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Intersection of algebraic space curves

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Abstract

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Bezout's theorem gives the degree of intersection of two properly intersecting algebraic varieties. As two irreducible algebraic space curves never intersect properly, Bezout's theorem cannot be directly used to bound the number of intersections of such curves. A general technique is developed in this paper for bounding the maximum number of intersection points of two irreducible space curves. The bound derived is a function of only the degrees of the respective curves. A number of special cases of this intersection problem for low degree curves are studied in some detail.

1. Introduction

Recent research in geometric modeling with curves and surfaces has focussed on the value of *algebro-geometric* techniques [5-7,10,11,13,15,18]. The early contributions in this context showed the applicability of *elimination techniques*, *Bezout's theorem*, and the *resolution of singularities* in the realization of improved

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algorithms for computing parametrizations, implicitizations, inversions and intersections of rational plane curves and rational surfaces.

Algebraic space curves are widely used in computer aided geometric design. These curves include Hermite interpolants, splines of various kinds and those arising from intersections of two or more algebraic surfaces. Two interrelated topics involving algebraic space curves that are of both mathematical and computational interest are *representation* and *intersection*.

Representation issues which have been addressed include problems such as finding the minimum number of equations needed to define an algebraic space curve in affine and projective three-space as well as the degrees of these defining equations. Also relevant are the problems of determining when there exist rational (polynomial) parametric representations of such curves. These parametrization issues are well solved for the case of algebraic plane curves and to a lesser extent for the case of algebraic surfaces [5–7]. The resolution of singularities of plane curves and surfaces [1] plays a key role in these solutions.

Consider the intersection of two nonoverlapping algebraic plane curves (in the projective plane). Bezout's theorem provides a complete answer to the problem of counting the number of intersection points since it implies that plane curves of degree m and n respectively intersect in exactly mn points (when counted appropriately). At present, no analogous theorems are known for the intersection of arbitrary algebraic space curves. For the special case of two rational cubic space curves it has been shown [10,13] that there are no more than five points of intersection and algorithms for determining the intersection set are given in the cited papers.

In a recent paper [8] we considered the general improper intersection of algebraic curves in k-dimensional space and obtained some bounds on the number of intersection points. In this paper we consider the problem of intersecting algebraic space curves, that is curves in 3-dimensional space, and present a general technique for bounding the number of intersections of two algebraic space curves of arbitrary degree.

The broad approach is to embed one of the space curves in appropriate low degree algebraic surfaces and then, using a version of Bezout's theorem, to bound the cardinality of the intersection set. The intersection bound theorems obtained are more general than those obtained in [8] because of the use of alternative proof techniques for curves in 3-space. The representation issues play an important role even in the problem of counting intersections. We believe that this approach could ultimately lead to analogues of Bezout's theorem for improper intersections of algebraic varieties.

The organization of the paper is as follows. Section 2 contains the definitions and related background results. In Section 3 we present the general technique for embedding a curve on a surface and for obtaining bounds on the number of intersection points. We discuss a technique for tightening these bounds in Section 4. Section 5 considers computational issues related to the constructions presented in earlier sections. Finally, in an appendix we use some of the ideas developed for the intersection problem to show that every irreducible space cubic can be constructed as the exact intersection of three quadric surfaces.

2. Definitions and background

2.1. Representation

We are concerned only with curves and surfaces that are algebraic.

Consider,

$$K\colon f(x,y)=0 \quad \text{where } f \text{ is a polynomial,}$$

$$S\colon g(x,y,z)=0 \quad \text{where } g \text{ is a polynomial.}$$

K and S represent a *plane curve* and a *surface* in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. K and S are *irreducible* if f and g respectively are irreducible polynomials. Equivalently K and S do not properly contain two or more curves or surfaces respectively of which they are the union.

The definition of a *space curve* and its irreducibility is not as straightforward. On the one hand, we may consider some space curves in rational parametric form expressed by

$$x=x(t), \quad y=y(t), \quad z=z(t)$$

where $x(\cdot)$, $y(\cdot)$, $z(\cdot)$ are rational functions.

However, not all space curves are rational. One precise definition of a space curve uses the idea of parametrizing it by a plane curve:

$$x=\lambda(s,t), \qquad y=\mu(s,t), \qquad y=\nu(s,t), \qquad \gamma(s,t)=0 \tag{$*$}$$

where λ, μ, ν are rational functions and γ is a polynomial. A space curve defined by $(*)$ is *irreducible* if the polynomial γ is irreducible.

All of the above definitions may also be applied to curves and surfaces in projective two- and three-space, i.e., $P^2(\mathbb{C})$ and $P^3(\mathbb{C})$ while keeping in mind that all defining polynomials would be of homogeneous degree in this case.

