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Theoretical macrobending loss in single-mode transmission through a uniform index fiber-optic cable

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Abstract: The macrobending loss of propagating waves in optical fibers is studied here by employing simple toroidal coordinates (r, θ, ϕ) for the governing Maxwell equations of electromagnetism, the solution for which is expressed by the central axial component Π_ϕ of the Hertz vector Π. The refractive indices of the core and its cladding are supposed uniform all along the length of the cable. Effects of coating, Jacketing and the effect of elastic strain in the fiber are not considered. Knowing that the bending losses are of different nature when the radius of curvature R exceeds or is less than a certain critical value R_c, the study is accordingly divided in to two parts. Firstly, when $R > R_c$ wave guide action holds, and it is shown through an example that it is small, varying linearly with the bending angle ϕ of the cable and practically independent of R. On the other hand, when $R < R_c$, leaking waves are propagated and the problem is remodeled by a leaking ring source of EM waves in a medium consisting of the cladding only. The analysis of the model results in a bending loss formula that varies as ϕ^3 and depends on R by a factor of the form $e^{-\alpha R}/R$.

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1. Introduction

It is well known in practice that when a fiber-optic cable is coiled or even bent, loss of power transmission takes place due to a curvature of the cable. This may lead to adverse effect on the performance of transmission of information through it. The physical effects of bending on signal propagation through the core is generally divided in to two categories: Microbending effects in which irregular small cracks may be formed and changes in the temperature of the core, and Macrobending effects due to change in the reflection pattern inside the bent core (Qiu *et al*. [1], Zendehnam *et al*. [2], Martins *et al*. [3]). In the latter case usually the radius of curvature of the bend is considered to be much greater than the radius of the core of the cable. This paper endeavors to address theoretically the problem from a new perspective. It is particularly well known that macrobending can significantly increase losses in fibers when a certain critical radius R_c of the curvature is reached. The value of R_c is often tens of centimeters for single mode fibers with large mode areas, compared to the small mode ones (Paschotta [4]). A theoretical formula by Marcuse [5] is often quoted for the bending loss when the radius of curvature is less than R_c; while it is assumed that the losses are small when the radius exceeds R_c. The formula is based on cylindrical radiation leak, as in a bent slab, detailed in the author's book [6]; where as the domain of the cable should be considered as a torus rather than a slab. Later Marcuse [7] has developed another theory of mode coupling for microbending losses. As alternative to these theories, numerical methods such as the beam propagation method in dielectric wave guides have also been proposed in combination with conformal mapping (Scarmozzino *et al*. [8], Heiblum and Harris [9], Scermer and Cole [10]). Other noteworthy treatments are those by Wang et. a. [11] and Wang *et al*. [12]. The above quoted papers report reasonable agreement with experimental results. Kaufman *et al*. [13] present an integral method to incorporate higher modes for multimode fibers.

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In this paper the macrobending problem is analyzed by assuming the bent cable as a circular tore employing simple toroidal coordinate system (r, θ, ϕ) (cf. Eq. (1)). Firstly, if it is assumed that the radius R of the tore is sufficiently large ($R>R_c$) such that the principle of total internal reflection in the core of the cable is not violated. As the core is assumed to be of single index principal mode wave guide, the solution of the Maxwell equations (Stratton [14], pp. 28–32) employed to express the associated electromagnetic field in terms of the single component Π_ϕ of the Hertz vector Π along the circular central axis of the tore. In order to check the effect of finite R, the solution for Π_ϕ is considered to be of the order of a/R, where $a\,(<<R)$ is the radius of the core of the cable. By application of the boundary conditions at the interface with the cladding at $r = a$, it is found that the dispersion equation of propagation remains substantially the same as in the case of a straight cable so that the dispersion equation remains as given in Bose [15]. Next using a techmique described in Stratton [14], pp. 542–544, a comparison of the energy flow in the axial ϕ and the radial r direction, leads to the attenuation coefficient of the propagating waves in the arc of the tore. A computation shows that the attenuation coefficient is practically proportional to ϕ, and of small amount independent of a/R. However, if there are a number of turns in a coiled cable, left in an installation, there will be multifold loss in transmission. In the other case of $R<R_c$, as there is leakage in the optical transmission from the cable in to its cladding, a model radiating ring of the cable is considered. By elaborate analysis as in the case of $R>R_c$, a formula for the attenuation coefficient is found, which has some similarity with that propounded by Marcuse [6]. The present formula for bend loss has a cubic dependence on ϕ. The formula has however limited applicability because of the coating and jacketing employed to protect the cable.

2. Formulation of the problem

The bent optical fiber is assumed to be a circular arc of central radius R, the core radius being a. The geometry of the problem is suggestive of the use of a simple toroidal system of coordinates (r, θ, ϕ), where (r, θ) are the polar coordinates of a field point P in a plane section of the cable and ϕ is is the angular sweep of circular arc of its length s through the point P so that $s = (R + r\cos\theta)\,\phi$. Suppose that the plane of the tore is taken as the x, y-plane with the center of the tore as the origin, and the coordinate z in the perpendicular direction, then one gets

$$x = (R + r\cos\theta)\cos\phi,\\ y = (R + r\cos\theta)\sin\phi,\ \ z = r\sin\theta \tag{1}$$

