

Convexity of integral transforms and function spaces

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For $\beta < 1$, let $\mathcal{P}_\gamma(\beta)$ denote the class of all normalized analytic functions f in the unit disc Δ such that

$$\operatorname{Re} \left\{ e^{i\varphi} \left((1-\gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right) \right\} > 0, \quad z \in \Delta,$$

for some $\varphi \in \mathbb{R}$. Let $\mathcal{S}^*(\mu)$, $0 \leq \mu < 1$, denote the usual class of starlike functions of order μ . Define $\mathcal{K}(\mu) = \{f: zf'(z) \in \mathcal{S}^*(\mu)\}$, the class of all convex functions of order μ . In this paper, we consider integral transforms of the form

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \quad \text{and} \quad \mathcal{V}_\lambda(f)(z) = z \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * f(z).$$

The aim of this paper is to find conditions on $\lambda(t)$ so that each of the transformations carries $\mathcal{P}_\gamma(\beta)$ into $\mathcal{S}^*(\mu)$ or $\mathcal{K}(\mu)$. A number of applications for certain special choices of $\lambda(t)$ are also established. These results extend the previously known results by a number of authors.

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1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions f in the open unit disk $\Delta = \{z \in \mathbb{C}: |z| < 1\}$ with the normalization $f(0) = 0 = f'(0) - 1$. Throughout this paper, we let $\lambda: [0, 1] \rightarrow \mathbb{R}$ be a nonnegative weight function normalized so that $\int_0^1 \lambda(t) dt = 1$. For $f \in \mathcal{A}$, we consider

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the integral transform $V_\lambda(f)$ defined by

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt. \quad (1)$$

This operator was introduced by Fournier and Ruscheweyh [1] mainly to find conditions on $\lambda(t)$ and β so that $V_\lambda(f)$ maps the class $\mathcal{P}(\beta)$ into $\mathcal{S}^*$, where

$$\mathcal{P}(\beta) = \{f \in \mathcal{A}: \operatorname{Re} e^{i\varphi} (f'(z) - \beta) > 0 \text{ for some } \varphi \in \mathbb{R} \text{ and for all } z \in \Delta\}$$

and $\mathcal{S}^*$ denote the class of all functions in $\mathcal{A}$, which maps Δ onto a domain that is starlike with respect to origin. The results of Fournier and Ruscheweyh [1] have been extended further by a number of authors [see, for example, refs. 2–5], and, in particular, these authors obtained interesting applications for various choices of $\lambda(t)$.

For $0 \leq \gamma \leq 1$ and $\beta < 1$, let $\mathcal{P}_\gamma(\beta)$ denote the class of all analytic functions f in $\mathcal{A}$ such that

$$\operatorname{Re} \left\{ e^{i\varphi} \left((1-\gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right) \right\} > 0$$

for some $\varphi \in \mathbb{R}$ and for all $z \in \Delta$. This class has been considered by several authors on different counts. One of the questions that we are interested in throughout this paper is the following.

PROBLEM 1.1 For $f \in \mathcal{P}_\gamma(\beta)$, find conditions on λ such that $V_\lambda(f) \in \mathcal{K}(\mu)$. Here, $\mathcal{K}(\mu)$ denotes the class of convex functions of order μ , $0 \leq \mu < 1$:

$$\mathcal{K}(\mu) = \left\{ f \in \mathcal{A}: \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \mu, \quad z \in \Delta \right\}.$$

We recall that $F(z) \in \mathcal{K}(\mu)$ if and only if $f(z) = zF'(z) \in \mathcal{S}^*(\mu)$. Here, $\mathcal{S}^*(\mu)$ denotes the class of all starlike functions of order μ , $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$, the class of all convex functions defined on the unit disc Δ . In a recent paper, Choi *et al.* [6] [see also ref. [2]] answered this problem for the case $\mu = 0$ with the help of duality theory of convolutions developed mainly by Ruscheweyh [7]. The same method has been used earlier to discuss similar problems involving the class $\mathcal{P}_\gamma(\beta)$ but with the assumption that $\gamma = 1$. In this paper, we discuss this problem for $\mu \in [0, 1/2]$ and hence our results extend and improve the work of Choi *et al.* [6] and some others [1, 3–5, 8, 16]. In addition to these results, we also find conditions on λ such that

$$\mathcal{V}_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}^*(\mu) \quad \text{or} \quad \mathcal{K}(\mu),$$

where $\mathcal{V}_\lambda(f)$ is the convolution operator defined by

$$\mathcal{V}_\lambda(f)(z) = z \int_0^1 \lambda(t) \frac{1 - \rho tz}{1 - tz} dt * f(z) \quad (\rho < 1).$$

2. Main results and their proofs

We start with the function $g_\gamma^\mu(t)$, which is the solution to the initial value problem

$$\frac{d}{dt} (t^{(1/\gamma)}(1 + g_\gamma^\mu(t))) = \frac{2}{\gamma} \frac{t^{(1/\gamma)-1}[1 - \mu(1+t)]}{(1-\mu)(1+t)^2}, \quad g_\gamma^\mu(0) = 1, \quad (2)$$

where $\mu \in [0, 1/2]$ and $\gamma \in [1/2, 1]$. Solving equation (2), we have

$$g_\gamma^\mu(t) = \frac{2}{\gamma(1-\mu)} t^{-1/\gamma} \int_0^t u^{(1/\gamma)-1} \frac{1 - \mu(1+u)}{(1+u)^2} du - 1. \quad (3)$$

Using the idea of the proof of Fournier and Ruscheweyh [1], the following lemmas were obtained by Balasubramanian *et al.* [4], where a number of new applications were established.

LEMMA 2.1 Assume that $\gamma > 0$,

$$\Lambda_\gamma(t) = \int_t^1 \frac{\lambda(s)}{s^{1/\gamma}} ds \quad (4)$$

and $\lim_{t \rightarrow 0^+} t^{1/\gamma} \Lambda_\gamma(t) = 0$ ($0 < t < 1$). Let $f \in \mathcal{P}_\gamma(\beta)$, $0 \leq \mu \leq 1/2$ and $\beta < 1$ with

$$\frac{\beta}{1-\beta} = - \int_0^1 \lambda(t) g_\gamma^\mu(t) dt,$$

where $g_\gamma^\mu(t)$ is given by equation (3). Then,

$$V_\lambda(f(z)) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt$$

is starlike of order μ if and only if $\mathcal{L}_{\Lambda_\gamma}(h) \geq 0$, where

$$\mathcal{L}_{\Lambda_\gamma}(h_\mu) = \operatorname{Re} \int_0^1 t^{1/(\gamma-1)} \Lambda_\gamma(t) \left[\frac{h_\mu(tz)}{tz} - \frac{1 - \mu(1+t)}{(1-\mu)(1+t)^2} \right] dt, \quad z \in \Delta \quad (5)$$

and

$$h_\mu(z) = \frac{z(1 + ((\epsilon + 2\mu - 1)/(2(1-\mu)))z)}{(1-z)^2}, \quad |\epsilon| = 1. \quad (6)$$

LEMMA 2.2 Let $\mu \in [0, 1/2]$ and h_μ be defined by equation (6). Assume Λ_γ is integrable on $[0, 1]$ and positive on $(0, 1)$. Assume further that

$$\frac{\Lambda_\gamma(t)}{(\log(1/t))^{1+2\mu}} \quad (7)$$

is decreasing on $(0, 1)$. Then, for $(1/2) \leq \gamma \leq 1$, we have $L_{\Lambda_\gamma}(h_\mu) \geq 0$.