Consider two polynomials $f(x,y,z)$ and $g(x,y,z)$ having no factor in common. The locus of common zeros of these two polynomials, i.e., the intersection of the two surfaces, is a finite union of irreducible space curves. The question arises whether each of these irreducible space curves is again the intersection of precisely two surfaces.

Around 1885, Kronecker proved that four surfaces are always enough to represent any irreducible space curve. In 1964, Kneser [16] sharpened this result by proving that in fact three surfaces suffice. The question as to whether two suffice remains open. A stronger version of this problem has been formulated in idealtheoretic terms and studied by Abhyankar [2–4], Abhyankar and Sathaye [9], Murthy and Towber [17] among others. The above question can be formulated for curves in projective three-space as well. There it is easy to see that a nonsingular, irreducible curve is not in general realized as the intersection of just two surfaces [3].

2.2. Degree

The *degree* (order) of a plane curve K defined by the root locus of the polynomial $f(x,y)$ is simply the degree of f. Alternatively,

$$\text{degree}(K) = \underset{(l \text{ is a line})}{\text{maximum}}\{\text{number of intersections of } l \text{ and } K \mid |l \cap K| \text{ is finite}\}.$$

The latter definition can be extended to define the degree of a space curve C as follows,

$$\text{degree}(C) = \underset{(P \text{ is a plane})}{\text{maximum}}\{\text{number of intersections of } P \text{ and } C \mid |P \cap C| \text{ is finite}\}.$$

We note that "most" planes will intersect C in degree(C) points. An algebraic definition of degree(C) can also be given in terms of the so-called Hilbert polynomial P_C of C. A theorem of Hilbert (cf. [1,23]) states that the Hilbert function $H_C(n)$ of C, which is the number of linearly independent surfaces of degree n containing C, is a polynomial $P_C(n)$ of the form $an+b$ (positive $a,b \in \mathbb{Z}$) for large n. The coefficient a of $P_C(n)$ is precisely degree (C).

So far we have discussed points, curves and surfaces in two- and three-dimensional spaces. Generalization of these concepts to higher dimensions leads to the abstract notion of an algebraic variety. An affine *algebraic variety* in $\mathbb{C}^n$ is simply defined as the set of all common solutions to a system of polynomial equations in n variables. In order to state Bezout's theorem we will need to make precise terms such as irreducible subvarieties, dimensions and proper intersections of varieties.

Let V be a variety of $\mathbb{C}^n$. By a *subvariety* of V we mean an algebraic variety W in $\mathbb{C}^n$ such that W is contained in V. V is said to be reducible if V can be expressed as the union of two subvarieties each of which is nonempty and is different from V. V is said to be *irreducible* if it is nonempty and not reducible. The *dimension* of V is the largest integer d such that there exists a strictly ascending sequence $V_0, V_1, V_2, \ldots, V_d$ of irreducible subvarieties of V. By strictly ascending we mean that for $i = 1, 2, \ldots, d$ we have that V_{i-1} is contained in V_i and different from V_i. We note that this definition is consistent with the geometric intuition that a point, a curve, and a surface are of dimension zero, one and two respectively. A *hypersurface* in n-space is a variety of dimension $n-1$. The *co-dimension* of a variety V in $\mathbb{C}^n$ is $n - \dim V$. A variety is said to be *pure* if all of its irreducible components have the same dimension. For example, a curve is a pure 1-dimensional object and a surface a pure 2-dimensional object. Suppose V is a pure d-dimensional variety in n-space. Consider the intersection of V with all linear spaces, L_{n-d} of dimension $n-d$. Then

$$\text{degree}(V) = \text{maximum}\{\text{number of intersections of } L_{n-d} \text{ and } V \mid |L_{n-d} \cap V| \text{ is finite}\}.$$

Two intersecting pure varieties V_1 and V_2 are said to *intersect properly* provided

$$\text{co-dim}(V_1 \cap V_2) = \text{co-dim}(V_1) + \text{co-dim}(V_2).$$

Some concrete examples of proper intersections are:

(a) $(P_1 \cap P_2)$ where P_1 and P_2 are irreducible plane curves that meet in a finite number of points.

(b) $(P \cap S)$ where P is an irreducible plane curve and S is an irreducible surface and they meet in a finite number of points.

(c) $(C \cap S)$ where C is an irreducible space curve and S an irreducible surface and they meet in a finite number of points.

(d) $(S_1 \cap S_2)$ where S_1 and S_2 are irreducible surfaces and they meet in a finite number of curves.

It is important to note that the intersection of two irreducible space curves C_1 and C_2 is never proper.

Bezout's theorem. *Let V_1 and V_2 be two pure varieties intersecting properly. Then*

$$\text{degree}(V_1 \cap V_2) \leq \text{degree}(V_1) \cdot \text{degree}(V_2)$$

(*and* $=$ *holds in* $P^n(\mathbb{C})$ *if intersections are counted with "appropriate" multiplicity*).

