Acta Math. Hung.
54 (3—4) (1989), 225—236.

ON THE MAXIMAL LENGTH OF TWO SEQUENCES OF CONSECUTIVE INTEGERS WITH THE SAME PRIME DIVISORS

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§ 1. Introduction

The initial motivation of the present paper is the following problem of Erdös and Woods, whose solution would be of interest in logic (see [3, 6, 11, 10, 16]):

Does there exist an integer $k \geqq 2$ with the following property: if x and y are positive integers such that for $1 \leqq i \leqq k$, the two numbers $x+i$ and $y+i$ have the same prime factors, then $x=y$.

For each integer $n \geqq 2$, let us denote by Supp (n) the set of prime factors of n. The only known examples of positive integers (x, y, k) with $1 \leqq x < y$, $k \geqq 2$ and

$$\text{Supp}(x+i) = \text{Supp}(y+i) \quad \text{for} \quad 1 \leqq i \leqq k \tag{1.1}$$

are given by

$$k = 2, \quad x = 2^h - 3, \quad y = 2^h(2^h - 2) - 1, \quad h \geqq 2$$

and

$$k = 2, \; x = 74, \quad y = 1214.$$

Under the assumption (1.1), we shall give an upper bound for k in terms of x:

$$\log k \leqq c_1 (\log x \log\log x)^{1/2} \quad \text{for} \quad x \geqq 3 \tag{1.2}$$

and lower bounds for $y-x$ either in terms of k:

$$y - x > \exp\left(c_2 k (\log k)^2 / \log\log k\right) \quad \text{for} \quad k \geqq 3 \tag{1.3}$$

or in terms of k and y:

$$y - x > (k \log\log y)^{c_3 k \log\log y (\log\log\log y)^{-1}} \quad \text{for} \quad y \geqq 27. \tag{1.4}$$

Here c_1, c_2 and c_3 are effectively computable absolute positive constants. The inequality (1.4) with $k=1$ is Theorem 4 of Erdös and Shorey [4]; in fact this will be used in the proof of (1.4).

We consider also some related problems, where the assumption (1.1) is replaced either by

$$P(x+i) = P(y+i) \quad \text{for} \quad 1 \leqq i \leqq k \tag{1.5}$$

where $P(n)$ denotes the greatest prime factor of n and $P(1)=1$, or by

$$\text{Supp}\Big(\prod_{i=1}^{k} (x+i)\Big) = \text{Supp}\Big(\prod_{i=1}^{k} (y+i)\Big). \tag{1.6}$$

The arrangement of this paper is as follows. We, first, collect in Section 2 several auxiliary lemmas from different sources. The main tool is a lower bound for linear forms in logarithms of rational numbers, due to Baker [1]. The proofs are via estimates for the greatest square free factors of products of integers from a bloc of consecutive integers.

In Section 3, we study the problem (1.5) related to the greatest prime divisors. In Section 4, we consider the problem (1.1) of Erdös—Woods. In Section 5, we deal with the assumption (1.6).

Throughout this paper, the letters n, x, y and k will denote positive integers with $n\geqq 2$ and with $x<y$. We denote by $Q(n)$ the greatest square-free factor of n and we write $\omega(n)$ for the number of distinct prime factors of n and $\omega(1)=1$. As usual, we recall that $\pi(n)$ is the number of primes less than or equal to n.

§ 2. Preliminary results

In this section, we give several auxiliary lemmas which will be used in the next sections.

LEMMA 2.1. *Let n be a positive integer.*

a) *There exists a prime p satisfying $n<p\leqq 2n$.*

b) *There exists an effectively computable absolute constant $c_4>0$ such that for $n>c_4$, there is a prime p satisfying*

$$n<p\leqq n+c_4 n^{1/2+1/21}.$$

Part a) is the well-known "postulat de Bertrand". Part b) is due to Iwaniec and Pintz [7].

COROLLARY 2.2. *If $P(x+i)=P(y+i)$ for $1\leqq i\leqq k$, then $y>k$. If, furthermore $k\geqq c_4$, then $k<c_4 y^{23/42}$.*

PROOF OF COROLLARY 2.2. Assume $y\leqq k$. By Lemma 2.1, Part a), there exists an i with $1\leqq i\leqq k$ such that $y+i$ is a prime number. Now $P(x+i)=P(y+i)$ implies that $y+i$ divides $x+i$, which is impossible since $y>x$.

