Improper Intersection of Algebraic Curves

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Bezout's theorem gives an upper bound on the degree of the intersection of properly intersecting algebraic varieties. In spaces of dimension higher than two, however, intersections between many algebraic varieties such as curves are improper. Bezout's theorem cannot be directly used to bound the number of points at which these curves intersect. In this paper an algebrogeometric technique is developed for obtaining an upper bound on the number of intersection points of two irreducible algebraic curves in k-dimensional space. The theorems obtained are applied to the specific case of intersecting algebraic space curves in three-dimensional space, and a number of examples are analyzed in this regard. The implications of the derived results for computer-aided geometric design are discussed.

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INTRODUCTION

Algebraic curves are widely used in geometric modeling. They include, as special cases, Bezier curves, Hermite interpolants, splines of various kinds, and intersection curves of algebraic surfaces. An important problem in computer-aided geometric design is to determine tight bounds on the number of intersection points between two algebraic space curves and to develop efficient algorithms for finding these points [13, 14, 17, 19]. Similar problems related to the intersection of trajectories in high-dimensional spaces frequently arise in computational geometry [12], dynamical systems, and control theory [11, 18].

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The intersection problem for algebraic plane curves has been elegantly resolved using classical algebrogeometric techniques [5–7, 19, 20]. These successes may be viewed as straightforward consequences of Bezout's theorem [2, 4, 20] applied to the "proper" intersection of algebraic plane curves. This theorem implies that two algebraic curves of degree m and n can intersect in no more than mn points on a plane. This, in general, provides the least upper bound for plane curves. Such well-defined bounds are hard to derive for curves in higher dimensional space, as Bezout's theorem does not extend to such "improper" intersections [20, 21].

Looking beyond plane curves, one considers the intersection of algebraic space curves, that is, curves in three-dimensional space. A simple example is the intersection of two nonoverlapping space cubics. Using planar projections (and Bezout's theorem) it follows that they can intersect in no more than nine points. Goldman [14] and Chandru and Kochar [10] showed that the actual number of points indeed is no greater than five. Exploiting the rational parameterizability of all space cubics, they also gave constructive methods for obtaining the intersection points.

More recently, Abhyankar, Chandrasekar, and Chandru [9] obtained "tight" bounds for the general problem of intersecting algebraic space curves of arbitrary degree. Asymptotically, they showed that two space curves of degree m and n intersect in $0(\min(m^{1/2}n, mn^{1/2}))$ points.

In this paper we consider the general problem of intersecting algebraic curves in k-dimensional space. "Tight" upper-bound results are obtained using algorithmic, algebrogeometric techniques. The aforementioned results for algebraic *space* curves are derived as specializations of the general results. The implications for computer-aided geometric design are also discussed.

1. DEFINITIONS AND BACKGROUND

We are concerned only with curves and hypersurfaces that are algebraic. Unless otherwise stated, the curves considered are in spaces of dimension higher than two. Consider

K: $f(x, y) = 0$, where f is a polynomial;

S: $g(x, y, z) = 0$, where g is a polynomial; and

H: $h(x_1, x_2, x_3, \ldots, x_k) = 0$, where h is a polynomial.

K and S represent a *plane curve* and a *surface* in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. H represents a hypersurface in $\mathbb{R}^k$. K, S, and H are *irreducible* if f, g, and h, respectively, are irreducible polynomials. Equivalently, K, S, and H do not properly contain two or more curves, surfaces, or hypersurfaces, respectively, of which they are the union.

The definition of an algebraic curve in k-space and its irreducibility is not as straightforward [1–3, 8]. It requires the abstract notion of an algebraic variety. An affine *algebraic variety* in $\mathbb{C}^k$ is simply defined as the set of all common solutions to a system of polynomial equations in k variables.