Bezout's theorem may be regarded as one of the central results of algebraic geometry. It has recently also been the focus of considerable interest in the area of computer aided geometric design and robotics [18]. For a discussion of this theorem including proofs, see [20]. Elimination techniques which played an important role in classical proofs of this theorem have enabled development of algorithmic techniques in these applied areas. As was noted above, intersections of space curves do not fall in the class of proper interections and Bezout's theorem therefore has little to say directly about them. An indirect approach is to project the two space curves C and D onto a common plane and then invoke Bezout's theorem for the "shadow" plane curves. As projection preserves intersection points we would obtain a valid upper bound on the number of intersection points of C and D. However, we may also expect this bound to be loose as many spurious intersection points result from projections. Thus for the example of two *space cubics* this technique yields a bound of *nine* whereas, as noted above, the true value is no larger than *five*. These observations provided the motivation for our investigation of the space curves' intersection problem.

3. Embedding a space curve in a surface

We first examine a classical combinatorial formula.

Proposition 3.1. *The minimum number of points needed to define a hyper-surface of degree d in n-space is* $[\binom{d+n}{n}-1]$.

Proof. The number of coefficients of the defining polynomial of a hyper-surface of degree d in n-space is equal to the number of monomials of exactly degree d in $n+1$ variables. This latter number equals the number of combinations of d elements that can be chosen from a selection of $n+1$ distinct elements *with replacement* permitted. This combinatorial identity is precisely $\binom{d+n}{n}$. Thus we may conclude that there are $\binom{d+n}{n}$ coefficients of the defining polynomial for our given hyper-surface. It follows that there is some selection of $[\binom{d+n}{n}-1]$ points on the hyper-surface which yields a system of $[\binom{d+n}{n}-1]$ homogeneous linear equations whose unique solution specifies all coefficient values in the polynomial. □

Other proofs of this proposition appear in standard algebraic geometry texts, see for example Griffiths and Harris [14], and Semple and Roth [19]. In particular, this proposition implies that there always exists a surface S_d of degree d in $P^3(\mathbb{C})$ containing any collection of $[\binom{d+3}{3}-1]$ points. The chosen points will, however, have to be in general position (i.e., the points define a linearly independent system of equations) to uniquely define S_d. For the proofs that follow this is not necessary. Consider now a curve C_m of degree m also in $P^3(\mathbb{C})$. By Bezout's theorem, $|C_m \cap S_d|$ is either md or C_m and S_d have a common component. Furthermore, if C_m is irreducible and $|C_m \cap S_d|$ is greater than md, then C_m lies on S_d. These observations lead to a general technique for embedding any curve in a suitably "low" degree surface.

Examples. (i) An irreducible C_2 can always be embedded in an S_1. By Proposition 3.1 there exists an S_1 containing any three points. Given C_2, we can choose any three distinct points on it and construct an S_1 containing them. Now C_2 intersects S_1 in at least three points. But by Bezout's theorem, if $|C_2 \cap S_1| > 2$, then C_2 lies on S_1 (for C_2 is irreducible). Hence the constructed S_1 contains C_2. This is a proof of the well-known fact that irreducible degree two space curves are actually conics.

(ii) An irreducible C_3 can always be embedded in an S_2. Again by Bezout's theorem, if $|C_3 \cap S_2| > 2 \cdot 3$, then C_3 lies on S_2. Of the nine points needed to construct S_2 we choose seven points on C_3. Thus a cubic space curve always lies on a quadric surface.

In general using the reasoning illustrated above, it is always possible to embed a curve C_m on a surface S_d by choosing the smallest integer d such that it satisfies the inequality

$$\binom{d+3}{3} > md + 1.$$

Remarks. (1) For "most" irreducible curves C_m this construction yields the minimum degree surface S_d containing them.

(2) The surfaces S_d so constructed may sometimes be reducible. In this case, of course, C_m lies on a surface of degree smaller than d.

We may now formulate a heuristic for bounding the number of intersections of two curves C_m and D_n in $P^3(\mathbb{C})$. Using the construction described above we would first obtain a surface S_d containing C_m. Applying Bezout's theorem we can determine the number of intersection points between S_d and D_n. This number will bound from above the number of intersection points between C_m and D_n. This heuristic may occasionally run into the difficulty that S_d also contains D_n whence we would obtain a trivial upper bound of infinity. In order to get around this difficulty we need to develop a technique for constructing S_d containing C_m such that S_d intersects D_n properly. Let

$$\alpha_{md} = \binom{d+3}{3} - md - 1.$$

In the discussion above we have always chosen d, the degree of S_d, to be such that α_{md} is a positive integer. A space curve C_m is said to be special if $C_m \subset S_{d'}$ for some $d' < d$. Most curves are nonspecial. Unless otherwise stated, the rest of this paper will be concerned with nonspecial irreducible curves.