Hence $y>k$. Now if $k\geqq c_4$, by Lemma 2.1, part b), there exists an i with $1\leqq i\leqq c_4 y^{23/42}$ such that $y+i$ is a prime number, and the same argument yields $k<c_4 y^{23/42}$.

LEMMA 2.3. *We put $S=\{x+1, \dots, x+k\}$. For every prime $p\leqq k$, we choose an $f(p)\in S$ such that p does not appear to a higher power in the factorisation of any other element of S:*

$$|f(p)|_p^{-1}=\max\{|n|_p^{-1},\ n\in S\}.$$

We denote by S_1 the subset of S obtained by deleting from S all $f(p)$ with $p\leqq k$. Then

$$\prod_{n\in S_1}\prod_{p\leqq k}|n|_p^{-1} \quad \textit{divides} \quad k!.$$

On the other hand, it may be noticed that $k!$ divides $\prod_{n\in S}\prod_{p\leqq k}|n|_p^{-1}$, since the binomial coefficient $\binom{x+k}{k}$ is an integer.

PROOF. This result is due to Erdös [2], Lemma 3. For the convenience of the reader, we give here a proof.

Let us write, for each $n \geqq 2$,

$$n = \prod_p p^{v_p(n)}$$

so that

$$|n|_p = p^{-v_p(n)}.$$

For each integer $j \geqq 1$, let us write

$$u_j = \mathrm{card}\,\{n \in S_1, v_p(n) = j\}$$

and

$$v_j = \sum_{i \geqq j} u_i = \mathrm{card}\,\{n \in S_1, v_p(n) \geqq j\}.$$

Therefore v_j is the number of $n \in S_1$ which is divisible by p^j. Notice that

$$\mathrm{Card}\,\{n \in S, v_p(n) \geqq j\} \leqq \left[\frac{k}{p^j}\right] + 1.$$

Further, if the set $\{n \in S,\ v_p(n) \geqq j\}$ is not empty, then it contains $f(p)$. Hence $v_j \leqq \left[\frac{k}{p^j}\right]$ and

$$\sum_{j \geqq 1} v_j \leqq \sum_{j \geqq 1} \left[\frac{k}{p^j}\right] = v_p(k!).$$

But

$$\sum_{n \in S_1} v_p(n) = \sum_{j \geqq 1} j u_j = \sum_{j \geqq 1} v_j,$$

and Lemma 2.3 follows.

We derive from Lemma 2.3 the following result. See Erdös and Turk [5].

COROLLARY 2.4. *Denote by t the number of integers $n \in S$ such that $P(n) \leqq k$. Then*

$$t \leqq k \frac{\log k}{\log x} + \pi(k). \tag{2.5}$$

PROOF. We consider the product of the $n \in S_1$ such that $P(n) \leqq k$. This product is at least $x^{t-\pi(k)}$, and divides $k!$ by Lemma 2.3. Hence $x^{t-\pi(k)} \leqq k^k$ which implies (2.5).

If k does not exceed a power of $\log x$, inequality (2.5) can be strengthened as follows.

LEMMA 2.6. *Let $\varepsilon > 0$. Assume that $x > e^{k^\varepsilon}$ and $k \geqq 3$. Let t be defined as in Corollary 2.4. Then*

$$t \leqq c_5 k (\log k)^{-2} \log\log k$$

where $c_5 > 0$ is an effectively computable number depending only on ε.

This is Theorem 2 of [12]. The proof of Lemma 2.6 depends on the theory of linear forms in logarithms. As stated in Section 1, this theory is the main tool for our investigations. Now we state a lower bound for linear forms in logarithms. This is a special case of [1] and [15] which is relevant here.