In the toroidal system the metric element ds of the fiber length is given by the expressions

$$h_r = 1,\quad h_\theta = r,\quad h_\phi = R + r\cos\theta \tag{2}$$

Let the refractive indices of the fiber and its cladding be respectively n_1 and n_2 ($<n_1$). A digitally coded optical pulse transmitted through the cable then travels as a totally reflected guided wave that can be decomposed in to a number of modes of propagating waves (Born and Wolf [15], p. 19). As higher order mode waves undergo rapid attenuation, the fundamental least frequency mode is often employed with advantage. The attenuation of such waves due to a bend in the fiber cable is studied here. Optical wave propagation is governed by Maxwell equations of electromagnetism in terms of electric and magnetic intensities $\mathbf{E}$ and $\mathbf{H}$ respectively. Alternately, an electromagnetic field can be expressed as a single vector field $\mathbf{\Pi}$ called the Hertz vector, such that

$$\mathbf{E} = \nabla\times\nabla\times\mathbf{\Pi},\quad \mathbf{H} = \epsilon_1\frac{\partial}{\partial \mathbf{t}}(\nabla\times\mathbf{\Pi}) \tag{3}$$

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(Stratton [14], p.32) taking into account the dielectric property of the glass core. The vector $\mathbf{\Pi}$, following the Maxwell equations is shown there to satisfy the equation

$$\nabla \times \nabla \times \mathbf{\Pi} - \nabla(\nabla \cdot \mathbf{\Pi}) + \mu_1 \epsilon_1 \frac{\partial^2 \mathbf{\Pi}}{\partial t^2} = 0 \tag{4}$$

where ϵ_1 and μ_1 are respectively the electric permittivity and the magnetic permeability of the glass material. As the propagation of waves along the axis of the cable is under study, the vector $\mathbf{\Pi}$ is considered to be aligned in that direction , so that it can be expressed as $\mathbf{\Pi} = \Pi_\phi \mathbf{i}_\phi$ where ($\mathbf{i}_r$, $\mathbf{i}_\theta$, $\mathbf{i}_\phi$) are unit vectors in the directions of r, θ and ϕ at the field point P. Accordingly, following Stratton [14], pp.49–50, using Eq. (3), to a first order in r/R, which is quite small for a fiber, the components of the vectors $\mathbf{E}$ and $\mathbf{H}$ in the core are respectively

$$\begin{aligned} E_r^{(1)} &= \frac{1}{R + r\cos\theta} \frac{\partial^2 \Pi_\phi}{\partial\phi \partial r}, \\ E_\theta^{(1)} &= \frac{1}{r(R + r\cos\theta)} \frac{\partial^2 \Pi_\phi}{\partial\phi \partial\theta}, \\ E_\phi^{(1)} &= -\Big[\frac{\partial^2 \Pi_\phi}{\partial r^2} + \frac{R}{R + r\cos\theta} \frac{1}{r} \frac{\partial \Pi_\phi}{\partial r} \\ &\quad + \frac{1}{r^2} \frac{\partial^2 \Pi_\phi}{\partial \theta^2}\Big] \end{aligned} \tag{5}$$

$$\begin{aligned} H_r^{(1)} &= \frac{\epsilon_1}{r} \frac{\partial}{\partial t}\Big(\frac{\partial \Pi_\phi}{\partial \theta}\Big), \\ H_\theta^{(1)} &= -\epsilon_1 \frac{\partial}{\partial t}\Big(\frac{\partial \Pi_\phi}{\partial r}\Big), \; H_\phi^{(1)} = 0 \end{aligned} \tag{6}$$

in which the component Π_ϕ of the vector $\mathbf{\Pi}$ satisfying Eq. (4), is governed by the toroidal wave equation having variable coefficients:

$$\begin{aligned} &\frac{\partial^2 \Pi_\phi}{\partial r^2} + \frac{R}{R + r\cos\theta} \frac{1}{r} \frac{\partial \Pi_\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Pi_\phi}{\partial \theta^2} \\ &+ \frac{1}{(R + r\cos\theta)^2} \frac{\partial^2 \Pi_\phi}{\partial \phi^2} = \frac{1}{c_1^2} \frac{\partial^2 \Pi_\phi}{\partial t^2} \end{aligned} \tag{7}$$

where $c_1 = 1/\sqrt{\mu_1 \epsilon_1}$ is the velocity of light for the glass medium core of the fiber. c_1 is related to the refractive index n_1 by the relation $c_1 = c/n_1$ where c is the velocity of light in vacuum. For simplicity in the present analysis, Eq. (7) is simplified in to the form

$$\begin{aligned} &\frac{\partial^2 \Pi_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Pi_\phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Pi_\phi}{\partial \theta^2} + \frac{\partial^2 \Pi_\phi}{\partial s^2} \\ &= \frac{1}{c_1^2} \frac{\partial^2 \Pi_\phi}{\partial t^2} \end{aligned} \tag{8}$$

as $r << R$. Equation (8) is a standard form cylindrical wave equation in (r, θ, s) coordinates.

In the cladding region of the cable, the electric and magnetic intensities $E_r^{(2)}$, $E_\theta^{(2)}$, $E_\phi^{(2)}$, $H_r^{(2)}$, $H_\theta^{(2)}$, $H_\phi^{(2)}$ are similarly given by the expressions appearing in Eqs. (5) and (6) subject to Eq. (7), with ϵ_1, μ_1, c_1 replaced by ϵ_2, μ_2, and $c_2 = 1/\sqrt{\mu_2 \epsilon_2} = c/n_2$ for the permittivity, permeability and light propagation velocity of the cladding material. As little penetration of energy takes place in to the cladding, its thickness can be taken infinite; so that its domain can be extended to $r \geq 0$. As $n_1 > n_2$, $c_1 < c_2$.

