Degree Complexity Bounds on the Intersection of Algebraic Curves

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ABSTRACT

The intersection of algebraic curves in three and higher dimensional spaces is considered. An algebrogeometric technique is developed for obtaining an upper bound on the number of intersection points of two irreducible algebraic curves. The asymptotic bounds are shown to be a function of only the degrees of the two intersecting curves. Some specific examples involving curves in 3-space are analyzed.

INTRODUCTION

An important problem in geometric modeling is to obtain tight bounds on the number of intersection points between two algebraic space curves and thence to develop efficient algorithms for finding these points [4,6,8]. Similar problems related to the intersection of trajectories in high dimensional spaces arise in dynamical systems and control theory [5].

Intersection problems for algebraic plane curves have been elegantly solved using classical algebrogeometric techniques [8]. These successes may be viewed as straightforward consequences of Bezout's theorem applied to the "proper" intersection of algebraic plane curves. This theorem implies that two algebraic curves, of degree m and n, intersect in no more than m n points on a plane, unless the curves overlap. This, in general, provides the least upper bound for the intersection of algebriac plane curves. Such well defined bounds are hard to derive for curves in higher dimensional space, as Bezout's theorem does not extend to such "improper" intersections [1,4,6].

In this paper we present a general technique for bounding the number of intersections of two algebraic space curves of arbitrary degree. The broad approach is to embed one of the space curves in appropriate low degree algebraic surfaces or hypersurfaces and then, using a version of Bezout's Theorem, to bound the cardinality of the intersection set. Asymptotic bounds are obtained for intersecting curves in both 3- and higher dimensional space. We believe that this approach could ultimately lead to analogues of Bezout's Theorem for improper intersections of algebraic varieties. Throughout this paper we are concerned only with curves and other varieties that are algebraic.

DEFINITIONS

Some terminology and propositions frequently used in this paper are briefly defined below. For detailed definitions refer to [1,3].

(i) An algebraic variety is said to be *pure* if all its irreducible components have the same dimension.

(ii) Two intersecting pure varieties V_1

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and V_2 are said to *intersect properly* provided co-dimension $(V_1 \cap V_2)$ equals the sum of the co-dimensions of V_1 and V_2. The intersection of two irreducible curves C and D in k-space is never proper (for $k \geq 3$).

(iii) For any given curve C_m, of degree m, a *minimal degree hypersurface* S_d of C_m is a hypersurface of degree d that contains C_m, where d is chosen to be the smallest positive integer such that $\alpha_{md} \triangleq \left[\binom{d+k}{k} - md - 1\right]$ evaluates to a positive value.

(iv) A curve C_m is called *special* if it can be embedded in a hypersurface $S_{d'}$ of degree d', for some d' smaller than d, where d is the degree of a minimal degree hypersurface of C_m.

(v) An irreducible curve D_n, which lies in the intersection of all the minimal degree hypersurfaces S_d of C_m, is called a *sibling* of C_m.

(vi) *Bezout's Theorem*

Let V_1 and V_2 be two pure varieties intersecting properly. Then

$$\text{degree}(V_1 \cap V_2) \leq \text{degree}(V_1) \cdot \text{degree}(V_2)$$

(and = holds in complex projective n-space $P_n(\mathbb{C})$ if intersections are counted with "appropriate" multiplicity).

RESULTS

We first examine a classical combinatorial formula.

Proposition

The minimal number of points needed to define a hyper-surface of degree d in k-space is $\left[\binom{d+k}{k} - 1\right]$.

The proof of this may be found in any of the classical algebraic geometry textbooks; see for example Ref. [12].

In particular, this proposition implies that there always exists a hypersurface S_d of degree d in $P^k(\mathbb{C})$ containing any collection of $\left[\binom{d+k}{k} - 1\right]$ points. Consider now a curve C_m of degree m also in $P^k(\mathbb{C})$. By Bezout's Theorem $|C_m \cap S_d|$ is either $\leq$ md or C_m and S_d have a common component. Furthermore, if C_m is irreducible and $|C_m \cap S_d|$ is greater than md, then C_m lies on S_d. These observations lead to a general technique for embedding any curve in a suitably "low" degree hypersurface. Some straightforward consequences of this discussion in $P^3(\mathbb{C})$ are that irreducible degree two space curves can be embedded in the plane and that an irreducible cubic space curve lies on a quadric surface.

