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ORIGINAL ARTICLE

Dynamical Study in Fuzzy Threshold Dynamics of a Cholera Epidemic Model

Prabir Panja · Shyamal Kumar Mondal · Joydev Chattopadhyay

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Abstract In this paper, a fuzzy mathematical model on cholera disease has been developed in which all parameters related to the disease have been considered as fuzzy numbers. Here, total human population is divided into three subpopulations such as susceptible persons, infected ones and recovered ones. Also, the bacterial population is the Vibrio Cholerae in the environment. Then the existence condition and boundedness of solution to our proposed mathematical model have been discussed. Also, the different equilibrium points and the stability condition of the system around these equilibrium points have been analyzed. The global stability condition of the proposed system around the endemic equilibrium point has been also discussed. Finally, some numerical simulations have been shown to test the theoretical results of the system.

Keywords Cholera disease · Fuzzy number · Triangular Fuzzy number · UFM method · Local stability · Global stability

1. Introduction

Cholera is an acute diarrhoeal infection caused by ingestion of food or water contaminated with the bacterium Vibrio Cholerae. It causes mortality, disability, social and economic damage for millions of people in the whole world specially in developing

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countries. So, it is a major threat to human beings. Two of the toxigenic V. Cholerae $O1$ and $O139$ are free-living bacterial organisms found in fresh and briny water. Frequently, they are found in association with zooplankton, shellfish and aquatic plants. It spreads through the contaminated food, drinking water and also from the feces of infected human. The main symptoms of cholera are watery diarrhea, vomiting, rapid dehydration, rapid heart rate, loss of skin elasticity, dry mucous membranes etc. Severe outbreaks usually occur in underdeveloped areas with inadequate sanitation, poor hygiene and limited access to safe water supplies. Although there are many recent progresses in medical sciences, cholera remains now as a global threat in some parts of the World.

Mathematical models have become more important tools for analyzing the spread of cholera disease. Basically, ordinary differential equation is used for formulation of this type of problem and provides some mathematical answer and explanation. A crisp mathematical model on cholera disease was described by Capasso [9]. It was consisted with two equations to follow the dynamics of infected individuals and the number of free-living infective stages. More recently Codeco [10] developed a more general model of cholera with an additional equation in the population. Modeling and analysis of the spread of carrier dependent infectious diseases with environmental effects was explored by Singh et al. [7]. There exist many mathematical models in crisp environment on cholera disease which explore the spread and control strategies of the disease such as [8, 11, 12, 14 – 20]. In 2016, Misra et al. [22] studied the effects of bacteriophage infection on the cholera disease dynamics. After that, Soufiane and Touaoula [26] investigated an epidemic model with infection age with the help of a set of nonlinear differential equation. Also, Nasr-Azadani et al. [23] explored the impact of climate change on cholera disease dynamics in the Ganges-Brahmaputra basin. Again, in 2017, Sun et al. [21] studied a mathematical model on cholera with different types of control strategies. After that, Cai et al. [24] and Lemos-Paio et al. [25] have investigated the cholera epidemic model in the presence of vaccination and treatment as control parameter.

The parameters involved in the mathematical models on cholera disease discussed above are crisp in nature. But it is found that the biological parameters involve in the differential equations are not always fixed. In the real world every community is changing with the varying environments. In the present time, the global warming is the main problem in the whole globe. It is the increase of earth's average surface temperature due to effect of greenhouse gases such as carbon dioxide emissions from burning fossil fuels or from deforestation which trap heat that would otherwise escape from earth. The change of temperature strongly effects on the reproduction rate of the bacterial population. Many parameters may oscillate with the change of environments in real world ecosystem. These parameters are also varying due to both natural and human activities such as earthquake, climate warming, financial crisis etc. Therefore, the interactions between the human and bacteria and the dynamics of cholera disease may be influenced by the environmental variations. In this respect, the fuzzy mathematical model is more meaningful than the crisp model. So, in this paper, we have used fuzzy set theory [5] to formulate this cholera model. There exist very few number of papers in infectious disease model in fuzzy environment [13]. The stability analysis of dynamical system with variables and parameters are uncertain in nature was explored by Mizukoshi et al. [3]. Fuzzy parameter based predator-prey mathematical models was presented by Peixto et al. [4] and Pal et al. [6]. The Table 1. gives the comparative study of the proposed model with the existing models in infectious cholera disease.

In this paper, a mathematical model on cholera disease has been developed considering all parameters to be fuzzy numbers. Here, total human population is classified into three subpopulations such as susceptible human, infected human and recovered human. Also, one bacterial population (Vibrio Cholerae) is considered in the this mathematical model. Existence condition and boundedness of solution of our proposed mathematical model have been discussed. Also, the different equilibrium points and the stability condition of the system around these equilibrium points have been analyzed. The global stability conditions of the proposed system around the positive equilibrium point have been also discussed. Ultimately, some numerical simulations have been given to verify our analytical findings.

Table 1: Comparison of proposed model with some existing models.

Reference No	Type of Model Parameters	Number of Human Compartments	Number of Bacterial Compartments
Lemos-Paio et al. [25]	Crisp	4 (Susceptible, Infectious, Quarantined, Recovered)	1 (Concentration of Bacterium)
Ponsey et al. [27]	Crisp	4 (Susceptible, Vaccinated, Infected, Recovered)	1 (Concentration of Vibrio Cholerae in Contaminated water)
Mwasa et al. [28]	Crisp	7 (Susceptible individuals, Educated individuals, Vaccinated individuals, Quarantined individuals, Infected individuals, Treated individuals, Removed individuals)	1 (Biomass level of Vibrio Cholerae)
Misra et al. [29]	Crisp	2 (Susceptible human, Infected human)	1 (Density of carrier population)
The proposed model	Fuzzy	3 (Susceptible human, Infected human, Recovered human)	1 (Concentration of Vibrio Cholerae in the Environment)

1. Preliminaries

Fuzzy Set

Fuzzy sets deals with objects that are 'matter of degree' with all possible grades of truth between yes or no. so a fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those do not. Let X be a collection of objects and x be an element of X. Then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}$ where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ which maps X to the membership space M which is considered as the closed interval $[0, u]$, where $0 \leq u \leq 1$.