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3. Solution to the first order in a/R (Case $R>R_c$)

In the core region of the cable $r \leq a$ whose solution for the fundamental mode of propagation is

$$\Pi_\phi = A\, J_0(\xi)\, e^{i(ks-\omega t)} \tag{9}$$

where $J_0(\xi),\ \xi = ur/a$ is the Bessel function of order zero and $u/a = \sqrt{\omega^2/c_1^2 = k^2}$, k and ω being the toroidal wave number and the angular frequency respectively. A is a constant. The phase velocity of the wave represented by Eq. (9) is $c_p = \omega/k$. For guided waves $c_1<c_p<c_2$, and so u is a real quantity. Using the solution (9) in Eqs. (5) and (6), the electric and magnetic intensities in the core of the fiber cable ($0 \leq r \leq a$) to first order in a/R (as that leads to finite order terms), turn out as

$$\begin{aligned} E_r^{(1)} = \frac{iku}{a}\Big[&-A\, J_1(\xi) \\ &+ \frac{a}{uR}\Big\{(1+iks)\, A\, J_0(\xi)\Big\} \times \cos\theta\Big]\, e^{i(ks-\omega t)} \end{aligned} \tag{10}$$

$$E_\theta^{(1)} = -\frac{ik}{R}\,(1+iks)\, A\, J_0(\xi)\sin\theta\, e^{i(ks-\omega t)} \tag{11}$$

$$\begin{aligned} E_\phi^{(1)} =& \frac{u^2}{a^2}\Big[A\, J_0(\xi) - \frac{a}{uR}\Big\{(1-2iks)\, A\, J_1(\xi)\Big\} \\ &\times \cos\theta\Big]\, e^{i(ks-\omega t)} \end{aligned} \tag{12}$$

$$H_r^{(1)} = i\omega\epsilon_1\, \frac{a}{uR}\Big[iks\xi\, A\, J_0(\xi)\Big]\sin\theta\, e^{i(ks-\omega t)} \tag{13}$$

$$\begin{aligned} H_\theta^{(1)} =& i\omega\epsilon_1\, \frac{u}{a}\Big[-A\, J_1(\xi) + \frac{a}{uR}\Big\{iks\, A\, J_0(\xi)\Big\} \\ &\times \cos\theta\Big]\, e^{i(ks-\omega t)} \end{aligned} \tag{14}$$

and $H_\phi^{(1)} = 0$. The magnetic intensities in Eqs. (13), (14) however are expressed in terms of the electric permittivity ϵ_1. In order to express the expressions in terms of the magnetic permeability μ_1, let k_1 be the wave number of a plane wave solution travelling in the s direction of Eq. (8); then $k_1^2 = \omega^2/c_1^2 = \mu_1\epsilon_1\omega^2$, so that the factor $\omega\epsilon_1$ can be replaced by $k_1^2/\mu_1\omega$ in the two Eqs. (13) and (14).

In the cladding material, modeled as an infinite medium $r \geq a$, the Hertz vector Π_ϕ satisfies a toroidal wave equation with c_1 replaced by $c_2 = 1/\sqrt{\mu_2\epsilon_2}$, ϵ_2, μ_2 being the electric permittivity and the magnetic permeability of the cladding material respectively. The model domain of the cladding domain can sustain only diverging waves from the axis of the cable. Hence, the solution for Π_ϕ is like that of Eq. (9) except that the Hankel functions of the first kind $H_0^{(1)}(wr/a)$ and $H_1^{(1)}(wr/a)$ appear in place of $J_0(ur/a)$ and $J_1(ur/a)$, where $w/a = \sqrt{\omega^2/c_2^2 - k^2}$ which is an imaginary quantity as $c_p = \omega/k<c_2$. Writing $w = iv$, the two Hankel functions of imaginary argument can be written in terms of the modified Bessel functions of the second kind $K_0(vr/a)$ and $K_1(vr/a)$ (Abramowitz and Stegun [17], section 9.6.1, p. 374). Thus as in the case of derivation of Eq. (9), the solution for the Hertz vector Π_ϕ in the cladding domain takes the form

$$\Pi_\phi = C\, K_0(\eta)\, e^{i(ks-\omega t)} \tag{15}$$

in which $\eta = vr/a$. The electric and magnetic intensity components in the cladding domain can be written down using Eqs. (5) and (6). The tangential components of these vectors in the present study are $E_\phi^{(2)}$ and $H_\theta^{(2)}$ given by the expressions

$$E_\phi^{(2)} = -\frac{v^2}{a^2}\left[C\,K_0(\eta) + \frac{a}{Rv}\left\{(1-2iks)\right.\right.$$
$$\left.\left.\times\, C\,K_1(\eta)\right\}\cos\theta\right] e^{i(ks-\omega t)} \tag{16}$$

and

$$H_\theta^{(2)} = \frac{ik_2^2}{\mu_2\omega}\,\frac{v}{a}\left[-C\,K_1(\eta)\right.$$
$$\left.+\frac{a}{Rv}\left\{iks\,C\,K_0(\eta)\right\}\cos\theta\right] e^{i(ks-\omega t)} \tag{17}$$

where $k_2^2 = \omega^2/c_2^2 = \mu_2\epsilon_2\omega^2$. Also $H_\phi^{(2)} = 0$, and

$$E_\theta^{(2)} = -\frac{ik}{R}\,(1+iks)\,C\sin\theta\,K_0(\eta)\,e^{i(ks-\omega t)} \tag{18}$$