Let V be a variety in $\mathbb{C}^k$. By a *subvariety* of V we mean an algebraic variety W in $\mathbb{C}^k$ such that W is contained in V. V is said to be reducible if V can be expressed as the union of two subvarieties each of which is nonempty and is different from V. V is said to be *irreducible* if it is nonempty and is not reducible. The *dimension* of V is the largest integer d such that there exists a strictly ascending sequence $V_0, V_1, V_2, \ldots, V_d$ of irreducible subvarieties of V. By strictly ascending we mean that for $i = 2, 3, \ldots, d$ we have that V_{i-1} is contained in V_i and is different from V_i. We note that this definition is consistent with the geometric intuition that a point, curve, and surface are of dimension zero, one, and two, respectively. A *hypersurface* in k-space is a variety of dimension $(k - 1)$. The *codimension* of a variety V in $\mathbb{C}^k$ is $(k - \dim V)$. A variety is said to be *pure* if all of its irreducible components have the same dimension. For example, a curve is a pure one-dimensional object; and a surface, a pure two-dimensional object. Suppose V is a pure d-dimensional variety in k-space. Consider the intersection of V with all linear spaces L_{k-d} of dimension $(k - d)$. Then,

$$\text{degree}(V) = \text{maximum}\left\{\begin{matrix}\text{number of intersections} \\ \text{of } L_{k-d} \text{ and } V\end{matrix} \;\middle|\; |L_{k-d} \cap V| \text{ is finite}\right\}.$$

In k-space this yields the following definition for the degree of a curve C:

$$\text{degree}(C) = \underset{(P \text{ is a hyperplane})}{\text{maximum}} \left\{\begin{matrix}\text{number of intersections} \\ \text{of } P \text{ and } C\end{matrix} \;\middle|\; |P \cap C| \text{ is finite}\right\}.$$

We note that "most" hyperplanes will intersect C in degree(C) points. A purely algebraic definition of degree(C) can also be given in terms of the so-called Hilbert polynomial P_C of C. The interested reader may refer to [1] or [21] for this definition.

Two intersecting pure varieties V_1 and V_2 are said to *intersect properly* provided that

$$co - \dim(V_1 \cap V_2) = co - \dim(V_1) + co - \dim(V_2).$$

Some concrete examples of proper intersections are

(1) $(P_1 \cap P_2)$ in 2-space, where P_1 and P_2 are irreducible plane curves that meet in a finite number of points (see Figure 1);
(2) $(P \cap S)$ in 3-space, where P is an irreducible plane curve, S is an irreducible surface, and they meet in a finite number of points (see Figure 2);
(3) $(C \cap S)$ in 3-space, where C is an irreducible space curve, S is an irreducible surface, and they meet in a finite number of points;
(4) $(S_1 \cap S_2)$ in 4-space, where S_1 and S_2 are irreducible surfaces of dimension two and they meet in a finite number of points; and
(5) $(C \cap H)$ in k-space, where C is a curve, H is a hypersurface, and they meet in a finite number of points.

It is important to note that the intersection of two irreducible curves C_1 and C_2 in k-space is never proper (for $k \geq 3$).

1.1 Bezout's Theorem

Let V_1 and V_2 be two pure varieties intersecting properly. Then,

$$\text{degree}(V_1 \cap V_2) \leq \text{degree}(V_1) \cdot \text{degree}(V_2),$$

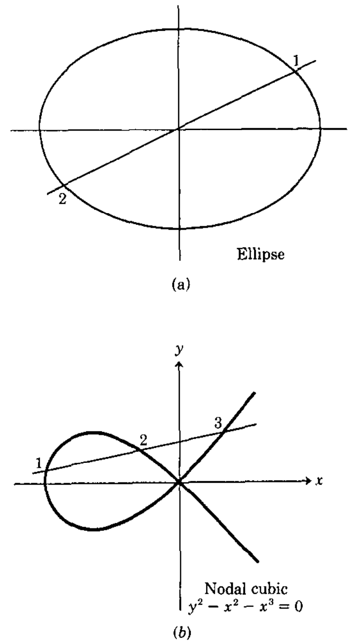

Fig. 1. Intersections between curves in the plane also illustrating Bezout's theorem. (a) A line (degree-one) intersects a degree-two curve, an ellipse, at two points. (b) A line intersects a degree-three curve, a nodal cubic, at three points. Note that some lines may intersect curves at infinity (e.g., the y-axis in (b)), in the complex plane, or more than once at a given point.