Now, we start developing a proof that applies to the integral transforms of form (1) carrying functions in $\mathcal{P}_\gamma(\beta)$ into convex functions of order μ , $\mu \in [0, 1/2]$. We need to introduce some

basic notation: let $q_\gamma^\mu(t)$ be the solution of the initial value problem

$$\frac{d}{dt} (t^{1/\gamma} q_\gamma^\mu(t)) = \frac{1}{\gamma} \frac{t^{(1/\gamma)-1} [(1-\mu) - (1+\mu)t]}{(1-\mu)(1+t)^3}, \quad q_\gamma^\mu(0) = 1. \quad (8)$$

Solving equation (8), we have

$$q_\gamma^\mu(t) = \frac{1}{\gamma(1-\mu)} t^{-1/\gamma} \int_0^t u^{(1/\gamma)-1} \frac{(1-\mu) - (1+\mu)u}{(1+u)^3} du. \quad (9)$$

For the function $\Lambda_\gamma(t)$, we define

$$M_{\Lambda_\gamma}^\mu(h_\mu) = \inf_{z \in \Delta} \int_0^1 t^{(1/\gamma)-1} \Lambda_\gamma(t) \left[\operatorname{Re} h'_\mu(zt) - \frac{(1-\mu) - (1+\mu)t}{(1+t)^3} \right] dt, \quad (10)$$

where $h_\mu(z)$ is defined by equation (6).

With this notation, we are in a position to state our first result, which generalizes Lemma 3 in ref. [6].

THEOREM 2.3 *Let $\gamma \in [1/2, 1]$ and $\Lambda_\gamma(t)$ be an integrable function on $[0, 1]$ defined by equation (4) such that*

$$\lim_{t \rightarrow 0^+} t^{1/\gamma} \Lambda_\gamma(t) = 0.$$

Suppose that $0 \leq \mu \leq 1/2$,

$$\frac{(1 - (1/\gamma))\Lambda_\gamma(t) - t\Lambda'_\gamma(t)}{(\log(1/t))^{1+2\mu}} = \frac{(1 - (1/\gamma))\Lambda_\gamma(t) + t^{1-(1/\gamma)}\lambda(t)}{(\log(1/t))^{1+2\mu}} \quad (11)$$

is decreasing on $(0, 1)$ and $\lambda(t)$ is decreasing on $[0, 1]$. Then, we have

- (i) $M_{\Lambda_\gamma}(h_\mu) \geq 0$
- (ii) $V_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{K}(\mu)$ if and only if $M_{\Lambda_\gamma}(h_\mu) \geq 0$, where $\beta < 1$ is given by

$$\frac{\beta - (1/2)}{1 - \beta} = - \int_0^1 \lambda(t) q_\gamma^\mu(t) dt. \quad (12)$$

Here, $q_\gamma^\mu(t)$ and $M_{\Lambda_\gamma}(h_\mu)$ are defined by equations (9) and (10), respectively.

Proof As in ref. [6], integration by parts shows that

$$\begin{aligned} I_\gamma^\mu(z) &= \int_0^1 t^{(1/\gamma)-1} \Lambda_\gamma(t) \left(\operatorname{Re} h'_\mu(zt) - \frac{1-\mu - (1+\mu)t}{(1-\mu)(1+t)^3} \right) dt \\ &= \int_0^1 t^{(1/\gamma)-1} \Lambda_\gamma(t) \frac{d}{dt} \left(\operatorname{Re} \frac{h_\mu(zt)}{z} - \frac{t(1-\mu(1+t))}{(1-\mu)(1+t)^2} \right) dt \\ &= - \int_0^1 t^{(1/\gamma)-1} \left[t\Lambda'_\gamma(t) - \left(1 - \frac{1}{\gamma} \right) \Lambda_\gamma(t) \right] \left(\operatorname{Re} \frac{h_\mu(zt)}{zt} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2} \right) dt. \end{aligned}$$

With our hypotheses here, $\lambda(t)$ is decreasing and nonnegative on $[0, 1]$ so that $\lambda(t) \geq \lambda(1) \geq 0$. Moreover, by equation (11), we have

$$-t\Lambda'_\gamma(t) = t^{1-(1/\gamma)}\lambda(t), \quad (13)$$

which shows that for $t \in (0, 1)$,

$$\begin{aligned} \frac{d}{dt} \left\{ -t \Lambda'_\gamma(t) + \left(1 - \frac{1}{\gamma}\right) \Lambda_\gamma(t) \right\} &= \frac{d}{dt} \left\{ t^{1-(1/\gamma)} \lambda(t) + \left(1 - \frac{1}{\gamma}\right) \Lambda'_\gamma(t) \right\} \\ &= t^{1-(1/\gamma)} \lambda'(t) \\ &< 0 \end{aligned}$$

because $\lambda(t)$ is decreasing on $[0, 1]$. By equations (11) and (13), this inequality implies that

$$-t \Lambda'_\gamma(t) + \left(1 - \frac{1}{\gamma}\right) \Lambda_\gamma(t) > 0, \quad t \in (0, 1).$$

In view of this observation and Lemma 2.1, it follows that $\inf_{z \in \Delta} I'_\gamma(z) \geq 0$, which is equivalent to $M_{\Lambda_\gamma}(h_\mu) \geq 0$ for each $0 \leq \mu \leq 1/2$. Part (i) follows.

(ii) For convenience, we let $F(z) = V_\lambda(f)(z)$. Then, by convolution theory [see ref. [7]], we have

$$F \in \mathcal{K}(\mu) \iff \frac{1}{z} (zf'(z) * h_\mu(z)) = \left(\int_0^1 \frac{\lambda(t)}{1-tz} dt \right) * \frac{f(z)}{z} * h'_\mu(z) \neq 0.$$

As $f \in \mathcal{P}_\gamma(\beta)$, $F \in \mathcal{K}(\mu)$ is equivalent to

$$\begin{aligned} 0 &\neq \left(\int_0^1 \frac{\lambda(t)}{1-tz} dt \right) * \left((1-\beta) \frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{1+xw}{1+yw} dw + \beta \right) * h'_\mu(z) \\ &= (1-\beta) \int_0^1 \lambda(t) \left(h'_\mu(tz) + \frac{\beta}{1-\beta} \right) dt * \frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{1+xw}{1+yw} dw \\ &= (1-\beta) \int_0^1 \lambda(t) \left(\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} h'_\mu(tw) dw + \frac{\beta}{1-\beta} \right) dt * \frac{1+xz}{1+yz}, \end{aligned}$$

which clearly holds if and only if

$$\operatorname{Re} \left[(1-\beta) \int_0^1 \lambda(t) \left(\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} h'_\mu(tw) dw + \frac{\beta}{1-\beta} \right) dt \right] > \frac{1}{2}$$

or

$$\operatorname{Re} \int_0^1 \lambda(t) \left(\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} h'_\mu(tw) dw + \frac{\beta - (1/2)}{1-\beta} \right) dt > 0.$$

By equation (12), the last condition is same as the inequality

$$\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{1/\gamma}} \left(\frac{\gamma t^{1/\gamma}}{z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} h'_\mu(tw) dw - t^{1/\gamma} q_\gamma^\mu(t) \right) dt > 0,$$

which is equivalent to

$$\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{(1/\gamma)-1}} \left(\frac{1}{\gamma z^{1/\gamma}} \int_0^{tz} u^{(1/\gamma)-1} h'_\mu(u) du - t^{1/\gamma} q_\gamma^\mu(t) \right) dt > 0.$$

Using the fact that $\Lambda'_\gamma(t) = -\lambda(t)t^{-1/\gamma}$ and the integration by parts, we obtain

$$\int_0^1 t^{(1/\gamma)-1} \Lambda_\gamma(t) \left(\operatorname{Re} h'_\mu(tz) - \frac{1-\mu-(1+\mu)t}{(1-\mu)(1+t)^3} \right) dt > 0,$$

which means that $M_{\Lambda_\gamma}^\mu(h_\mu) > 0$. This completes the proof of part (ii). ■

THEOREM 2.4 *Let $\mu \in [0, 1/2]$ and $\gamma \in [1/2, 1]$ and let $\Lambda_\gamma(t)$ be defined as in equation (4). For $\rho < 1$ and $f \in \mathcal{A}$, let*

$$\mathcal{V}_\lambda(f)(z) = z \int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt * f(z)$$

and let β be given by

$$\frac{1}{2(1 - \rho)(1 - \beta)} = \int_0^1 \lambda(t) \left(\frac{1 - g_\gamma^\mu(t)}{2} \right) dt,$$

where $g_\gamma^\mu(t)$ is defined by equation (3). Then, $\mathcal{V}_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}^*(\mu) \iff \mathcal{L}_{\Lambda_\gamma}(h_\mu) \geq 0$.

