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**Abstract**

We consider the random walk in an independent and identically distributed (i.i.d.) random environment on a Cayley graph of a finite *free product* of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$. Such a Cayley graph is readily seen to be a regular tree. Under a uniform elipticity assumption on the i.i.d. environment we show that the walk has positive speed and establish the annealed central limit theorem for the graph distance of the walker from the starting point.

*Keywords*: Random walk on free group, random walk in random environment, trees, transience, Central Limit Theorem, Positive Speed

## 1 Introduction

In this article, we consider a *random walk in random environment (RWRE)* model on a regular tree, which was introduced in [ABD14]. Like in any other (static) RWRE model, in our model also, we first choose an *environment* by some random mechanism and keep it fixed throughout the time evolution. A walker then moves randomly on the vertex set of a regular tree in such a way that given the environment, its position forms a time homogeneous Markov chain whose transition probabilities depend only on the environment. RWRE model on the one dimensional integer lattice $\mathbb{Z}$ was first introduced by Solomon in [Sol75] where he gave explicit criteria for the recurrence and transience of the walk for *independent and identically distributed (i.i.d.)* environment distribution. Perhaps the earliest known results for RWRE on trees is by Pemantle and Lyons [LP92], where they consider a model on rooted tress, which later got to known as *random conductance model*. In their model, the random conductances along each path from vertices to the root are assumed to be independent and identically distributed. The random walk is then shown to be recurrent or transient depending on how large is the value of the average conductance. Motivated by these, [ABD14], considered a RWRE model on a regular tree, where the environment (or rather the

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transition laws) at each vertex are independent and also "identically" distributed. However, unlike in the usual RWRE models on integer lattices, such as on $\mathbb{Z}$ as introduced by [Sol75], or the random conductance models on trees [LP92], it is not entirely obvious how to make the random transition laws on the vertices of a tree "identically" distributed. To make this notion of i.i.d.-ness of the environment rigorous in [ABD14] defined the model on a the finite free product of copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ and then transfers it back to an appropriate degree regular tree which is essentially same as a Cayley graph associated with the group. A more detailed description is given in the following subsection. A similar model was also considered in [Roz01].

In both [Roz01, ABD14] under differing mild non-degeneracy assumptions the authors proved that the RWRE is transient with probability one. In this work, we extend their result and prove that under *uniform ellipticity* assumption on the i.i.d. environment (which is stronger assumption than the one made in [ABD14]), the walk has a positive drift away from the starting point and admits an *annealed Central Limit Theorem* under linear centring and square-root scaling. Our proofs are motivated by the work [ABD14].

### 1.1 Model and Main Results

Even though our framework is same as in [ABD14], but for sake of completeness, we begin by providing a detailed description of the model below.

**Group structure:** Following [ABD14] we will also consider a group $G$ which is a free product of finitely many groups, say, $G_1, G_2, \cdots, G_k$ and $H_1, H_2, \cdots, H_r$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$. Let $d = 2k + r$.

**Cayley graph:** Let $G$ be a group defined above. Suppose $G_i = \langle a_i \rangle$ for $1 \leq i \leq k$ and $H_j = \langle b_j \rangle$ where $b_j^2 = e$ for $1 \leq j \leq r$. Here by $\langle a \rangle$ we mean the group generated by a single element $a$. Let $S := \{a_1, a_2, \ldots, a_k\} \cup \left\{a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1}\right\} \cup \{b_1, b_2, \ldots, b_r\}$ be a generating set for $G$. We note that $S$ is a symmetric set, that is, $s \in S \iff s^{-1} \in S$.

We now define a graph $\bar{G}$ with vertex set $G$ and edge set $E := \left\{\{x, y\} \,\middle|\, yx^{-1} \in S\right\}$. We will say $x \sim y$ whenever $\{x, y\} \in E$. Such a graph $\bar{G}$ is called a *(left) Cayley Graph* of $G$ with respect to the generating set $S$. Since $G$ is a free product of groups which are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, it is easy to see that $\bar{G}$ is a graph with no cycles and is regular with degree $d$, thus it is isomorphic to the $d$-regular infinite tree which we will denote by $\mathbb{T}_d$. We will abuse the terminology a bit and will write $\mathbb{T}_d$ for the Cayley graph of the group $G$. This way we essentially endow the $d$-regular tree, $\mathbb{T}_d$ a *group structure*, which we will make use to define an *i.i.d.* environment.

Note that for the $d$-dimensional Euclidean lattice, such a group structure is automatic, which is the product of $d$ copies of the abelian free group $\mathbb{Z}$. In our case, all the difference appears due to the fact that on $\mathbb{T}_d$, a group can only be obtained through free product of several copies of $\mathbb{Z}$ and also with possible free product of groups generated by *torsion* elements.

We will consider the identity element $e$ of $G$ as the root of $\mathbb{T}_d$. We will write $N(x) := \left\{y \in G \,\middle|\, yx^{-1} \in S\right\}$ for the set of all neighbors of a vertex $x \in \mathbb{T}_d$.

Observe that from definition $N(e) = S$.

For $x \in G$, define the mapping $\theta_x : G \to G$ by $\theta_x(y) = yx$, then $\theta_x$ is an automorphism of $\mathbb{T}_d$. We will call $\theta_x$ the *translation by $x$*. For a vertex $x \in \mathbb{T}_d$ and $x \neq e$, we denote by $|x|$, the length of the unique path from the root $e$ to $x$ and $|e| = 0$. Further, if $x \in \mathbb{T}_d$ and $x \neq e$ then we define

$\overleftarrow{x}$ as the *parent* of $x$, that is, the penultimate vertex on the unique path from $e$ to $x$.