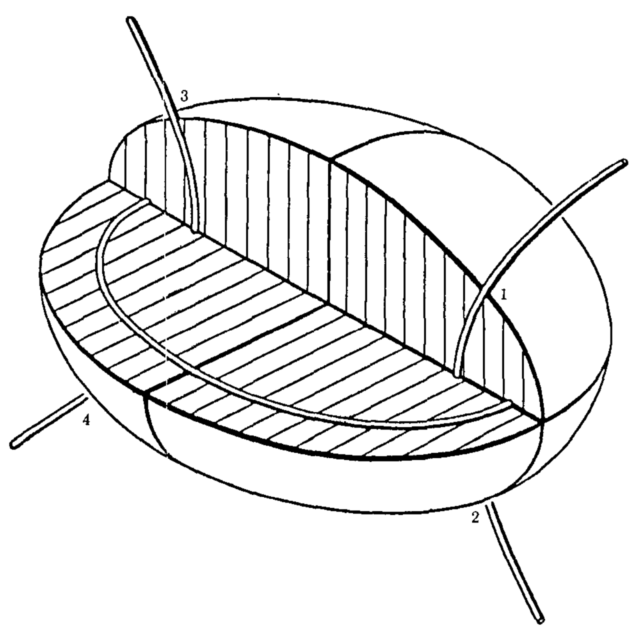

Fig. 2. The proper intersection between a hyperbola (degree two) and an ellipsoid (degree two) in 3-space (after [16]). Note that there are four points of intersection between the curve and the surface, which is equal to the product of their degrees, again illustrating Bezout's theorem.

and = holds in complex projective k-space, $P^k(\mathbb{C})$, if the intersection degree is counted with "appropriate" multiplicity.

When V_1 and V_2 are solids (hypersurfaces) in 4-space, then degree($V_1 \cap V_2$) is, in general, the degree of the intersection surface of codimension two. If V_1 and V_2 are hypersurfaces in k-space, then degree($V_1 \cap V_2$) is the degree of the intersection variety of codimension two.

Bezout's theorem may be regarded as one of the central results of algebraic geometry. It has recently also been the focus of considerable interest in the area of computer-aided geometric design and robotics [13, 14, 17]. Elimination techniques that played an important role in classical proofs of this theorem have enabled development of algorithmic techniques in these applied areas.

As was noted above, intersections of curves in n-space do not fall in the class of proper intersections, and Bezout's theorem therefore has little to say directly about them. In k-space an indirect approach is to project the two space curves C and D onto a common plane and then invoke Bezout's theorem for the "shadow" plane curves. As projection preserves intersection points, we would obtain a valid upper bound on the number of intersection points of C and D. However, we may also expect this bound to be loose, as many spurious intersection points result from projections. Thus, for example, for two space quintics (degree five) in 3-space this technique yields a bound of 25 on the intersection points, whereas the true value can be no greater than 13 [9]. Other such examples are discussed in [9], where "tight" bounds are derived by the authors for the intersection of curves in 3-space. This motivated the generalization to curves in k-space, which is the focus of this paper.