Proof Clearly, for $\rho < 1$, we can write

$$\mathcal{V}_\gamma(f)(z) = \rho z + (1 - \rho)V_\lambda(f)(z) =: F(z),$$

where $V_\lambda(f)$ is defined by equation (1). Let $f \in \mathcal{P}_\gamma(\beta)$ and G be defined by

$$G(z) = \frac{1}{1 - \beta} \left[(1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right]$$

so that $f \in \mathcal{P}_\gamma(\beta)$ is equivalent to $\operatorname{Re}(e^{i\varphi} G(z)) > 0$ for some $\varphi \in \mathbb{R}$. As usual, by duality principle, we may restrict our attention to the function $f \in \mathcal{P}_\gamma(\beta)$ for which

$$G(z) = \frac{1 + xz}{1 + yz}, \quad |x| = |y| = 1$$

so that

$$\frac{f(z)}{z} = \beta + (1 - \beta) \frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{1/\gamma-1} \left(\frac{1 + xw}{1 + yw} \right) dw.$$

It follows that

$$\frac{F(z)}{z} = \int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt * \frac{1}{z} \int_0^z f'(w) dw = \int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt * \frac{f(z)}{z}.$$

Let $h_\mu(z)$ be given by equation (6). Therefore, to show that $F \in \mathcal{S}^*(\mu)$, it suffices to prove that

$$0 \neq \frac{1}{z} (F(z) * h_\mu(z)),$$

which is equivalent to

$$\begin{aligned} 0 &\neq \int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt * \frac{f(z)}{z} * \frac{h_\mu(z)}{z} \\ &= \left(\int_0^1 \lambda(t) \frac{1 - \rho t z}{1 - t z} dt \right) * \left(\beta + (1 - \beta) \frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{1 + xw}{1 + yw} dw \right) * \frac{h_\mu(z)}{z} \\ &= \int_0^1 \lambda(t) \left[(1 - \beta)(1 - \rho) \left\{ \frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{h_\mu(tw)}{tw} dw \right. \right. \\ &\quad \left. \left. + \frac{1}{(1 - \beta)(1 - \rho)} - 1 \right\} \right] dt * \left(\frac{1 + xz}{1 + yz} \right). \end{aligned}$$

Using another well-known result from the convolution theory [7], we conclude that the above holds if and only if

$$\operatorname{Re} \int_0^1 \lambda(t) \left[\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{h_\mu(tw)}{tw} dw - \left(1 - \frac{1}{2(1-\beta)(1-\rho)} \right) \right] dt > 0.$$

Substituting the value of β in this inequality, we get

$$\operatorname{Re} \int_0^1 \lambda(t) \left[\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{h_\mu(tw)}{tw} dw - \frac{1 + g_\gamma^\mu(t)}{2} \right] dt > 0.$$

By the change of variable $u = tw$, the last inequality is equivalent to

$$\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{1/\gamma}} \left[\frac{1}{\gamma z^{1/\gamma}} \int_0^{tz} u^{(1/\gamma)-1} \frac{h_\mu(u)}{u} du - t^{1/\gamma} \left(\frac{1 + g_\gamma^\mu(t)}{2} \right) \right] dt > 0$$

and an integration by parts gives

$$\int_0^1 t^{(1/\gamma)-1s} \Lambda_\gamma(t) \left[\operatorname{Re} \frac{h_\mu(tz)}{tz} - \frac{1 - \mu(1+t)}{(1-\mu)(1+t)^2} \right] dt > 0,$$

which is same as $\mathcal{L}_{\Lambda_\gamma}(h_\mu) > 0$ and the proof is complete. ■

THEOREM 2.5 Assume the hypotheses of Theorem 2.3. For $\rho < 1$ and $f \in \mathcal{A}$, define

$$\mathcal{V}_\lambda(f)(z) = z \int_0^1 \lambda(t) \frac{1 - \rho tz}{1 - tz} dt * f(z).$$

Then, for each $\gamma \in [1/2, 1]$, we have $\mathcal{V}_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{K}(\mu) \Leftrightarrow \mathcal{M}_{\Lambda_\gamma}(h_\mu) \geq 0$, where $\beta < 1$ is given by

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t) [1 - q_\gamma^\mu(t)] dt,$$

$q_\gamma^\mu(t)$ and $\mathcal{M}_{\Lambda_\gamma}(h_\mu)$ are defined by equations (9) and (10), respectively.

Proof We follow the idea of the proof of the previous theorem. According to this, $F = \mathcal{V}_\lambda(f)$ is convex function of order μ if and only if $0 \neq (F(z) * H_\mu(z))/z$, $H_\mu(z) = zh'_\mu(z)$. To complete the proof, in view of this observation, it suffices to verify

$$\operatorname{Re} \int_0^1 \lambda(t) \left[\frac{1}{\gamma z^{1/\gamma}} \int_0^z w^{(1/\gamma)-1} \frac{H_\mu(tw)}{tw} dw - \left(1 - \frac{1}{2(1-\beta)(1-\rho)} \right) \right] dt > 0.$$

If we substitute the value of β together with the change of variable $u = tw$, then integration by parts clearly shows that the last inequality is equivalent to

$$\operatorname{Re} \int_0^1 t^{(1/\gamma)-1} \Lambda_\gamma(t) \left[\operatorname{Re} h'_\mu(tz) - \frac{1 - \mu - t(1+\mu)}{(1-\mu)(1+t)^3} \right] dt > 0,$$

which is same as $\mathcal{M}_{\Lambda_\gamma}(h_\mu) > 0$. The proof follows. ■

3. Applications to special integral operators

For our applications, we need some preparation and then we apply Theorem 2.3. In order to apply Theorem 2.3 with $\mu \in [0, 1/2]$, the hardest part is to show that the function g defined by

$$g(t) = \frac{t^{1-(1/\gamma)}\lambda(t) + (1 - 1/\gamma)\Lambda_\gamma(t)}{(\log(1/t))^{1+2\mu}}$$

is decreasing on $(0, 1)$, where $\Lambda_\gamma(t)$ is given by equation (4). Note that in our applications, each $\lambda(t)$ that we have chosen in this section satisfies the condition $\lambda(1) = 0$ and hence in our discussion here, we assume that this condition holds. Using the fact that

$$\Lambda'_\gamma(t) = -\frac{\lambda(t)}{t^{1/\gamma}},$$

it can be easily seen that

$$g'(t) = \frac{\psi(t)}{t(\log(1/t))^{1+2\mu}},$$

where

$$\psi(t) = \left(\log\left(\frac{1}{t}\right)\right) t^{2-(1/\gamma)}\lambda'(t) + (1 + 2\mu) \left(t^{1-(1/\gamma)}\lambda(t) + \left(1 - \frac{1}{\gamma}\right)\Lambda_\gamma(t)\right).$$

As $\lambda(1) = 0$, we see that $\psi(1) = 0$. Thus, $g'(t) \leq 0$ for $t \in (0, 1)$ is equivalent to show that $\psi(t)$ is increasing on $(0, 1)$. Now, we compute

$$\psi'(t) = t^{1-(1/\gamma)} \left[\log\left(\frac{1}{t}\right) t\lambda''(t) + \left(\left(2 - \frac{1}{\gamma}\right) \log\left(\frac{1}{t}\right) + 2\mu \right) \lambda'(t) \right]$$

and $\psi'(t) \geq 0$ for $t \in (0, 1)$ if and only if

$$t\lambda''(t) + \left(2 - \frac{1}{\gamma} + \frac{2\mu}{\log(1/t)}\right)\lambda'(t) \geq 0. \quad (14)$$