Triangular Fuzzy Number

A Triangular fuzzy number $\tilde{A}$ is specified by the triplet (a, b, c) and is defined by its continuous membership function $\mu_{\tilde{A}}(x) : X \to [0, 1]$ as follows

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a}, & if\, a \leq x \leq b, \\ \dfrac{c-x}{c-b}, & if\, b \leq x \leq c, \\ 0, & \text{otherwise.} \end{cases}$$

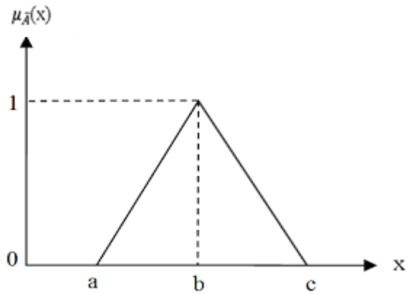

Fig.1 Triangular fuzzy number

α-cut of a Fuzzy Number

A α- cut of a fuzzy number $\tilde{A}$ in X is denoted by A_α and is defined as the following crisp set

$$A_\alpha = \{x|\mu_{\tilde{A}}(x) \geq \alpha, x \in X\} \text{ where } \alpha \in [0, 1],$$

A_α is a non-empty bounded closed interval contained in X and it can be denoted by $A_\alpha = [A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha)$ and $A_R(\alpha)$ are the lower and upper bounds of the closed interval respectively.

It is clear that α-cut of triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is a closed and bounded interval $[A_L(\alpha), A_R(\alpha)]$, where $A_L(\alpha) = a_1 + \alpha(a_2 - a_1)$ and $A_R(\alpha) = a_3 - \alpha(a_3 - a_2)$.

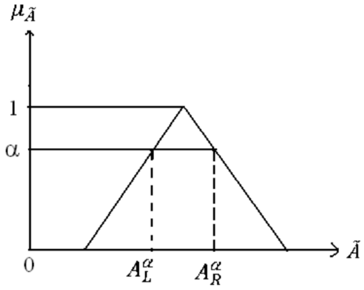

Fig.2 $\alpha-$ cut of a Triangular fuzzy number

Interval Arithmetic of a Fuzzy Number

Let $[P_L, P_R]$ and $[Q_L, Q_R]$ be two interval numbers. The addition and subtraction of two interval numbers are given by

$$[P_L, P_R] + [Q_L, Q_R] = [P_L + Q_L, P_R + Q_R],$$
$$[P_L, P_R] - [Q_L, Q_R] = [P_L - Q_R, P_R - Q_L].$$

Utility Function Method(UFM)

In UFM, a utility function is defined for each of the objectives g_i according to their relative importance. A simple utility function may be defined as $w_i g_i$ for $i-th$ objective, where w_i is a scalar and represent the weight assigned to the corresponding objective. Then the total utility defined as the weighted sum of the objectives as follows

$$U = \sum_{i=1}^{n} w_i g_i, w_i \geq 0$$

subject to the condition $\sum_{i=1}^{n} w_i = 1$.

2. Model Formulation

We have considered $S(t)$, $I(t)$, $R(t)$ and $V_E(t)$ as the population densities of the susceptible human, infected human, recovered human and Vibrio Cholerae in the environment at time t respectively. Let $\widetilde{A}$, $\widetilde{\mu_d}$, $\widetilde{\beta}$ and $\widetilde{\delta}$ be the fuzzy intrinsic growth rate of susceptible human, fuzzy natural death rate of susceptible human, fuzzy transmission rate of susceptible human to infected human and fuzzy rate of loose of natural immunity respectively. Let $\widetilde{\mu_d}$, $\widetilde{m}$, $\widetilde{\alpha_1}$ and $\widetilde{\gamma}$ be the fuzzy natural death rate of infected human, fuzzy disease related death rate of infected human, fuzzy rate of recovered human and fuzzy rate of excretion of Vibrio Cholerae in the environment by vomiting, feces etc. by infected human. Let $\widetilde{\mu_d}$ be the fuzzy natural death rate of recovered human and fuzzy natural death rate of Vibrio Cholerae in the environment. By using the concept on the fuzzy initial value problem [1] and differentials of fuzzy functions [2] and considering the above assumptions, a set of fuzzy differential equations regarding the cholera disease has been developed as follows:

$$
\begin{aligned}
\frac{\widetilde{dS}}{dt} &= \widetilde{A} - \widetilde{\mu_d} S - \widetilde{\beta} S V_E + \widetilde{\delta} R,\\
\frac{\widetilde{dI}}{dt} &= \widetilde{\beta} S V_E - \widetilde{\mu_d} I - \widetilde{m} I - \widetilde{\alpha_1} I - \widetilde{\gamma} I,\\
\frac{\widetilde{dR}}{dt} &= \widetilde{\alpha_1} I - \widetilde{\mu_d} R - \widetilde{\delta} R,\\
\frac{\widetilde{dV_E}}{dt} &= \widetilde{\gamma} I - \widetilde{\mu_{V_E}} V_E.
\end{aligned}
\tag{1}
$$

To find the solution of (1) let

$$
\left[\frac{\widetilde{dx}}{dt}\right]_\alpha = \left[\left(\frac{dx}{dt}\right)^\alpha_L, \left(\frac{dx}{dt}\right)^\alpha_R\right].
$$

The deterministic system of the model (1) is given by

$$
\begin{aligned}
\left(\frac{dS}{dt}\right)^\alpha_L &= (A_L)^\alpha - (\mu_{d_R})^\alpha S - (\beta_R)^\alpha S V_E + (\delta_L)^\alpha R,\\
\left(\frac{dS}{dt}\right)^\alpha_R &= (A_R)^\alpha - (\mu_{d_L})^\alpha S - (\beta_L)^\alpha S V_E + (\delta_R)^\alpha R,\\
\left(\frac{dI}{dt}\right)^\alpha_L &= (\beta_L)^\alpha S V_E - [(\mu_{d_R})^\alpha + (m_R)^\alpha + (\alpha_{1_R})^\alpha + (\gamma_R)^\alpha] I,\\
\left(\frac{dI}{dt}\right)^\alpha_R &= (\beta_R)^\alpha S V_E - [(\mu_{d_L})^\alpha + (m_L)^\alpha + (\alpha_{1_L})^\alpha + (\gamma_L)^\alpha] I,\\
\left(\frac{dR}{dt}\right)^\alpha_L &= (\alpha_{1_L})^\alpha I - [(\mu_{d_R})^\alpha + (\delta_R)^\alpha] R,\\
\left(\frac{dR}{dt}\right)^\alpha_R &= (\alpha_{1_R})^\alpha I - [(\mu_{d_L})^\alpha + (\delta_L)^\alpha] R,\\
\left(\frac{dV_E}{dt}\right)^\alpha_L &= (\gamma_L)^\alpha I - \left(\mu_{V_{E_R}}\right)^\alpha V_E,\\
\left(\frac{dV_E}{dt}\right)^\alpha_R &= (\gamma_R)^\alpha I - \left(\mu_{V_{E_L}}\right)^\alpha V_E.
\end{aligned}
$$