Proposition 3.2. *Let C_m and D_n be two distinct, irreducible, algebraic space curves in $P^3(\mathbb{C})$, with C_m a nonspecial curve. If conditions* (a) *and* (b) *below hold, then there always exists a surface S_d of degree d such that S_d contains C_m and S_d intersects D_n properly.*

(a) $\alpha_{md} \geq 2$,

(b) $n > d^2 - m$.

Consequently, $|C_m \cap D_n| \leq |S_d \cap D_n| = nd$.

Proof. Consider the vector space of all surfaces of degree d in $P^3(\mathbb{C})$ that contain C_m. The rank of this space is precisely α_{md}. Therefore condition (a) implies that there exist at least two linearly independent surfaces S_d^1 and S_d^2 that contain C_m. If neither S_d^1 nor S_d^2 intersects D_n properly, then D_n lies on both (since D_n is irreducible). Therefore $(S_d^1 \cap S_d^2)$ is of degree at least $m+n$. However, Bezout's theorem implies that the degree of $(S_d^1 \cap S_d^2)$ is no larger than d^2. These two observations are in conflict since (b) implies that $m+n$ is larger than d^2. □

It is necessary that C_m be nonspecial for Proposition 3.2 to hold. For if C_m is special, then the surface S_d constructed above may be reducible and a component of S_d could contain both C_m and D_n; this would make the intersection between S_d and D_n improper.

Examples (*Space cubic* ($m=3$)). For an irreducible and nonplanar cubic space curve C_3 it follows that the minimum degree surface in which it can be embedded is a quadric, i.e., S_2. Since for this case α_{32} equals 3 and d^2-m equals 1 we can apply Proposition 3.2 to choose an S_2 that intersects properly with D_n for n greater than or equal to 2. Hence C_3 and D_n will intersect in no more than $2n$ points for $n \geq 2$. In particular, C_3 and D_3 meet in no more than six points.

(*Space quintic* ($m=5$)). In this case C_5 can be embedded in a cubic surface S_3 ($d=3$) such that α_{53} equals 4. Proposition 3.2 applies as long as n is 5 or larger (d^2-m is 4). Thus two space quintics intersect in no more than 15 points.

The proposition gives us *sufficient conditions* under which we obtain a bound of nd on the number of intersection points of C_m and D_n. The asymptotic effects of this bound will be discussed below. First, however, let us examine the assumptions (a) and (b) in that order. As we shall see, the former is not restrictive at all and the latter is only mildly so.

Lemma 3.3. $\alpha_{md}=1$ *if and only if one of the following holds,*
(δ) $m=2$; $d=1$ (*conic* C_2 *on plane* S_1),
(β) $m=4$; $d=2$ (*quartic* C_4 *on quadric* S_2),
(γ) $m=6$; $d=3$ (*sextic* C_6 *on cubic* S_3).

Proof. The definition of α_{md} yields the following equation that is equivalent to fixing α_{md} at 1.

$$6md = d^3 + 6d^2 + 11d - 6.$$

The left-hand side is integer and hence so is the right-hand side. Further the left-hand side is divisible by d and so are the first three terms of the sum on the right-hand side. Hence six must be divisible by the positive integer d. This yields $d=1$, 2, 3 or 6 and the first three possibilities define the three cases (δ), (β) and (γ) of the lemma. To see that $d=6$ is impossible note that $\alpha_{m6}=1$ yields a nonintegral value for m. □

We note that the cases (δ), (β) and (γ) of the lemma are amenable to direct analysis even though Proposition 3.2 does not apply. In case (δ) if the curve D_n happens not to lie on S_1, then $(S_1 \cap D_n)$ is a proper intersection. If D_n lies on S_1, then C_m and D_n are both curves in the same plane and Bezout's theorem can be directly applied to bound their intersection cardinality. In case (β), $\alpha_{4,2} = 1$ and this means that the vector space of linearly independent quadric surfaces S_2 that contain the nonspecial curve C_4 is 1. Since $\alpha_{43} = 7$ there exist seven linearly independent cubic surfaces that contain C_4. If we can show that at least two of these seven surfaces are irreducible, then we have an embedding of C_4 in a cubic surface which does not contain D_n ($n \geq 6$) and a bound of $3n$ for the cardinality of $(C_4 \cap D_n)$ is obtained. Suppose at least six of the above seven cubic surfaces are reducible. Since C_4 is nonspecial and the least degree surface on which it lies is a quadric surface, each of these six linearly independent reducible cubic surfaces contain a plane and a quadric as their irreducible components, with the C_4 lying on the quadric. But the vector space of planes in 3-space has dimension 3 and $\sigma_{42} = 1$ and therefore the above six cubic surfaces cannot be linearly independent; a contradiction. Hence it follows that the number of linearly independent reducible cubic surfaces that contain C_4 is strictly less than six. So C_4 lies on at least two irreducible distinct cubic surfaces and by looking at the degrees of the intersection of these two surfaces it is easily seen that D_n ($n \geq 6$) does not lie completely on at least one of these surfaces. In fact since C_4 lies on at least one irreducible quadric and an irreducible cubic surface, D_n does not lie on at least one of these surfaces for $n \geq 3$. Using Bezout's theorem a bound of $3n$ is obtained for $|C_4 \cap D_n|$ for $n \geq 3$. The argument is exactly the same for case (γ) where a bound of $4n$ follows for $|C_6 \cap D_n|$, $n \geq 7$. It is also possible to arrive at similar conclusions using the classification of quartic and sextic curves given in [19]. But that approach works only for nonsingular curves.