Thus, to prove that g is decreasing on $(0, 1)$, it suffices to look at the inequality (14). Now, we define ϕ by

$$\phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n, \quad t \in (0, 1),$$

where $b_n \geq 0$ for each $n \geq 1$, and consider

$$\lambda(t) = Kt^{B-1}(1-t)^{C-A-B}\phi(1-t), \quad (15)$$

where K is a constant satisfying the normalization condition that $\int_0^1 \lambda(t)dt = 1$. With the $\lambda(t)$ given by equation (15), by a simple computation, we find that

$$\begin{aligned} \lambda'(t) = & Kt^{B-2}(1-t)^{C-A-B-1}[(B-1)(1-t) \\ & - (C-A-B)t]\phi(1-t) - t(1-t)\phi'(1-t) \end{aligned}$$

and

$$\begin{aligned} t\lambda''(t) = & Kt^{B-2}(1-t)^{C-A-B-2}[\{(B-1)(B-2)(1-t)^2 \\ & - 2(B-1)(C-A-B)t(1-t) + (C-A-B)(C-A-B-1)t^2\}\phi(1-t) \\ & + \{2(C-A-B)t - 2(B-1)(1-t)\}t(1-t)\phi'(1-t) + t^2(1-t)^2\phi''(1-t)]. \end{aligned}$$

If we substitute the values of $\lambda'(t)$ and of $t\lambda''(t)$ in equation (14), then, after simplifying the expression, we obtain that g is decreasing on $(0, 1)$ if and only if

$$\Psi(t) \geq 0, \quad (16)$$

where

$$\begin{aligned} \Psi(t) = & \left[\log\left(\frac{1}{t}\right) X(t) - 2\mu(1-t)Y(t) \right] \phi(1-t) + Z(t)t(1-t)\phi'(1-t) \\ & + \log\left(\frac{1}{t}\right) t^2(1-t)^2\phi''(1-t) \end{aligned}$$

with

$$\begin{aligned} X(t) = & (1-B)\left(\frac{1}{\gamma} - B\right)(1-t)^2 + \left(\frac{1}{\gamma} - 2B\right)(C-A-B)t(1-t) \\ & + (C-A-B)(C-A-B-1)t^2 \\ Y(t) = & (1-B)(1-t) + (C-A-B)t \\ Z(t) = & \log\left(\frac{1}{t}\right) \left\{ 2(C-A-B)t - \left(2B - \frac{1}{\gamma}\right)(1-t) \right\} - 2\mu(1-t). \end{aligned}$$

Now we are in a position to state and prove a general result.

THEOREM 3.1 *Let $(1/2) \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by (9). Suppose that β is given by*

$$\frac{\beta - (1/2)}{1 - \beta} = -K \int_0^1 t^{B-1}(1-t)^{C-A-B} \phi(1-t) q_\gamma^\mu(t) dt,$$

(where $A, B, C > 0$, and K is a constant such that $K \int_0^1 t^{B-1}(1-t)^{C-A-B} \phi(1-t) dt = 1$). Then, for $f \in P_\gamma(\beta)$, the function

$$V_\lambda(f(z)) = K \int_0^1 t^{B-1}(1-t)^{C-A-B} \phi(1-t) \frac{f(tz)}{t} dt$$

belongs to $K(\mu)$ whenever A, B, C, γ and μ satisfy the conditions

$$0 < B \leq \frac{1}{2\gamma} - \mu \quad \text{and} \quad C \geq A + B + 1 + 2\mu. \quad (17)$$

Proof To establish the theorem, it is enough to show that the inequality (16) holds under the stated conditions. As $\phi(1-t) = 1 + \sum_{n=1}^{\infty} b_n(1-t)^n$ with $b_n \geq 0$ for all $n \geq 1$, each of the functions $\phi(1-t)$, $\phi'(1-t)$ and $\phi''(1-t)$ in the expression of $\Psi(t)$ is nonnegative for

$t \in (0, 1)$. Therefore, to complete the proof, it suffices to show the following two inequalities for $t \in (0, 1)$:

$$\log\left(\frac{1}{t}\right) X(t) \geq 2\mu(1-t)Y(t) \quad (18)$$

and

$$Z(t) \geq 0, \quad (19)$$

where $X(t)$, $Y(t)$ and $Z(t)$ are defined as earlier. Clearly,

$$2(1-t) \leq (1+t) \log \frac{1}{t} \quad (20)$$

holds for $t \in (0, 1)$. The truth of this inequality may be seen by comparing the coefficients of $(1-t)^n$ in the series expansion of both side functions in

$$\frac{1}{1 - ((1-t)/2)} \leq \frac{-\log(1 - (1-t))}{1-t}.$$

For convenience, we let $\alpha = C - A - B$. To prove equation (18), in view of equation (20), it is enough to show that $X(t) \geq \mu(1+t)Y(t)$. That is,

$$\begin{aligned} (1-B) \left(\frac{1}{\gamma} - B \right) (1-t)^2 + \left(\frac{1}{\gamma} - 2B \right) \alpha t (1-t) + \alpha(\alpha-1)t^2 \\ \geq \mu \{ (1-B)(1-t) + \alpha t \} (1-t+2t), \end{aligned}$$

which may be rewritten as

$$\begin{aligned} (1-t)^2 \left\{ (1-B) \left(\frac{1}{\gamma} - B - \mu \right) \right\} + t(1-t) \left\{ \alpha \left(\frac{1}{\gamma} - 2B - \mu \right) - 2\mu(1-B) \right\} \\ + \alpha(\alpha-1-2\mu)t^2 \geq 0. \end{aligned}$$

We claim that this inequality holds for all $t \in (0, 1)$. To do this, we observe that under the conditions in equation (17), both the coefficients of $(1-t)^2$ and t^2 in the left-hand expression of the last inequality are nonnegative. The coefficients of $t(1-t)$ are also nonnegative. Indeed,

$$\begin{aligned} \alpha \left(\frac{1}{\gamma} - 2B - \mu \right) - 2\mu(1-B) &= (\alpha - \mu) \left(\frac{1}{\gamma} - 2B - \mu \right) - \left[\mu^2 + \left(2 - \frac{1}{\gamma} \right) \mu \right] \\ &\geq (1 + \mu) \left(\frac{1}{\gamma} - 2B - \mu \right) - \left[\mu^2 + \left(2 - \frac{1}{\gamma} \right) \mu \right] \\ &\quad (\text{as } \alpha \geq 1 + 2\mu) \\ &= (1 + \mu) \left[\frac{1}{\gamma} - 2B - 2\mu + \left(\frac{1}{\gamma} - 1 \right) \frac{\mu}{1 + \mu} \right] \\ &\geq 0 \quad (\text{as } \gamma \leq 1 \text{ and } \frac{1}{\gamma} - 2B - \mu \geq 0). \end{aligned}$$

Therefore, equation (18) holds for $t \in (0, 1)$. It remains to verify the inequality (19) for $t \in (0, 1)$. Again, in view of equation (20), the inequality (19) holds if

$$2\alpha t - \left(2B - \frac{1}{\gamma} \right) (1-t) \geq \mu(1+t)$$

is true for $t \in (0, 1)$. We may rewrite this inequality as

$$[2(\alpha - 2\mu) + \mu]t + 2\left(\frac{1}{2\gamma} - B - \mu\right)(1 - t) + \mu \geq 0,$$

which is clearly true under condition (17). Thus, we obtain that $\Psi(t) \geq 0$ under the stated conditions and hence we complete the proof by applying Theorem 2.3. ■

The presence of the factor $\phi(1 - t)$ in the last result is important in the sense that special cases give a number of interesting applications as we see in the following two theorems. Before we state these special results, it is necessary to introduce some notation. For $a \neq 0$ and n a nonnegative integer, let $(a, n) := a(a + 1) \cdots (a + n - 1)$ if $n > 0$ and $(a, 0) = 1$. Now, the Gaussian hypergeometric function is defined by

$$F(a, b; c; z) \equiv {}_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)(1, n)} z^n (|z| < 1),$$

where a, b and c are complex numbers with $c \neq 0, -1, -2, \dots$. A number of authors [9–13] discussed the univalence, starlikeness and the convexity of the normalized hypergeometric functions $zF(a, b; c; z)$ and $(c/ab)[F(a, b; c; z) - 1]$, under a suitable restriction on the parameters a, b and c .

