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Coalescence in the recent past in rapidly growing populations

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Abstract

In a rapidly growing population one expects that two individuals chosen at random from the nth generation are unlikely to be closely related if n is large. In this paper it is shown that for a broad class of rapidly growing populations this is not the case. For a Galton–Watson branching process with an offspring distribution $\{p_j\}$ such that $p_0 = 0$ and $\psi(x) = \sum_j p_j I_{\{j \geq x\}}$ is asymptotic to $x^{-\alpha} L(x)$ as $x \to \infty$ where $L(\cdot)$ is slowly varying at ∞ and $0 < \alpha < 1$ (and hence the mean $m = \sum jp_j = \infty$) it is shown that if X_n is the generation number of the coalescence of the lines of descent backwards in time of two randomly chosen individuals from the nth generation then $n - X_n$ converges in distribution to a proper distribution supported by $\mathbb{N} = \{1, 2, 3, \ldots\}$. That is, in such a rapidly growing population coalescence occurs in the recent past rather than the remote past. We do show that if the offspring mean m satisfies $1 < m \equiv \sum jp_j < \infty$ and $p_0 = 0$ then coalescence time X_n does converge to a proper distribution as $n \to \infty$, i.e., coalescence does take place in the remote past.

Keywords: Coalescence; Recent past; Rapidly growing populations; Branching processes; Regular variation

1. Introduction

If one considers a rapidly growing population then two individuals chosen at random from the nth generation are very unlikely to be closely related as n gets large. For example, in a deterministic binary tree the *coalescence time* X_n for two individuals chosen at random from the nth generation (X_n is the generation number where the lines of descent of these two chosen

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individuals going backwards in time meet) has the probability distribution $P(X_n < k) = \frac{1-2^{-k}}{1-2^{-n}}$, $k \geq 0$ which converges to $1 - 2^{-k}$ as $n \to \infty$. Thus, X_n converges in distribution to a geometric $(\frac{1}{2})$ distribution on $\mathbb{N}^+ \equiv \{0, 1, 2, \ldots\}$.

A similar assertion holds for any regular m-nary tree where from each vertex at any grown level there are m branches coming out for the next level. So it is somewhat surprising and counter intuitive that if the population grows too rapidly this need not hold. In this paper we provide a broad class of such growing populations for which the coalescence time X_n does not stay reasonable but goes to ∞ and indeed $n - X_n$ stays reasonable, i.e., coalescence tends to be very recent.

We show that in a Galton–Watson branching process if the offspring distribution is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$ (necessarily with infinite mean) then X_n not only does not converge to a proper distribution but does converge to infinity in distribution and in fact, $n - X_n$ converges to a proper distribution (Theorem 1). That is, coalescence of two lines of descent indeed occurs in the very recent past. This somewhat counter intuitive result may be explained by the fact (see Grey [6], Davies [4] and Schuh and Barbour [9]) that if the offspring distribution is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$ then between any two lines of descent one outpaces the other (i.e., the ratio of their population sizes in nth generation goes to zero as n goes to infinity). We do show also that if the offspring mean $m = \sum_{j=1}^{\infty} j p_j$ satisfies $1 < m < \infty$ and if $p_0 = 0$ then coalescence does take place in the *remote past* (Theorem 2).

2. Main results

Let $\{p_j\}_{j\geq 0}$ be a probability distribution with $p_0 = 0$ and in the domain of attraction of a stable law of order α, $0 < \alpha < 1$. An equivalent condition (see Feller [5], Bingham et al. [3]) is that there exists a function $L(\cdot)$ from $[1, \infty)$ to $\mathbb{R}^+$ such that for any $1 < c, x < \infty$, $\frac{L(cx)}{L(x)} \to 1$ as $x \to \infty$ (called slowly varying at ∞) and

$$\frac{\sum_{j\geq x} p_j}{x^{\alpha} L(x)} \to 1 \quad \text{as } x \to \infty. \tag{1}$$

Let $\{Z_n\}_{n\geq 0}$ be a Galton–Watson branching process with the offspring distribution $\{p_j\}_{j\geq 0}$ and initial population size $Z_0 < \infty$. That is, there exist i.i.d. rv $\{\xi_{n,i} : i \geq 1, n \geq 0\}$ with distribution $\{p_j\}_{j\geq 0}$ such that for each $n = 0, 1, 2, \ldots$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}.$$

Thus $\xi_{n,i}$ can be thought of as the number of offsprings of the ith parent of the nth generation. Let $\mathbb{T}$ denote a typical tree generated by the above process. Now pick two individuals from the nth generation and trace their lines of descent back in time till a common ancestor is found, i.e. the lines *coalesce*. Let X_n denote the generation number of their *last common ancestor*.