LEMMA 2.7. *Let $p_1, \dots, p_n$ be distinct prime numbers with $n \geqq 2$ and let $b_1, \dots, b_n$ be rational integers which are not all zero. Define*

$$P = \max_{1 \leqq i \leqq n} p_i \quad \text{and} \quad B = \max_{1 \leqq i \leqq n} |b_i|.$$

Then

$$|p_1^{b_1} \dots p_n^{b_n} - 1| > \exp\left(-(c_6 n)^n (\log P)^{n+1} (\log B + \log\log P)\right)$$

with $c_6 = 2^{24}$.

The following result of Størmer is well-known, but we give a proof via Lemma 2.7 to illustrate the way we shall use Lemma 2.7.

COROLLARY 2.8. *When $m \to \infty$, $P(m(m+1))$ tends to infinity effectively.*

In particular for each $x \geqq 1$, there are only finitely many $y > x$ such that

$$P(x+1) = P(y+1) \quad \text{and} \quad P(x+2) = P(y+2)$$

and these y can be effectively determined. It follows that for each $x_0 \geqq 1$, it is a finite problem to determine all x and y with $x \leqq x_0$ and $y > x$ such that

$$\operatorname{Supp}(x+1) = \operatorname{Supp}(y+1) \quad \text{and} \quad \operatorname{Supp}(x+2) = \operatorname{Supp}(y+2).$$

Lehmer [9] applied Størmer's method on Pellian equations to investigate this problem.

PROOF OF COROLLARY 2.8. We decompose m and $m+1$ in prime factors and we write

$$\frac{1}{m} = \frac{m+1}{m} - 1 = \Big(\prod_{p \leqq P} p^{b_p}\Big) - 1$$

where $P = P(m(m+1))$, $2^{b_p} \leqq m+1$. We apply Lemma 2.7 with $B = 3\log m$ (for $m \geqq 2$) and $n \leqq \pi(P)$. We conclude that for each $\varepsilon > 0$

$$P(m(m+1)) > (1-\varepsilon)\log\log m \quad \text{for} \quad m > m_0(\varepsilon)$$

where $m_0(\varepsilon)$ is an effectively computable number depending only on ε.

We shall need the following refinement of Corollary 2.8.

LEMMA 2.9. *Let $A > 0$. Let $f(X)$ be a polynomial with integer coefficients and with at least two distinct roots. Then there exists an effectively computable number $c_7 > 0$ depending only on f and A such that for every integer $Y \geqq 27$ with $f(Y) \neq 0$, the inequality*

$$\log P(|f(Y)|) \leqq (\log\log Y)^A$$

implies that

$$\omega(|f(Y)|) \geqq c_7 \frac{\log\log Y}{\log\log\log Y}.$$

This is Theorem 2 of [13]. For an account and refinements of the results of this section, see Chapter 7 of the book by T. N. Shorey and R. Tijdeman: *Exponential Diophantine Equations*, Cambridge Tracts in Mathematics, **87** (1986).

§ 3. Sequences of consecutive integers with the same greatest prime factor

In this section, we denote by x, y and k three positive integers satisfying $x<y$, $k\geqq 1$ and

$$P(x+i)=P(y+i) \quad \text{for} \quad 1\leqq i\leqq k. \tag{3.1}$$

It has been checked on a computer that (3.1) has no solution with $y<5000$ and $k\geqq 3$. It follows from a result of Tijdeman [14] that (3.1) implies that

$$y-x \geqq 10^{-6}\log\log x.$$

Our aim is to prove the bounds

$$y-x > k^{k/2} \quad \text{and} \quad y-x > (\log\log y)^{k/2}$$

for k sufficiently large. Our actual result is sharper: the exponent $k/2$ is replaced by $k(1-\varepsilon)$ and we can even improve the ε.

PROPOSITION 3.2. *There exist effectively computable absolute positive constants c_8 and c_9 such that for $k\geqq 3$, we have*

$$y-x > k^k \exp\left(-c_8 k\,\frac{\log\log k}{\log k}\right) \tag{3.3}$$

and

$$y-x > \left(\frac{2}{3}\log\log y\right)^{k(1-c_9(\log k)^{-2}\log\log k)}. \tag{3.4}$$

PROOF OF (3.3). We choose a sufficiently large absolute constant $c_8>0$. If $(\log k)^2<c_8\log\log k$, then the right hand side of (3.3) is less than one, while $y-x\geqq 1$. Therefore we may assume that k is sufficiently large. From Corollary 2.2, we derive $y>k^{9/5}$.