THEOREM 3.2 *Let $a, b, c > 0$, $(1/2) \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by equation (9). Suppose that $\beta = \beta_{a,b,c,\gamma,\mu}$ is given by*

$$\frac{\beta - (1/2)}{1 - \beta} = -\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - a - b + 1)} \int_0^1 t^{b-1} (1 - t)^{c-a-b} F\left(\frac{c-a}{c-a-b+1}, \frac{1-a}{c-a-b+1}; 1-t\right) q_\gamma^\mu(t) dt.$$

Then for $f \in \mathcal{P}_\gamma(\beta)$, the function $zF(a, b; c; z) * f(z)$ belongs to $\mathcal{K}(\mu)$ whenever a, b, c, γ and μ are related by

$$0 < a \leq 1, \quad 0 < b \leq \frac{1}{2\gamma} - \mu \quad \text{and} \quad c \geq a + b + 1 + 2\mu.$$

Proof The desired conclusion follows from Theorem 3.1 if we choose

$$\phi(1 - t) = F\left(\frac{c-a}{c-a-b+1}, \frac{1-a}{c-a-b+1}; 1-t\right)$$

and substitute A, B and C in Theorem 3.1 by a, b and c , respectively. ■

We recall the following integral representation from refs. [5, 8]:

$$V_\lambda f(z) = zF(a, b; c; z) * f(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt, \quad \text{with}$$

$$\lambda(t) = K t^{b-1} (1 - t)^{c-a-b} \phi(1 - t),$$

where we assume that $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$, $\operatorname{Re}(c + 1) > \operatorname{Re}(a + b)$ and the normalization constant K in this choice is

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c - a - b + 1)}.$$

We remark that the operator $zF(1, 1 + \gamma; 2 + \gamma; z) * f(z)$ is known as the Bernardi operator.

THEOREM 3.3 Let $1/2 \leq \gamma \leq 1$, $\mu \in [0, 1/2]$ and $q_\gamma^\mu(t)$ be defined by equation (9). Suppose that $\beta = \beta(a, p, \gamma, \mu)$ is given by

$$\frac{\beta - (1/2)}{1 - \beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \left(\log \left(\frac{1}{t} \right) \right)^{p-1} q_\gamma^\mu(t) dt.$$

Then, for $f \in \mathcal{P}_\gamma(\beta)$, the Hadamard product function $\Phi_p(a; z) * f(z)$ defined by

$$\Phi_p(a; z) * f(z) = \left(\sum_{n=1}^{\infty} \frac{(1+a)^p}{(n+a)^p} z^n \right) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log \frac{1}{t} \right)^{p-1} t^{a-1} f(tz) dt$$

belongs to $\mathcal{K}(\mu)$ whenever $p \geq 2(1 + \mu)$ and $0 \leq a + 1 \leq (1/2\gamma) - \mu$. The value of β is sharp. (This convolution operator is called the Komatu operator [14].)

Proof Choose

$$\phi(1-t) = \left(\frac{\log(1/t)}{1-t} \right)^{p-1} = \left(\frac{-\log(1-(1-t))}{1-t} \right)^{p-1},$$

$C - A - B = p - 1$ and $B = a + 1$ so that $\lambda(t)$ defined by equation (15) takes the form

$$\lambda(t) = K t^a (1-t)^{p-1} \phi(1-t), \quad K = \frac{(1+a)^p}{\Gamma(p)}.$$

The desired conclusion follows from Theorem 3.1. ■

Our final application concerns the integral operator introduced by Ponnusamy [15] and studied later by Ponnusamy and Rønning [3], and by Balasubramanian *et al.* [8] [see also refs. 2, 4]. Define

$$\lambda(t) = \begin{cases} (a+1)(b+1) \left(\frac{t^a(1-t^{b-a})}{b-a} \right) & \text{for } b \neq a, a > -1, b > -1, \\ (a+1)^2 t^a \log \left(\frac{1}{t} \right) & \text{for } b = a, a > -1. \end{cases} \quad (21)$$

With this $\lambda(t)$, we have the integral transform

$$G_f(a, b; z) := \left(\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n \right) * f(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.$$

In view of the symmetry between a and b , without loss of generality, we assume that $b > a$ in the case $b \neq a$.

THEOREM 3.4 Let $b > -1$, $a > -1$ and $\gamma \in [1/2, 1]$. Suppose that a, b and γ are related by any one of the following conditions:

- (i) $-1 < a \leq 0$, and $b = a$
- (ii) $-1 < a \leq 0$, and $-1 < b \leq (1/\gamma) - 1$.

Let $q_\gamma^0(t)$ and $\lambda(t)$ be defined by equations (9) and (21), respectively. If β is given by

$$\frac{\beta - (1/2)}{1 - \beta} = - \int_0^1 \lambda(t) q_\gamma^0(t) dt,$$

then for $f \in \mathcal{P}_\gamma(\beta)$, the function $G_f(a, b; z) \in \mathcal{K}$. The value of β is sharp.

Proof Clearly, as in the proof of our earlier theorems, it suffices to verify the inequality (14) for the $\lambda(t)$ defined by equation (21). Now, we have

$$\lambda'(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a} t^{a-1} (a - bt^{b-a}) & \text{for } b > a > -1 \\ (a+1)^2 \left(-1 + a \log \left(\frac{1}{t} \right) \right) t^{a-1} & \text{for } b = a > -1 \end{cases}$$

and

$$t\lambda''(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a} t^{a-1} (a(a-1) - b(b-1)t^{b-a}) & \text{for } b > a > -1 \\ (a+1)^2 \left(1 - 2a + a(a-1) \log \left(\frac{1}{t} \right) \right) t^{a-1} & \text{for } b = a > -1. \end{cases}$$

Case (i). Let $b = a > -1$. If we substitute the expressions for $\lambda'(t)$ and the $t\lambda''(t)$ in equation (14), then inequality (14) is seen to be equivalent to

$$\left(\log \left(\frac{1}{t} \right) \right)^2 a \left(a + 1 - \frac{1}{\gamma} \right) + \log \left(\frac{1}{t} \right) \left(\frac{1}{\gamma} - 1 - 2a + 2a\mu \right) - 2\mu \geq 0. \quad (22)$$

As $t \rightarrow 1^-$, this inequality implies that $-\mu \geq 0$ and hence the only possible case is when $\mu = 0$. So, substituting $\mu = 0$ in equation (22) gives

$$a \left(a + 1 - \frac{1}{\gamma} \right) \log \left(\frac{1}{t} \right) + \left(\frac{1}{\gamma} - 1 - 2a \right) \geq 0,$$

which holds for $t \in (0, 1)$ whenever $-1 < a \leq \min\{0, (1/\gamma) - 1, (1/2)((1/\gamma) - 1)\} = 0$.

Case (ii). Let $b > a > -1$ with $a \in (-1, 0]$ and $-1 < b \leq (1/\gamma) - 1$. In this case, equation (14) with $\mu = 0$ is seen to be equivalent to $\psi_t(a) \geq \psi_t(b)$, where

$$\psi_t(a) = a(a-1)t^a + \left(2 - \frac{1}{\gamma} \right) at^a.$$

We observe that case $\mu \neq 0$ is not possible and can be verified by letting $t \rightarrow 1^-$ in the inequality (14). So, for each fixed $t \in (0, 1)$, we first claim that $\psi_t(a)$ is a decreasing function of a for $a \in (-1, 0)$. Differentiating $\psi_t(a)$ respect to a , we find that

$$\begin{aligned} \psi'_t(a) &= (2a-1)t^a - a(a-1)t^a \log(1/t) + \left(2 - \frac{1}{\gamma} \right) \left(t^a - at^a \log \left(\frac{1}{t} \right) \right) \\ &= t^a \left[2a + 1 - \frac{1}{\gamma} - a^2 \log \left(\frac{1}{t} \right) + a \left(\frac{1}{\gamma} - 1 \right) \log \left(\frac{1}{t} \right) \right], \end{aligned}$$

which shows that $\psi'_t(a) \leq 0$ for $a \in (-1, 0)$. Thus, $\psi_t(a)$ is a decreasing function of a for $a \in (-1, 0)$. In particular, for $b > a$ with $b \in (-1, 0)$ and $a \in (-1, 0)$, inequality (14) holds. When $b > a$ with $0 < b \leq (1/\gamma) - 1$, we have $\psi_t(a) \geq \psi_t(0) = 0$ for each fixed $t \in (0, 1)$. But, for $0 < b \leq (1/\gamma) - 1$, $\psi_t(b) \leq 0$ is equivalent to the inequality

$$bt^b \left(b + 1 - \frac{1}{\gamma} \right) \leq 0,$$

which is clearly true. Hence, in this case, the inequality $\psi_t(a) \geq 0 \geq \psi_t(b)$ holds for each $t \in (0, 1)$. Thus, $\psi_t(a)$ is a decreasing function of a for $b > a > -1$ with $a \in (-1, 0]$ and $b \in (-1, (1/\gamma) - 1]$ and the desired conclusion follows. ■

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On some formulas for Weber $E_\nu(z)$ and Anger $J_\nu(z)$ functions

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Closed expressions are obtained for the Weber $E_{\pm n}(z)$, $E_{1/2 \pm n}(z)$ and Anger $J_{\pm n}(z)$, $J_{1/2 \pm n}(z)$ functions ($n = 0, 1, \dots$), for the n th derivatives of $E_\nu(z)$ and $J_\nu(z)$ with respect to the argument and for the first derivatives with respect to the order ν at the points $\nu = \pm n$.