Theorem 1. *Assume* $\{p_j\}_{j\geq 0}$ *satisfies* (1) *with* $0 < \alpha < 1$. *Then*

(i) *for almost all family trees* $\mathbb{T}$ *and* $k = 1, 2, \ldots$,

$$P(X_n < k|\mathbb{T}) \to 0 \quad \textit{as } n \to \infty;$$

(ii) *for* $k = 1, 2, \ldots,$

$$\lim_{n\to\infty} P(n - X_n \le k) = \pi(k) \quad \textit{exists};$$

(iii) *let* $\{\eta_i\}_{i\ge1}$ *be i.i.d.* $\exp(1)$ *random variables; let* $\Gamma_k \equiv \sum_{i=1}^{k} \eta_i$, $k \ge 1$; *for* $0 < \alpha < 1$, *let*

$$Y_\alpha \equiv \frac{\left(\sum\limits_{k=1}^{\infty} \Gamma_k^{-\frac{2}{\alpha}}\right)^{\frac{1}{2}}}{\left(\sum\limits_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}}\right)}.$$

Then, $\pi(k) = E(Y_{\alpha^k})$, $k \ge 1$ *and* $\pi(k) \uparrow 1$ *as* $n \uparrow \infty$.

Remark 1. Conclusion (i) above is often referred to as the "quenched" version (i.e. given the tree $\mathbb{T}$). It implies the "annealed" version, i.e., for $k = 1, 2, \ldots,$

$$P(X_n < k) \to 0 \quad \text{as } n \to \infty.$$

Remark 2. By the strong law, $\frac{\Gamma_k}{k} \to 1$ as $k \to \infty$ and since $0 < \alpha < 1$, $\sum_{k=1}^{\infty} \Gamma_k^{-\frac{p}{\alpha}}$ converges with probability 1 for $p = 1, 2$. Thus, for $0 < \alpha < 1$, Y_α is a well-defined random variable. Since $Y_\alpha = \left(\sum_{k=1}^{\infty} \xi_k^2\right)^{\frac{1}{2}}$ where $\xi_k \equiv \frac{\Gamma_k^{-\frac{1}{\alpha}}}{\sum_{j=1}^{\infty} \Gamma_j^{-\frac{1}{\alpha}}}$ satisfies $0 < \xi_k < 1$, $\forall k$, and $\sum_{k=1}^{\infty} \xi_k = 1$, it follows that $0 < Y_\alpha < 1$ with probability 1.

Remark 3. Note that (ii) and (iii) in Theorem 1 above shows that coalescence in this case takes place in the very recent past. Also, (iii) gives an explicit expression for $\pi(k)$. It can be shown (see Proposition 4) that $E(Y_\alpha)$ is strictly increasing in α and have $\pi(k) - \pi(k-1) > 0$ for all $k \ge 1$.

Remark 4. There is no quenched version of Theorem 1 (ii). This is so since as shown in the proof the random quantity $P(n - X_n \le k|\mathbb{T})$ given by (6) does not converge as $n \to \infty$ for each tree $\mathbb{T}$ but does converge only in distribution.

Under the hypothesis of Theorem 1 the offspring mean $m = \sum_{j=1}^{\infty} jp_j$ is necessarily infinite. If on the other hand the mean m is finite then the coalescence time X_n does indeed converge to a proper distribution.