**Random Environment:** Let $\mathcal{S} := \mathcal{S}_e$ be a collection of probability measures on the $d$ elements of $N(e) = S$. To simplify the presentation and avoid various measurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $x \in \mathbb{T}_d$, $\mathcal{S}_x$ is the the push-forward of the space $\mathcal{S}$ under the translation $\theta_x$, that is, $\mathcal{S}_x := \mathcal{S} \circ \theta_x^{-1}$. Note that the probabilities on $\mathcal{S}_x$ have support on $N(x)$. That is to say, an element $\omega(x, \cdot)$ of $\mathcal{S}_x$, is a probability measure satisfying $\omega(x,y) \geq 0 \ \forall \ y \in \mathbb{T}_d$ and $\sum\limits_{y \in N(x)} \omega(x,y) = 1$.

Let $\mathcal{B}_{\mathcal{S}_x}$ denote the Borel $\sigma$-algebra on $\mathcal{S}_x$. The *environment space* is defined as the measurable space $(\Omega, \mathcal{F})$ where

$\Omega := \prod\limits_{x \in \mathbb{T}_d} \mathcal{S}_x$ and

$\mathcal{F} := \bigotimes\limits_{x \in \mathbb{T}_d} \mathcal{B}_{\mathcal{S}_x}$.

An element $\omega \in \Omega$ will be written as $\left\{ \omega(x, \cdot) \,\middle|\, x \in \mathbb{T}_d \right\}$. An environment distribution is a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We will denote by $E$ the expectation taken with respect to the probability measure $\mathbb{P}$.

**Random Walk:** Given an environment $\omega \in \Omega$, a random walk $(X_n)_{n \geq 0}$ is a time homogeneous Markov chain taking values in $\mathbb{T}_d$ with transition probabilities given by $(\omega(x,y))_{x,y \in \mathbb{T}_d}$. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^x$ the law induced by $(X_n)_{n \geq 0}$ on $\left( (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{G} \right)$, where $\mathcal{G}$ is the $\sigma$-algebra generated by the cylinder sets, such that $\mathbf{P}_\omega^x(X_0 = x) = 1$. The probability measure $\mathbf{P}_\omega^x$ is called the *quenched law* of the random walk $(X_n)_{n \geq 0}$, starting at $x$. We will use the notation $\mathbf{E}_\omega^x$ for the expectation under the quenched measure $\mathbf{P}_\omega^x$.

Following Zeitouni [Zei04], we note that for every $B \in \mathcal{G}$, the function $\omega \mapsto \mathbf{P}_\omega^x(B)$ is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left( \Omega \times (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G} \right)$ by the relation

$$\mathbb{P}^x(A \times B) = \int_A \mathbf{P}_\omega^x(B) \, \mathbb{P}(d\omega), \quad \forall \ A \in \mathcal{F}, \, B \in \mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{T}_d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $(X_n)_{n \geq 0}$, starting at $x$. We will use the notation $\mathbb{E}^x$ for the expectation under the annealed measure $\mathbb{P}^x$

**Assumptions:** Throughout this paper we will assume the following hold,

(E1) $\mathbb{P}$ is a product measure on $(\Omega, \mathcal{F})$ with "*identical*" marginals, that is, under $\mathbb{P}$ the random probability laws $\left\{ \omega(x, \cdot) \,\middle|\, x \in \mathbb{T}_d \right\}$ are independent and "identically" distributed in the sense that

$$\mathbb{P} \circ \theta_x^{-1} = \mathbb{P}, \tag{1}$$

for all $x \in G$.

(E2) There exists $\epsilon > 0$ such that

$$\mathbb{P}(\omega(e, s_i) > \epsilon \ \forall \ 1 \leq i \leq d) = 1. \tag{2}$$

We are now ready to state our main results. We begin with a law of large numbers result for $| X_n |$ which also establishes that the speed of walk is positive.

**Theorem 1.1.** *Assume (E1) and (E2). Then there exists $v > 0$, such that,*

$$\lim_{n\to\infty} \frac{|X_n|}{n} = v, \tag{3}$$

*almost surely with respect to $\mathbb{P}^e$*

In [ABD14], it was pointed out that under (E1), (E2), $\liminf_{n\to\infty} \frac{|X_n|}{n} > 0$, if $\epsilon > \frac{1}{2(d-1)}$. The above result not only establishes that the walk on $\mathbb{T}_d$ has a *positive speed*, but also shows that the corresponding limit exits almost surely. Our next result is an annealed central limit theorem for $| X_n |$.

**Theorem 1.2.** *Assume (E1)and (E2). Then there exists $\sigma^2 > 0$ such that, under $\mathbb{P}^e$,*

$$\sqrt{n}\left(\frac{|X_n|}{n} - v\right) \overset{d}{\to} Z, \tag{4}$$

*with $Z$ Normal$(0, \sigma^2)$.*

We note here that although we define the walk starting at $X_0 = e$, the root, results hold for starting at any vertex $x$ of $\mathbb{T}_d$. This is because the environment is invariant under the translation by the group $G$. Indeed it will be evident from the proofs that the constants $v$ and $\sigma^2$ are also independent of the starting position. Thus the Theorems 1.1 and 1.2 hold for any initial distribution of $X_0$ on the vertex set of $\mathbb{T}_d$.

The basic framework of proof is inspired by the arguments laid out in [CS11]. Formally speaking we begin by defining regeneration times as first time the walk reaches a new level and never visits it again. We then prove moment bounds for these regeneration times and followed established techniques laid out in the RWRE literature to obtain our results. However, we note that the nature of i.i.d in the environment (assumption (E1)) in this paper is derived from the group structure of the graph and is a different from that of the environment in [CS11]. So, even though the basic framework was available we needed different techniques for executing the same.