Now let us examine assumption (b) of Proposition 3.2. It dictates that the intersection bound of nd for C_m and D_n is valid when n is chosen larger than $d^2 - m$. For small values of m the resulting value of d (so that $\alpha_{md} \geq 2$) is such that this choice of n is not restrictive. However, a simple asymptotic analysis of ($\alpha_{md} \geq 2$) shows that d grows as $(6m)^{1/2}$. Therefore asymptotically, Proposition 3.2 applies only for situations where n is larger than $5m$. However, it is important to note that for "most" choices of S_d, D_n will meet it in a proper intersection.

We define a *sibling* of C_m to be an irreducible curve, distinct from C_m, which lies in the intersection of all degree d surfaces containing C_m. In view of the discussions following Lemma 3.3, the number of linearly independent degree d surfaces containing C_m can be taken to be at least two except for the specific cases covered by Lemma 3.3. The degree of the intersection of two of these surfaces is d^2. Therefore $d^2 - m$ is an upper bound on the sum of the degrees of the siblings of C_m. Hence, the number of siblings of C_m is finite. We have proved the following.

Theorem 3.4. *Let C_m be any nonspecial and irreducible space curve of degree m. Then all irreducible space curves D_n, distinct from C_m and its siblings, intersect C_m in $O(m^{1/2}n)$ points.*

This is really a Bezout-type theorem for algebraic space curves.

4. Tighter bounds

In the previous sections we showed that two distinct irreducible space curves C_m and D_n (with minor restrictions) can intersect in no more than nd points, where d is the smallest positive integer satisfying the inequality

$$\binom{d+3}{3} > md + 1.$$

We now refine some of the techniques discussed above to obtain tighter bounds on the number of intersection points between space curves C_m and D_n meeting the assumptions of Proposition 3.2. Two examples involving space cubics and quintics will be used to motivate the general discussion.

4.1. Space cubics ($m = n = 3$)

Consider two irreducible curves C_3 and D_3. As shown earlier there exists a quadric surface $S_2 \supseteq C_3$ which intersects D_3 properly. Furhermore, the vector space of quadric surfaces that contain C_3 has dimension $\alpha_{32} = 3$ (see proof of Proposition 3.2). This implies that there exist three independent quadric surfaces S_2^1, S_2^2 and S_2^3 such that $C_3 \subseteq S_2^1 \cap S_2^2 \cap S_2^3$.

Proposition 3.2 implies that $|C_3 \cap D_3| \leq 6$. Suppose $|C_3 \cap D_3| = 6$. Let q be a point on D_3 that is not on C_3. Since $\alpha_{32} = 3$ there exist constants a_1 and b_1 such that q lies on the quadric surfaces

$$T_2^1 = S_2^1 + a_1 S_2^2$$

and

$$T_2^2 = S_2^2 + b_1 S_2^3.$$

$T_2^1 \neq T_2^2$ since S_2^1, S_2^2 and S_2^3 are linearly independent surfaces. Certainly $C_3 \subseteq T_2^1 \cap T_2^2$ and both these surfaces T_2^1 and T_2^2 intersect D_3 in at least seven points (the six points on $C_3 \cap D_3$ and q). Therefore, by Bezout's theorem, $D_3 \subseteq T_2^1 \cap T_2^2$. Furthermore $C_3 \cup D_3$, which is of degree 6, is contained in $T_2^1 \cap T_2^2$ which is at most of degree 4. This is a contradiction. Hence $|C_3 \cap D_3| \leq 5$ refining our earlier bound of 6 (Proposition 3.2).