Theorem 2. *Let the offspring distribution* $\{p_j\}_{j\ge0}$ *satisfy* $p_0 = 0$, $1 < m = \sum_{j=1}^{\infty} jp_j < \infty$. *Then, for almost all family tree* $\mathbb{T}$

(i) *for* $1 \le k < \infty$

$$\lim_{n\to\infty} P(X_n < k|\mathbb{T}) = \pi_k(\mathbb{T}) \quad \textit{exists};$$

(ii) $\pi_k(\mathbb{T}) \uparrow 1$ *as* $k \uparrow \infty$;

(iii) *for* $1 \le k < \infty$,

$$\lim_{n\to\infty} P(X_n < k) \equiv \pi_k \quad \textit{exists}$$

and $\pi_k \uparrow 1$ *as* $k \uparrow \infty$.

Remark 5. In Theorem 2, (i) and (ii) do yield the quenched version. The annealed version (iii) follows easily from (i) and (ii) by the bounded convergence theorem.

In the next section we collect some results needed to prove the above two theorems. Some of these are known in the literature. For others the proofs are given in the Appendix.

3. Some useful results

The proofs of Propositions 1–3 are given in the Appendix.

Proposition 1 (*Grey [6] and Davies [4]*)**.** *Let $\{Z_n\}_{n\geq 0}$ and $\{Z_n^*\}_{n\geq 0}$ be two independent copies of a Galton–Watson branching process with $Z_0 = 1 = Z_0^*$ and the offspring distribution $\{p_j\}_{j\geq 0}$ with $p_0 = 0$ and $m \equiv \sum_{j=1}^{\infty} jp_j = \infty$. Let $\{p_j\}_{j\geq 0}$ be in the domain of attraction of a stable law of order α, $0 < \alpha < 1$. Then*

$$P\left(\frac{Z_n^*}{Z_n} \to 0\right) = \frac{1}{2} = P\left(\frac{Z_n}{Z_n^*} \to 0\right).$$

Proposition 2. *Let $\{Z_n\}_{n\geq 0}$ be a Galton–Watson branching process with the offspring distribution $\{p_j\}_{j\geq 0}$ and $Z_0 = 1$. Suppose $\{p_j\}_{j\geq 0}$ is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$. Then for each $1 \leq k < \infty$, Z_k is in the domain of attraction of a stable law of order α^k.*

Proposition 3. *Let $\{X_i\}_{i\geq 1}$ be i.i.d. rv such that $P(0 < X_1 < \infty) = 1$ and X_1 is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$. Let $\{\eta_i\}_{i\geq 1}$ be i.i.d.* exp(1) *random variables. Let $\Gamma_k \equiv \sum_{i=1}^{k} \eta_i$, $k \geq 1$. Let*

$$Y_\alpha \equiv \frac{\left(\sum_{k=1}^{\infty} \Gamma_k^{-\frac{2}{\alpha}}\right)^{\frac{1}{2}}}{\left(\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}}\right)}.$$

Then

(i)

$$\frac{\sum_{i=1}^{n} X_i^2}{\left(\sum_{i=1}^{n} X_i\right)^2} \xrightarrow{d} Y_\alpha \quad \text{as } n \to \infty;$$

(ii) $\forall 0 < \alpha < 1$, $P(0 < Y_\alpha < 1) = 1$ *and* $E(Y_\alpha) \uparrow 1$ *as* $\alpha \downarrow 0$.

4. Proofs of main results

4.1. Proof of Theorem 1

Fix $0 \leq k < \infty$. Then

$$P(X_n \geq k|\mathbb{T}) = \frac{\sum_{i=1}^{Z_k} \binom{Z_{n-k,i}^{(k)}}{2}}{\binom{Z_n}{2}} \tag{2}$$

where $\{Z^{(k)}_{j,i} : j \geq 0\}$ is the branching process initiated by the ith individual in the kth generation.

The right side of (2) can be written as

$$\frac{\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i}(Z^{(k)}_{n-k,i}-1)}{\left(\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i}\right)\left(\sum_{i=1}^{Z_k} Z^{(k)}_{n-k,i}-1\right)}. \tag{3}$$

By the result of Grey [6] and Davies [4] (Proposition 1) quoted earlier it follows that for almost all trees $\mathbb{T}$, given Z_k, there exists an i_0 (depending on the tree $\mathbb{T}$) such that $1 \leq i_0 \leq Z_k$ and for $i \neq i_0$

$$\frac{Z^{(k)}_{n-k,i}}{Z^{(k)}_{n-k,i_0}} \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Thus the expression in (3) converges with probability 1 to 1 as $n \to \infty$.