# 2 Regeneration Times

In this section we shall introduce a sequence of *regeneration times* and provide moment bounds for them. We begin with some notation. For any $x, y \in \mathbb{T}_d$, recall that $[x, y] = \{\{x_i\}_{i=0}^n \mid x_0 = x, x_n = y, x_i^{-1}x_{i-1} \in S, 1 \leq i \leq n\}$. Let $T_d(y)$ be the sub-tree rooted at $y$, i.e $\{x \in \mathbb{T}_d : y \in [e, x]\}$ and $\mathbb{T}_d^n = \{x \in \mathbb{T}_d :| x |= n\}$. The type of $x \in \mathbb{T}_d, x \neq e$ is $s \in S$ if $\overleftarrow{x}^{-1}x = s$ and we shall denote it by $s_x$.

$$T(y) := \inf\{n \geq 0 \colon X_n = y\} \tag{5}$$

and

$$R(y) := \inf\{n \geq 1 \colon X_{n-1} \in \mathbb{T}_d(y),\, X_n = y\}, \tag{6}$$

be the hitting time of $y$ and the return time to $y$, respectively. We also define,

$$\mathbb{T}_n := \inf\{k \geq 0 \colon X_k \in \mathbb{T}_d^n\} \tag{7}$$

and

$$R := \inf\{n \geq 1\colon X_n = X_0\}, \tag{8}$$

to be the hitting time of level $n$ and return time of the walk to its starting point, respectively.

The *first regeneration level* is then defined as $l_1 := \inf\{k \geq 1 : R(X_{T_k}) = \infty\}$, and the *$n-th$ regeneration level* for $n \geq 2$ is defined recursively as $l_n := \inf\{k \geq l_{n-1} : R(X_{T_k}) = \infty\}$. Regeneration times for $n \geq 1$, are defined by

$$\tau_n := \begin{cases} T_{l_n} & \text{if } l_n < \infty; \\ \\ \infty & \text{otherwise.} \end{cases} \tag{9}$$

### 2.1 Tail bounds for the first regeneration time

We begin by proving tail bounds for first regeneration level, followed by moment bounds on number of visits of the walk to the root and number of distinct vertices visited before first regeneration. We conclude this section with the required moment bounds on the regeneration times defined above (See Proposition 2.4).

Let $\{h_n(x,y) | n \geq 1, x, y \in \mathbb{T}_d,$ and $x \sim y\}$ be i.i.d Exponential random variables with mean 1. Suppose $X_0 = x$, it is easy to verify that $X_{n+1} = \underset{y \sim X_n}{\operatorname{argmin}} \dfrac{h_{n+1}(X_n, y)}{\omega(X_n, y)}$. Fix a finite sub-tree $\mathcal{C}$ of $\mathbb{T}_d$ with $x \in \mathcal{C}$. Let $\{Y_n^{\mathcal{C}}\}_{n\geq 0}$ be such that $Y_0^{\mathcal{C}} = x$ and $Y_{n+1}^{\mathcal{C}} = \underset{y \sim Y_n^{\mathcal{C}}, y \in \mathcal{C}}{\operatorname{argmin}} \frac{h_{n+1}(Y_n^{\mathcal{C}}, y)}{\omega(Y_n^{\mathcal{C}}, y)}$. It is easy to see that $\{Y_n^{\mathcal{C}}\}_{n\geq 0}$ is a Markov chain on $\mathcal{C}$. We shall construct a coupling with the walk $\{X_n : n \geq 1\}$ so that $Y^{\mathcal{C}}$ has the same law as $X$ whenever it visits the sub-tree $\mathcal{C}$. Define $\lambda_1 = \inf\{n \geq 0 : X_n \notin \mathcal{C}\}$, and recursively for $i \geq 1$,

$$\mu_i = \inf\{n \geq \lambda_i : X_n \in \mathcal{C}\} \text{ followed by } \lambda_{i+1} = \inf\{n \geq \mu_i : X_n \notin \mathcal{C}\}.$$

Define for each $k \geq 1$,

$$W_k = \begin{cases} X_k & \text{if } k \leq \lambda_1 - 1; \\ X_{\mu_j + k} & \text{if } \mu_j < k \leq \lambda_j - 1, \text{for some } j \geq 1. \end{cases}$$

Note that as $\mathbb{T}_d$ is a tree, for all $i \geq 1$, $X_{\lambda_i - 1} = X_{\mu_i} \in \mathcal{C}$. Further $X_{\mu_j + k} \in \mathcal{C}$ if $\mu_j < k \leq \lambda_j - 1$ for some $j \geq 1$. It is easy to see that $\{W_n\}_{n\geq 0}$ is a Markov chain on $\mathcal{C}$ and has the same law as $\{Y_n^{\mathcal{C}}\}_{n\geq 0}$.

**Colouring scheme :** We begin by colouring the root $e$ as red. Let $k \geq 1$ and $\psi \geq 1$. A vertex $y \in \mathbb{T}_d^{k\psi}$ is coloured red if and only if

- its ancestor at level $(k-1)\psi$, say $x$, is coloured red, and
- $\{Y_n^{[x,y]}\}_{n\geq 0}$, started at $x$, hits $y$ before returning to $x$.