Denote by t_1 the number of i with $1\leqq i\leqq k$ and $P(y+i)\leqq k$. We apply Corollary 2.4 with x replaced by y to conclude that

$$t_1 < \frac{5}{9}k+\pi(k) \leqq \frac{5}{9}k+\frac{2k}{\log k} \leqq \frac{3}{5}k.$$

Consequently, there are at least $[2k/5]+1$ integers i with $1\leqq i\leqq k$ and $P(y+i)>k$. By (3.1), we see that for distinct integers $d_1, \dots, d_v$ with $1\leqq d_s\leqq k$ $(1\leqq s\leqq v)$ and $P(y+d_s)>k$ $(1\leqq s\leqq v)$,

$$P(y+d_1)\dots P(y+d_v)|(y-x). \tag{3.5}$$

Since $v>\frac{2}{5}k$, we get $y-x>k^{2k/5}$, which implies that $k<\log y$. We now apply Lemma 2.6 to conclude that

$$t_1 \leqq c_{10}k(\log k)^{-2}\log\log k \tag{3.6}$$

where $c_{10}>0$ is an effectively computable absolute constant. Now in (3.5) we have $v\geqq k-t_1$, hence

$$k^{k-t_1} < y-x. \tag{3.7}$$

We combine (3.7) and (3.6) to complete the proof of (3.3).

PROOF OF (3.4). It is easy to check that the assumption (3.1) with $k \geqq 3$ implies that $y > 100$. Therefore $\log\log y > \frac{3}{2}$ and this enables us to assume that $(\log k)^2 > c_9 \log\log k$. Thus we may assume that k is sufficiently large.

If $\log\log y \leqq \frac{3}{2} k$, then (3.3) implies (3.4). Therefore we may assume that $k < \frac{2}{3}\log\log y$. Finally, we may assume that $y-x < \left(\frac{2}{3}\log\log y\right)^k$, for otherwise (3.4) is clear.

We fix an integer i, $1 \leqq i \leqq k$, and we use Lemma 2.7 for

$$\frac{y-x}{y+i} = 1 - \frac{x+i}{y+i}$$

with

$$n \leqq \frac{5}{4} P(\log P)^{-1}, \quad B \leqq (\log y)/\log 2$$

where $P = P(x+i) = P(y+i)$. We use the upper bound

$$\frac{y-x}{y} < \left(\frac{2}{3}\log\log y\right)^{(2/3)\log\log y} y^{-1} < y^{-1/2},$$

and we get $P \geqq \frac{2}{3}\log\log y$. Then we see from (3.5) with $v \geqq k - t_1$:

$$y-x > \left(\frac{2}{3}\log\log y\right)^{k-t_1}.$$

Combining this estimate with (3.6), we obtain (3.4).

REMARK. The assumption (3.1) implies that

(3.8) $$\omega(y-x) > k - c_{10} k (\log k)^{-2} \log\log k.$$

Indeed, (3.3) shows that $k < \log y$; and we apply (3.6) with

$$\omega(y-x) \geqq \sum_{\substack{1 \leqq i \leqq k \\ P(y+i) > k}} 1 \geqq k - t_1$$

to complete the proof of (3.8).

§ 4. Sequences of consecutive integers with the same prime factors

Throughout this section, we assume that x, y, k are integers satisfying $0 < x < y$, $k \geqq 1$ and

(4.1) $$\mathrm{Supp}\,(x+i) = \mathrm{Supp}\,(y+i) \quad \text{for} \quad 1 \leqq i \leqq k.$$

Let $\varepsilon > 0$. From Section 3, we deduce that for $k \geqq k_0 = k_0(\varepsilon)$

(4.2) $$k \log k < (1+\varepsilon) \log (y-x)$$

and

$$k \log \log \log y < (1+\varepsilon) \log (y-x). \tag{4.3}$$

We sharpen (4.2) and (4.3) as follows.