4.2. Space quintics ($m = n = 5$)

Let C_5 and D_5 be distinct irreducible space quintics. $\alpha_{53} = 4$ and hence C_5 can be embedded in a cubic surface S_3 that does not contain D_5. Furthermore, there exist four linearly independent cubic surfaces S_3^1, S_3^2, S_3^3 and S_3^4 containing C_5. Suppose

$|C_5 \cap D_5| \geq 14$. Let q_1 and q_2 be two points belonging to $D_5 \setminus C_5$. Then there exist constants a_1, a_2 and b_1, b_2 such that q_1 and q_2 lie on both the cubic surfaces

$$T_3^1 = S_3^1 + a_1 S_3^2 + a_2 S_3^3$$

and

$$T_3^2 = S_3^2 + b_1 S_3^3 + b_2 S_3^4.$$

Since $|D_5 \cap T_3^1|$ and $|D_5 \cap T_3^2|$ are both at least 16, Bezouts's theorem implies that $D_5 \subseteq T_3^1 \cap T_3^2$. In fact $D_5 \cup C_5$ (degree 10) $\subseteq T_3^1 \cap T_3^2$ (degree 9), which is a contradiction. Therefore $|C_5 \cap D_5| \leq 13$, a smaller bound than the 15 implied by Proposition 3.2.

The results for the cubics and quintics may be generalized as follows. Let C_m and D_n be distinct irreducible space curves of degree m and n repsectively. C_m can be embedded in a suitably "low" degree surface S_d, whose intersection with D_n is proper, if m, n and d satisfy the conditions (a) and (b) of Proposition 3.2. Since the vector space of surfaces of degree d in $P^3(\mathbb{C})$ containing C_m is α_{md}, there exist linearly independent surfaces $S_d^1, S_d^2, \ldots, S_d^{\alpha_{md}}$ such that

$$C_m \subseteq S_d^1 \cap S_d^2 \cap \cdots \cap S_d^{\alpha_{md}}.$$

Suppose $|C_m \cap D_n| \geq nd - (\alpha_{md} - 3)$. Let $q_1, q_2, \ldots, q_{(\alpha_{md}-2)}$ be a set of points belonging to $D_n \setminus C_m$. Again, using the fact that the surfaces $S_d^1, S_d^2, \ldots, S_d^{\alpha_{md}}$ are linearly independent, it is possible to find constants $a_1, a_2, \ldots, a_{(\alpha_{md}-2)}$ and $b_1, b_2, \ldots, b_{(\alpha_{md}-2)}$ such that the above set $\{q_i\}$ of points lie on each of the following degree d surfaces:

$$T_d^1 = S_d^1 + \sum_{i=2}^{\alpha_{md}-1} a_i S_d^i$$

$$T_d^2 = S_d^2 + \sum_{i=3}^{\alpha_{md}} b_i S_d^i,$$

where T_d^1 is not equal to T_d^2. Now $|D_n \cap T_d^1|$ and $|D_n \cap T_d^2|$ are both at least $[nd - (\alpha_{md} - 3) + (\alpha_{md} - 2)]$, that is $nd + 1$. Bezout's theorem implies that $D_n \subseteq T_d^1 \cap T_d^2$. Certainly $C_m \subseteq T_d^1 \cap T_d^2$, therefore $C_m \cup D_n \subseteq T_d^1 \cap T_d^2$. $C_m \cup D_n$ is of degree $m + n$ whereas $T_d^1 \cap T_d^2$ is at most of degree d^2, by Bezout's theorem. Therefore $m + n \leq d^2$. But m, n and d satisfy Proposition 3.2 and this implies that $m + n > d^2$, thereby leading to a contradiction. Hence our assumption that $|C_m \cap D_n| \geq nd - (\alpha_{md} - 3)$ is wrong. Therefore, $|C_m \cap D_n| \leq nd - (\alpha_{md} - 2)$.

We have proved the following upper bound theorem for space curve intersections.

Theorem 4.1. *Let C_m and D_n be distinct irreducible space curves in $P^3(\mathbb{C})$, C_m being nonspecial, satisfying*

(a) $n > d^2 - m$,

(b) $\alpha_{md} \geq 2$,

where d is the smallest positive integer satisfying the inequality

$$\binom{d+3}{3} > md + 1.$$

Then C_m and D_n intersect in at most $[nd-(\alpha_{md}-2)]$ points.

Remarks. (1) For the intersection of two irreducible space cubics, the upper bound of 5 given by Theorem 4.1 is also the least upper bound. This is because a minimum of six points are needed to define a unique rational space cubic [21]. Given C_3, a rational D_3 can always be constructed to pass through five points of C_3 and by construction we have realized two space cubics which intersect at five points. In order to extend this argument to the intersection of any C_m and D_n ($m \leq n$), let us consider the equations which specify a rational D_n given below:

$$x(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0,$$

$$y(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0,$$

$$z(t) = c_n t^n + c_{n-1} t^{n-1} + \cdots + c_1 t + c_0,$$

$$w(t) = d_n t^n + d_{n-1} t^{n-1} + \cdots + d_1 t + d_0,$$

where a_i, b_i, c_i and d_i are real constants. Using the Lagrange interpolation formula it can be shown that at least $n+1$ points are needed to define the constants in each of these polynomials. Furthermore, using some relationships that exist between polynomials defining a rational curve D_n, it is conjectured that a minimum of $n+3$ points are required to define a unique D_n. If this were proved to be true, then using our previous argument it is seen that the bound on the intersection cardinality of $(C_m \cap D_n)$ can never be made smaller than $n+2$.