For each $\psi \geq 1$ and $k \geq 1$, $s \in S$ let $Z_\psi(k, s)$ be the number of red vertices at level $k\psi$ of type $s$. Let $Z_\psi(0) = \{e\}$ and for $k \geq 1$, define

$$Z_\psi(k) := \{Z_\psi(k, s) : s \in S\}.$$

Under the annealed measure, $\{Z_\psi\}$ is a multi-type Branching process with expected offspring matrix $M = (m_{Sue})_{s\in S, u\in S}$ is given by

$$m_{su} = \mathbb{E}\left[\sum_{x_\psi \in \mathbb{T}_d^\psi} \left(\sum_{m=1}^{\psi-1} \prod_{j=1}^{m} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}\right)^{-1}\right],$$

where $s, u \in S$ and $s_{x_1} = s,$ and $s_{x_\psi} = u.$

**Proposition 2.1.** *Assume (E1) and (E2). There exists $\psi \geq 1$ such that the $Z_\psi$ is supercritical.*

*Proof.* We will show that the largest eigenvalue, $\rho$, of the offspring matrix $M$ is larger than 1. We observe that for $1 \leq i \leq n-1$

$$\mathbf{P}_\omega(Y_n^{[x,y]} = x_{i-1} \mid Y^{[x,y]} = x_i) = \frac{\omega(x_i, x_{i-1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_{i-1})}$$

and

$$\mathbf{P}_\omega(Y_n^{[x,y]} = x_{i+1} \mid Y^{[x,y]} = x_i) = \frac{\omega(x_i, x_{i+1})}{\omega(x_i, x_{i+1}) + \omega(x_i, x_{i-1})}.$$

Using a standard gambler's ruin chain argument we can conclude

$$\mathbf{P}_\omega(Y^{[x,y]} \text{ hits } y \text{ before returning to } x\ ) = \left(\sum_{m=0}^{n-1} \prod_{j=0}^{m} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}\right)^{-1}.$$

From the arguments in proof of Theorem 1 in [ABD14], it is easy to see that, for $1 < c_1 < d-1$, there exists $n \geq 1$, such that,

$$\frac{1}{d(d-1)^{n-1}} \# \left\{\sigma_n \in \mathbb{T}_d^n : \left(\sum_{m=1}^{n-1} \prod_{j=1}^{m} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}\right)^{-1} \geq c_1^{-(n-1)}\right\} \geq \frac{1}{2} \text{ a.s. } \mathbb{P}.$$

Therefore, for all $s \in S$, and large enough $\psi$

$$\sum_{u\in S} m_{su} = \mathbb{E}\left[\sum_{x_\psi \in \mathbb{T}_d^\psi} \left(\sum_{m=1}^{\psi-1} \prod_{j=1}^{m} \frac{\omega(x_j, x_{j-1})}{\omega(x_j, x_{j+1})}\right)^{-1}\right] \geq \frac{d(d-1)^{\psi-1}}{2} c_1^{-(\psi-1)} > 1. \tag{10}$$

As the row sums of $M$ are larger than 1, this implies that the largest eigenvalue $\rho$ is bigger than 1 and this implies the process is super-critical.

□

Now, for $x \in \mathbb{T}_d$, $y \in \mathbb{T}_d(x)$ is called a *first child* if it is (almost surely) the minimiser of

$$\min_{z\sim x, z\neq \overleftarrow{x}} \frac{h_1(x,z)}{\omega(x,z)} \tag{11}$$

For $m \geq 1$,

$$F_{m,x} = \{y \in \mathbb{T}_d(y) :\mid y \mid - \mid x \mid = m \text{ and } y \text{ is a } \textit{first child}\}.$$

Let $\psi \geq 1, \zeta \geq 1$

$$\Sigma_x = \mathbb{T}_d(x) \cap Z_\psi \cap_{k=1}^{\infty} F^c_{k\zeta\psi,x}$$

$$B(x) = \{\Sigma_x \text{ is finite}\}, B_0 = B(e), \text{ and } B_k = B(X_{T_{k\psi\zeta}}),\ k \geq 1.$$

**Lemma 2.2.** *The collection of events $\{B_i, i \geq 1\}$, are independent.*

*Proof.* The event $B(x) \in \sigma\{h_n(z,y) : z, y \in \mathbb{T}_d(x) n \geq 1\}$. Let $i_1 < i_2 < \ldots < i_k$ be positive integers. Note, as observed,

$$B_{i_j} \in \sigma\{h_n(z,y) : z, y \in \mathbb{T}_d(X_{i_j\zeta\psi}) n \geq 1\}.$$

Note however that $X_{i_j\zeta\psi}$ is a first child at level $i_j\zeta\psi$. and this implies $\mathbb{T}_d(X_{i_j\zeta\psi}) \cap \mathbb{T}_d(X_{i_l\zeta\psi}) = \emptyset$. Hence $\{B_{i_j} : 1 \leq j \leq k\}$ are mutually independent. □

**Proposition 2.3.** $\exists\, \gamma < 1$ *such that for $n \geq 1$, we have*

$$\mathbb{P}(l_1 \geq n\psi\zeta) \leq \gamma^{n-1}.$$

*Proof.* Note that $B^c_i \subseteq \{\text{level } i\psi\zeta \text{ is a regeneration level}\}$. Hence, using that $B_i$ are independent we have,

$$\mathbb{P}(l_1 \geq n\psi\zeta) \leq \mathbb{P}(\bigcap_{i=1}^{n-1} B_i) = \prod_{i=1}^{n-1} \mathbb{P}(B_i).$$

For $s \in S$, let

$$B^s_i := \{\overleftarrow{X}^{-1}_{T_{i\zeta\psi}} X_{T_{i\zeta\psi}} = s\} \cap B_i.$$

Note that for $s \in S$, $\mathbb{P}(B^s_i) = \mathbb{P}(B^s_j)$ for all $1 \leq i, j \leq n$. Hence ,

$$\mathbb{P}(B_i) = \mathbb{P}(\cup_{s\in S} B^s_i) = \sum_{s \in S} \mathbb{P}(B^s_i) = \gamma$$

Therefore $\mathbb{P}(l_1 \geq n\psi\zeta) \leq \prod_{i=1}^{n-1} \left(\sum_{s\in S} \mathbb{P}(B^s_i)\right) = \gamma^{n-1}$.