4. Boundary conditions

At the interface of the core and the cable, the tangential components of the electric and magnetic intensity must equal, that is, $E_\phi^{(1)} = E_\phi^{(2)}$, and $H_\theta^{(1)} = H_\theta^{(2)}$ at $r = a$, or $\xi = u$ and $v = \eta$. In satisfying these conditions, great deal of simplification occurs in the expressions since the properties of the core and the cladding hardly differ so that one may assume $\mu_2 \approx \mu_1$, $k_2 \approx k_1$. In this way, Eqs. (12), (16), and (14). Equation (17) yields the pair of equations

$$u^2 J_0(u)\,A + v^2 K_0(v)\,C = 0 \tag{19}$$

$$uJ_1(u)\,A - vK_1(v)\,C = 0 \tag{20}$$

Equations (19) and (29) on elimination of A and C results in the dispersion equation of propagation

$$\frac{1}{u}\,\frac{J_1(u)}{J_0(u)} + \frac{1}{v}\,\frac{K_1(v)}{K_0(v)} = 0 \tag{21}$$

a result obtained in Bose [15] for a straight fiber-optic cable without any bend. It may be noted that there are two other boundary conditions at $r = a$, viz. $E_\theta^{(1)} = E_\theta^{(2)}$ and $H_\phi^{(1)} = H_\phi^{(2)}$. While the latter condition is satisfied exactly, the two sides being zero, the condition on the transversal component of the electric intensity is satisfied only approximately, the two sides of the equation being infinitesimally small for large values of R. The dispersion equation is approximate in that sense.

5. Bend attenuation

The flow of electromagnetic energy per unit area per unit time in space is represented by the Poynting vector $\mathbf{S} = \mathbf{E}\times\mathbf{H}$. For a monochromatic field, the time average of this vector becomes $\bar{\mathbf{S}} = \frac{1}{2}\,\mathbf{E}\times\mathbf{H}^*$, in which the asterisk denotes complex conjugation (Born and Wolf [16], p. 34). The real part of $\bar{\mathbf{S}}$ actually represents the physical time averaged energy flow vector. The average energy in the core per unit area along the length of the cable is thus

$$\bar{S}_\phi = \frac{1}{2}\left[E_r^{(1)}H_\theta^{(1)*} - E_\theta^{(1)}H_r^{(1)*}\right] \tag{22}$$

where the elecromagnetic field components in the above equation are given by the Eqs. (10), (11) and (13), (14). Substitution of the expressions given in these four equations leads to its real part as

$$Re(\bar{S}_\phi) = \frac{1}{2}\,\omega k\epsilon_1\,A^2\,\frac{u^2}{a^2}\left[J_1^2(\xi)\right.$$
$$\left. - \frac{a}{uR}\,J_1(\xi)\,J_0(\xi) + \frac{a}{uR}\,k^2\phi^2\xi\,J_0^2(\xi)\,\frac{\sin^2\theta}{r}\right] \tag{23}$$

Therefore the total energy flow across a section of the core of the fiber is

$$\begin{aligned}\bar{W}_\phi &= \int_0^a\int_0^{2\pi} Re(\bar{S}_\phi)\,r\,d\theta\,dr\\ &= \omega k\epsilon_1\pi\,A^2\int_0^u\left[J_1^2(\xi) - \frac{a}{uR}\,J_0(\xi)J_1(\xi)\right.\\ &\left.+\frac{1}{2}\,\frac{k^2a^2}{u^2}\,\phi^2\,J_0^2(\xi)\right]\xi\,d\xi\end{aligned} \tag{24}$$

Next, consider the average energyflow $\bar{S}_r$ per unit area in the outward radial direction at the boundary of the core. This quantity is given by the simple expression

$$\bar{S}_r = -\frac{1}{2}\,E_\phi^{(1)}H_\theta^{(1)*} \tag{25}$$

as $H_\phi^{(1)*} = 0$. In the left hand side of the above equation, the full form of $E_\phi^{(1)}$ given by

$$\begin{aligned}E_\phi^{(1)} = \frac{u^2}{a^2}&\left[A\,J_0(\xi)\left(1 + \frac{k^2a^2}{u^2}\,\phi^2\right)\right.\\ &- \frac{a}{uR}\,A\,J_1(\xi)(1-2iks)\;\cos\theta\\ &\left.+\frac{ika^2}{u^2R^2}\,\phi\,A\,J_0(\xi)\cos^3\theta\right]e^{i(ks-\omega t)}\end{aligned} \tag{26}$$

is used. Equation (26) reduces to Eq. (12) when ϕ is approximated as s/R and terms of the order of $1/R^2$ are neglected. The corresponding expression for $H_\theta^{(1)}$ is given in Eq. (14). Using these expressions in Eq. (25), the real part of $\bar{S}_r$ is given by

$$\begin{aligned}Re(\bar{S}_r) &= \omega k\epsilon_1\,\frac{u^2}{2a^2}\,\phi\,A^2\left[2J_1^2(\xi)\cos\theta\right.\\ &+ J_1(\xi)\,\frac{a}{uR}\,J_0(\xi)\cos^3\theta + J_0(\xi)\cos\theta\\ &\left.\times\left\{J_0(\xi)\left(1+\frac{k^2a^2}{u^2}\phi^2\right) - \frac{a}{uR}\,J_1(\xi)\cos\theta\right\}\right]\end{aligned} \tag{27}$$

Now, due to total internal reflection towards the axis of the fiber, the quantity of interest is the energy flow towards the central axis, which per unit length of the fiber is given by the expression

$$\begin{aligned}\bar{W}_r &= -\int_{-\pi/2}^{\pi/2} Re(\bar{S}_r)\Big|_{r=a}\,a\,d\theta\\ &+\int_{\pi/2}^{3\pi/2} Re(\bar{S}_r)\Big|_{r=a}\,a\,d\theta\end{aligned} \tag{28}$$

where $r = a$ means that $\xi = u$. Using Eq. (37) in (38), one obtains

$$\begin{aligned}\bar{W}_r = -\frac{2u^2}{a^2}\,\omega k\epsilon_1 a\,\phi\,A^2&\left[\left(1+\frac{k^2a^2}{u^2}\,\phi^2\right)\right.\\ &\left.\times J_0^2(u) + 2J_1^2(u) + \frac{2}{3}\,\frac{a}{uR}\,J_0(u)J_1(u)\right]\end{aligned} \tag{29}$$

The negativity of $\bar{W}_r$ implies that the energy flows radially outwards.

