 + k_2} R, \tag{6.7}$$

where  $r_{1,2}$  denote the radii of the patches to the left and right of the intermediate brane, respectively. Explicitly, for  $z_p \neq \pi R/2$  and unequal values of the parameter  $k$  about  $z_p$ , we obtain the solution

$$\sigma(z) = \begin{cases} \alpha_1 \frac{\Lambda_2}{3M_5^3} (z - z_p) & z < z_p, \\ -\alpha_2 \frac{\Lambda_2}{3M_5^3} (z - z_p) & z > z_p \end{cases} \tag{6.8}$$

with the 4D CC tuning now amounting to  $\Lambda_2 = 2M_5^3(k_1 + k_2) = -\Lambda_1/2\alpha_1 = -\Lambda_3/2\alpha_2$ . Here,  $k_1$  and  $k_2$  denote the constant<sup>26</sup> mass parameters for  $z < z_p$  and  $z > z_p$ , respectively, and  $\alpha_{1,2}$  are defined as

$$\alpha_1 \equiv \frac{2k_1 + k_2}{3(k_1 + k_2)} \quad \text{and} \quad \alpha_2 \equiv \frac{k_1 + 2k_2}{3(k_1 + k_2)}. \tag{6.9}$$

In other words, we have

$$\sigma(z) = \begin{cases} \frac{2}{9}(2k_1 + k_2)(z - z_p) & z < z_p, \\ -\frac{2}{9}(k_1 + 2k_2)(z - z_p) & z > z_p. \end{cases} \tag{6.10}$$

The relation which the VEVs must satisfy for radius stabilization now becomes,

$$\mathcal{V}_1 - \mathcal{V}_3 = \frac{2\pi}{3} [(k_1 + 2k_2)r_2 - (2k_1 + k_2)r_1] = \frac{2\pi}{3}(k_1 - k_2)R. \tag{6.11}$$

where, the last equality is obtained using the relation in eq. (6.7). The equality of the potentials on the brane, viz.  $\mathcal{V}_1 = \mathcal{V}_3$ , therefore, can be understood as being tantamount to a fine-tuning required to preserve the  $\mathbb{Z}_2$  parity.

It should be remembered that, with the  $\mathbb{Z}_2$  now having been explicitly broken, there is no notion of even and odd solutions, far less mass degeneracy. Indeed, the only permissible nontrivial solutions to the equation of motion (with similar forms of the BCs as assumed originally) are those that have support on only one of the two domains, left or right. This is not difficult to understand as, on breaking the  $\mathbb{Z}_2$ , all that we have are two contiguous, yet disparate patches, that have been sewed together. In particular, the solutions *have to vanish* (namely a Dirichlet boundary condition) at the boundary between the two patches. The number of independent solutions remains the same though, the erstwhile even and odd pairs being replaced by the pair of solutions over the individual patches.

The mass splitting, in this case, is easy to ascertain, being given by the expressions in eq. (5.6), but computed with differing  $k_{1,2}$ .

<sup>25</sup>The ensuing arguments are qualitatively valid for  $\mathbb{Z}_2$  asymmetric  $m$  and  $f$  parameters as well in the discrete theory.

<sup>26</sup>Field equations of the LD theory only allow for flat variations of  $k$  in the two patches around the central fixed point. A generic variation  $\delta k(z)$  would, perhaps, require venturing beyond the linear dilaton paradigm and introducing additional degrees of freedom in the bulk.

### 6.3 The generic case

The most general situation (where both the  $\mathbb{Z}_2$  is broken and a complicated warping is in place) can be qualitatively understood in terms of the two simplified discussions above. The Barbieri-Giudice measure for fine-tuning is expected to remain tolerable (as long as it is not applied to the issue of the vanishing cosmological constant, in common to all other such warped models). Furthermore, the features alluded to above would continue to remain applicable.

## 7 Summary

The clockwork mechanism provides a novel way of generating hierarchical couplings without resorting to large fine tunings. The fact that the original class of clockwork theories also finds a correspondence with the 5D linear dilaton model makes it a compelling avenue to be explored further. In this paper, we substitute the original open ended construction by a closed chain structure of the near-neighbour interactions. The resulting  $\mathbb{Z}_2$  symmetry under field exchanges about the centre-most site gives rise to two sets of physical states, one even and the other odd under an exchange parity. The even massless state is localised towards one of the  $\mathbb{Z}_2$  fixed points (the pivot sites) reflecting the clockwork mechanism. Keeping this symmetry intact then renders the lightest odd state absolutely stable and, hence, a potential dark matter candidate.