Now we will show that we can choose $\zeta > 0$, such that, $\gamma < 1$. It is enough to show that we can choose $\zeta > 0$, such that, $\mathbb{P}(B_1) < 1$. For this we follow an argument similar to the proof of Lemma 3.3 of [CS11]. From definition, it is clear that the vertices which belong to $\Sigma_{X_{\psi\zeta}}$, are obtained as follows. The vertices at level $(\zeta - 1)\psi$ are of $d$-types, and has a distribution with mean matrix $M^{(\zeta-1)\psi}$. Further, the vertices at level $\zeta\psi$ has a number of various types of coloured vertices, and we have deleted the first child, thus the expectation matrix of such vertices is $M - A$, where $A$ is a $d \times d$-matrix with $0 \leq A_{su} \leq 1$ and $A\mathbf{1} = \mathbf{1}$. Thus, $\gamma = \mathbb{P}(B_1)$ is at most as large as, the extinction probability of a multi-type branching process with mean matrix $\tilde{M}_\zeta := M^{(\zeta-1)\psi}(M - A)$. But, from equation (10), it follows that

$$m_0 := \min_{s\in S} \sum_{u \in S} m_{su} > 1.$$

So for any $s \in S$, the $s$-th row sum of $\tilde{M}_\zeta$ is at least as large as $m_0^{(\zeta-1)\psi}(m_0 - 1)$. Now select $\zeta \geq 1$, such that, $m_0^{(\zeta-1)\psi}(m_0 - 1) > 1$. From the argument above then we can conclude that $\gamma < 1$. □

### 2.2 Moment bounds

For $x \in \mathbb{T}_d$, let $T(x)$ be as in (5) and $\tau_1$ be as in (9). Further let

$$L(x) := \sum_{j=0}^{\infty} I_{\{X_j = x\}} \text{ and } \mathcal{D} := \sum_{x \in \mathbb{T}_d} I_{\{T(x) \leq \tau_1\}} \tag{12}$$

be total number visits to $x$ and the number of distinct vertices visited before $\tau_1$ respectively.

**Proposition 2.4.** *Assume (E1) and (E2). Then for $p \geq 1$,*

*(a)* $\mathbb{E}[L(e)^p] < \infty$*;*

*(b)* $\mathbb{E}[\mathcal{D}^p] < \infty$*; and*

*(c)* $\mathbb{E}[\tau_1^p] < \infty$*.*

*Proof.* (a) For $n \geq 1$, let

$$\mathcal{U}_n = \{\{x^i\}_{i=1}^d : x_i \in \mathbb{T}_d^n \text{ and } [x^i, e] \cap [x^j, e] = \{e\} \text{ for all } 1 \leq i \neq j \leq d\}.$$

We will denote any element of $U_n$ by $\mathcal{A}_n$ and the smallest sub-tree in $\mathbb{T}_d$ containing $\mathcal{A}_n$ will be denoted by $\mathfrak{T}_n$. Consider the walk $\{Y_k^{\mathfrak{T}_n} : k \geq 1\}$. Define

$$\tilde{T}_{\mathcal{A}_n} = \inf\{k \geq 1 : Y_k^{\mathfrak{T}_n} \in \mathcal{A}_n\}, \text{ and } \tilde{L}(e, \tilde{T}_{\mathcal{A}_n}) = \sum_{i=0}^{\infty} I_{\{Y_i^{\mathfrak{T}_n} = e, i < \tilde{T}_{\mathcal{A}_n}\}}$$

to be the hitting time of $\mathcal{A}_n$ and the number of visits of $Y^{\mathfrak{T}_n}$ to $e$ before the walk $Y^{\mathfrak{T}_n}$ hits $\mathcal{A}_n$ respectively. Define $\tilde{R}_n = \inf\{k \geq 1 : Y_k^{\mathfrak{T}_n} = e\}$ return time to $e$. Under the quenched law, $\tilde{L}(e, \tilde{T}_{\mathcal{A}_n})$ is a geometric random variable with parameter $q_\omega$ given by

$$q_w = \mathbf{P}_\omega(\tilde{T}_{\mathcal{A}_n} < \tilde{R}_n) = \sum_{i=1}^{d} w(e, s_i) \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{-1}.$$

Using standard results about geometric random variables we know that for $p > 1$, $\exists\, c_p > 0$, such that ,

$$\begin{aligned}
\mathbb{E}[\tilde{L}(e, \tilde{T}_{\mathcal{A}_n})^p] &\leq c_p \mathbb{E}[q_\omega^{-p}] = c_p \mathbb{E}\left[ \left( \sum_{i=1}^{d} w(e, s_i) \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{-1} \right)^{-p} \right] \\
&\leq c_p\, d^{-p}\, \mathbb{E}\left[ \left( \min_{1 \leq i \leq d} \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{-1} \right)^{-p} \right] \\
&= c_p\, d^{-p}\, \mathbb{E}\left[ \max_{1 \leq i \leq d} \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{p} \right].
\end{aligned} \tag{13}$$

For $x \in \mathbb{T}_d$, define $R^x := \inf\{n \geq 1 : Y_n^{\mathbb{T}_d(x)}) = x\}$, and

$$H \quad := \quad \inf\{k \geq 1 : \exists \mathcal{A}_k \in \mathcal{U}_k \text{ such that } \mathbf{P}_\omega^x(R^x = \infty) = 1 \text{ for all } x \in A_k\}.$$