(2) If either C_m or D_n is reducible, then the techniques are applied to the intersection of their irreducible components.

(3) We always choose d to be the minimum value such that α_{md} is positive (barring the exceptional cases (β) and (γ) of Section 3). Intuitively it seems possible therefore to find an irreducible surface S_d on which we may embed C_m. In the case where C_m does not lie on a surface of degree less than d, it is obvious that the chosen S_d is irreducible (for example a nonplanar cubic always lies on an irreducible quadric).

(4) The central idea behind the technique used in this section was to exploit the fact that in most cases the curve C_m can be embedded in many linearly independent surfaces of degree d. This naturally leads us to questions as how many of these surfaces are needed to precisely obtain C_m as their intersection. This is akin to the representation problems adressed in the introduction [3,9,16,17] with the added caveat that we are controlling the degrees of the defining equations. In the appendix we present a solution for the case of space cubics by proving that three quadrics suffice.

(5) The asymptotic analysis presented in Section 3 is unaffected by Theorem 4.1.

5. Computational issues

It is of both practical and theoretical interest to examine the possibility of making all of the constructions presented in this paper completely algorithmic. The fact that the representation of the given space curves C_m and D_n is not uniformly specified makes it difficult to present a totally unified discussion of the computational issues. However, at an abstract level it is clear that the main steps of an algorithm would be to:

Step 1. Generate a requisite number of points on C_m.
Step 2. Construct one or more surfaces S_d^i to contain C_m (and not D_n).
Step 3. Compute the intersection points in $(D_n \cap S_d^i)$.
Step 4. Parse the candidates from Step 3 to obtain the true intersection points in $(C_m \cap D_n)$.

5.1. Rational parametric space curves

Suppose C_m and D_n are represented as rational parametric curves

$$C_m : (x_C(t),\ y_C(t),\ z_C(t)),$$

$$D_n : (x_D(s),\ y_D(s),\ z_D(s)).$$

In this case the computations are easily carried out. To generate points on C_m we simply choose values of the parameter t. To construct the S_d^i we solve systems of linear equations. To compute $(D_n \cap S_d^i)$ we substitute the parametric forms of D_n in the equation for S_d^i and solve numerically for the roots of the resulting univariate polynomial in s. To detect true intersection points we solve inversion problems on the parametric representation of C_m. There are well-known techniques for all of these steps [18].

5.2. Implicit space curves

In some applications (for example in computer aided geometric design) each of the space curves C_m and D_n may be given as the intersection of two or more surfaces. In such a situation we may avoid Steps 1 and 2 altogether and choose one of the given surfaces as S_d^i. However, if we want a minimum degree surface the main difficulty is in generating the requisite points on C_m. One approach would be to use an arbitrary rational parametric surface and compute intersections of this surface with the ones defining C_m. By substituting the parametrizations and then eliminating a parameter using resultants we could obtain points on C_m.

A more elegant (and perhaps more efficient) approach may be to realize a plane curve parametrization of C_m. The general technique would be to take a planar projection of C_m (via elimination) and then to identify the appropriate irreducible plane curve component that is birationally related to C_m. Some results along these lines are discussed in Hoffmann [15] and Garrity and Warren [12] for special cases. The details of a general algorithm are yet to be worked out and we pose it as a problem for further study. Such a parametrization will be useful in Step 1 for generating points on C_m and also in Step 3 for computing $(D_n \cap S_d^i)$ if D_n is given in implicit form.

6. Conclusion

One may raise the issue as to why the problem of intersecting space curves is new. This is probably because geometric intuition is that most space curves do not intersect at all. However, in computer aided geometric design, where solid models are often constructed using surface patches, the selection of edges (space curves) is such that they meet at vertices (intersection points). In this context the "improper" intersection of space curves is quite natural. In other applications some other improper intersections such as that of a curve and a surface in four-dimensional space may be relevant. We conclude with the hope that the preliminary investigations of intersecting algebraic space curves, reported in this paper, will lead to further study of improper intersections of algebraic varieties.

Appendix

We show here how any space cubic can be obtained as the complete intersection of three quadric surfaces. Furthermore, it is also possible to obtain by explicit computation the equations of these defining quadric surfaces.