2. EMBEDDING A CURVE IN A HYPERSURFACE

We first examine some implications of the following combinatorial formula:

PROPOSITION 1. *The minimum number of points required to define a hypersurface of degree d in k-space is*

$$\left[\binom{d+k}{k} - 1\right].$$

PROOF. See, for example, [15] and [20]. □

In particular, this proposition implies that there always exists a hypersurface S_d of degree d in $P^k(\mathbb{C})$ containing any collection of

$$\left[\binom{d+k}{k} - 1\right]$$

points. Consider now a curve C_m of degree m also in $P^k(\mathbb{C})$. By Bezout's theorem, $| C_m \cap S_d |$ is either $\leq md$, or C_m and S_d have a common component. Furthermore, if C_m is irreducible and $| C_m \cap S_d |$ is greater than md, then C_m lies on S_d. These observations lead to a general technique for embedding any curve in a suitably "low"-degree hypersurface.

2.1 Examples

(1) In $P^3(\mathbb{C})$, an irreducible C_3 can always be embedded in an S_2. By Proposition 1 there exists an S_2 containing any seven points. Given C_3 we can therefore choose any seven distinct points on it and construct an S_2 containing them. Now C_3 intersects this S_2 in at least seven points. But by Bezout's theorem if $|C_3 \cap S_2| > 6$ then C_3 lies on S_2, since C_3 is irreducible. Hence, the constructed S_2 contains C_3. All irreducible space cubics in 3-space therefore lie on a quadric surface.

(2) In $P^4(\mathbb{C})$, an irreducible C_3 can always be embedded in a hypersurface S_1. Proposition 1 implies that in 4-space the number of points required to define an S_1 is four. Again, by Bezout's theorem if $|C_3 \cap S_1| > 3 \cdot 1$ then C_3 lies on S_1. We can choose all the four points required to construct an S_1 on C_3. Thus, a cubic curve in 4-space always lies on a hyperplane.

In general, using the reasoning illustrated above, it is always possible to embed a curve C_m on a hypersurface S_d in $P^k(\mathbb{C})$ by choosing d to be the smallest positive integer satisfying the inequality

$$\binom{d+k}{k} > md + 1. \tag{1}$$

LEMMA 1. *In k-space any irreducible curve of degree less than k lies on a hyperplane.*

PROOF. Consider an irreducible curve C_m of degree m in k-space. If m satisfies inequality (1) above with d equal to one, then C_m lies on a hyperplane. The inequality with d equal to one is

$$\binom{k+1}{k} > m \cdot 1 + 1;$$

that is, $k > m$. So any irreducible curve C_m in $P^k(\mathbb{C})$ lies on a hyperplane, if $m < k$. □

Similar proofs appear in basic algebraic geometry texts; see, for example, [15].

Lemma 1 includes the well-known fact that all irreducible degree-two space curves in 3-space are actually conics.

A stronger lemma along the same lines is as follows:

LEMMA 2. *Let V be an irreducible algebraic variety in $P^k(\mathbb{C})$. Then the following relation holds:*

$$degree(V) + dimension(V) \geq r(V),$$

where $r(V)$ = dimension of the least-dimensional linear subspace containing V.

PROOF. See [1]. □

Remarks

(1) For "most" irreducible curves C_m, the construction using inequality (1) yields the minimum-degree hypersurface S_d containing them.

(2) The hypersurfaces S_d so constructed may sometimes be reducible. In this case, of course, C_m lies on a hypersurface of degree smaller than d.
(3) Lemma 1 implies that in high-dimensional space "many" curves lie on hyperplanes.

We may now formulate a heuristic for bounding the number of intersections of two curves C_m and D_n in $P^k(\mathbb{C})$. Using the construction described above, we would first obtain a hypersurface S_d containing C_m. Applying Bezout's theorem we can determine the number of intersection points between S_d and D_n. This number will bound from above the number of intersection points between C_m and D_n. This heuristic may run into the difficulty that S_d also contains D_n, whence we would obtain a trivial upper bound of infinity. In order to get around this difficulty, we need to develop a technique for constructing S_d containing C_m such that S_d intersects D_n properly. In the given space $P^k(\mathbb{C})$, let

$$\alpha_{md} = \binom{d+k}{k} - md - 1.$$

In the discussion above, we have always chosen d, the degree of S_d, to be the smallest positive integer such that α_{md} is positive. The natural number α_{md} then represents the dimension of the vector space of hypersurfaces of degree d that contain the given curve C_m of degree m. Such an S_d is said to be a *minimal-degree surface* for C_m. A curve C_m is said to be *special* if $C_m \subset S_{d'}$, for some $d' < d$. Most curves are *nonspecial*. Unless otherwise stated, the rest of this paper will be concerned with nonspecial irreducible curves in $P^k(\mathbb{C})$.