Observe that for any $n \geq 1$,

$$L(e) I_{\{H=n\}} \leq \tilde{L}(e, \tilde{T}_{\mathcal{A}_n}) I_{\{H=n\}}.$$

By Hölder's inequality for $\epsilon > 0$, with $a = 1 + \epsilon/p$, $b = 1 + p/\epsilon$, and using (13) we have

$$\begin{aligned}
\mathbb{E}[L(e)^p, I_{\{H=n\}}] &\leq \mathbb{E}[(\tilde{L}(e, \tilde{T}_{\mathcal{A}_n}))^p I_{\{H=n\}}] \\
&\leq \mathbb{E}[\tilde{L}(e, \tilde{T}_{\mathcal{A}_n})^{pa}]^{1/a} \mathbb{P}(H = n)^{1/b} \\
&\leq \left( c_{pa}\, d^{-pa}\, \mathbb{E}\left[ \max_{1 \leq i \leq d} \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{pa} \right] \right)^{1/a} \mathbb{P}(H = n)^{1/b}.
\end{aligned} \tag{14}$$

Now,

$$\begin{aligned}
\mathbb{P}(H = n) &\leq \mathbb{P}\left( \bigcup_{s \sim e} \bigcap_{y \in \mathbb{T}_d(s) \cap \mathbb{T}_d^{n-1}} \{R^y < \infty\} \right) \leq \sum_{s \sim e} \mathbb{P}\left( \bigcap_{y \in \mathbb{T}_d(s) \cap \mathbb{T}_d^{n-1}} \{R^y < \infty\} \right) \\
&= \sum_{s \sim e} \prod_{y \in \mathbb{T}_d(s) \cap \mathbb{T}_d^{n-1}} \mathbb{P}\left(R^y < \infty\right), \leq d \left( \max_{s \in S, y \in \mathbb{T}_d(s) \cap \mathbb{T}_d^{n-1}} \mathbb{P}\left(s_y = s, R^y < \infty\right) \right)^{(d-1)^{n-2}}.
\end{aligned} \tag{15}$$

Let $q(s) = \mathbb{P}(R^y < \infty : y \in \mathbb{T}_d^1, s_y = s)$ and $q = \max_{s \in S} q(s)$. From (14) and (15) we have,

$$\begin{aligned}
\mathbb{E}[L(e)^p, I_{\{H=n\}}] &\leq \left( c_{pa}\, d^{-pa}\, \mathbb{E}\left[ \max_{1 \leq i \leq d} \left( \sum_{j=1}^{n} \prod_{k=1}^{j-1} \frac{\omega(x_k^i, x_{k-1}^i)}{\omega(x_k^i, x_{k+1}^i)} \right)^{pa} \right] \right)^{1/a} \left( dq^{(d-1)^{n-2}} \right)^{\frac{1}{b}} \\
&\leq c_1 c_2^n q^{\frac{(d-1)^{n-2}}{b}}
\end{aligned} \tag{16}$$

Now it easily follow that $\mathbb{E}[L(e)^p] = \sum_{n=1}^{\infty} \mathbb{E}[L(e)^p, I_{\{H=n\}}] < \infty$.

(b) From the definition of $\mathcal{D}$, we have

$$\begin{aligned}
\mathcal{D} &= \sum_{x \in \mathbb{T}_d} I_{\{T(x) \leq \tau_1\}} = 1 + \sum_{x \neq e, x \in \mathbb{T}_d} \sum_{n=1}^{\infty} I_{\{T(x) \leq T_n\}} I_{\{l_1 = n\}} \\
&= 1 + \sum_{n=1}^{\infty} \sum_{x \neq e, x \in \mathbb{T}_d} I_{\{T(x) \leq T_n\}} I_{\{l_1 = n\}} \\
&= 1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \sum_{x \in \mathbb{T}_d^k} I_{\{T(x) \leq T_n\}} \right) I_{\{l_1 = n\}} \\
&\leq 1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \sum_{x \in \mathbb{T}_d^k} I_{\{T(x) < \infty\}} \right) I_{\{l_1 = n\}}.
\end{aligned}$$

For each $k \geq 1$, we may dominate the random variable $\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}$ by a Geometric $(1-q)$ random variable (where $q$ was defined in the previous proof). This implies

$$\mathbb{E}\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}\right)^{2p} \leq c_p(1-q)^{-2p}.$$

Using this, Jensen's inequality, followed by Hölder's inequality we have for all $n \geq 1$,

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}\right)^{p} I_{\{l_1=n\}}\right) &\leq n^{p-1}\sum_{k=1}^{n}\mathbb{E}\left(\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}\right)^{p} I_{\{l_1=n\}}\right)\\
&\leq n^{p-1}\sum_{k=1}^{n}\sqrt{\mathbb{E}\left(\sum_{x\in\mathbb{T}_d^k} I_{\{T(x)<\infty\}}\right)^{2p}}\sqrt{\mathbb{P}(l_1=n)}\\
&\leq c_p(1-q)^{-2p}n^p\sqrt{\mathbb{P}(l_1=n)}
\end{aligned}$$

Then,

$$\mathbb{E}[\mathcal{D}^p] \leq c_p(1-\mathbb{P}(R<\infty))^{-p}\sum_{n=1}^{\infty} n^p\,\mathbb{P}(l_1=n)^{1/2}. \tag{17}$$

The result follows from Proposition 2.3.

(c) Let $\{x_i : 1 \leq i \leq \mathcal{D}\}$ be an enumeration of the vertices visited by the walk $X$ before time $\tau_1$. It is easy to see that

$$\tau_1 = \sum_{i=1}^{\mathcal{D}} L(x_i).$$

So,

$$\mathbb{E}[\tau_1^p] = \mathbb{E}\left[\left(\sum_{i=1}^{\mathcal{D}} L(x_i)\right)^p\right] \leq \mathbb{E}\left[\mathcal{D}^{p-1}\sum_{i=1}^{\mathcal{D}} L(x_i)^p\right]. \tag{18}$$

Now

$$\mathcal{D}^{p-1}\sum_{i=1}^{\mathcal{D}} L(x_i)^p = \sum_{i=1}^{\infty}\mathcal{D}^{p-1}L(x_i)^p I_{\{\mathcal{D}\geq i\}}.$$