This yields from (2) that for each $1 \leq k < \infty$

$$P(X_n \geq k|\mathbb{T}) \to 1 \quad \text{as } n \to \infty, \text{ establishing (i).}$$

Next, for $1 \leq k < \infty$

$$P(X_n \geq n-k) = E\left(\frac{\sum_{i=1}^{Z_{n-k}} Z^{(n-k)}_{k,i}(Z^{(n-k)}_{k,i}-1)}{Z_n(Z_n-1)}\right). \tag{5}$$

From Proposition 2, it follows that for each $k < \infty$, $\{Z^{(n-k)}_{k,i} : 1 \leq i \leq Z_{n-k}\}$ are conditionally (given Z_{n-k}) i.i.d. distributed as Z_k with $Z_0 = 1$ and hence in the domain of attraction of a stable law of order α^k. Also $Z_{n-k} \to \infty$ with probability 1. So by Proposition 3(i) the selfnormalized sum

$$\frac{\sum_{i=1}^{Z_{n-k}} Z^{(n-k)}_{k,i}(Z^{(n-k)}_{k,i}-1)}{\left(\sum_{i=1}^{Z_{n-k}} Z^{(n-k)}_{n,i}\right)\left(\sum_{i=1}^{Z_{n-k}} Z^{(n-k)}_{n,i}-1\right)} \tag{6}$$

converges in distribution to a random variable Y_{α^k} (where Y_α is as in Proposition 3).

By the bounded convergence theorem (5) implies that

$$P(X_n \geq n-k) \to E(Y_{\alpha^k}) = \pi(k), \quad \text{say.}$$

This in turn implies that

$$P(n - X_n \leq k) \to \pi(k) \quad \text{as } n \to \infty.$$

Since $0 < \alpha < 1$, $\alpha^k \to 0$ as $k \to \infty$. Hence, by Proposition 3(ii) $\pi(k) = E(Y_{\alpha^k}) \uparrow 1$ as $k \to \infty$. This completes the proof of Theorem 1. $\square$

4.2. Proof of Theorem 2

From the Seneta–Heyde results (see Athreya and Ney [1]) and (3) there exists a sequence $\{c_n\}_{n\geq 0}$ of constants such that for each k, $0 \leq k < \infty$

$$P(X_n \geq k|\mathbb{T}) = \frac{\sum_{i=1}^{Z_k}\left(\frac{Z_{n-k,i}^{(k)}(Z_{n-k,i}^{(k)}-1)}{c_{n-k}^2}\right)}{\left(\sum_{i=1}^{Z_k}\frac{Z_{n-k,i}^{(k)}}{c_{n-k}}\right)\left(\sum_{i=1}^{Z_k}\frac{Z_{n-k,i}^{(k)}}{c_{n-k}} - \frac{1}{c_{n-k}}\right)}$$

$$\to \frac{\sum_{i=1}^{Z_k} W_{k,i}^2}{\left(\sum_{i=1}^{Z_k} W_{k,i}\right)^2} \equiv \tilde{\pi}_k(\mathbb{T}) \quad \text{say, as } n \to \infty,$$

where conditioned on Z_k, $\{W_{k,i}\}_{i\geq 1}$ are i.i.d. random variables distributed as $W \equiv \lim_{n\to\infty} \frac{Z_n}{c_n}$ (which exists with probability 1 and $P(0 < W < \infty) = 1$). This proves (i).

Since $P(X_n \geq k|\mathbb{T})$ is non-increasing in k, it follows that

$$\tilde{\pi}_k(\mathbb{T}) \downarrow \tilde{\pi}(\mathbb{T}) \quad \text{as } k \uparrow \infty.$$

It remains to show that $\tilde{\pi}(\mathbb{T}) = 0$ with probability 1.

Now if $\{W_i\}_{i\geq 1}$ are i.i.d. distributed as W then for any n

$$\frac{M_n^2}{S_n^2} \leq \frac{\sum_{i=1}^{n} W_i^2}{\left(\sum_{i=1}^{n} W_i\right)^2} \leq \frac{M_n}{S_n}$$

where $M_n = \max\{W_i : 1 \leq i \leq n\}$ and $S_n = \sum_{i=1}^{n} W_i$.