This new class of theories can also be obtained as a deconstruction of a bulk 5D theory in a linear dilaton background invariant under a  $\mathbb{Z}_2$  transformation about a fixed point identified with the centre of the compact extra dimension. The discretization prescription, however, needs to be amended so as to preserve the  $\mathbb{Z}_2$  symmetry. The underlying  $\mathbb{Z}_2$  invariant warped background metric is obtained as the *exact* (i.e., including backreaction) solution of an extended linear dilaton theory on an  $S^1/\mathbb{Z}_2$  orbifold with three equidistant 3-branes. The theory, therefore, potentially provides two physically inequivalent setups, namely, the IR-UV-IR and the UV-IR-UV scenarios, both admitting different, yet viable solutions to the EW-Planck hierarchy problem.

Bulk field theories (whether scalar or vector) in this background, on compactification, are characterised by a physical spectrum consisting of two KK towers with states that are even and odd under the  $\mathbb{Z}_2$  parity, mimicking the notion of KK parity in typical UED models with a flat metric. With identical boundary conditions at the orbifold fixed points (NN and DD), the theory admits a massless mode and the two KK towers (beyond a mass gap  $\sim k$ ) turn out to be doubly degenerate at each level with mass eigenvalues that match those in the discrete theory in the large  $N$  limit at leading order in  $(1/N)$ . This scenario, therefore, elucidates the correspondence between the discrete and the continuum clockwork theories. On the other hand, with non-identical boundary conditions, we obtain a non-degenerate spectrum of heavy KK states along with a light mode that has a small, albeit nonzero, mass. While this picture does not immediately relate to the discrete theories that we introduced, it is potentially interesting in its own right. In particular, it demonstrates that the choice of boundary conditions is germane to the aspect of establishing a correspondence between a discrete CW theory and a bulk 5D theory in the linear dilaton background.

In short, this study demonstrates that the clockwork mechanism and its correspondence with the linear dilaton model need not be associated with only a specific topology of the discrete and continuum theories and that extensions within this CW/LD framework are possible, like the one we have discussed in this work which projects a richer phenomenology and, presumably, new and interesting model building applications.

## A Mass degeneracy

### A.1 Degeneracy in the continuum CW

Reexamining the equation of motion (eq. (5.5)) for the bulk modes of the scalar theory, namely,

$$\left(\partial_z^2 - k^2 + m_n^2\right) \left(e^{3\sigma(z)/2} f_n(z)\right) = 0, \tag{A.1}$$

we immediately see that, if  $f_n(z)$  satisfies the equation, so does  $f_n'(z)$  and for the same  $m_n$ . In other words, the two wavefunctions, if admissible, are degenerate, and if they are  $\mathbb{Z}_2$  eigenstates, they must correspond to opposite eigenvalues. It is easy to verify that the odd (**DD**) solutions are indeed derivatives of the even (**NN**) solutions, viz.

$$f_n^{(-)}(z) = -m_n^{-1} \partial_z f_n^{(+)}(z). \tag{A.2}$$

Naively, it might seem that further derivatives would produce more candidate eigenstates for a given  $m_n$ . However, note that the boundary conditions restrict the admissible solutions. As for the remaining, these can be written as linear combinations of the (fundamental) solutions  $f_n^{(\pm)}$  for each KK level  $n$  and, hence, are not independent. This is expected as the solutions  $f_n^{(\pm)}$  span the two-dimensional degenerate space that is stipulated by the  $\mathbb{Z}_2$  symmetry. Clearly, the same argument also holds for the vector field theory.

In contrast, for the mixed BC case (**DN** and **ND**), the BCs at  $z = 0$  corresponding to the even and odd modes of the same KK level  $n$  are incompatible with a relation like eq. (A.2) which serves to explain the non-degeneracy encountered in that case.