Using the concept of UFM, we can write the above system of differential equations as follows:

$$\begin{aligned}\frac{dS}{dt} &= w_1\left(\frac{dS}{dt}\right)_L^\alpha + w_2\left(\frac{dS}{dt}\right)_R^\alpha,\\ \frac{dI}{dt} &= w_1\left(\frac{dI}{dt}\right)_L^\alpha + w_2\left(\frac{dI}{dt}\right)_R^\alpha,\\ \frac{dR}{dt} &= w_1\left(\frac{dR}{dt}\right)_L^\alpha + w_2\left(\frac{dR}{dt}\right)_R^\alpha,\\ \frac{dV_E}{dt} &= w_1\left(\frac{dV_E}{dt}\right)_L^\alpha + w_2\left(\frac{dV_E}{dt}\right)_R^\alpha,\end{aligned} \tag{2}$$

where w_1 and w_2 are two weight functions such that $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$. Then the equation (2) can be written as

$$\begin{aligned}\frac{dS}{dt} &= a_{11} - a_{12}S - a_{13}SV_E + a_{14}R,\\ \frac{dI}{dt} &= a_{21}SV_E - a_{22}I,\\ \frac{dR}{dt} &= a_{31}I - a_{32}R,\\ \frac{dV_E}{dt} &= a_{41}I - a_{42}V_E,\end{aligned} \tag{3}$$

where

$a_{11} = w_1(A_L)^\alpha + w_2(A_R)^\alpha, a_{12} = w_1(\mu_{d_R})^\alpha + w_2(\mu_{d_L})^\alpha, a_{13} = w_1(\beta_R)^\alpha + w_2(\beta_L)^\alpha,$
$a_{14} = w_1(\delta_L)^\alpha + w_2(\delta_R)^\alpha, a_{21} = w_1(\beta_L)^\alpha + w_2(\beta_R)^\alpha,$
$a_{22} = w_1\left[(\mu_{d_R})^\alpha + (m_R)^\alpha + (\alpha_{1_R})^\alpha + (\gamma_R)^\alpha\right] + w_2\left[(\mu_{d_L})^\alpha + (m_L)^\alpha + (\alpha_{1_L})^\alpha + (\gamma_L)^\alpha\right],$
$a_{31} = w_1(\alpha_{1_L})^\alpha + w_2(\alpha_{1_R})^\alpha, a_{32} = w_1\left[(\mu_{d_R})^\alpha + (\delta_R)^\alpha\right] + w_2\left[(\mu_{d_L})^\alpha + (\delta_L)^\alpha\right],$
$a_{41} = w_1(\gamma_L)^\alpha + w_2(\gamma_R)^\alpha, a_{42} = w_1\left(\mu_{V_{E_R}}\right)^\alpha + w_2\left(\mu_{V_{E_L}}\right)^\alpha.$

3. Boundedness of Solution

In this section, the boundedness of all solutions of the proposed system (3) has been shown. Before proving the boundedness at first Lemma 1 has been proved.

Lemma 1 *Here,* $(a_{22} - a_{31}) > 0, (a_{32} - a_{14}) > 0$ *and* $(a_{21} - a_{13}) > 0$ *provided that* $\left[(\mu_{d_L})^\alpha + (m_L)^\alpha + (\gamma_L)^\alpha\right] + (\alpha_{1_L})^\alpha > (\alpha_{1_R})^\alpha$, $(\mu_{d_L})^\alpha + (\delta_L)^\alpha > (\delta_R)^\alpha$ *and* $0 \leq w_1 \leq 0.5$*, respectively.*

Proof We have

$$
\begin{aligned}
a_{22}-a_{31} &= w_1\,[(\mu_{d_R})^{\alpha}+(m_R)^{\alpha}+(\alpha_{1_R})^{\alpha}+(\gamma_R)^{\alpha}]+w_2[(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}\\
&\quad +(\gamma_L)^{\alpha}]\\
&= w_1\Big\{[(\mu_{d_R})^{\alpha}+(m_R)^{\alpha}+(\alpha_{1_R})^{\alpha}+(\gamma_R)^{\alpha}]-[(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}\\
&\quad +(\gamma_L)^{\alpha}]+(\alpha_{1_R})^{\alpha}-(\alpha_{1_L})^{\alpha}\Big\}\\
&\quad +[(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}+(\gamma_L)^{\alpha}]-(\alpha_{1_R})^{\alpha}\,,\ \text{since } w_1+w_2=1.
\end{aligned} \tag{4}
$$

Now, when $w_1 = 0$, we have

$$
a_{22}-a_{31} = [(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}+(\gamma_L)^{\alpha}]-(\alpha_{1_R})^{\alpha}\,.
$$

Then, $a_{22} - a_{31}$ will be positive for $w_1 = 0$ if

$$
[(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}+(\gamma_L)^{\alpha}] > (\alpha_{1_R})^{\alpha}\,. \tag{5}
$$

Therefore, from equation (4) it is obtained that

$$
\begin{aligned}
\frac{d(a_{22}-a_{31})}{dw_1} &= [(\mu_{d_R})^{\alpha}+(m_R)^{\alpha}+(\alpha_{1_R})^{\alpha}+(\gamma_R)^{\alpha}]-[(\mu_{d_L})^{\alpha}+(m_L)^{\alpha}+(\alpha_{1_L})^{\alpha}\\
&\quad +(\gamma_L)^{\alpha}]+(\alpha_{1_R})^{\alpha}-(\alpha_{1_L})^{\alpha} > 0, \forall w_1 \in [0,1].
\end{aligned}
$$

So, $(a_{22} - a_{31})$ is an increasing function with respect to w_1 and it will be positive if condition (5) holds.