We may now formulate a heuristic for bounding the number of intersections of two curves C_m and D_n in $P^k(\mathbb{C})$. Using the construction described above we would first obtain a hypersurface S_d containing C_m. Applying Bezout's Theorem we can determine the number of intersection points between S_d and D_n. This number will bound from above the number of intersection points between C_m and D_n. This heuristic may run into the difficulty that S_d also contains D_n whence we would obtain a trivial upper bound of infinity. In order to get around this difficult we need to develop a technique for constructing S_d containing C_m such that S_d intersects D_n properly. In the given space $P^k(\mathbb{C})$ let

$$\alpha_{md} = \binom{d+k}{k} - md - 1 \qquad (1)$$

d is chosen as the smallest positive integer such that α_{md} is positive. The number α_{md} represents the dimension of the vector space of hypersurfaces of degree d that contain the given curve C_m of degree m. Such an S_d is said to be a *minimal degree surface* for C_m. A curve C_m is said to be *special* if $C_m \subset S_{d'}$, for some $d' < d$. Most curves are *non-special.* Unless otherwise stated, the rest of this paper will be concerned with non-special irreducible curves.

Theorem 1

Let C_m and D_n be two distinct,

irreducible, non-special algebraic curves in k-space with $m<n$. Then there exists a hypersurface S_d of degree d such that S_d contains C_m and S_d intersects D_n properly. Consequently, $|C_m \cap D_n| \leq |S_d \cap D_n| \leq nd$.

Proof: Let d and d′ be the smallest integers for which α_{md} and $\alpha_{nd'}$ are positive. The dimensions of the vector spaces of hypersurfaces of degree d and d′ which contain C_m and D_n are respectively at least equal to α_{md} and $\alpha_{nd'}$. Consider the following two cases

Case 1: d is not equal to d′.

There exists a hypersurface S_d of degree d which contains C_m. D_n does not lie on S_d because it is non-special and the least degree hypersurface $S_{d'}$ on which it lies has degree d′, $d'>d$. Hence this S_d intersects D_n properly.

Case 2: d is equal to d′

α_{md} is greater than α_{nd} because $m<n$. That is, the dimension of the vector space of hypersurfaces S_d of degree d which contain C_m is greater than the dimension of the corresponding vector space of degree d hypersurfaces containing D_n. Hence, there certainly exists at least one hypersurface S_d which contains C_m and which intersects D_n properly. ∎

The hypersurfaces S_d and $S_{d'}$ considered above are in fact irreducible since both C_m and D_n are non-special curves. Furthermore, the theorem holds even if m is equal to n provided C_m and D_n are not siblings.

Asymptotic Bound

We know that in k-space any curve C_m of degree m can be embedded in a hypersurface S_d of minimal degree d. Recall that d is the smallest positive integer such that $\alpha_{md} \geq 1$, that is

$$\frac{(d+1)(d+2)\ldots(d+k-1)(d+k)}{k!} - 1 - md \geq 1$$

A simple asymptotic analysis shows that d grows as $(k!m)^{\frac{1}{(k-1)}}$. All curves D_n of degree n, where n is greater than m, will intersect this minimal degree hypersurface, S_d, properly. In fact any D_m which is not a *sibling* of C_m will also intersect S_d properly. We have proved the following:

Corollary

For C_m and D_n meeting the conditions in theorem 1, $|C_m \cap D_n| = O(m^{(\frac{1}{k-1})} n)$.

This corollary provides a succinct description of the upper bound on the number of intersection points. Notice that as k, the dimension of the space, increases the bound decreases and ultimately becomes linear in n. This is the algebraic manifestation of the simple geometric intuition that as the dimension of the space k increases, the degree d of the minimal degree hypersurface containing C_m must decrease. Eventually, for $k > m$, a hyperplane will suffice, i.e. $d=1$. If one is interested in the actual number of intersections, rather than the asymptotic behaviour, the following improvement of Theorem 1 is possible.

Theorem 2

Let C_m and D_n be distinct, irreducible curves in $P^k(\mathbb{C})$ with m, the degree of C_m, less than n, the degree of D_n. Let d and d′ be the degrees of the minimal degree hypersurfaces containing C_m and D_n respectively. Then,

$$|C_m \cap D_n| \leq \begin{cases} nd - (\alpha_{md} - 1) & \text{if } d \neq d' \\ nd - (\alpha_{md} - \alpha_{nd} - 1) & \text{if } d = d' \end{cases}$$

Proof: Abhyankar et al. [3].

Some further refinements of this theorem for curve intersection in 3-dimensional space may be found in [2].