So, it suffices to show that $\frac{M_n}{S_n} \xrightarrow{P} 0$ as this will imply by the bounded convergence theorem $E\tilde{\pi}_k(\mathbb{T}) \to 0$ as $k \to \infty$ implying $E\tilde{\pi}(\mathbb{T}) = 0$ and hence $\tilde{\pi}(\mathbb{T}) = 0$ with probability 1.

Since

$$\tilde{\pi}_k(\mathbb{T}) \equiv \frac{\sum_{i=1}^{Z_k} W_{k,i}^2}{\left(\sum_{i=1}^{Z_k} W_{k,i}\right)^2} \leq \frac{\max\{W_{k,i} : 1 \leq i \leq Z_k\}}{\sum_{i=1}^{Z_k} W_{k,i}} \equiv \tilde{\tilde{\pi}}_k(\mathbb{T}), \quad \text{say}$$

it suffices to show that $\tilde{\tilde{\pi}}_k(\mathcal{T}) \xrightarrow{P} 0$.

Athreya and Schuh [2] have shown that the Seneta–Heyde limit W satisfies $\psi(x) = E(W : W \leq x)$ is slowly varying at ∞, i.e., $\frac{\psi(cx)}{\psi(x)} \to 1$ as $x \to \infty$ for all $0 < c < \infty$.

Also O'Brien [8] has established the following result.

Let $\{X_i\}_{i\geq 1}$ be i.i.d. rv with $P(0 < X_1 < \infty) = 1$. Then the following are equivalent.

(i) $\psi(x) = E(X_1 : X_1 \le x)$ is slowing varying at ∞,

(ii) $\frac{M_n}{S_n} \xrightarrow{P} 0$ where $M_n = \max_{1\le i\le n} X_i$ and $S_n = \sum_{i=1}^{n} X_i$, $n \ge 1$.

Now conditioned on Z_k, $\{W_{k,i} : 1 \le i \le Z_k\}$ are i.i.d. and $Z_k \to \infty$ with probability one, it follows from Athreya and Schuh [2] and O'Brien [8] results that $\tilde{\tilde{\pi}}_k(\mathcal{T}) \to 0$ with probability one. So the proof is complete. $\square$

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Appendix

Proof of Proposition 1.

Remark 6. This proposition is due to Grey [6] and can be deduced from Davies [4]. For completeness, the proof is outlined below.

Since

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}$$

for all $n \ge 0$ as in Section 2, it follows that

$$\frac{Z_{n+1}}{(n+1)^{\frac{1}{\alpha}}} = \left(\frac{1}{Z_n^{\frac{1}{\alpha}} \sum_{i=1}^{Z_n} \xi_{n,i}} \right) \left(\frac{Z_n}{n}\right)^{\frac{1}{\alpha}} \left(\frac{n}{n+1}\right)^{\frac{1}{\alpha}}$$

implying

$$V_{n+1} = \log U_n + \frac{1}{\alpha} V_n + \frac{1}{\alpha} \log\left(1 + \frac{1}{n}\right)^{-1}$$

where $V_n = \ln\left(\frac{Z_n}{n}\right)$, $U_n = \frac{1}{Z_n^{\frac{1}{\alpha}}} \sum_{i=1}^{Z_n} \xi_{n,i}$.

This in turn implies

$$\alpha^{n+1} V_{n+1} = \alpha^{n+1} \log U_n + \alpha^n V_n + \alpha^n \log\left(1 + \frac{1}{n}\right)^{-1}.$$