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The attenuation coefficient due to the bend is now estimated from the expressions for $\bar{W}_\phi$ and $\bar{W}_r$ given in Eqs. (24) and (29) following an argument given in Stratton [14], pp. 543–544. The two expressions are derived on the premise that the wave number k of the waves propagating in the s-direction is lossless. However, since damping is present due to the bend in the cable, the quantity k is complex with an imaginary part say k', which is an attenuation measure. Thus the right hand sides of the expressions for $\bar{S}_\phi$ and $\bar{S}_r$ and their real parts $Re(\bar{S}_\phi)$ and $Re(\bar{S}_r)$ would contain the factor $\exp(-2k's)$. The rate at which $\bar{W}_\phi$ decreases along the length s is therefore

$$\frac{d\bar{W}_\phi}{ds} = -2k'\,\bar{W}_\phi = \bar{W}_r \tag{30}$$

by radial out flow of energy. Hence,

$$k' = -\frac{1}{2}\,\frac{\bar{W}_r}{\bar{W}_\phi} \tag{31}$$

Thus, using the Eqs. (24) and (29), the nondimensional attenuation coefficient due to the bend of the cable is given by the formula

$$\frac{k'}{k} = \frac{\phi}{\pi\,ka}\,\frac{(1+\frac{k^2a^2}{u^2}\,\phi^2)J_0^2(u) + 2J_1^2(u) + \frac{2}{3}\,\frac{a}{uR}\,J_0(u)J_1(u)}{\int_0^1[J_1^2(uz)+\frac{1}{2}\,\frac{k^2a^2}{u^2}\,\phi^2J_0^2(uz)-\frac{a}{uR}J_0(uz)J_1(uz)]zdz} \tag{32}$$

where the change of variable $\xi = uz$ is made in the integral appearing in Eq. (24).

As a representative numerical example, a fiber radius of $a = 0.2$ mm and a bend radius of $R = 30$ mm are chosen, with refractive indices of $n_1 = 1.5$ and $n_2 = 1.48515$ respectively for the fiber glass and its cladding respectively. The wave length of the transmitted light is taken near to that of the infrared portion of the spectrum viz. $2\pi/k = 1.5\ \mu$m so that $ka = 837.758$. For calculating u for the above stated data, it is noted that $u = ka\sqrt{c_p^2/c_1^2 - 1}$ and $v = ka\sqrt{1 - c_p^2/c_2^2}$, where $c_p = \omega/k$ is the phase velocity of propagation in the fiber core. Eliminating c_p from the preceding two equations and using the relation $c_1^2/c_2^2 = n_2^2/n_1^2$, one has

$$v = ka\sqrt{1 - \frac{n_2^2}{n_1^2}\left(1 + \frac{u^2}{k^2a^2}\right)} \tag{33}$$

Now using Eq. (33) in the dispersion Eq. (21), an equation for u is obtained that can be numerically solved by the bisection method. The root of the equation for which c_p is closest to c_1, the value of u is obtained as $u = 3.79939$. For these data, k'/k given by Eq. (32) computed by using the approximation formulae for the Bessel functions given in Abramowitz and Stegun [16] is plotted against ϕ in Fig. 1, for a complete loop of the cable. It is noteworthy that the plot is practically linear with a slope of 0.15178×10^{-2}. For a 90^o bend $\phi = \pi/2$, the nondimensional loss is 0.23842×10^{-2}, and that for a turn around $\phi = \pi$, its value is 0.47676×10^{-2}. For a complete coil $\phi = 2\pi$, for which the value is 0.95370×10^{-2}. In decibels, the bending loss are respectively 0.43373×10^5, 0.86732×10^5 and 0.17350×10^6. The dB values are obtained by multiplying the k' values by the factor 4.343.

The linearity of the plot in Fig. 1 is due to the fact that the coefficients of the ϕ^2 term in both the numerator and denominator of Eq. (32) are of the same order k^2a^2/u^2 which is large. Evidently this linearity of attenuation would prevail for other fiber specifications also because of largeness of the value of the parameter ka for the spectra used in general.

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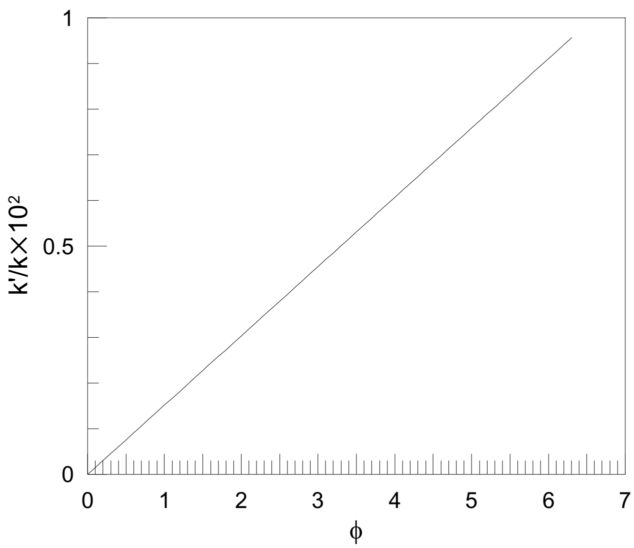

Fig. 1. Attenuation versus bend angle for $R>R_c$.