In the similar way, it can be proved that $(a_{32} - a_{14})$ is an increasing function with respect to w_1 and it will be positive if condition

$$
(\mu_{d_L})^{\alpha}+(\delta_L)^{\alpha} > (\delta_R)^{\alpha}
$$

holds, and also $(a_{21} - a_{13})$ is an increasing function with respect to w_1 and it will be positive if condition $0 \le w_1 \le 0.5$ holds.

Theorem 1 *All solutions of the system* (3) *are bounded in the region* R^4_+ *provided that* $a_{21} \ge a_{13}$ *and* $\sigma = \min\{a_{12}, (a_{22} - a_{31}), (a_{32} - a_{14})\}$.

Proof Let us define a function

$$
W = S + I + R. \tag{6}
$$

Now, differentiating (6) with respect to time t and simplifying we have

$$
\frac{dW}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt},
$$

i.e.,

$$
\frac{dW}{dt} = a_{11} - a_{12}S + (a_{31} - a_{22})I + (a_{14} - a_{32})R + (a_{21} - a_{13})S\,V_E. \tag{7}
$$

For a positive real number σ, multiplying σ on the both sides of (6) and then adding with Equation (7) we have

$$\frac{dW}{dt} + \sigma W = a_{11} + (\sigma - a_{12})S + (\sigma - (a_{22} - a_{31}))I + (\sigma - (a_{32} - a_{14}))R$$
$$+(a_{21} - a_{13})S V_E.$$

Now according to Lemma 1, if $\sigma = \min\{a_{12}, (a_{22} - a_{31}), (a_{32} - a_{14})\}$, then the above equation reduces to in the following form

$$\frac{dW}{dt} + \sigma W \leq a_{11}.$$

Solving the above we get

$$W \leq \frac{a_{11}}{\sigma} + c_1 e^{-\sigma t}.$$

Taking t tends to infinity, we have

$$W \leq \frac{a_{11}}{\sigma}.$$

So, it can be written as $S(t) \leq \frac{a_{11}}{\sigma}$, $I(t) \leq \frac{a_{11}}{\sigma}$ and $R(t) \leq \frac{a_{11}}{\sigma}$.

Now, from the fourth equation of System (3) we have

$$\frac{dV_E}{dt} = a_{41} I - a_{42} V_E,$$
$$\frac{dV_E}{dt} \leq \frac{a_{11} a_{41}}{\sigma} - a_{42} V_E,$$
$$\frac{dV_E}{dt} + a_{42} V_E \leq \frac{a_{11} a_{41}}{\sigma}.$$

Solving the above equation, we have

$$V_E \leq \frac{a_{11} a_{41}}{\sigma a_{42}} + c_2 e^{-a_{42} t}.$$

Taking t tends to infinity, it is obtained that

$$V_E \leq \frac{a_{11} a_{41}}{\sigma a_{42}}.$$

This proves that the solutions of the system are bounded.

4. Equilibrium Points

The system (3) has two equilibrium points such as

(i) Disease free equilibrium point (E_0)=($S_0, 0, 0, 0$), where $S_0 = a_{11}/a_{12}$.

(ii) Endemic equilibrium point $(E^*) = (S^*, I^*, R^*, V_E^*)$, where

$$S^* = \frac{a_{22}a_{42}}{a_{21}a_{41}},$$
$$I^* = \frac{a_{11}a_{21}a_{41}a_{32} - a_{22}a_{32}a_{12}a_{42}}{a_{22}a_{32}a_{13}a_{41} - a_{21}a_{41}a_{14}a_{31}},$$
$$R^* = \frac{a_{31}I^*}{a_{32}}, V_E^* = \frac{a_{41}I^*}{a_{42}}.$$

The basic reproduction number is $R_0 = (a_{11}a_{21}a_{41})/(a_{22}a_{12}a_{42})$.

5. Stability Analysis

The jacobian matrix of the System (3) is given by

$$J(S, I, R, V_E) = \begin{pmatrix} -a_{12} - a_{13}V_E & 0 & a_{14} & -a_{13}S \\ a_{21}V_E & -a_{22} & 0 & a_{21}S \\ 0 & a_{31} & -a_{32} & 0 \\ 0 & a_{41} & 0 & -a_{42} \end{pmatrix}.$$

Theorem 2 *The system* (3) *is locally asymptotically stable at* E_0 *if* $(a_{11}a_{21}a_{41})/(a_{22}a_{12}a_{42}) < 1$.

Proof The characteristic equation at E_0 of the System (3) is given by

$$(a_{32} + x)(x + a_{12})\left[x^2 + b_1 x + b_2\right] = 0,$$

where $b_1 = (a_{22} + a_{42})$, $b_2 = a_{22}a_{42} - (a_{11}a_{21}a_{41})/a_{12}$.

The roots of the characteristic equation are $x = -a_{32} < 0$, $x = -a_{12} < 0$, since a_{12}, a_{32} are positive and also by Routh-Hurwitz criteria the roots of the quadratic equation will be negative real number or complex with negative real parts if $b_1 > 0, b_2 > 0$ and $b_1^2 - 4b_2 < 0$. Since $a_{22} > 0, a_{42} > 0$, then obviously $b_1 > 0$ and b_2 will be positive if

$$a_{22}a_{12}a_{42} - a_{11}a_{21}a_{41} > 0,$$
i.e., $$\frac{a_{11}a_{21}a_{41}}{a_{22}a_{12}a_{42}} < 1.$$

Theorem 3 *The system* (3) *is locally asymptotically stable at* E^* *if* $c_i > 0$ *for* $i = 1, 2, 3, 4$ *and* $c_1c_2 > c_3, c_1c_2c_3 > c_3^2 + c_1^2c_4$.