Let C_3 be any irreducible space cubic in $P^3(\mathbb{C})$. C_3 can be embedded in two quadric surfaces S_2^1 and S_2^2 $(S_2^1 \neq S_2^2)$ using the techniques outlined in the main paper. Now $C_3 \subseteq S_2^1 \cap S_2^2$ and $S_2^1 \cap S_2^2 = C_3 \cup L$, where L is a line. This follows from Bezout's theorem. L meets C_3 in at least one point since a connectedness theorem due to Zariski [22] states that the intersection of two surfaces is connected. Moreover L meets C_3 in at most two points. For if $|L \cap C_3| = 3$, then we can choose a point $q \in C_3 \setminus L$ and construct a plane S, containing L and q. Then the plane S_1 intersects the space curve C_3 in at least four points contradicting Bezout's theorem which states that $|S_1 \cap C_3| = 3$. This proof carries over for the intersection of any line with a space cubic (their intersection cannot exceed 2). Suppose there are two distinct lines, each of which intersect the space cubic in two distinct points. These two lines cannot intersect each other. For if they intersect each other, we can construct a plane containing these two lines and this plane will intersect the space cubic in at least four points which leads to a contradiction (by Bezout's theorem). With these preliminaries established we can prove the following proposition.

Proposition A.1. *Any irreducible space cubic C_3 in $P^3(\mathbb{C})$ is the exact intersection of three quadric surfaces.*

Proof. Figure 1 is useful in visualizing some of the details of the proof.

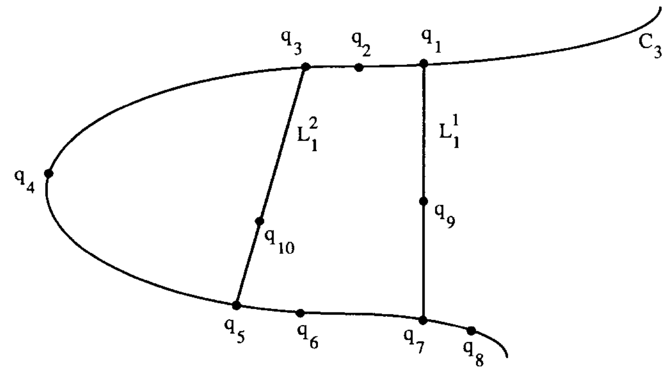

Fig. 1.

C_3 is an irreducible space cubic and we choose eight distinct points $q_1, q_2, \ldots, q_8$ on C_3. This can be done using one of the methods discussed in Section 5 of the paper. L_1^1 and L_1^2 are two lines passing through points q_1, q_7 and q_3, q_5 respectively, q_9 and q_{10} are points on $L_1^1 \setminus C_3$ and $L_1^2 \setminus C_3$ respectively. From the results obtained earlier we know that L_1^1 and L_2^1 do not intersect each other, nor do they intersect C_3 in any other point besides those shown in Fig. 1. Construct a quadric surface S_2^1 containing the points $q_1, q_2, \ldots, q_7$ and q_9, q_{10}. It is easily seen, as a consequence of Bezout's theorem, that $S_2^1 \supseteq C_3 \cup L_1^1 \cup L_1^2$. Let S_2^2 be a quadric surface defined by the nine points $q_1, q_2, \ldots, q_7, q_9$ and a point not on S_2^1. It is obvious that $S_2^2 \neq S_2^1$ and $S_2^2 \supseteq C_3 \cup L_1^1$. $C_3 \cup L_1^1 \subseteq S_2^1 \cup S_2^2$ and in fact as a consequence of having equal degrees, by Bezout's theorem $C_3 \cup L_1^1 = S_2^1 \cup S_2^2$. Let S_2^3 be a quadric surface containing the nine points $q_1, q_2, q_3, \ldots, q_7$ on C_3, q_{10} on L_1^2 and a point not on $S_1 \cup S_2$. $S_2^3 \supset C_3 \cup L_1^2$ and $S_2^1 \cap S_2^3 = C_3 \cup L_1^2$ as a result of Bezout's theorem. By construction $S_2^2 \neq S_2^3$ and $S_2^1 \neq S_2^3$.

Certainly $C_3 \subseteq S_2^1 \cap S_2^2 \cap S_2^3$,

$$S_2^1 \cap S_2^2 \cap S_2^3 \subseteq (S_2^1 \cap S_2^2) \cap (S_2^2 \cap S_2^3) \subseteq (C_3 \cup L_1) \cap (C_3 \cup L_2)$$
$$\subseteq C_3 \cup (L_1 \cap L_2) = C_3$$

(since $|L_1 \cap L_2| = \emptyset$). Hence $C_3 = S_2^1 \cap S_2^2 \cap S_2^3$. $\square$

In fact it is easily seen that equality is stronger than just set-theoretic.

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