6. High curvature bend loss (Case $R<R_c$)

When the radius of curvature R is less than a certain critical value R_c, the phenomenon of total internal reflection in the core of the cable breaks down and part of the energy is leaked through the cladding material (Que *et al.* [1]). For such leaking mode of propagation, consider a complete ring of the optical fiber in the x, y-plane given by Eq. (1), where $0 \leq \phi \leq 2\pi$. The energy flowing along the axis of the fiber $\bar{W}_\phi$ by total internal reflection is given by Eq. (24), as argued in the preceding sections. But, for leaked energy in case of high curvature of the fiber, one may view the fiber ring as a source of radiating energy outwards in an infinite space composed of the cladding material. Thus, one can employ Eq. (8) for the Hertz vector in the medium with the modification that

$$\begin{aligned} &\frac{\partial^2 \Pi_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Pi_\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Pi_\phi}{\partial \theta^2} + \frac{\partial^2 \Pi_\phi}{\partial s^2} \\ &- \frac{1}{c_2^2}\frac{\partial^2 \Pi_\phi}{\partial \phi^2} = C'\,\delta(x-R)\,\delta(z) \end{aligned} \tag{34}$$

where $s = (R + r\cos\theta)\,\phi$ and C' is the strength of the source with $\delta(\cdot)$ as the Dirac delta function representing its distribution. In order to find a suitable particular solution of Eq. (34), let $\Pi_\phi = \psi\, e^{i(ks-\omega t)}$ and consider the transformation $x = R + r\cos\theta$, $z = r\sin\theta$. This leads to the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{v^2}{a^2}\psi = C'\,\delta(x-R)\,\delta(z) \tag{35}$$

where $v/a = \sqrt{k^2 - \omega^2/c_2^2} = k\sqrt{1 - c_p^2/c_2^2} = k\sqrt{1 - 1/n_2^2}$ is a real quantity as $n_2>1$. Introducing the double cosine Fourier transform

$$\begin{aligned} \bar{\psi} &= \int_0^\infty\!\!\int_0^\infty \psi(x,\,z)\,\cos\xi x\,\cos\zeta z\,dx\,dz; \\ \psi &= \frac{4}{\pi^2}\int_0^\infty\!\!\int_0^\infty \bar{\psi}(\xi\,\zeta)\,\cos x\xi\,\cos z\zeta\,d\xi\,d\zeta \end{aligned} \tag{36}$$

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Equation (45) yields

$$\begin{aligned} &-\left(\xi^2+\zeta^2+\frac{v^2}{a^2}\right)\bar{\psi} \\ &= C'\int_0^\infty\int_0^\infty \delta(x-R)\delta(z)\cos\xi x\cos\zeta z\,dx\,dz \\ &= \frac{C'}{2}\cos\xi R \end{aligned} \tag{37}$$

Hence,

$$\psi = -\frac{2C'}{\pi^2}\int_0^\infty\int_0^\infty \frac{\cos\xi R\cos x\xi\cos z\zeta d\xi\,d\zeta}{\xi^2+\zeta^2+v^2/a^2} \tag{38}$$

The double integral of Eq. (38) can be evaluated exactly by first evaluating the ζ-integral and then the ξ-integral by the use of formulae 3.732, p. 406 and 3.961 (2), p. 498 given in Gradshteyn and Ryzhik [18]. In this way one gets

$$\begin{aligned} \psi &= -\frac{C'}{\pi}\left[K_0\left\{\frac{v}{a}\sqrt{(R+x)^2+z^2}\right\}\right. \\ &\left.+K_0\left\{\frac{v}{a}\sqrt{(R-x)^2+z^2}\right\}\right] \end{aligned} \tag{39}$$

or,

$$\begin{aligned} \psi &= -\frac{C'}{\pi} \\ &\times\left[K_0\left\{\frac{v}{a}\sqrt{(2R+r\cos\theta)^2+r^2\sin^2\theta}\right\}\right. \\ &\left.+K_0\left(\frac{v}{a}r\right)\right] \end{aligned} \tag{40}$$

In the above Eq. (40), it is recognized that the second term $K_0(vr/a)$ is a solution obtained earlier in Eq. (15) and so comparing coefficients of the two terms it follows that

$$C' = -\pi\,C \tag{41}$$

This particular term being independent of R is not important for the present discussion, and so the particular solution represented by the first term of the equation leads to the consideration of the particular solution

$$\psi = C\,K_0\left\{\frac{v}{a}\sqrt{4R^2+r^2+4\,Rr\cos\theta}\right\} \tag{42}$$

or, as $K_0(x) = \frac{\pi i}{2}H_0^{(1)}(ix)$, $i=\sqrt{-1}$, $H_0^{(1)}(\cdot)$ being the Hankel function of the first kind,

$$\begin{aligned} \psi &= C\frac{\pi i}{2}H_0^{(1)}\left\{\frac{iv}{a}\sqrt{4R^2+r^2+4\,Rr\cos\theta}\right\} \\ &= C\sum_{n=-\infty}^{\infty}K_n\left(\frac{2Rv}{a}\right)J_n\left(\frac{ivr}{a}\right),\ \text{for } R>r \end{aligned} \tag{43}$$

by Graf's addition theorem of Bessel functions (Abramowitz and Stegun [17] formula 9.1.79, p.363). Now for large E, one has the asymptotic expansion (Gradshteyn and Ryzhik [18] formula