PROPOSITION 2. *Let C_m and D_n be two distinct, irreducible, nonspecial algebraic curves in k-space with $m < n$. Then there exists a hypersurface S_d of degree d such that S_d contains C_m and S_d intersects D_n properly. Consequently,* $|C_m \cap D_n| \leq |S_d \cap D_n| \leq nd$.

PROOF. Let d and d' be the smallest integers for which α_{md} and $\alpha_{nd'}$ are positive. The dimensions of the vector spaces of hypersurfaces of degree d and d' that contain C_m and D_n are, respectively, at least equal to α_{md} and $\alpha_{nd'}$. Consider the following two cases:

Case 1. *d is not equal to d'.* There exists a hypersurface S_d of degree d that contains C_m. D_n does not lie on S_d because it is nonspecial and the least-degree hypersurface $S_{d'}$ on which it lies has degree d', $d' > d$. Hence, this S_d intersects D_n properly.

Case 2. *d is equal to d'.* α_{md} is greater than α_{nd} because $m < n$. That is, the dimension of the vector space of hypersurfaces S_d of degree d that contain C_m is greater than the dimension of the corresponding vector space of degree-d hypersurfaces containing D_n. Hence, there certainly exists at least one hypersurface S_d that contains C_m and intersects D_n properly. □

Remarks

(1) Since C_m and D_n are nonspecial curves, the hypersurfaces S_d and $S_{d'}$ considered above are both irreducible.

(2) The proposition also holds if m and n are equal, provided D_n does not lie in the intersection of all degree-d hypersurfaces containing C_m. An irreducible curve D_n that lies in the intersection of all degree-d hypersurfaces containing C_m is said to be a *sibling* of C_m. Thus, Proposition 2 holds even if $m = n$ provided that C_m and D_n are not siblings.

(3) In 3-space, that is, when C_m is a space curve, the number of siblings of C_m is finite whenever $\alpha_{md} \geq 2$. In particular, the total degree of the siblings can be no larger than $(d^2 - m)$ [9]. Finding similar bounds on the number of siblings in higher dimensions ($k \geq 4$) appears to be a challenging problem.

(4) Even when C_m and D_n are siblings, it is always possible to find a hypersurface S containing C_m and not D_n. This follows from the ideal theoretical definition of these curves. A purely geometric construction of such a hypersurface can also be given. It is possible to construct a cone K that contains C_m and not D_n, with degree(K) a factor of m. Pick a point x on D_n, but not on C_m. Define K', the cone with apex at x and containing C_m. Now pick a point y outside K'. Construct the cone with apex at y and containing C_m. This cone K cannot contain D_n, since if it did the line xy would be a line of both K and K'. This contradicts our choice of y outside K'. It can be shown that degree(K) is a factor of m [20, 21]. Finding a minimal-degree hypersurface S containing C_m but not D_n is an interesting problem for future research.

Examples

(1) *Space quintic* ($m = 5$). Consider a quintic space curve, C_5, in 3-space. C_5 can be embedded in a cubic surface $S_3(d = 3)$. C_5 will therefore intersect any space curve, D_n, of degree n ($n > 5$) in no more than $3n$ points. This bound is significantly lower than the bound of $5n$ derivable from projection arguments using Bezout's theorem.