Proof The characteristic equation of the system (3) at the equilibrium point E^* is

$$x^4 + c_1x^3 + c_2x^2 + c_3x + c_4 = 0,$$

where

$$
\begin{aligned}
c_1 &= a_{32} + a_{42} + a_{12} + a_{13}V_E^* + a_{22},\\
c_2 &= a_{32}a_{42} + a_{22}a_{12} + a_{22}a_{13}V_E^* - a_{21}a_{41}S^*,\\
c_3 &= a_{32}a_{42}a_{12} + a_{32}a_{42}a_{13}V_E^* + a_{22}a_{32}a_{42} + a_{22}a_{32}a_{12} + a_{22}a_{32}a_{13}V_E^* + a_{22}a_{42}a_{12}\\
&\quad + a_{22}a_{42}a_{13}V_E^* - a_{14}a_{21}a_{31}V_E^* - a_{12}a_{21}a_{41}S^* - a_{21}a_{41}a_{32}S^*,\\
c_4 &= a_{22}a_{32}a_{42}a_{12} + a_{22}a_{32}a_{42}a_{13}V_E^* - a_{14}a_{21}a_{31}a_{42}V_E^* - a_{12}a_{21}a_{41}a_{32}S^*.
\end{aligned}
$$

Now, according to Routh-Hurwitch criteria the roots of a biquadratic equation will be negative or have a negative real parts i.e., the System (3) will be locally asymptotically stable at E^* if each $c_i > 0$ for $i = 1, 2, 3, 4$ and $c_1c_2 > c_3, c_1c_2c_3 > c_3^2 + c_1^2c_4$.

Theorem 4 *The system* (3) *is globally asymptotically stable around* E^* *if* $\mu_3 > 0$ *where* $\mu_3 = a_{21}\mu_2 - a_{22} + \min[(a_{12} + a_{22} + a_{13}\mu_2) - \max(a_{21}\mu_2, a_{13}\mu_2), a_{42} - a_{41} + \min\{a_{21} + \mu_2(a_{13} - a_{21}), a_{22}\}]$.

Proof Let us consider the subsystem of the System (3) as

$$
\begin{aligned}
\frac{dS}{dt} &= a_{11} - a_{12}S - a_{13}SV_E,\\
\frac{dI}{dt} &= a_{21}SV_E - a_{22}I,\\
\frac{dV_E}{dt} &= a_{41}I - a_{42}V_E.
\end{aligned}
\tag{8}
$$

The jacobian matrix of the System (8) is given by

$$
J = \begin{pmatrix} -a_{12} - a_{13}V_E & 0 & -a_{13}S \\ a_{21}V_E & -a_{22} & a_{21}S \\ 0 & a_{41} & -a_{42} \end{pmatrix}.
$$

Now, the second additive matrix is given by

$$
J^{[2]} = \begin{pmatrix} -a_{12} - a_{22} - a_{13}V_E & a_{21}S & a_{13}S \\ a_{41} & -a_{12} - a_{13}V_E - a_{42} & 0 \\ 0 & a_{21}V_E & -a_{22} - a_{42} \end{pmatrix}.
$$

We have

$$
P(x) = P(S, I, V_E) = diag\left(\frac{S}{I}, \frac{S}{I}, \frac{S}{I}\right),
$$

$$
P_f = \frac{\partial P}{\partial x} = diag\left(\frac{\dot{S}}{I} - \frac{S\dot{I}}{I^2}, \frac{\dot{S}}{I} - \frac{S\dot{I}}{I^2}, \frac{\dot{S}}{I} - \frac{S\dot{I}}{I^2}\right).
$$

Now, it follows that $P_fP^{-1} = diag\left(\frac{\dot{S}}{S} - \frac{\dot{I}}{I}, \frac{\dot{S}}{S} - \frac{\dot{I}}{I}, \frac{\dot{S}}{S} - \frac{\dot{I}}{I}\right)$ and $PJ^{[2]}P^{-1} = J^{[2]}$, so that

$$B = P_f P^{-1} + PJ^{[2]}P^{-1} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$
$$B_{11} = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - (a_{12} + a_{22} + a_{13}V_E),\ B_{12} = \begin{pmatrix} a_{21}S & a_{13}S \end{pmatrix},\ B_{21} = \begin{pmatrix} a_{41} & 0 \end{pmatrix}^t,$$
$$B_{22} = \begin{pmatrix} \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - (a_{12} + a_{13}V_E + a_{42}) & 0 \\ a_{21}V_E & \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - (a_{22} + a_{42}) \end{pmatrix}.$$

Now, $\mu_1(B_{11}) = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - (a_{12} + a_{13}V_E + a_{42})$, $|B_{12}| = \max\{a_{21}S, a_{13}S\}$, $B_{21} = a_{41}$, $\mu_1(B_{22}) = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - a_{42} - \min\{a_{21} + V_E(a_{13} - a_{21}), a_{22}\}$.

Therefore,

$$\mu(B) \leq \sup\{p_1, p_2\}, \tag{9}$$

where

$$p_1 = \mu_1(B_{11}) + |B_{12}| = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - (a_{12} + a_{22} + a_{13}V_E) + max\{a_{21}S, a_{13}S\},$$
$$p_2 = \mu_1(B_{22}) + |B_{21}| = \frac{\dot{S}}{S} - \frac{\dot{I}}{I} - a_{42} + a_{41} - min\{a_{21} + V_E(a_{13} - a_{21}), a_{22}\}.$$

We have

$$\dot{I} = a_{21}SV_E - a_{22}I,$$
$$\frac{\dot{I}}{I} = \frac{a_{21}SV_E}{I} - a_{22}.$$

If there exists $t_1 > 0$ such that $\inf\{S(t), I(t), R(t), V_E(t)\} = \mu_2$, i.e., if $R_0 > 1$, then from the above we have

$$p_1 = \frac{\dot{S}}{S} - a_{21}\mu_2 + a_{22} - (a_{12} + a_{22} + a_{13}\mu_2) + max\{a_{21}\mu_2, a_{13}\mu_2\},$$
$$p_2 = \frac{\dot{S}}{S} - a_{21}\mu_2 + a_{22} - a_{42} + a_{41} - min\{a_{21} + \mu_2(a_{13} - a_{21}), a_{22}\}.$$

Then from the equation (9) it is obtained that

$$\begin{aligned} \mu(B) \leq & \frac{\dot{S}}{S} - a_{21}\mu_2 + a_{22}) \\ & - \min[(a_{12} + a_{22} + a_{13}\mu_2 - \max(a_{21}\mu_2, a_{13}\mu_2), \\ & a_{42} - a_{41} + \min\{a_{21} + \mu_2(a_{13} - a_{21}), a_{22}\}], \end{aligned}$$

i.e.,
$$\mu(B) \leq \frac{\dot{S}}{S} - \mu_3, \tag{10}$$

where

$$\mu_3 = a_{21}\mu_2 - a_{22} + \min[(a_{12} + a_{22} + a_{13}\mu_2) - \max(a_{21}\mu_2, a_{13}\mu_2),$$

$a_{42} - a_{41} + \min\{a_{21} + \mu_2(a_{13} - a_{21}), a_{22}\}]$.