8.451 (6), p.963)

$$K_n\left(\frac{2R\,v}{a}\right) \sim \sqrt{\frac{\pi a}{4Rv}}\, e^{-2Rv/a} \tag{44}$$

while (Abramowitz and Stegun [17] formula 9.1.46, p.361)

$$\sum_{n=-\infty}^{\infty} J_n\left(\frac{ivr}{a}\right) = 1 \tag{45}$$

Insertion of Eqs. (44) and (45) in Eq. (43), thus leads to the particular solution

$$\Pi_\phi = C\sqrt{\frac{\pi a}{4Rv}}\; e^{-2Rv/a}\; e^{i(ks-\omega t)} \tag{46}$$

The component of the electromagnetic field, for calculating the bend loss due to radiation leak are therefore after simple calculation that

$$\begin{aligned} E_\phi^{(2)} &= -\left[\frac{\partial^2\Pi_\phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial \Pi_\phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\Pi_\phi}{\partial\theta^2}\right] \\ &= C\sqrt{\frac{\pi a}{4Rv}}\, e^{-2Rv/a}\, k^2\, \phi^2\, e^{i(ks-\omega t)} \end{aligned} \tag{47}$$

$$\begin{aligned} H_\theta^{(2)} &= -\epsilon_2\,\frac{\partial}{\partial t}\left(\frac{\partial \Pi_\phi}{\partial r}\right) \\ &= -\omega\epsilon_2 kC\sqrt{\frac{\pi a}{4Rv}}\, e^{-2Rv/a}\cos\theta\,\phi\, e^{i(ks-\omega t)} \end{aligned} \tag{48}$$

where the relation $s = (R + r\cos\theta)\,\phi$ is used. Hence, as in Eq. (25), the average energy flow in the radial direction of the cable is

$$\begin{aligned} S_r^{(2)} &= -\frac{1}{2}\, E_\phi^{(2)}\, H_\theta^{(2)*} \\ &= C^2\,\frac{\pi a}{8Rv}\, e^{-4Rv/a}\,\omega\epsilon_2 k^3\,\phi^3\cos\theta \end{aligned} \tag{49}$$

and so as in the development of Eq. (28), the inward energy flow towards the axis of the core per unit length of the fiber is

$$\bar{W}_r = -C^2\,\frac{\pi a}{4Rv}\, e^{-4Rv/a}\, a\omega\epsilon_2 k^3\,\phi^3 \tag{50}$$

The internal axial flow in the bent cable however remains the same as due to total internal reflection. As such the total energy flow across a section of the fiber is again given by Eq. (34), with the provision that the term of the order of a/R is dropped as has been done in this section. Hence this energy flux is expressed as

$$\begin{aligned} \bar{W}_\phi = \omega k\epsilon_1\,\pi A^2 \int_0^u &\Big[J_1^2(\xi) \\ &+ \frac{1}{2}\,\frac{a^2k^2}{u^2}\,\phi^2\, J_0^2(\xi)\Big]\,\xi\, d\xi \end{aligned} \tag{51}$$

Hence, following the arguments of the preceding section 5, the attenuation coefficient k' is estimated as

$$k'/k = -\frac{\bar{W}_r}{\bar{W}_\phi} = \frac{e^{-4Rv/a}}{4Rv/a}\;\frac{aku^2\, J_0^2(u)\,\phi^3}{v^4\, K_0^2(v)\,\int_0^1 [J_1^2(uz) + \frac{1}{2}\,\frac{k^2a^2}{u^2}\,\phi^2\, J_0^2(uz)]\, z\, dz} \tag{52}$$

where it is taken that $\epsilon_1 \approx \epsilon_2$ and the value of C/A is obtained from Eq. (21). The bending loss formula (52) differs substantially from that of Marcuse [5], principally in respect of dependence on R and dependence on the angle ϕ - the angle subtended by the bent circular cable at the center of the circle. There is no dependence on ϕ in the Marcuse formula. Comparing Eq. (52) with Eq. (32) it is noticed that the bending loss varies faster as ϕ^3 when there is leakage compared to linear variation with ϕ when there is none.

For a numerical study of the bending loss (52), the data used in the preceding section 5 is again used, with $n_1 = 1.5$ and $n_2 = 1.48515$ with nondimensional wave number 837.758, and $u = 3.79939$. The corresponding value of v from Eq. (33) is $v = 117.53080$. As v is large, one can use the asymptotic approximation $K_0(v) = \sqrt{\pi/2v}\, e^{-v}$, so that Eq. (52) becomes

$$e^{-2v}\,\frac{k'}{k} = \frac{1}{2\pi}\,\frac{e^{-4Rv/a}}{Rv/a}\,\frac{ka\,u^2\,J_0^2(u)\,\phi^3}{v^3\int_0^1 [J_1^2(uz) + \frac{1}{2}\frac{k^2a^2}{u^2}\,\phi^2\,J_0^2(uz)]\,z\,dz} \tag{53}$$

The computed values of the bending attenuation is presented in Fig. 2, for $\phi = \pi/2,\ \pi,\ 2\pi$, the last value being that for a full ring of the cable. The scaling of both the abscissa and the ordinate is done for computationally presentable results. The curves are similar to those of Marcuse [5], showing steep fall in the attenuation with increasing values of R. However, in the Marcuse formula the dependence of k'/k is of the form $e^{-\beta R}/\sqrt{R}$ where β is independent of R. Evidently, for other specifications of the core radius and spectrum value, the attenuation property will be of similar nature.