(2) *Cubic curves* ($m = 3$). It is instructive to look at the intersection of a cubic curve, C_3, with curves of degree n ($n > 3$), D_n, in 2-, 3-, and 4-space. In 2-space, that is, in the plane, C_3 intersects any D_n also lying in the same plane in $3n$ points. This follows from Bezout's theorem for the plane. In 3-space C_3 can be embedded in a degree-two surface S_2, and therefore, D_n intersects C_3 in no more than $2n$ points. In 4-space any C_3 lies on a hyperplane (Lemma 1) and therefore intersects D_n in no more than n points. This example brings out the fact that, as we go to higher dimensions, two space curves will tend to intersect less and less.

Proposition 2 gives us *sufficient conditions* under which we can obtain a bound of (nd) on the number of intersection points between C_m and D_n in k-space. We now consider some of the asymptotic effects of this bound.

2.2 Asymptotic Analysis

We know that in k-space any curve C_m of degree m can be embedded in a hypersurface S_d of minimal degree d. Recall that d is the smallest positive integer such that $\alpha_{md} \geq 1$; that is,

$$\frac{(d+1)(d+2)\cdots(d+k-1)(d+k)}{k!} - 1 - md \geq 1.$$

A simple asymptotic analysis shows that d grows as $(k!\ m)^{1/(k-1)}$. All curves D_n of degree n, where n is greater than m, will intersect this minimal-degree hypersurface, S_d, properly. In fact, any D_m that is not a *sibling* of C_m will also intersect S_d properly. We have proved the following:

THEOREM 1. *Let C_m be any nonspecial and irreducible curve of degree m in k-space. Then all irreducible nonspecial curves D_n of degree n, where n is larger than m, will intersect C_m in $O(m^{1/(k-1)}n)$ points. The result is also true when n is equal to m, provided D_n is not a sibling of C_m. In the limit, as k tends to infinity, this bound tends to n.*

The theorem brings out clearly the intuitive observation that the number of possible intersections must decrease as the dimension of the space increases. A challenging problem is to construct greatest lower bounds on the number of intersections. In [9] we argue that $0(n)$ is a valid lower bound for curves in 3-space. The argument extends to k-space for arbitrary k.

3. TIGHTER INTERSECTION BOUNDS

In the previous sections, we showed that two distinct irreducible curves C_m and D_n in k-space (with minor restrictions) can intersect each other in no more than nd points, where d is the smallest positive integer satisfying the inequality

$$\binom{d+k}{k} > md + 1.$$

This is really a Bezout-type theorem for algebraic curves and generalizes the results derived in [9] for space curves.

It is possible to obtain tighter bounds on the number of intersection points between curves C_m and D_n satisfying the assumptions of Proposition 2, by exploiting further the techniques discussed above. We illustrate the approach used to tighten the intersection bounds with an example. The general theorem is then derived.

Example. Space quintic ($m = 5$). In 3-space consider the intersection of an irreducible quintic space curve, C_5, with an irreducible space curve, D_7, of degree seven. $\alpha_{53} = 4$ and C_5 can be embedded in a cubic hypersurface S_3. Since D_7 is a nonspecial curve, the minimal-degree hypersurface on which it lies is an S_4. Hence, there exist at least four linearly independent cubic hypersurfaces S_3^1, S_3^2, S_3^3, and S_3^4 that contain C_5 and that intersect D_7 properly. Proposition 2 implies that $|C_5 \cap D_7| \le 21$. Suppose $|C_5 \cap D_7| \ge 19$. Let q_1, q_2, and q_3 be three points belonging to $D_7 \backslash C_5$. Then there exist constants a_1, a_2, a_3, and a_4 such that q_1, q_2, and q_3 lie on the cubic hypersurface

$$T_3 = a_1 S_3^1 + a_2 S_3^2 + a_3 S_3^3 + a_4 S_3^4.$$

Now $|D_7 \cap T_3|$ is at least 22, and therefore, Bezout's theorem implies that $D_7 \subseteq T_3$. This yields a contradiction since the minimal-degree hypersurface containing D_7 has degree equal to 4. Therefore, our assumption that $(D_7 \cap C_5)$ is at least 19 is not possible. Hence, $|D_7 \cap C_5| \le 18$.