PROPOSITION 4.4. *There exists an effectively computable absolute constant* $c_{11}>0$ *such that*

$$y-x \geqq (k \log \log y)^{c_{11} k (\log \log y)(\log \log \log y)^{-1}} \tag{4.5}$$

for $y \geqq 27$.

Further, we prove

PROPOSITION 4.6. *There exists an effectively computable absolute constant* $c_{12}>0$ *such that*

$$\log x > c_{12} (\log k)^2/\log \log k \tag{4.7}$$

for $k \geqq 3$.

By (4.5) and (4.2), we obtain

$$y-x > \exp \left(c_{13} k (\log k)^2 (\log \log k)^{-1}\right) \quad \text{for} \quad k \geqq 3.$$

PROOF OF PROPOSITION 4.4. Denote by c_{14}, c_{15} and c_{16} effectively computable absolute positive constants. In view of a result of Erdös and Shorey [4] mentioned in Section 1, we may assume that $k \geqq c_{14}$ with c_{14} sufficiently large. Then we see from (4.2) that

$$k < \log y. \tag{4.8}$$

Further, we observe from (4.1) that

$$Q((y+1) \dots (y+k)) \mid (y-x). \tag{4.9}$$

Suppose that there are $d_1, \dots, d_\nu$ integers between 1 and k such that $\nu = [k/8]+1$ and

$$P(y+d_j) \geqq \exp \left((\log \log y)^{100}\right), \quad 1 \leqq j \leqq \nu.$$

In view of (4.8), observe that the right hand side of the above inequality exceeds k. Therefore, by (4.9), we see that

$$P(y+d_1) \dots P(y+d_\nu) \mid (y-x)$$

and hence,

$$y-x \geqq \exp \left(\frac{k}{8} (\log \log y)^{100}\right)$$

which, together with (4.8), implies (4.5).

Thus we may assume that there are at least $[7k/8]$ integers i with $1 \leqq i \leqq k$ and

$$P(y+i) < \exp \left((\log \log y)^{100}\right).$$

Consequently, there are at least $[k/4]$ integers i with $1 \leqq i < k$ and

$$P\left((y+i)(y+i+1)\right) < \exp \left((\log \log y)^{100}\right).$$

Now we apply Lemma 2.9 with $A=101, f(X)=X(X+1)$ and $Y=y+i$ to conclude that

$$R := \sum_{i=1}^{k} \omega(y+i) \geqq c_{15} k \frac{\log\log y}{\log\log\log y}. \tag{4.10}$$

Further

$$\omega((y+1)\dots(y+k)) \geqq R - \sum_{p \leqq k} \left(\left[\frac{k}{p}\right]+1\right) \geqq R - c_{16} k \log\log k$$

which, together with (4.10) and (4.8), implies that

$$\omega((y+1)\dots(y+k)) > \frac{c_{15} k}{2} \frac{\log\log y}{\log\log\log y}.$$

Now we multiply the prime factors of $(y+1)\dots(y+k)$ to derive (4.5) from (4.9). This completes the proof of Proposition 4.4.

Now we turn to the proof of Proposition 4.6. We shall derive Proposition 4.6 from the following result.

PROPOSITION 4.11. *There exists an effectively computable absolute constant $c_{17}>0$ such that*

$$\log x > c_{17} \log k \frac{\log\log y}{\log\log\log y} \tag{4.12}$$

for $y \geqq 27$ and $k \geqq 2$.

PROOF OF PROPOSITION 4.11. Denote by $c_{18}, c_{19}, \dots, c_{22}$ effectively computable absolute positive constants. Suppose that $x=1$. Then (4.1) with $k \geqq 2$ implies $y+1$ is a power of 2 and $y+2$ is a power of 3. Then it is well-known that $y=7$ which is a contradiction, since $y \geqq 27$. Thus we may assume that $x>1$. Further we see from Corollary 2.2 that $y>k$. Therefore we may assume that $y>c_{18}$ with c_{18} sufficiently large. Then we see from (4.2) that (4.8) is valid. Let $0<\varepsilon_1<1$. We assume that

$$\log x < \varepsilon_1 \log k \frac{\log\log y}{\log\log\log y} \tag{4.13}$$

and we shall arrive at a contradiction for a suitable value of ε_1.