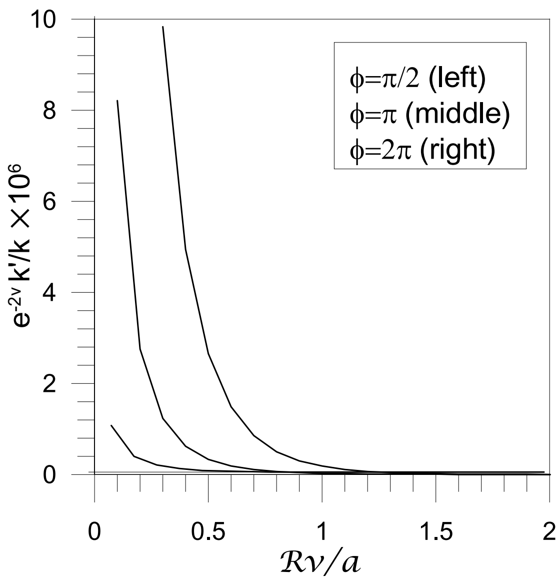

Fig. 2. Bend loss versus Rv/a for $R<R_c$.

7. Conclusion

The energy and power loss in a bent fiber-optic cable is theoretically studied in this paper. The bend of the cable is assumed to be of circular shape as in the existing literature. Moreover, only macrobending effects are only considered affecting total internal reflection through the core covered by a slightly lower refractive index. The effects of coating and jacketing of the cable, nor the small effect of strain on the refractive indices considered in the analysis. It is known in the literature that when the radius of curvature of the bent cable R is greater than a critical value R_c, the loss in total internal reflection is significantly small, but when $R<R_c$, significant

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loss due to leaked propagation takes place. Marcuse [5] has developed a formula for the loss from models different from the ones presented here.. Modeling a bent cable as a tore of radius R, generated by revolving a circle of radius a about an in-plane axis of revolution, simple toroidal coordinates (r, θ, ϕ) are employed in the analysis of the Maxwell equations of electromagnetism for the optical propagation. Working in terms of the Hertz vector $\mathbf{\Pi}$, only the component Π_ϕ is required for the axial propagation along the axis of the cable. In a detailed analysis to be reported elsewhere it is shown that the approximation of Eq. (7) to Eq. (8) to a first order in r/R does not change the final results presented here. Treating first the case of $R>R_c$ when no leakage occurs, the analysis is carried out to a first order in a/R. Involving the energy of propagation in the axial ϕ direction and the perpendicular radial r direction a formula for the bend attenuation k' is obtained as given in Eq. (32). The formula shows that the attenuation is small and practically varies as ϕ, with the implication that if the cable is coiled a number of times, then the loss will be that many fold. Next, the case $R<R_c$ is treated, where in leakage of energy takes place in the radial direction. Modeling the leaking optical tore as a ring of radiation in a medium of the cladding material, an elaborate analysis leads to a formula for the attenuation coefficient k' as in the first case. The bend loss formula Eq. (53) is akin to the formula of Marcuse [5] but with significant differences in the functional dependence on R and independence from ϕ. But for the fact that the affects of the coating and jacketing of the cable are not included, an estimate of R_c is not possible from the present results by means of comparing the two formulas (32) and (53).

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Data availability. No data were generated or analyzed in the presented research.

References

1. J. Qiu, D. Zheng, K. Zhu, B. Fang, and L. Cheng, "Optical fiber sensor experimental research based on the theory of bending loss applied to monitoring differential settlement at the earth-rock junction," J. Sensors **2015**, 1–13 (2015).
2. A. Zendehnam, M. Mirzaei, A. Farashiani, and L. H. Farahan, "Investigation of bending loss in single-mode optical fiber," Pramana **74**(4), 591–603 (2010).
3. A. Martins, A. M. Rocha, B. Neto, A. L. J. Teixeira, M. Facao, R. V. Nogueira, M. J. Lima, and P. S. Andre, "Modeling of bend losses in single-mode optical fibers," in *the 7th Conference on Telecomunnications CONFTELE 2009* (2009).
4. R. Paschotta, "Bend losses," in *Encyclopedia of Laser Physics and Technology* (RP Photonics, 2009), https://www.rp-photonics.com/bend_losses.html.
5. D. Marcuse, "Curvature loss formula for optical fibers," J. Opt. Soc. Am. **66**(3), 216–220 (1976).
6. D. Marcuse, *Light transmission optics*, (Van Nostrand Reinhold, 1982), pp. 406–414.
7. D. Marcuse, "Microdeformation losses of single-mode fibers," Appl. Opt. **23**(7), 1082–1091 (1984).
8. R. Scarmozzino, A. Gopinath, R. Prega, and S. Helfert, "Numerical techniques for modelling guided-wave photonic devices," IEEE J. Sel. Top. Quantum Electron. **6**(1), 150–162 (2000).
9. M. Heiblum and J. H. Harris, "Analysis of optical wave guides by conformal transformation," IEEE J. Quantum Electron. **11**(2), 75–83 (1975).
10. R. T. Schermer and J. H. Cole, "Improved bend loss formula verified for optical fiber by simulation and experiment," IEEE J. Quantum Electron. **43**(10), 899–909 (2007).
11. P. F. Wang and Q. Wang, "Investigation of microbending losses of standard single mode fiber with small bend radii," Microw. Opt. Technol. Lett. **49**(9), 2133–2138 (2007).
12. Q. Wang, G. Farrel, and T. Freir, "Theoretical and experimental investigation of micro-bend losses for standard single mode fibers," Opt. Express **13**(12), 4476–4484 (2005).
13. K. S. Kaufman, R. Terras, and R. F. Mathis, "Curvature loss in multimode optical fibers," J. Opt. Soc. Am. **71**(12), 1513–1518 (1981).
14. J. A. Stratton, *Electromagnetic Theory* (John Wiley & Sons, 2007).
15. S. K. Bose, "On the transmission of data packets through giber-optic cables of uniform index," J. Opt. Commun. **2021**, 000010151520210031 (2021).
16. M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, 1999).
17. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, 1972).
18. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals Series and Products* (Academic, 1965).