Integrating the above equation (10) from 0 to t we have

$$\int_0^t \mu(B)ds \leq log\frac{S(t)}{S(0)} - \mu_3 t,$$
$$\frac{1}{t}\int_0^t \mu(B)ds \leq \frac{1}{t}log\frac{S(t)}{S(0)} - \mu_3,$$
$$\lim_{t\to\infty} \sup\sup \frac{1}{t}\int_0^t \mu(B)ds < -\mu_3 < 0, \text{ if } \mu_3 > 0.$$

Now, if $\mu_3 > 0$, then, the interior equilibrium point $E^*(S^*, I^*, R^*, V_E^*)$ will be globally asymptotically stable.

6. Numerical Simulations

To study the feasibility of the fuzzy model about the cholera disease, all biological parameters are hypothesized in imprecise nature which are considered here triangular fuzzy number.

Table 2. Values of the Parameters.

Parameter Name	Value	References
A	50 – 100	[8, 10, 17]
μ_d	0.2 – 0.4	[8, 12]
β	0.001 – 0.04	[16, 17, 18]
δ	0.001 – 0.01	[7, 8]
m	0.005 – 0.007	[10, 11, 17]
α_1	0.1 – 0.4	[17, 18, 19]
γ	0.02 – 0.03	Assumed
μ_{V_E}	0.02 – 0.04	[8, 12, 17]

To discuss the dynamical numerically following problems have been considered.

Problem 1: In this Problem the following hypothetical data of all parameter involved in the model are considered as:

$\tilde{A} = (50, 60, 70), \tilde{\beta} = (0.001, 0.0011, 0.0012), \tilde{\mu_d} = (0.2, 0.3, 0.4),$

$\tilde{\delta} = (0.001, 0.002, 0.003), \tilde{m} = (0.005, 0.006, 0.007),$

$\tilde{\alpha_1} = (0.2, 0.3, 0.4), \tilde{\gamma} = (0.02, 0.021, 0.022), \tilde{\mu_{V_E}} = (1/50, 1/40, 1/30).$

For the above set of parametric values with $\alpha = 0.1, w_1 = 0.2, w_2 = 0.8$ the Fig.3 has been drawn. From this figure, it is observed that the system is free of disease. Here, the disease free equilibrium point is $E_0(265.8537, 0, 0, 0)$. From Theorem 2, it

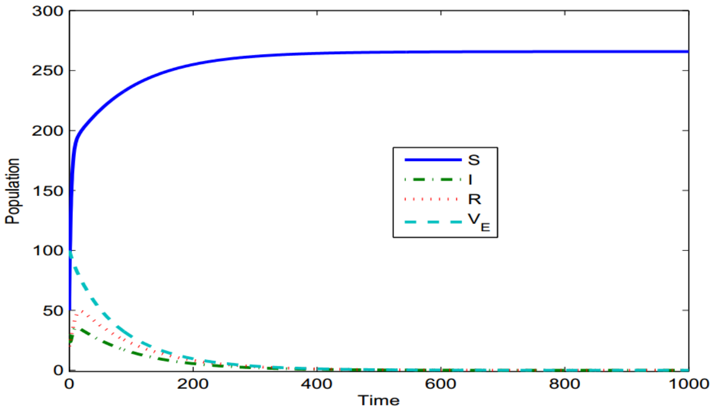

Fig.3 Local stability of disease free equilibrium point

is also seen that $R_0 = (a_{11}a_{21}a_{41})/(a_{22}a_{12}a_{42}) = 0.5572 < 1$, so the system (3) is locally asymptotically stable around the equilibrium point $E_0(265.8537, 0, 0, 0)$.

Problem 2: In this problem, we consider the same set of parametric values in Problem 1 except $\beta = (0.01, 0.02, 0.03)$. Using this data Fig.4 has been drawn, from which this figure it is observed that the system is endemic and the endemic equilibrium point is given by $E^* = (21.68, 204.30, 292.20, 192.10)$.

From Theorem 3, it is also seen that

$$c_1 = 3.8404 > 0,$$

$$c_2 = 1.5746 > 0,$$

$$c_3 = 0.4374 > 0,$$

$$c_4 = 0.0081 > 0,$$

$$c_1c_2 - c_3 = 5.6096 > 0$$

and

$$c_1c_2c_3 - (c_3)^2 - (c_1)^2c_4 = 2.3334 > 0,$$

so the system (3) is locally asymptotically stable around the equilibrium point

$$E^*(21.68, 204.30, 292.20, 192.10).$$

Problem 3: In this case following data set has been considered as:

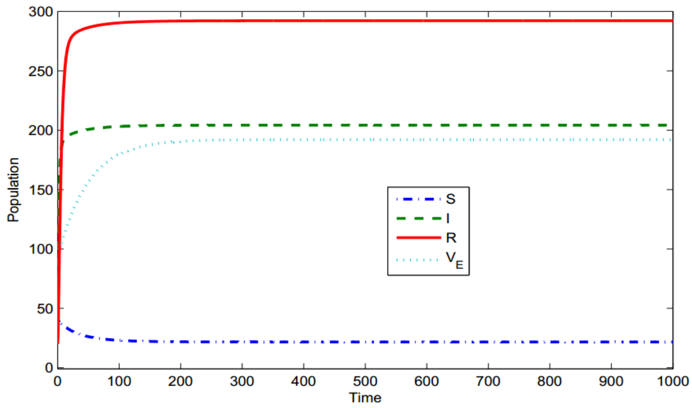

Fig.4 Local stability of endemic equilibrium point

$\tilde{A} = (50, 60, 70), \tilde{\beta} = (0.01, 0.02, 0.03),$
$\tilde{\mu_d} = (0.2, 0.3, 0.4), \tilde{\delta} = (0.001, 0.002, 0.003),$
$\tilde{m} = (0.005, 0.006, 0.007), \tilde{\alpha_1} = (0.1, 0.2, 0.3),$
$\tilde{\gamma} = (0.02, 0.021, 0.022), \tilde{\mu_{V_E}} = (1/50, 1/40, 1/30).$

For this data set considering different weight w_1, w_2 different equilibrium points have been computed for $\alpha = 0.0, 0.6, 1.0$ in Table 3.