Since $\{\xi_{n,i}\}_{i\ge1}$ belong to the domain of attraction of a stable law of order α, $0 < \alpha < 1$, $\log U_n$ converges in distribution as $n \to \infty$ (conditioned on Z_n). This can be used to show that $\sum_{j=1}^{\infty} \alpha^j \log U_j$ converges with probability one yielding the result (proved first by Davies [4]) that $\{\alpha^n \log Z_n\}$ converges with probability one to a continuous random variable V, say. Thus, in the explosive case, Z_n grows *super exponentially* fast. For two independent copies of $\{Z_n\}$, $\alpha^n \log Z_n^{(1)} - \alpha^n \log Z_n^{(2)}$ converges with probability one to a random variable $V_1 - V_2$ where V_1

and V_2 are i.i.d. distributed as V. Now, $V_1 - V_2$ is a symmetric continuous random variable and hence $P(V_1 - V_2 > 0) = \frac{1}{2} = P(V_1 - V_2 < 0)$. This in turns yields

$$\log\left(\frac{Z_n^{(1)}}{Z_n^{(2)}}\right) \to \infty \quad \text{or} \quad -\infty$$

with probability $\frac{1}{2}$ each. This is Grey's [6] result quoted in Proposition 1. □

Proof of Proposition 2. Let $f(s) \equiv E(s^{Z_1}|Z_0 = 1) \equiv \sum_{j=0}^{\infty} p_j s^j$, $0 \le s \le 1$. Then $f_2(s) \equiv f(f(s)) = E(s^{Z_2}|Z_0 = 1)$.

From Feller [5, pp. 447] and Bingham et al. [3] it is known that $\{p_j\}_{j\ge 0}$ is in the domain of attraction of a stable law of order α, $0 < \alpha < 1$, iff

$$1 - f(s) \sim (1-s)^{\alpha} L\left(\frac{1}{1-s}\right) \quad \text{as } s \uparrow 1 \tag{A.1}$$

where $L : \mathbb{R}^+ \to \mathbb{R}^+$ is a slowly varying (at ∞) function.

Now,

$$E(s^{Z_2}) = 1 - f_2(s) = 1 - f(f(s)).$$

From (A.1) this is

$$\begin{aligned} &\sim (1 - f(s))^{\alpha} L\Big(\frac{1}{1-f(s)}\Big) \quad \text{as } s \uparrow 1 \\ &\sim (1-s)^{\alpha^2} \left(L\Big(\frac{1}{1-s}\Big)\right)^{\alpha} L\left(\frac{1}{1-f(s)}\right) \quad \text{as } s \uparrow 1 \\ &\sim (1-s)^{\alpha^2} \left(L\Big(\frac{1}{1-s}\Big)\right)^{\alpha} L\left(\frac{1}{(1-s)^{\alpha} L(\frac{1}{1-s})}\right). \end{aligned} \tag{A.2}$$

Let $\tilde{L}(x) = \Big(L(x)\Big)^{\alpha} L\left(x^{\alpha} \frac{1}{L(x)}\right)$ for $x > 0$.

Then for $0 < c < \infty$

$$\frac{\tilde{L}(cx)}{\tilde{L}(x)} = \left(\frac{L(cx)}{L(x)}\right)^{\alpha} \frac{L\big(c^{\alpha} x^{\alpha} \frac{1}{L(cx)}\big)}{L\big(x^{\alpha} \frac{1}{L(x)}\big)}.$$

Since $L(\cdot)$ is slowly varying at ∞, for $0 < c < \infty$,

$$\frac{L(cx)}{L(x)} \to 1 \quad \text{as } x \to \infty$$

$$\frac{L\left(\big(x^{\alpha} \frac{1}{L(x)}\big) c^{\alpha} \frac{L(x)}{L(cx)}\right)}{L\left(x^{\alpha} \frac{1}{L(x)}\right)} \to 1 \quad \text{as } x \to \infty$$

implying

$$\frac{\tilde{L}(cx)}{\tilde{L}(x)} \to 1 \quad \text{as } x \to \infty$$

i.e., $\tilde{L}(\cdot)$ is slowly varying at ∞.

Thus, from Feller [5, pp. 447], it follows that Z_2 (given $Z_0 = 1$) is in the domain of attraction of a stable law of order α^2.

The case of $Z_k | Z_0 = 1$ for $k > 2$ follows by induction. $\square$

Proposition 3 is a special case of the following more general result.