We end this discussion by reexamining the source of this degeneracy. Consider any deformation along the  $z$ -dimension, e.g., a different value of the parameter  $k$  over a patch in the bulk (and, to ensure  $\mathbb{Z}_2$  symmetry, a corresponding patch in the opposite region). This would introduce continuity conditions on the derivatives of the solutions that, e.g., are  $\mathbb{Z}_2$ -odd for even solutions. The mass eigenvalues would have a dependence on these derivative conditions, thereby, breaking the degeneracy between the even and the odd modes. Thus, it is the  $\mathbb{Z}_2$  symmetry, the boundary conditions and the universality of the parameters in the theory that, *together*, facilitate the emergence of a degeneracy in the physical mass spectrum.

### A.2 Degeneracy in discrete CW

While the degeneracy in this case could be motivated from that for the continuum, we examine the issue on its own. As previously mentioned, the degeneracy in the spectrum emerges by virtue of the three characteristic features of the theory, namely, the universality of the parameters along the sites, the  $\mathbb{Z}_2$  symmetry and the specific form of interactions

involving the pivot sites. It is, perhaps, easier to trace the consequences of each of the properties individually in order to verify the preceding assertion. To start with, the universal nature of the parameters essentially leads to a (pseudo-)tridiagonal mass matrix with deviations only in the 0-th,  $p$ -th and  $(2p - 1)$ -th rows (which, for brevity, we address as the *pivot* rows) with these being a consequence of the  $\phi_0 - \phi_{2p-1}$  and the  $\phi_{p-1} - \phi_p$  mixings. The resulting eigenvalue equations, barring those involving the pivot ones, are second order linear difference equations of the form

$$-qa_{j-1,n} + (1 + q^2 - \lambda_n)a_{j,n} - qa_{j+1,n} = 0, \tag{A.3}$$

with equal coefficients for  $a_{j-1,n}$  and  $a_{j+1,n}$ . Here,  $\lambda_n$  denotes the  $n$ -th eigenvalue. This equation is nothing but a second-order  $q$ -difference equation or, in other words, the discretized version of the harmonic equation with ordinary derivatives replaced by  $q$ -derivatives. Consequently, the solutions (away from the pivot points) are given by

$$a_{nj} = \mathcal{N}_n \begin{cases} \sin j\theta_n + B \cos j\theta_n, & j \leq p \\ C \sin j\theta_n + D \cos j\theta_n, & j > p \end{cases}, \tag{A.4}$$

where  $B$ ,  $C$  and  $D$  are constant coefficients,  $\mathcal{N}_n$  are the normalisation factors and  $\theta_n$  is an *a priori* undetermined function of  $n$ . Reverting to the eigenvalue equation for the pivot rows, we have

$$\begin{aligned} 2q^2 a_{n,0} - qa_{n,1} - qa_{n,N-1} &= \lambda_n a_{n,0} \\ -qa_{n,p-1} + 2a_{n,p} - qa_{n,p+1} &= \lambda_n a_{n,p}. \end{aligned} \tag{A.5}$$

Furthermore, the eigenvalues have the form,

$$\lambda_n = 1 + q^2 - 2q \cos \theta_n. \tag{A.6}$$

Thus, in effect, the universal parametrisation fixes the form of the eigensystem to the simple expressions given in eq. (A.4) and (A.6).

Appealing, now, to the  $\mathbb{Z}_2$  symmetry, it stipulates the existence of even and odd solutions. Consequently, one obtains for the odd eigenvectors  $a_{k,p}^- = 0$  which implies  $B = -\tan p\theta_n$ .

Finally, with the odd solutions satisfying the second equation in (A.5) trivially, we see that the first pivot equation determines the exact form of the eigenvalues by specifying  $\theta_n = n\pi/p$ , or, alternatively,

$$\lambda_n = 1 + q^2 - 2q \cos \frac{n\pi}{p}, \tag{A.7}$$

which clearly are the doubly-degenerate eigenvalues discussed before in section 3.1. The pivot equations, therefore, play the role of boundary conditions about the  $\mathbb{Z}_2$  fixed points as encountered in the continuum picture. In summary, the preceding arguments establish that the structural properties of the theory intricately conspire to give rise to the degenerate mass spectrum and that a deformity in any of the three properties would break the degeneracy.