Table 3. Equilibrium points for different α and w_1, w_2.

w_1	w_2	$\alpha = 0.0$	$\alpha = 0.6$	$\alpha = 1.0$
0.0	1.0	(9.85,655.00,977.70,719.90)	(20.00,199.20,185.30,185.8)	(31.39,96.24,63.73,80.83)
0.2	0.8	(16.41,287.70,309.90,273.40)	(24.22,148.90,120.10,131.40)	(31.39,96.24,63.73,80.83)
0.4	0.6	(26.47,137.70,107.50,115.20)	(29.28,111.20,78.65,93.21)	(31.39,96.24,63.73,80.83)
0.6	0.4	(42.47,64.15,35.83,47.60)	(35.42,82.25,50.91,65.64)	(31.39,96.24,63.73,80.83)
0.8	0.2	(69.70,24.03,9.28,15.96)	(45.45,59.73,32.23,45.46)	(31.39,96.24,63.73,80.83)
1.0	0.0	(119.50,0.99,0.2483,0.608)	(52.28,41.96,19.61,30.51)	(31.39,96.24,63.73,80.83)

Again, from Table 3, it is seen that for fixed α as the weight w_1 increases and w_2 decreases then the equilibrium levels of susceptible human gradually increases and the equilibrium level of infected human, recovered human and Vibrio Cholerae in the environment gradually decrease. It is also seen that when $\alpha = 1$ and for different combinations of w_1 and w_2 the equilibrium levels of all the populations remain same. These happen because for $\alpha = 1$, then the left and right intervals of triangular fuzzy number coincide with each other.

For this data set Fig.5 has been drawn for $\alpha = 0.6$. From this figure, as the weight w_1 increases and w_2 decreases then the equilibrium level of susceptible human gradually increases and the other three populations such as infected human, recovered human and Vibrio Cholerae in the environment gradually decrease.

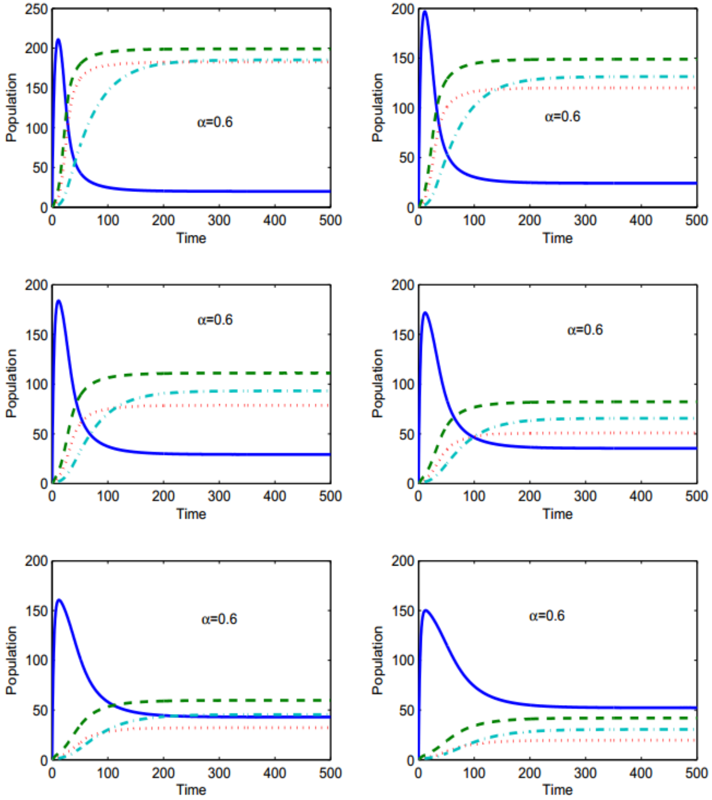

Fig.5 Solid line represents susceptible human, dash line represents infected human, dotted line represents recovered human and dash dot line represents Vibrio Cholerae in the environment

Also, for the same set of parametric values used in Problem 3 considering fixed weight values such as $w_1 = 0.4, w_2 = 0.6$ and for different values of α the Fig.6 has been drawn. From this figure, it is seen that as the value of α increases then all the human and bacterial populations gradually decrease. Now, from the above discussion, it is concluded that the interaction between human and bacterial population depends on the imprecise nature of the biological parameters.

Now, for the same set of parametric values used in Problem 3 taking different values of α, w_1, w_2 Fig.7 has been drawn. In this figure, infected human and recovered human be plotted with the change of Vibrio Cholerae in the environment. From this figure, it is seen that the intersecting points of infected human and recovered human is influenced by the imprecise value of the parameters. This supports that the impreciseness of parameters included in our proposed model.

Using the above parametric values used in Problem 3, Fig.8 has been drawn from which it is seen that the endemic equilibrium value will be changed with the change of α. So, it is concluded that the endemic equilibrium point is influenced by the imprecise values of the parameters.