For each $i \geq 1$, using Hölder's inequality twice (first with with $q = 1+\delta/p$, and $q' = 1+p/\delta$) and Chebychev's inequality , we have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{D}^{p-1}L(x_i)^p I_{\{\mathcal{D}\geq i\}}\right] &\leq \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{(p-1)q'}I_{\{\mathcal{D}\geq i\}}]\right]^{1/(q')}\\
&\leq \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{2(p-1)q'}]\right]^{1/(2q')}\mathbb{P}(\mathcal{D}\geq i)^{1/(2q')}\\
&\leq \left[\mathbb{E}[L(x_i)^{p+\delta}]\right]^{1/q}\left[\mathbb{E}[\mathcal{D}^{2(p-1)q'}]\right]^{1/(2q')}\left[\mathbb{E}(\mathcal{D}^{4q'})\right]^{1/(2q')}\frac{1}{i^2}.
\end{aligned} \tag{19}$$

By definition of $L$ we have

$$\mathbb{E}[L(x_i)^{p+\delta}] \quad \le \quad \mathbb{E}[L(x_i)^{p+\delta})|X_0 = x_i] = \mathbb{E}[L(e)^{p+\delta}]. \tag{20}$$

Using (18), (19), (20), the fact that $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, along with part (a) and (b), we have the result. $\square$

# 3 Proof of Main Results

*Proof of Theorem 1.1:* Let $\{\tau_n\}_{n\geq 1}$ be the sequence of regeneration times defined in (9). By Proposition (2.4), $\mathbb{E}(\tau_1) < \infty$. So for all $n \geq 1$, there exists a (random) subsequence $\{k_n\}_{n\geq 1}$ such that

$$\tau_{k_n} < n \leq \tau_{k_n+1}. \tag{21}$$

It is then readily seen that

$$\frac{\mid X_n \mid}{n} = \frac{\mid X_{\tau_1} \mid + \sum_{l=1}^{k_n-1}(\mid X_{\tau_{l+1}} \mid - \mid X_{\tau_l} \mid) + \mid X_n \mid - \mid X_{\tau_{k_n}} \mid}{\tau_1 + \sum_{l=1}^{k_n-1}(\tau_{l+1} - \tau_l) + n - \tau_{k_n}}. \tag{22}$$

For any $s \in S$, define

$$Y_i(s) = \begin{cases} \tau_1 I_{\{s_{X_{\tau_1}} = s\}} & i = 1 \\ (\tau_i - \tau_{i-1}) I_{\{s_{X_{\tau_i}} = s\}} & i > 1 \end{cases}, \quad Z_i(s) = \begin{cases} \mid X_{\tau_1} \mid I_{\{s_{X_{\tau_1}} = s\}} & i = 1 \\ (\mid X_{\tau_i} \mid - \mid X_{\tau_{i-1}} \mid) I_{\{s_{X_{\tau_i}} = s\}} & i > 1 \end{cases}. \tag{23}$$

Then, for each $s \in S$, $\{Y_i(s)\}_{i\geq 1}$ and $\{Z_i(s)\}_{i\geq 1}$ are i.i.d. Using Proposition 2.4,

$$\mathbb{E}[Y_i(s)] = \mathbb{E}[Y_1(s)] = \mathbb{E}[\tau_1 I_{\{s_{X_{\tau_1}} = s\}}] \leq \mathbb{E}[\tau_1] < \infty,$$

and

$$\mathbb{E}[Z_i(s)] = \mathbb{E}[Z_1(s)] = \mathbb{E}[\mid X_{\tau_1} \mid I_{\{s_{X_{\tau_1}} = s\}}] \leq \mathbb{E}[\mid X_{\tau_1} \mid] \leq c_1 \mathbb{E}[\tau_1] < \infty.$$

By strong of law of large numbers for each $s \in S$,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} Y_i(s)}{n} \to \mathbb{E}[Y_1(s)] \text{ and } \lim_{n\to\infty} \frac{\sum_{i=1}^{n} Z_i(s)}{n} \longrightarrow \mathbb{E}[Z_1(s)]$$

almost surely $\mathbb{P}$, as $n \to \infty$. Consequently,

$$\frac{\sum_{l=1}^{k_n-1} \tau_{l+1} - \tau_l}{k_n - 1} = \sum_{s\in S} \sum_{i=1}^{k_n-1} \frac{Y_i(s)}{k_n - 1} \longrightarrow \sum_{s\in S} \mathbb{E}[Y_1(s)] = \mathbb{E}[\tau_1], \tag{24}$$

and

$$\frac{\sum_{l=1}^{k_n-1} \mid X_{\tau_{l+1}} \mid - \mid X_{\tau_l} \mid}{k_n - 1} = \sum_{s\in S} \sum_{i=1}^{k_n-1} \frac{Z_i(s)}{k_n - 1} \longrightarrow \sum_{s\in S} \mathbb{E}[Z_1(s)] = \mathbb{E}[\mid X_{\tau_1} \mid], \tag{25}$$

almost surely $\mathbb{P}$, as $n \to \infty$. Now,

$$0 \le n - \tau_{k_n} \le \tau_{k_n+1} - \tau_{k_n}, \tag{26}$$

$$\mathbb{E}[\tau_{k_n} - \tau_{k_n-1}] = \sum_{s\in S} \mathbb{E}[\tau_{k_n} - \tau_{k_n-1} I_{\{s_{X_{\tau_{k_n}}} = s\}}]$$

$$= \sum_{s\in S} \mathbb{E}[I_{\{s_{X_{\tau_{k_n}}} = s\}} \mathbb{E}[\tau_{k_n} - \tau_{k_n-1} \mid s_{X_{\tau_{k_n}}} = s]] = \sum_{s\in S} \mathbb{E}[I_{\{s_{X_{\tau_{k_n}}} = s\}} \mathbb{E}[\tau_2 - \tau_1 \mid s_{X_{\tau_1}} = s]]$$