## B Dynamical dilaton modes and IR-UV-IR/UV-IR-UV

In section 4.2 we had discussed briefly the emergence of two kinds of scenarios in our triple brane theory, namely, the IR-UV-IR and the UV-IR-UV setups,<sup>27</sup> corresponding to  $k > 0$  and  $k < 0$  respectively. Each of these provides possible explanations of the hierarchy problem. Here, we delineate the stability of the two cases with regards to the physicality of the dynamical modes generated therein. Concentrating on scalars, the said modes can be identified by considering, at the linear order, the scalar fluctuations<sup>28</sup>  $\Psi(x, z)$  and  $\Phi(x, z)$  in our CW metric [48], viz.

$$ds^2 = F(z)^2 [(1 + 2\Psi(x, z))\eta_{\mu\nu}dx^\mu dx^\nu + (1 + \Phi(x, z))], \quad F(z) \equiv e^{\sigma(z)}. \quad (\text{B.1})$$

Furthermore, the perturbation of the linear dilaton field  $S$  in eq. (4.10) is denoted by  $\delta S(x, z)$ . On linearization of the equations of motion, the aforementioned separation of the scalar mode from the others is encapsulated in the constraint equations

$$\begin{aligned} \Phi - \frac{F}{F'}\Psi' - \frac{F}{9F'}S'\delta S &= 0, \\ \Phi &= -2\Psi \end{aligned} \quad (\text{B.2})$$

where the primes denote  $z$ -derivatives. On using these, only one independent bulk scalar remains, and its dynamical equation reduces to

$$\left(\partial_x^2 + \partial_z^2 - k^2\right) \left[e^{3\sigma(z)/2}\Phi(x, z)\right] = 0. \quad (\text{B.3})$$

On effecting the decomposition

$$\Phi(x, z) = \sum_n Q_n(x)\tilde{\Phi}_n(z), \quad (\text{B.4})$$

equations of motion are decoupled, and

$$\begin{aligned} \left(\partial_x^2 - m_n^2\right) Q_n(x) &= 0 \\ \left(\partial_z^2 - k^2 + m_n^2\right) \left[e^{3\sigma(z)/2}\tilde{\Phi}_n(z)\right] &= 0. \end{aligned} \quad (\text{B.5})$$

Additionally, by integrating the linearised Einstein equations (including the boundary terms) over a small interval about the fixed points, we obtain two junction conditions (JC) at each brane for the perturbations. One of these is equivalent to eq. (B.2) when evaluated at the branes and, hence, imposes no new constraints. The remaining condition, namely,

$$\delta S' - S'\Phi \Big|_{z_\alpha - \epsilon}^{z_\alpha + \epsilon} = 6e^{\sigma(z)}\lambda''_\alpha(S)\delta S \Big|_{z_\alpha}, \quad (\text{B.6})$$

<sup>27</sup>We specify an IR brane as one where any localised theory, as viewed in the effective 4D metric, would have its physical mass parameters set by the TeV scale. Similarly a UV brane is one where a theory would have all its mass parameters, ostensibly, near the Planck scale. Hence, for  $k > 0$  the IR branes are situated at the boundaries of the orbifold with the UV brane located in the middle, and the other way around for  $k < 0$ .

<sup>28</sup>Note that the vector fluctuations of the metric, alongwith one combination of the two scalar fluctuations, disappear in the unitary gauge to resurface as the longitudinal modes of the massive tensor modes.

relates the perturbation  $\delta S$  to the parameters at the branes. This, in turn, can be substituted in eq. (B.2) to obtain JCs operative directly on  $\Phi$ , one at each fixed point.<sup>29</sup> Here,  $\lambda_\alpha(S)$  are the brane potentials, as defined in eq. (4.22) and the primes on  $\lambda_\alpha$  denote differentiations with respect to  $S$ . The JC at the  $z = z_p$  brane generates the solutions

$$\begin{aligned}\tilde{\Phi}_0(z) &\propto e^{-3\sigma/2} \left\{ \sinh \left[ \tilde{\beta} |z - z_p| \right] + \tilde{\gamma} \cosh \left[ \tilde{\beta} (z - z_p) \right] \right\} \\ \tilde{\Phi}_{n>0}(z) &\propto e^{-3\sigma/2} \left\{ \sin \left[ \beta_n |z - z_p| \right] + \gamma_n \cos \left[ \beta_n (z - z_p) \right] \right\},\end{aligned}\tag{B.7}$$

where  $n = 0$  corresponds to the radion mode and

$$\begin{aligned}\tilde{\beta}^2 &\equiv k^2 - m_0^2 & \tilde{\gamma} &\equiv \frac{3\tilde{\beta}\mu_p}{(k\mu_p - k^2 + \tilde{\beta}^2)} \\ \beta_n^2 &\equiv m_n^2 - k^2 & \gamma_n &\equiv \frac{3\beta_n\mu_p}{(k\mu_p - k^2 - \beta_n^2)}.\end{aligned}\tag{B.8}$$