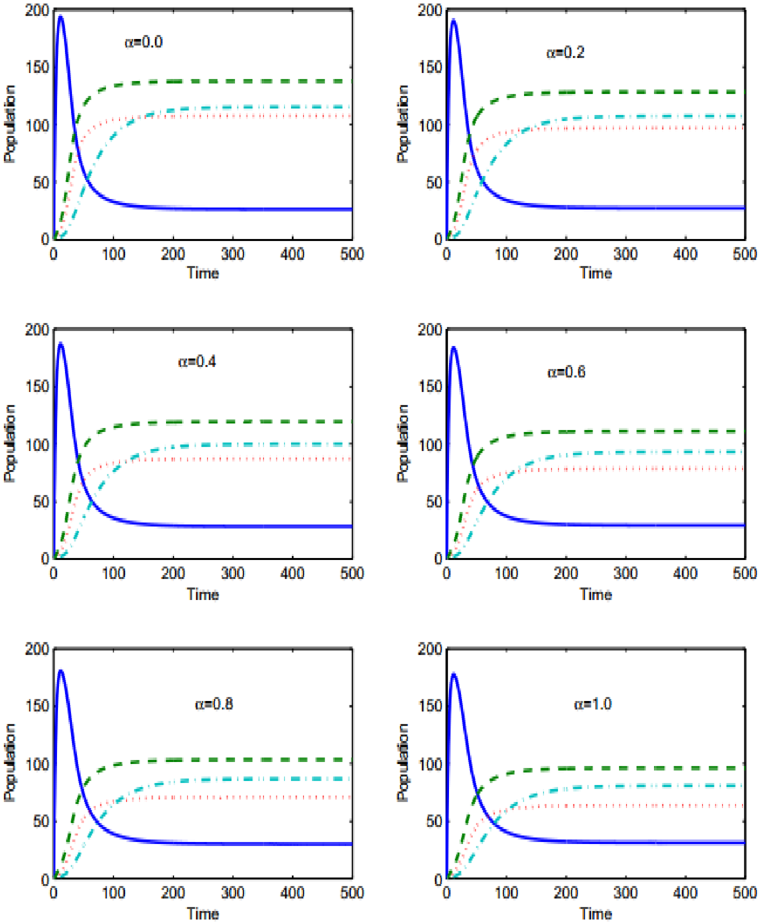

Fig.6 Solid line represents susceptible human, dash line represents infected human, dotted line represents recovered human and dash dot line represents Vibrio Cholerae in the environment

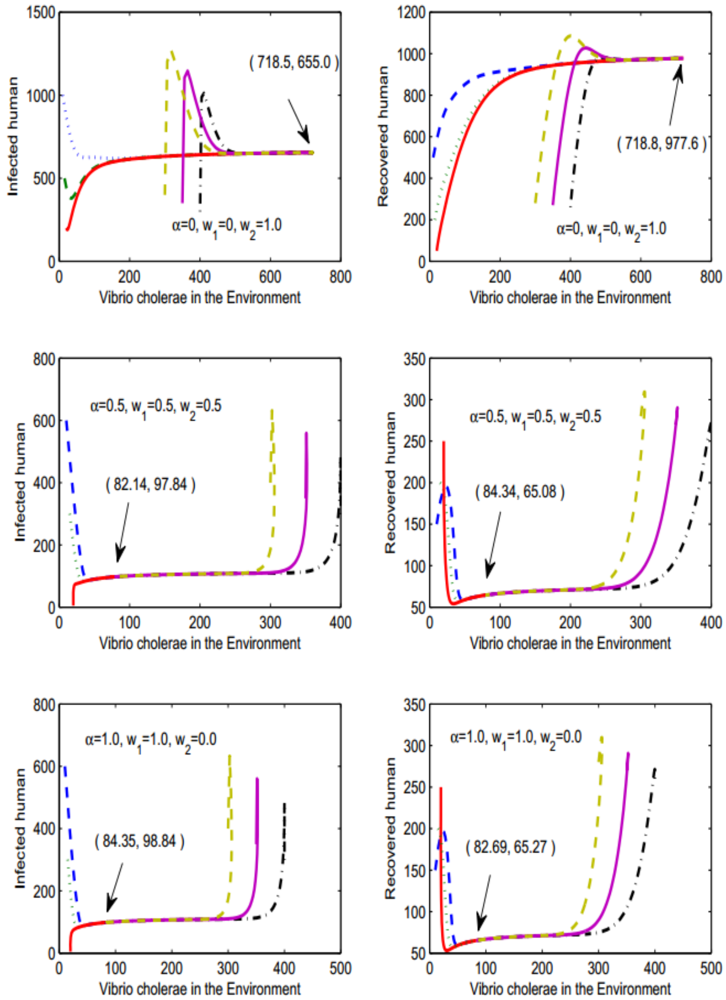

Fig.7 Phase space trajectories of infected human and recovered human with respect to Vibrio Cholerae in the environment

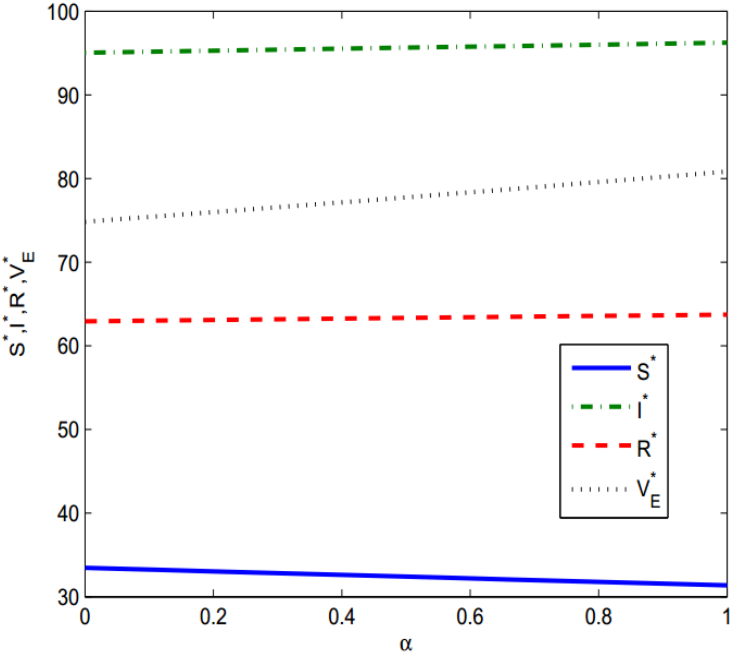

Fig.8 Change of endemic equilibrium point with respect to α

7. Conclusion

In this paper, a cholera model has been considered incorporating the fuzzyness in all biological parameters due to its natural variability. Here, total human population is divided into three subpopulations such as susceptible people, infected ones, recovered ones and a bacterial population consist of Vibrio Cholerae in the environment. The stability analysis of this fuzzy cholera disease model has been done and it is shown that the system will be disease free and endemic for certain value is less than one or grater than one. The global stability analysis of the fuzzy cholera disease model has been included in this paper around the endemic equilibrium point. In numerical simulations, the disease free and endemic equilibrium points have been computed. For fixed value of α and different combinations of weight values w_1 and w_2 we draw different figures from which, it is concluded that human and bacterial populations have been greatly influenced by imprecise value of the parameters. So, we can say that the fuzzy model is more realistic than the corresponding crisp model since crisp ones are the particular case of fuzzy model.

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