Proposition 4. *Let* $\{X_i\}_{i \geq 1}$ *be i.i.d. random variables such that*

1) $P(0 < X_1 < \infty) = 1$;
2) *for some* $0 < \alpha < 1$ *and a function* $L(\cdot) : [1, \infty) \to (0, \infty)$ *slowly varying at* ∞, *i.e.*

$$\frac{L(cx)}{L(x)} \to 1 \quad \text{as } x \to \infty, \forall 0 < c < \infty, \qquad \lim_{x \to \infty} \frac{P(X > x)}{x^{-\alpha} L(x)} = 1.$$

Let

$$T_{n,r} \equiv \frac{\left(\sum_{i=1}^{n} X_i^r \right)^{\frac{1}{r}}}{\left(\sum_{i=1}^{n} X_i \right)}, \quad 0 < r < \infty, n \geq 1.$$

Then

(i) *for each* $0 < r < \infty$, $0 < \alpha < 1$, *as* $n \to \infty$

$$T_{n,r} \xrightarrow{d} \frac{\left(\sum_{k=1}^{\infty} \Gamma_k^{-\frac{r}{\alpha}} \right)^{\frac{1}{r}}}{\left(\sum_{k=1}^{\infty} \Gamma_k^{-\frac{1}{\alpha}} \right)} \equiv T_\alpha^{(r)}, \qquad \textit{say},$$

where $\Gamma_k \equiv \sum_{i=1}^{k} \eta_i$, $k \geq 1$, $\{\eta_i\}_{i \geq 1}$ *are i.i.d.* exp(1) *random variables;*

(ii) $\forall 0 < r < \infty$, $T_\alpha^{(r)} \to 1$ *with probability* 1 *as* $\alpha \downarrow 0$.

Proof. (i) is proved by Lepage et al. [7, Theorem 1, Corollary 1].

It remains only to prove (ii).

Let $\xi_{r,\alpha} = \sum_{k=1}^{\infty} \Gamma_k^{-\frac{r}{\alpha}}$, where $\{\Gamma_k\}_{k \geq 1}$ is as in (i).

Clearly,

$$\frac{\xi_{r,\alpha}}{\Gamma_1^{-\frac{r}{\alpha}}} = 1 + \sum_{k=2}^{\infty} \left(\frac{\Gamma_1}{\Gamma_k} \right)^{\frac{r}{\alpha}}.$$

Now $0 < \frac{\Gamma_1}{\Gamma_k} < 1$ for all k. So, $\forall k$, $\left(\frac{\Gamma_1}{\Gamma_k} \right)^{\frac{r}{\alpha}} \downarrow 0$ as $\alpha \downarrow 0$.

Further, $0 < \alpha < \frac{r}{2}$,

$$\left(\frac{\Gamma_1}{\Gamma_k} \right)^{\frac{r}{\alpha}} < \left(\frac{\Gamma_1}{\Gamma_k} \right)^2.$$

By s. l. l. n. $\frac{\Gamma_k}{k} \to 1$ with probability 1.

So, $\sum_{k=1}^{\infty} \left(\frac{\Gamma_1}{\Gamma_k} \right)^2 < \infty$ with probability 1.

So by the dominated convergence theorem

$$\lim_{\alpha \to 0} \sum_{k=2}^{\infty} \left(\frac{\Gamma_1}{\Gamma_k} \right)^{\frac{r}{\alpha}} = 0 \quad \text{with probability 1.}$$

Then for $0 < r < \infty$,

$$\frac{\xi_{r,\alpha}}{\Gamma_1^{-\frac{r}{\alpha}}} \to 1 \quad \text{with probability 1 as } \alpha \downarrow 0.$$

Since

$$T_\alpha^{(r)} = \frac{\left(\xi_{r,\alpha}\right)^{\frac{1}{r}}}{\xi_{1,\alpha}} = \left(\frac{\xi_{r,\alpha}}{\Gamma_1^{-\frac{r}{\alpha}}} \right)^{\frac{1}{r}} \frac{1}{\left(\frac{\xi_{1,\alpha}}{\Gamma_1^{-\frac{1}{\alpha}}} \right)}$$

it follows that $T_\alpha^{(r)} \to 1$ with probability 1 as $\alpha \downarrow 0$. $\square$

Remark 7. Proposition 3 is a special case of Proposition 4 with $r = 2$.

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