For an integer i with $1 \leqq i \leqq k$, we denote by $\omega'(y+i)$ the number of prime divisors of $y+i$ which are greater than k. For $1 \leqq i \leqq k$, we see from (4.1) and (4.13) that

$$\omega'(y+i) \leqq \varepsilon_1 \frac{\log\log y}{\log\log\log y} + 1 \tag{4.14}$$

and

$$\log P(y+i) \leqq 2 \log k \frac{\log\log y}{\log\log\log y}. \tag{4.15}$$

Now, as pointed out to us by K. Alladi, we observe that

$$\Delta := \sum_{i=1}^{k} \omega(y+i) \leqq \sum_{p \leqq k} \left(\left[\frac{k}{p}\right]+1\right) + \sum_{i=1}^{k} \omega'(y+i)$$

which, together with (4.14), implies that

$$\Delta \leqq c_{19} k \log\log k + \varepsilon_1 k \frac{\log\log y}{\log\log\log y}. \tag{4.16}$$

By (4.15) and (4.8),

$$\log P(y+i) < (\log\log y)^2, \quad 1 \leqq i \leqq k.$$

Now we apply Lemma 2.9 with $A=3$, $f(X)=X(X+1)$ and $Y=y+i$ to conclude that

$$\omega\big((y+i)(y+i+1)\big) \geqq c_{20} \frac{\log\log y}{\log\log\log y'}, \quad 1 \leqq i < k.$$

Consequently, we have

$$\Delta \geqq c_{21} k \frac{\log\log y}{\log\log\log y}. \tag{4.17}$$

Let $\varepsilon_1 = (2c_{21})^{-1}$. We combine (4.17) and (4.16) to conclude that

$$c_{21} \frac{\log\log y}{\log\log\log y} \leqq 2c_{19} \log\log k$$

which, together with (4.8), implies that $y \leqq c_{22}$. This is not possible if $c_{18} > c_{22}$. This completes the proof of Proposition 4.11.

Proof of Proposition 4.6. Since $k \geqq 3$, it is easy to see that (4.1) implies that $x > 1$. Then we may assume that k exceeds a sufficiently large effectively computable absolute constant. Then we derive (4.8) from (4.2). In particular, $y \geqq 27$. Now (4.7) follows immediately from (4.12) and (4.8). This completes the proof of Proposition 4.6.

Remarks (i). Let $x > 1$ and $k \geqq 2$. If

$$\omega(y-x) < 2 \log\log (y-x),$$

then $k/\log x$ is bounded by an effectively computable absolute constant.

Proof. We may assume that k exceeds a sufficiently large effectively computable absolute constant. Then we see from (4.1), (4.8) and Lemma 2.6 with x replaced by y that

$$\omega(y-x) \geqq \sum_{\substack{1 \leqq i \leqq k \\ P(y+i) > k}} 1 \geqq k/2.$$

Hence $k < 4 \log\log y$. Now the assertion follows from Proposition 4.11.

(ii) Woods proved that Hall's conjecture (see [8]) implies that $k \leqq 20$ under the assumption (4.1). It is remarked in [8] that Hall's conjecture would follow from a very sharp refinement of Lemma 2.7. (See M. Langevin, C. R. Acad. Sc. Paris, **281** (1975), 491—493).

§ 5. Two sequences of products of k consecutive integers with the same prime factors

Throughout this section, we assume that x, y and k are positive integers satisfying $y-x\geqq k$ and

(5.1) $$\operatorname{Supp}\Big(\prod_{i=1}^{k}(x+i)\Big) = \operatorname{Supp}\Big(\prod_{i=1}^{k}(y+i)\Big).$$

We prove

PROPOSITION 5.2. *There exists an effectively computable absolute constant* $c_{23}>0$ *such that*

(5.3) $$x > c_{23}k^{61/42}.$$

PROOF OF PROPOSITION 5.2. Denote by $c_{24}, \ldots, c_{27}$ effectively computable absolute positive constants. We may assume that $k\geqq c_{24}$ with c_{24} sufficiently large, otherwise (5.3) follows immediately. Since $y-x\geqq k$, we see from (5.1) that none of $y+1, \ldots, y+k$ is a prime number. Consequently, we derive from Lemma 2.1 that