The above argument can be generalized as follows: Let C_m and D_n be distinct irreducible curves of degree m and n, respectively, in k-space, with $n > m$. C_m can be embedded in a minimal-degree hypersurface of degree d that intersects D_n properly. α_{md} is the dimension of the vector space of degree-d hypersurfaces containing C_m. In a similar manner, D_n can be embedded in a minimal-degree hypersurface of degree d'. Two cases need to be considered:

Case 1. *d is not equal to d'.* In fact, here, $d < d'$ for $m < n$. Since the dimension of the vector space of hypersurfaces of degree d in $P^k(\mathbb{C})$ containing C_m is α_{md}, there exist linearly independent hypersurfaces $S_d^1, S_d^2, S_3^3, \ldots, S_d^{\alpha_{md}}$, such that

$$C_m \subseteq S_d^1 \cap S_d^2 \cap S_d^3 \cap \cdots \cap S_d^{\alpha_{md}}.$$

Each of these hypersurfaces intersects with D_n properly. Suppose $|C_m \cap D_n| \geq nd - (\alpha_{md} - 2)$. Consider a set of points $q_1, q_2, \ldots, q_{(\alpha_{md}-1)}$ belonging to $D_n \backslash C_m$. There exist constants $a_1, a_2, \ldots, a_{\alpha_{md}}$ such that the above set of points is contained in the following degree-d hypersurface:

$$T_d = \sum_{i=1}^{\alpha_{md}} a_i S_d^i.$$

C_m is certainly contained in T_d. Therefore, $|D_n \cap T_d|$ is at least equal to $(nd - (\alpha_{md} - 2) + (\alpha_{md} - 1))$, that is, $(nd + 1)$ points. By Bezout's theorem, $D_n \subseteq T_d$. But the minimal-degree hypersurface containing D_n has degree d' greater than d. This yields a contradiction. Therefore, $|C_m \cap D_n| \leq (nd - (\alpha_{md} - 1))$.

Case 2. *d is equal to d'.* Let α_{md} and α_{nd} be the dimension of the vector space of hypersurfaces containing C_m and D_n, respectively. Since $n > m$, $\alpha_{md} > \alpha_{nd}$. Therefore, there exist at least $(\alpha_{md} - \alpha_{nd})$ linearly independent hypersurfaces of degree d that contain C_m and that intersect D_n properly. Following through the arguments presented in Case 1, with the number $(\alpha_{md} - \alpha_{nd})$ playing a role similar to α_{md} in Case 1, we find that

$$|C_m \cap D_n| \leq nd - (\alpha_{md} - \alpha_{nd} - 1).$$

We have proved the following tighter bound theorem for the intersection of algebraic curves:

THEOREM 2. *Let C_m and D_n be distinct irreducible algebraic curves in $P^k(\mathbb{C})$ with $m < n$. Let d and d' be the degree of the minimal-degree hypersurfaces containing C_m and D_n, respectively.*

Case 1. *If d is not equal to d', then C_m and D_n can intersect in at most $(nd - (\alpha_{md} - 1))$ points.*

Case 2. *If d is equal to d', then C_m and D_n can intersect in at most $(nd - (\alpha_{md} - \alpha_{nd} - 1))$ points.*

3.1 Space Curves

We prove the following theorem for the intersection of space curves in [9]:

Let C_m and D_n be distinct irreducible space curves in $P^3(\mathbb{C})$ satisfying (*)

(*a*) $n > d^2 - m$ *and*
(*b*) $\alpha_{md} \geq 2$,

where d is the minimal degree of a surface S_d containing C_m. Then C_m and D_n intersect in at most $(nd - (\alpha_{md} - 2))$ points.