Here,  $\mu_\alpha$  are the mass parameters in the brane potentials  $\lambda_\alpha(S)$ , as detailed in eq. (4.22). Solving the equations emerging from the JC at  $z = 0$ , on the other hand, specifies the mass spectrum which, in the limit  $\epsilon_\alpha \equiv |k|/\mu_\alpha \ll 1$  and upto linear order in  $\epsilon_\alpha$ , are given by

$$\begin{aligned}m_0^2 &\simeq \begin{cases} \frac{8k^2}{9} \left( 1 - \frac{1}{9} \left[ (\epsilon_0 - \epsilon_p) + (\epsilon_0 + \epsilon_p) \coth \frac{k\pi R}{6} \right] \right) \simeq \frac{8k^2}{9} \left( 1 - \frac{2\epsilon_0}{9} \right) & k > 0 \\ \frac{8k^2}{9} \left( 1 - \frac{1}{9} \left[ (\epsilon_p - \epsilon_0) + (\epsilon_0 + \epsilon_p) \coth \frac{k\pi R}{6} \right] \right) \simeq \frac{8k^2}{9} \left( 1 - \frac{2\epsilon_p}{9} \right) & k < 0 \end{cases} \\ m_n^2 &\simeq k^2 + \frac{4n^2}{R^2} \left[ 1 - \frac{12(4n^2 + k^2 R^2)}{|k|\pi R(36n^2 + k^2 R^2)} (\epsilon_0 + \epsilon_p) \right].\end{aligned}\tag{B.9}$$

At the second order in the scalar perturbations the dynamical field, after diagonalisation of the action, is specified by a combination of  $\Phi$  and  $\delta S$ , namely,

$$\xi(x, z) \equiv \left( \frac{\Phi}{2} - \frac{\delta S}{3} \right) = \sum_n Q_n(x) \tilde{\xi}_n(z),\tag{B.10}$$

which satisfies the dynamical equation given in eq. (B.5).

The free 4D actions for the physical modes are, then, given by

$$S_n = \mathcal{C}_n \int d^4x Q_n \left( \partial_x^2 - m_n^2 \right) Q_n,\tag{B.11}$$

where

$$\mathcal{C}_n \equiv \frac{3M_5^3}{2} \int_0^{\pi R} dz F^{-1} \left[ \frac{9}{2S'^2} \left( F^2 \tilde{\Phi}_n \right)'^2 + \left( F^2 \tilde{\Phi}_n \right)^2 \right] > 0.\tag{B.12}$$

The positivity of  $\mathcal{C}_n$  ensures that the theory is free of negative kinetic terms (and, hence, ghosts) for either sign of  $k$ . It is easy to ascertain that, for,  $\epsilon_\alpha \ll 1$ , the theory is free of tachyonic modes (as exemplified by a negative mass-squared) as well. For large

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<sup>29</sup>Note that on account of the  $\mathbb{Z}_2$  symmetric structure of the theory about the central fixed point, the junction condition at one of the orbifold boundaries trivially fixes that on the opposite boundary.

values of  $\epsilon_\alpha$ , however, one needs to perform the stability analysis numerically as closed-form analytical expressions for the KK masses are not straightforward. Moreover, invoking non-minimalities, such as assuming a Higgs-curvature interaction on a brane, could lead to a ghost radion field depending on the value of the mixing parameter [35]. In the spirit of the RS gravity models [49, 50], adding brane-localised curvature terms could potentially introduce a ghost radion as well and, hence, would constrain their coefficients. Thus, in such cases, the stability of the two scenarios ought to be inspected again thoroughly. It should also be borne in mind that a possible embedding of linear dilaton models in a string theoretic setting [34] offers a way to ameliorate putative instabilities arising on account of negative tension branes.

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