Keywords: Weber function; Anger function

MSC (2000): C33

1. Introduction

The Weber $E_\nu(z)$ and Anger $J_\nu(z)$ functions are defined by the formulas

$$E_\nu(z) = \frac{1}{\pi} \int_0^\pi \sin(\nu t - z \sin t) dt, \quad (1)$$

$$J_\nu(z) = \frac{1}{\pi} \int_0^\pi \cos(\nu t - z \sin t) dt, \quad (2)$$

respectively. The following relations:

$$2\nu E_\nu(z) = z [E_{\nu-1}(z) + E_{\nu+1}(z)] + \frac{2}{\pi} [1 - \cos(\nu\pi)], \quad (3)$$

$$2\nu J_\nu(z) = z [J_{\nu-1}(z) + J_{\nu+1}(z)] + \frac{2}{\pi} \sin(\nu\pi), \quad (4)$$

$$\sin(\nu\pi) E_\nu(z) = J_{-\nu}(z) - \cos(\nu\pi) J_\nu(z), \quad (5)$$

$$\sin(\nu\pi) J_\nu(z) = \cos(\nu\pi) E_\nu(z) - E_{-\nu}(z), \quad (6)$$

$$2D[E_\nu(z)] = E_{\nu-1}(z) - E_{\nu+1}(z), \quad (7)$$

$$2D[J_\nu(z)] = J_{\nu-1}(z) - J_{\nu+1}(z), \quad (8)$$

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$$\mathbf{E}_n(z) = \frac{2z^{n-1}}{(2n-1)!!\pi} \sum_{k=0}^{[(n-1)/2]} \left(\frac{1}{2}\right)_k \left(\frac{1}{2} - n\right)_k \left(-\frac{4}{z^2}\right)^k - \mathbf{H}_n(z), \quad (9)$$

$$\mathbf{J}_n(z) = J_n(z), \quad (10)$$

where $D = d/dz$, are known [1, 2]. From equation (1), (2) and (5), (6) we immediately get

$$\mathbf{E}_v(z) = -\mathbf{E}_{-v}(-z), \quad \mathbf{J}_v(z) = \mathbf{J}_{-v}(-z). \quad (11)$$

$$\mathbf{E}_{n+1/2}(z) = (-1)^n \mathbf{J}_{-n-1/2}(z) = (-1)^n \mathbf{J}_{n+1/2}(-z), \quad (12)$$

$$\mathbf{E}_{-n+1/2}(z) = (-1)^n \mathbf{J}_{n-1/2}(z) = (-1)^n \mathbf{J}_{-n+1/2}(-z). \quad (13)$$

Here and in all subsequent formulas, n is a nonnegative integer or zero. Subsequently, we use the following notations:

$$\begin{aligned} \text{Si}(z) &= \int_0^z \frac{\sin t}{t} dt, \quad \text{Shi}(z) = \int_0^z \frac{\sinh t}{t} dt, \\ \text{ci}(z) &= \int_0^z \frac{\cos t - 1}{t} dt + \ln z + \mathbf{C}, \quad \text{chi}(z) = \int_0^z \frac{\cosh t - 1}{t} dt + \ln z + \mathbf{C}, \end{aligned}$$

where $\mathbf{C}$ is the Euler constant.

2. Derivatives with respect to the argument

PROPOSITION 1 *The following formulas of differentiation are valid:*

$$\begin{aligned} D^n [z^{v/2} \mathbf{J}_v(a\sqrt{z})] &= \left(\frac{a}{2}\right)^n z^{(v-n)/2} \mathbf{J}_{v-n}(a\sqrt{z}) - \left(-\frac{a}{2}\right)^n \frac{\sin(v\pi)}{\pi a} z^{(v-n-1)/2} \\ &\quad \times \sum_{k=0}^{n-1} \left(\frac{n-k-v+1}{2}\right)_k \left(\frac{2}{a\sqrt{z}}\right)^k. \end{aligned} \quad (14)$$

$$\begin{aligned} D^n [z^{-v/2} \mathbf{J}_v(a\sqrt{z})] &= \left(-\frac{a}{2}\right)^n z^{(-v-n)/2} \mathbf{J}_{v+n}(a\sqrt{z}) - \left(\frac{a}{2}\right)^n \frac{\sin(v\pi)}{\pi a} z^{-(v+n+1)/2} \\ &\quad \times \sum_{k=0}^{n-1} (-1)^k \left(\frac{n-k+v+1}{2}\right)_k \left(\frac{2}{a\sqrt{z}}\right)^k. \end{aligned} \quad (15)$$

$$\begin{aligned} D^n \left[z^{n-v/2-1} \mathbf{J}_v\left(\frac{a}{\sqrt{z}}\right) \right] &= \left(-\frac{a}{2}\right)^n z^{(-v-n)/2-1} \mathbf{J}_{v-n}\left(\frac{a}{\sqrt{z}}\right) - \left(\frac{a}{2}\right)^n \frac{\sin(v\pi)}{\pi a} z^{-(v+n+1)/2} \\ &\quad \times \sum_{k=0}^{n-1} \left(\frac{n-k-v+1}{2}\right)_k \left(\frac{2\sqrt{z}}{a}\right)^k. \end{aligned} \quad (16)$$

$$\begin{aligned} D^n \left[z^{n+v/2-1} \mathbf{J}_v \left(\frac{a}{\sqrt{z}} \right) \right] &= \left(\frac{a}{2} \right)^n z^{-(v+n)/2-1} \mathbf{J}_{v+n} \left(\frac{a}{\sqrt{z}} \right) - \left(-\frac{a}{2} \right)^n \frac{\sin(v\pi)}{\pi a} z^{(v-n-1)/2} \\ &\quad \times \sum_{k=0}^{n-1} (-1)^k \left(\frac{n-k+v+1}{2} \right) \left(\frac{2\sqrt{z}}{a} \right)^k. \end{aligned} \quad (17)$$

$$\begin{aligned} D^n [z^{v/2} \mathbf{E}_v(a\sqrt{z})] &= \left(\frac{a}{2} \right)^n z^{(v-n)/2} \mathbf{E}_{v-n}(a\sqrt{z}) + \frac{1}{\pi a} \left(\frac{a}{2} \right)^n z^{(v-n-1)/2} \\ &\quad \times \sum_{k=0}^{n-1} [(-1)^k + (-1)^n \cos(v\pi)] \left(\frac{n-k-v+1}{2} \right)_k \left(\frac{2}{a\sqrt{z}} \right)^k. \end{aligned} \quad (18)$$

$$\begin{aligned} D^n [z^{-v/2} \mathbf{E}_v(a\sqrt{z})] &= \left(-\frac{a}{2} \right)^n z^{-(v+n)/2} \mathbf{E}_{v+n}(a\sqrt{z}) + \frac{1}{\pi a} \left(\frac{a}{2} \right)^n z^{-(v+n+1)/2} \\ &\quad \times \sum_{k=0}^{n-1} [(-1)^n + (-1)^k \cos(v\pi)] \left(\frac{n-k+v+1}{2} \right)_k \left(\frac{2}{a\sqrt{z}} \right)^k. \end{aligned} \quad (19)$$