$$\le \sum_{s\in S} \mathbb{E}[\mathbb{E}[\tau_2 - \tau_1 \mid s_{X_{\tau_1}} = s]] = \mid S \mid \mathbb{E}[\tau_2 - \tau_1], \tag{27}$$

and

$$0 \le \mid X_n \mid - \mid X_{\tau_{k_n}} \mid \le \mid X_{\tau_{k_n+1}} \mid - \mid X_{\tau_{k_n}} \mid \text{ and } n - \mid X_{\tau_{k_n}} \mid \le c_1(n - \tau_{k_n}) \tag{28}$$

for some $c_1 > 0$. Using (24)-(28) along with simple elementary algebra on (22) yields

$$\frac{\mid X_n \mid}{n} \longrightarrow \frac{\mathbb{E}[\mid X_{\tau_1} \mid]}{\mathbb{E}[\tau_1]}$$

almost surely as $n \to \infty$. □

*Proof of Theorem 1.2:* Recall from (23) and (21), $k_n, \tau_\cdot$, $Z_\cdot(\cdot)$ and $Y_\cdot(\cdot)$. Let

$$v = \frac{\mathbb{E}[X_{\tau_1}]}{\mathbb{E}[\tau_1]},\ W_k(s) = Z_k(s) - Y_k(s)v,\ S_n(s) = \sum_{k=1}^{n} W_k(s), \text{ and } S_n = \sum_{s\in S} S_n(s).$$

Observe that,

$$\sqrt{n}\left(\frac{|X_n|}{n} - v\right) = \sqrt{n}\left(\frac{|X_n|}{n} - S_{k_n} - v\right) + \frac{S_{k_n}}{\sqrt{n}}.$$

As,

$$\frac{1}{\sqrt{n}} \mid \mid X_n \mid - S_{k_n} - nv \mid \le \max_{1\le i\le k_n} \frac{\tau_i - \tau_{i-1}}{\sqrt{n}},$$

our result will follow if we establish that as $n \to \infty$

$$\frac{S_{k_n}}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \quad \text{for some } \sigma^2 > 0 \tag{29}$$

and for all $\delta > 0$

$$\mathbb{P}\left(\max_{0\le i\le k_n} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta\right) \longrightarrow 0. \tag{30}$$

*Proof of (29):* Note that it is easy to check that $s_{X_{\tau_i}}$ is uniform on $S$ (see [ABD14, Section 2])and thus the vector $(W_k(s))_{s\in S}$ form an i.i.d sequence with

$$\begin{aligned}
\mathbb{E}[W_1(s)] &= \mathbb{E}[X_{\tau_1} I_{\{s_{X_{\tau_1}} = s\}}] - \mathbb{E}[\tau_1 I_{\{s_{X_{\tau_1}} = s\}}]v \text{ for each } s \in S \text{ and} \\
\sigma_{s_1 s_2} &= \mathbb{E}\prod_{i=1}^{2}[\Big(\mid X_{\tau_1} \mid I_{\{s_{X_{\tau_1}} = s_i\}} - \tau_1 I_{\{s_{X_{\tau_1}} = s_i\}} v - \mathbb{E}[W_1(s_i)]\Big)] \text{ for } s_1, s_2 \in S.
\end{aligned}$$

Therefore by the Multivariate central limit theorem, we have

$$\frac{(S_n(s) - n\mathbb{E}[W_1(s)])_{s\in S}}{\sqrt{n}} \xrightarrow{d} N(0,\Sigma)$$

where $\Sigma = (\sigma_{s_i s_j})_{s_i,s_j\in S}$. The continuous map theorem then implies that for each $s \in S$,

$$\frac{S_n(s) - n\mathbb{E}[W_1(s)]}{\sqrt{n}} \xrightarrow{d} N(0,\sigma_s^2)$$

with $\sigma_s^2 = \sigma_{ss}$. If $\sigma = 1^t\Sigma 1$ then it is immediate that as $n \to \infty$

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0,\sigma^2). \tag{31}$$

Note that there is no centering because $\sum_{s\in S}\mathbb{E}[W_1(s)] = 0$. From proof of Theorem 1.1, we know that $\frac{k_n}{n} \to \frac{1}{\mathbb{E}[\tau_1]}$. Using this and (31) we are done.

*Proof of (30):* Using Proposition 2.4 (c) the proof is standard (See [BZ06, Proof of Theorem 2.3]). Since $k_n \leq n$, we have that for any $\delta > 0$,

$$\mathbb{P}\left(\max_{0\leq i\leq k_n} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta\right) \leq \sum_{i=1}^{n} \mathbb{P}(\tau_1 > \delta\sqrt{n}). \tag{32}$$

Note that, since $\mathbb{E}\left[\tau_1^2\right] < \infty$, one has that

$$\sum_{i=1}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) = \sum_{i=1}^{\infty} \mathbb{P}(\tau_1^2 > \frac{\delta^2 i}{T}) < \infty\,.$$

Hence, for each $\epsilon > 0$ there is a deterministic constant $N \equiv N(d,\delta,\epsilon)$ such that

$$\sum_{i=N}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) < \epsilon\,.$$

Therefore,

$$\limsup_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(\tau_1 > \delta\sqrt{n}) \leq \limsup_{n\to\infty} \left(\sum_{i=1}^{N} \mathbb{P}(\tau_1 > \delta\sqrt{n}) + \sum_{i=N+1}^{\infty} \mathbb{P}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}})\right) \leq \epsilon\,.$$

As $\epsilon > 0$ was arbitrary, one concludes from the last limit and (32) that (30) holds. □

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