(5.4) $$k < c_{25}\, y^{23/42}.$$

Denote by t_2 the number of integers i with $1\leqq i\leqq k$ and $P(y+i)\leqq k$. Then we see from Corollary 2.4 and (5.4) that

$$t_2 \leqq (23/42)k + c_{26}\pi(k).$$

Therefore the number of distinct prime factors $>k$ of $(y+1)\ldots(y+k)$ is at least $(19/42)k-c_{26}\pi(k)$. Consequently, by (5.1), the number of distinct prime factors $>k$ of $(x+1)\ldots(x+k)$ is at least $(19/42)k-c_{26}\pi(k)$. Hence

(5.5) $$\binom{x+k}{k} > c_{27}^{k}\, k^{(19/42)k}.$$

On the other hand,

(5.6) $$\binom{x+k}{k} \leqq (x+k)^k/k!.$$

We combine (5.5) and (5.6) to obtain (5.3).

Finally we give an upper bound for y in terms of x, assuming $k\geqq 2$. For small values of k the next result is sharper than (4.12) and (5.3).

PROPOSITION 5.7. *There exists an effectively computable absolute constant* c_{28} *such that*

$$\log x > c_{28} \log\log y$$

for $x\geqq 2$ *and* $k\geqq 2$.

PROOF OF PROPOSITION 5.7. Denote by $c_{29}, \ldots, c_{36}$ effectively computable absolute constants. We may assume that $y>c_{29}$ with c_{29} sufficiently large, otherwise the assertion follows immediately.

Let us assume that

(5.8) $$\log x < \log\log y.$$

By (5.1) we have for $1 \leqq i \leqq k$

$$P(y+i) \leqq P\Big(\prod_{j=1}^{k}(y+j)\Big) = P\Big(\prod_{j=1}^{k}(x+j)\Big) \leqq x+k,$$

and from (5.3) and (5.8) we deduce

$$\log P(y+i) \leqq (\log\log y)^2.$$

Now we apply Lemma 2.9 with $A=3$, $f(X)=X(X+1)$ and $Y=y+i$ to conclude that

$$\omega\big((y+i)(y+i+1)\big) \geqq c_{30} \log\log y/\log\log\log y \quad (1 \leqq i < k).$$

Consequently, if we define

$$\Delta := \sum_{i=1}^{k} \omega(y+i),$$

we have

(5.9)
$$\Delta \geqq c_{31}\, k \log\log y/\log\log\log y.$$

For an integer $n>0$ we denote by $\omega'(n)$ the number of prime divisors of n which are greater than k. By (5.1) we have

$$\sum_{i=1}^{k} \omega'(y+i) = \omega'\Big(\prod_{i=1}^{k}(y+i)\Big) \leqq \omega\Big(\prod_{i=1}^{k}(x+i)\Big).$$

For all integers $n \geqq 3$ we have

$$\omega(n) \leqq c_{32} \log n/\log\log n,$$

hence (5.3) yields

$$\omega\Big(\prod_{i=1}^{k}(x+i)\Big) < c_{33}\, k \log x/\log\log x.$$

Therefore

$$\Delta \leqq \sum_{p \leqq k}\left(\left[\frac{k}{p}\right]+1\right) + \sum_{i=1}^{k} \omega'(y+i) \leqq$$

$$\leqq c_{34}\, k \log\log k + c_{33}\, k \log x/\log\log x \leqq c_{35}\, k \log x/\log\log x,$$

by (5.3). Therefore, we conclude from (5.9)

$$\log\log y/\log\log\log y < c_{36} \log x/\log\log x,$$

which completes the proof of Proposition 5.7.

REMARKS. The restriction (5.1) can be relaxed to

$$\operatorname{Supp}\Big(\prod_{i=1}^{k}(x+i)\Big) \supseteqq \operatorname{Supp}\Big(\prod_{i=1}^{k}(y+i)\Big)$$

in Propositions 5.2 and 5.7.

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(Received July 3, 1986)

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