$$\begin{aligned} D^n \left[z^{n-v/2-1} \mathbf{E}_v \left(\frac{a}{\sqrt{z}} \right) \right] &= \left(-\frac{a}{2} \right)^n z^{-(v+n)/2-1} \mathbf{E}_{v-n} \left(\frac{a}{\sqrt{z}} \right) + \frac{1}{\pi a} \left(-\frac{a}{2} \right)^n z^{-(v+n+1)/2} \\ &\quad \times \sum_{k=0}^{n-1} [(-1)^k + (-1)^n \cos(v\pi)] \left(\frac{n-k-v+1}{2} \right)_k \left(\frac{2\sqrt{z}}{a} \right)^k. \end{aligned} \quad (20)$$

$$\begin{aligned} D^n \left[z^{n+v/2-1} \mathbf{E}_v \left(\frac{a}{\sqrt{z}} \right) \right] &= \left(\frac{a}{2} \right)^n z^{(v-n)/2-1} \mathbf{E}_{v+n} \left(\frac{a}{\sqrt{z}} \right) + \frac{1}{\pi a} \left(-\frac{a}{2} \right)^n z^{(v-n-1)/2} \\ &\quad \times \sum_{k=0}^{n-1} [(-1)^n + (-1)^k \cos(v\pi)] \left(\frac{n-k+v+1}{2} \right)_k \left(\frac{2\sqrt{z}}{a} \right)^k. \end{aligned} \quad (21)$$

The relations (14), (15) and (18), (19) can be easily proved by induction. Other formulas follow from the equation

$$D^n \left[z^{n-1} f \left(\frac{1}{z} \right) \right] = (-1)^n z^{-n-1} F \left(\frac{1}{z} \right) \quad \text{where } D^n [f(z)] = F(z).$$

Besides, the following formulas can be easily deduced from equations (7) and (8):

$$D^n [\mathbf{E}_v(z)] = \left(\pm \frac{1}{2} \right)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \mathbf{E}_{v \pm 2k \mp n}(z), \quad (22)$$

$$D^n [\mathbf{J}_v(z)] = \left(\pm \frac{1}{2} \right)^n \sum_{k=0}^n (-1)^k \binom{n}{k} \mathbf{J}_{v \pm 2k \mp n}(z). \quad (23)$$

More differential formulas can be obtained by using the equality

$$D^n [f(z^2)] = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(2z)^{n-2k}}{k!(n-2k)!} f^{(n-k)}(z^2), \quad (24)$$

where

$$f^{(n-k)}(z^2) = D_u^{n-k}[f(u)]|_{u=z^2}.$$

We have

$$D^n[\mathbf{E}_v(\sqrt{z})] = D^n[z^{-v/2} z^{v/2} \mathbf{E}_v(\sqrt{z})] = \sum_{k=0}^n \binom{n}{k} D^{n-k}[z^{-v/2}] D^k[z^{v/2} \mathbf{E}_v(\sqrt{z})].$$

Using the relation

$$D^{n-k}[z^{-v/2}] = (-1)^{n-k} \left(\frac{v}{2}\right)_{n-k} z^{-v/2-n+k}$$

and the formulas (18) and (24), we obtain after some calculations

$$\begin{aligned} D^n[\mathbf{E}_v(z)] &= n! \left(-\frac{z}{2}\right)^{-n} \sum_{k=0}^{[n/2]} \frac{(-1/4)^k}{k!(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(\frac{v}{2}\right)_{n-k-p} \left(-\frac{z}{2}\right)^p \\ &\quad \times \left\{ \mathbf{E}_{v-p}(z) + \frac{1}{\pi z} \sum_{r=0}^{p-1} [(-1)^r + (-1)^p \cos(v\pi)] \left(\frac{p-r-v+1}{2}\right)_r \left(\frac{2}{z}\right)^r \right\}. \end{aligned} \quad (25)$$

This formula contains only $\mathbf{E}_{v-p}(z)$ with $p \geq 0$. Similarly, using formula (19), we obtain

$$\begin{aligned} D^n[\mathbf{E}_v(z)] &= n! \left(-\frac{z}{2}\right)^{-n} \sum_{k=0}^{[n/2]} \frac{(-1/4)^k}{k!(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(-\frac{v}{2}\right)_{n-k-p} \left(\frac{z}{2}\right)^p \\ &\quad \times \left\{ \mathbf{E}_{v+p}(z) + \frac{1}{\pi z} \sum_{r=0}^{p-1} [1 + (-1)^{p+r} \cos(v\pi)] \left(\frac{p-r+v+1}{2}\right)_r \left(\frac{2}{z}\right)^r \right\}. \end{aligned} \quad (26)$$

For $\mathbf{J}_v(z)$, we have

$$\begin{aligned} D^n[\mathbf{J}_v(z)] &= n! \left(-\frac{z}{2}\right)^{-n} \sum_{k=0}^{[n/2]} \frac{(-1/4)^k}{k!(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(\frac{v}{2}\right)_{n-k-p} \left(-\frac{z}{2}\right)^p \\ &\quad \times \left\{ \mathbf{J}_{v-p}(z) - (-1)^p \frac{\sin(v\pi)}{\pi z} \sum_{r=0}^{p-1} \left(\frac{p-r-v+1}{2}\right)_r \left(\frac{2}{z}\right)^r \right\}, \end{aligned} \quad (27)$$

$$\begin{aligned} D^n[\mathbf{J}_v(z)] &= n! \left(-\frac{z}{2}\right)^{-n} \sum_{k=0}^{[n/2]} \frac{(-1/4)^k}{k!(n-2k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} \left(-\frac{v}{2}\right)_{n-k-p} \left(\frac{z}{2}\right)^p \\ &\quad \times \left\{ \mathbf{J}_{v+p}(z) - (-1)^p \frac{\sin(v\pi)}{\pi z} \sum_{r=0}^{p-1} \left(\frac{p-r+v+1}{2}\right)_r \left(-\frac{2}{z}\right)^r \right\}. \end{aligned} \quad (28)$$

3. Values of $\mathbf{E}_\nu(z)$ and $\mathbf{J}_\nu(z)$ for $\nu = \pm n + 1/2$

Applying the differentiation formulas (14), (15) and (18), (19) to the functions $\mathbf{E}_{\pm 1/2}(z)$ and $\mathbf{J}_{\pm 1/2}(z)$ [1],

$$\mathbf{E}_{1/2}(z) = \mathbf{J}_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \{ \cos z [C(z) + S(z)] - \sin z [C(z) - S(z)] \},$$

$$\mathbf{E}_{-1/2}(z) = -\mathbf{J}_{1/2}(z) = -\sqrt{\frac{2}{\pi z}} \{ \sin z [C(z) + S(z)] + \cos z [C(z) - S(z)] \},$$

we obtain

$$\begin{aligned} \mathbf{E}_{-n-1/2}(z) &= \frac{1}{2\pi} \sum_{k=0}^{n-1} \left(\frac{2n-2k+1}{4} \right)_k \left(\frac{2}{z} \right)^{k+1} + \frac{(-1)^n}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{2}{z} \right)^{k-1} \\ &\quad \times \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\frac{3}{4} \right)_{k-m-1} \left(\frac{z}{2} \right)^m \{ J_{m-1/2}(z) [(-1)^k J_{k-n-1/2}(z) \\ &\quad + (-1)^n J_{n-k-1/2}(z)] + (-1)^m J_{1/2-m}(z) [(-1)^k J_{k-n+1/2}(z) \\ &\quad - (-1)^n J_{n-k-1/2}(z)] \} - [J_{n-1/2}(z) - (-1)^n J_{1/2-n}(z)] S(z) \\ &\quad + [(-1)^n J_{n-1/2}(z) + J_{1/2-n}(z)] C(z). \end{aligned} \quad (29)$$