Note that (*) is strictly subsumed by Theorem 2 with two exceptions: (1) $C_3 \cap D_3$ and (2) $C_5 \cap D_5$. Furthermore, the bound in Theorem 2 is smaller than that of (*) by 1.

4. IMPLICATIONS FOR COMPUTER-AIDED GEOMETRIC DESIGN

The efficient computation of intersections of curves and surfaces in two- and three-dimensional spaces is of fundamental importance in computer-aided geometric design. The constructions presented thus far in this paper were used for proving *upper bounds* on the number of intersection points of two curves. We now discuss the possibility of using these constructive arguments to explicitly compute the intersection set of two algebraic space curves.

The fact that the representation of the given algebraic space curves C_m and D_n is not uniformly specified makes it difficult to present a totally unified discussion of the computational issues. At an abstract level, however, it is clear that the main steps of an algorithm would be to

(a) generate a requisite number of points on C_m,
(b) construct one or more of the surfaces S_d^i to contain C_m (and not D_n),
(c) compute the intersection points in $(D_n \cap S_d^i)$, and
(d) parse the candidates from step (c) to obtain the true intersection points in $(C_m \cap D_n)$.

The two representation schemes most commonly used for space curves are (1) rational (polynomial) parametric and (2) implicit. Let us first consider the rational parametric case. Assume that C_m and D_n are defined by rational parametric forms in parameter t and s, respectively. To generate points on C_m (step (a)), we simply choose specific values of the parameter t. To construct the surfaces S_d^i (step (b)), we solve systems of linear equations, whose size depends of course on the degree d. To compute $(D_n \cap S_d^i)$ (step (c)), we substitute the parametric form of D_n in the equation for S_d^i and solve for the roots of the resulting univariate polynomial in s. To detect the true intersection points (step (d)), we solve inversion problems on the parametric representation of C_m. There are well-known techniques for all of these computations (cf. [19]).

In some applications, each of the space curves C_m and D_n may be given in implicit form as the intersection of two or more surfaces. In this case there are several possibilities for carrying out the steps of the algorithm. We may decide to avoid steps (a) and (b) altogether by using the defining surfaces in selecting S_d^i. Alternatively, we can use a parametric plane (surface) along with the defining equation and apply elimination techniques to generate points on C_m. The final choice, and perhaps the best one, would be to realize a plane curve parameterization of the space curve (one always exists [3]). This parametric form can be used to generate the requisite points on C_m. Hoffman [17] discusses an algorithm for realizing the plane curve parameterization of a space curve. This parameterization technique would also be useful in computing $(D_n \cap S_d^i)$ in step (c) if D_n is given in implicit form.

The attentive reader may have noticed that we have glossed over certain technical issues in the description above. We have, for example, not discussed how to ensure that the constructed S_d^i intersect D_n properly and that C_m and D_n are nonspecial and irreducible. In most practical situations, these considerations may be irrelevant. However, we would like to propose that some of these algorithmic issues are worthy of attention in future research. Finally, we would also like to point out that the algorithmic realization of a "low-degree" surface embedding of a given algebraic space curve (steps (a) and (b) above) is of independent interest.

5. CONCLUSION

Improper intersections between algebraic varieties are quite common in high-dimensional spaces and therefore necessitate study. In the present study, we have obtained an upper bound on the number of intersection points of irreducible algebraic curves in k-dimensional space ($k > 2$) where their intersection is always improper. The bound derived is only a function of the degrees of the individual curves. Our earlier results for algebraic space curves [9] have been easily derived from the present results. Many of the ideas used in the proofs are quite algorithmic in nature, and therefore, several of the steps involved are amenable to explicit computation. This is especially useful for computer-aided geometric design. It is our hope that these ideas developed for curves can be made completely algorithmic and can also be extended to the study of other types of improperly intersecting algebraic varieties.

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