Examples

1. Consider the intersection of a ***cubic curve*** C_3 with curve $D_n (n \geq 4)$ in 2-, 3- and 4-dimensional space. In 2-space, that is in the plane, C_3 can intersect D_n in as many as 3n points (Bezout's Theorem). In 3-space, C_3 can be embedded in a quadric surface S_2 and therefore D_n intersects C_n in no more than 2n points. In 4-space, any C_3 lies on a hyperplane and therefore intersects D_n in no more than n points.

2. *Space Quintic (m=5)*

In 3-space consider the intersection of an irreducible quintic space curve, C_5, with an irreducible space curve, D_7 of degree seven. $\alpha_{53}=4$ and C_5 can be embedded in a cubic hypersurface S_3. Since D_7 is a non-special curve, the minimal degree hypersurface on which it lies is an S_4. Hence there exists at least four linearly independent cubic hypersurfaces S_3^1, S_3^2, S_3^3 and S_3^4 which contain C_5 and which intersect D_7 properly. Theorem 1 implies that $|C_5 \cap D_7| \leq 21$. If we apply Theorem 2 to this curve intersection problem, then we get a bound $|C_5 \cap D_7| = (7)(3) - (\alpha_{53}-1) = 18$.

APPLICATIONS

Some applications of the above results are outlined in this section.

1. Embeddings

A space curve can be embedded in a surface of appropriately "low" degree. This is based on the following two facts:

a) The minimal number of points needed to define a hypersurface of degree d in k-space is $\left[\binom{d+k}{k}-1\right]$.

b) An irreducible curve C_m of degree m and a surface S_d of degree intersect in no more than md points, unless C_m lies on S_d.

The technique we outlined to obtain the embedding also gives the equation of the surface. Embeddings are useful in geometric modelling and interpolation problems.

Another specific embedding result is

Lemma In k-space any irreducible curve of degree less than k lies on a hyperplane.

Proof: Consider an irreducible curve C_m of degree m in k-space. If m satisfies inequality (1) above with d equal to one, then C_m lies on a hyperplane. The inequality with d equal to one is

$$\binom{k+1}{k} > m \cdot 1 + 1.$$

that is $k>m$.

So any irreducible curve C_m in $P^k(\mathbb{C})$ lies on a hyperplane, if $m<k$. ■

2. Computations with Space Curves

The proof techniques developed in this paper can be used algorithmically in several ways. In particular, they can be used in the explicit computation of the intersection points and in realizing implicit and parametric representations of curves. These applications are of particular significance (in computer aided geometric design) when the curves are taken to be space curves, i.e. curves in $P^3(\mathbb{C})$.

Intersections

In order to mechanise the proofs of theorems 1 and 2, we need to

- Generate a requisite number of points on C_m
- Construct surfaces S_d^i to contain C_m (and not D_n)
- Compute $(D_n \cap S_d^i)$, points on D_n
- Select points on C_m

These steps are easily accomplished if C_m and D_n are given in rational parametric form. If they are given in implicit form, we can obtain a plane curve parametrization (using the technique of Hoffmann [7]) which can be the basis of explicit computation.

Representations

The surface embedding of a space curve, discussed above, can be the basis of new implicit representations. For example, it is a classical result that every space cubic can be expressed as the solution to three degree two equations (ie the intersection of three quadrics). Using a variation on the embedding proof of theorem 2, we can obtain an explicit construction of the three defining quadric equations for any given space cubic [2].

3. Combinatorial Complexity of Arrangements

A unifying theme in computational geometry is the complexity of arrangements of objects in the plane, in space, etc. (Edelsbrunner [10]). In this approach one considers the complexity of incidences of several objects of bounded algebraic complexity (points, lines, circles, spheres, ...). The asymptotic bounds on this combinatorial complexity have been the focus of considerable attention in recent years (Aranov and Sharir [9], Clarkson et al [11]).

In the problems considered in this paper, we have exactly the reverse situation. We consider the intersection of only two objects at a time. However, each of the objects (curves) is of high algebraic complexity. A natural connection between these problems of improperly intersecting curves and other varieties, on one hand, and combinatorial arrangements, on the other, is obtained by considering a degree m curve to roughly correspond to m lines in space. In this light, the results of this paper may be viewed as an "algebriac" approach to analyzing the complexity of arrangements.

CONCLUSION

The problem of improperly intersecting algebraic curves commonly arises in three and higher dimensional spaces. This has been studied in the present paper with the objective of obtaining bounds on the number of intersection points. Such bounds have been derived using algorithmic algebraic geometry methods. The results should be useful for explicit computations with space curves, which arise in computer aided geometric design.

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