$$\begin{aligned} \mathbf{E}_{-n-1/2}(z) &= -\frac{1}{2\pi} \sum_{k=0}^{n-1} (-1)^k \left(\frac{2n-2k+3}{4} \right)_k \left(\frac{2}{z} \right)^{k+1} + \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{2}{z} \right)^{k-1} \\ &\quad \times \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\frac{3}{4} \right)_{k-m-1} \left(\frac{z}{2} \right)^m \{ J_{m-1/2}(z) [(-1)^{k+n} J_{k-n-1/2}(z) \\ &\quad + J_{n-k+1/2}(z)] - (-1)^m J_{1/2-m}(z) [(-1)^{k+n} J_{k-n-1/2}(z) - J_{n-k+1/2}(z)] \} \\ &\quad - [(-1)^n J_{n+1/2}(z) - J_{-n-1/2}(z)] S(z) + [(-1)^n J_{n+1/2}(z) + J_{-n-1/2}(z)] C(z). \end{aligned} \quad (30)$$

$$\begin{aligned} \mathbf{J}_{n+1/2}(z) &= \frac{(-1)^n}{2\pi} \sum_{k=0}^{n-1} (-1)^k \left(\frac{2n-2k+3}{4} \right)_k \left(\frac{2}{z} \right)^{k+1} - \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{2}{z} \right)^{k-1} \\ &\quad \times \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\frac{3}{4} \right)_{k-m-1} \left(\frac{z}{2} \right)^m \{ J_{m-1/2}(z) [(-1)^{k+n} J_{k-n-1/2}(z) \\ &\quad + J_{n-k+1/2}(z)] - (-1)^m J_{1/2-m}(z) [(-1)^{k+n} J_{k-n-1/2}(z) - J_{n-k+1/2}(z)] \} \\ &\quad + [J_{n+1/2}(z) - (-1)^n J_{-n-1/2}(z)] S(z) + [J_{n+1/2}(z) + (-1)^n J_{-n-1/2}(z)] C(z). \end{aligned} \quad (31)$$

$$\begin{aligned} \mathbf{J}_{1/2-n}(z) &= \frac{(-1)^n}{2\pi} \sum_{k=0}^{n-1} \left(\frac{2n-2k+1}{4} \right)_k \left(\frac{2}{z} \right)^{k+1} - \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} \left(\frac{2}{z} \right)^{k-1} \\ &\quad \times \sum_{m=0}^{k-1} \binom{k-1}{m} \left(\frac{3}{4} \right)_{k-m-1} \left(\frac{z}{2} \right)^m \{ J_{m-1/2}(z) [(-1)^k J_{1/2-n+k}(z) \end{aligned}$$

$$\begin{aligned}
& + (-1)^n J_{n-k-1/2}(z)] + (-1)^m J_{1/2-m}(z)[(-1)^k J_{1/2-n+k}(z) \\
& - (-1)^n J_{n-k-1/2}(z)] + [J_{1/2-n}(z) - (-1)^n J_{n-1/2}(z)]S(z) \\
& + [J_{1/2-n}(z) + (-1)^n J_{n-1/2}(z)]C(z),
\end{aligned} \tag{32}$$

where

$$S(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\sin t}{\sqrt{t}} dt, \quad C(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \frac{\cos t}{\sqrt{t}} dt$$

are the Fresnel integrals and $J_\nu(z)$ is the Bessel function.

4. Formulas for $\mathbf{E}_{v\pm n}(z)$ and $\mathbf{J}_{v\pm n}(z)$

Starting from formulas (3) and (4), we can construct the relations expressing $\mathbf{E}_{v\pm n}(z)$ and $\mathbf{J}_{v\pm n}(z)$ in terms of $\mathbf{E}_v(z)$, $\mathbf{E}_{v+1}(z)$ and $\mathbf{J}_v(z)$, $\mathbf{J}_{v+1}(z)$, respectively. These relations can be proved by induction.

$$\begin{aligned}
\mathbf{E}_{v+n}(z) &= (\nu+1)_{n-1} \left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, \nu+1, 1-n-\nu; -z^2\right) \mathbf{E}_v(z) \\
&- (\nu+2)_{n-2} \left(\frac{2}{z}\right)^{n-2} {}_2F_3\left(1-\frac{n}{2}, \frac{3-n}{2}; 2-n, \nu+2, 1-n-\nu; -z^2\right) \mathbf{E}_{v+1}(z) \\
&+ \frac{1}{\pi} \sum_{k=0}^{n-2} [1 + (-1)^{k+n} \cos(\nu\pi)] (1-\nu-n)_k \\
&\times \left(-\frac{2}{z}\right)^{k+1} {}_2F_3\left(-\frac{k}{2}, \frac{1-k}{2}; -k, \nu-k+n, 1-n-\nu; -z^2\right), \quad n \geq 2,
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathbf{E}_{v-n}(z) &= (-\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; -n, -\nu, \nu-n+1; -z^2\right) \mathbf{E}_v(z) \\
&- (1-\nu)_{n-1} \left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, 1-\nu, \nu-n+1; -z^2\right) \\
&\times \mathbf{E}_{v+1}(z) - \frac{1}{\pi} \sum_{k=0}^{n-1} [1 + (-1)^{k+n} \cos(\nu\pi)] (\nu-n+1)_k \left(\frac{2}{z}\right)^{k+1} \\
&\times {}_2F_3\left(-\frac{k}{2}, \frac{1-k}{2}; -k, n-k-\nu, \nu-n+1; -z^2\right), \quad n \geq 1,
\end{aligned} \tag{34}$$

$$\begin{aligned}
\mathbf{J}_{v+n}(z) &= (\nu+1)_{n-1} \left(\frac{2}{z}\right)^{n-1} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, \nu+1, 1-n-\nu; -z^2\right) \mathbf{J}_v(z) \\
&- (\nu+2)_{n-2} \left(\frac{2}{z}\right)^{n-2} {}_2F_3\left(1-\frac{n}{2}, \frac{3-n}{2}; 2-n, \nu+2, 1-n-\nu; -z^2\right) \\
&\times \mathbf{J}_{v+1}(z) + (-1)^n \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{n-2} (1-\nu-n)_k \left(\frac{2}{z}\right)^{k+1} \\
&\times {}_2F_3\left(-\frac{k}{2}, \frac{1-k}{2}; -k, \nu-k+n, 1-n-\nu; -z^2\right), \quad n \geq 2,
\end{aligned} \tag{35}$$

$$\begin{aligned}
\mathbf{J}_{\nu-n}(z) &= (-\nu)_n \left(-\frac{2}{z}\right)^n {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; -n, -\nu, \nu-n+1; -z^2\right) \mathbf{J}_\nu(z) \\
&\quad - (1-\nu)_{n-1} \left(-\frac{2}{z}\right)^{n-1} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, 1-\nu, \nu-n+1; -z^2\right) \\
&\quad \times \mathbf{J}_{\nu+1}(z) - (-1)^n \frac{\sin(\nu\pi)}{\pi} \sum_{k=0}^{n-1} (\nu-n+1)_k \left(-\frac{2}{z}\right)^{k+1} {}_2F_3 \\
&\quad \times \left(-\frac{k}{2}, \frac{1-k}{2}; -k, n-k-\nu, \nu-n+1; -z^2\right), \quad n \geq 1.
\end{aligned} \tag{36}$$

From these formulas, the alternative expressions for $\mathbf{E}_{\pm n \pm 1/2}(z)$ and $\mathbf{J}_{\pm n \pm 1/2}(z)$ can be obtained:

$$\begin{aligned}
\mathbf{J}_{n+1/2}(z) &= \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)_n \left(\frac{2}{z}\right)^{n+1/2} {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; -n, \frac{1}{2}-n, \frac{1}{2}; -z^2\right) \\
&\quad \times \left[\sin\left(z - \frac{\pi}{4}\right) S(z) + \sin\left(z + \frac{\pi}{4}\right) C(z) \right] \\
&\quad - \sqrt{\frac{2}{\pi}} \left(\frac{3}{2}\right)_{n-1} \left(\frac{2}{z}\right)^{n-1/2} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, \frac{1}{2}-n, \frac{3}{2}; -z^2\right) \\
&\quad \times \left[\sin\left(z + \frac{\pi}{4}\right) S(z) - \sin\left(z - \frac{\pi}{4}\right) C(z) \right] \\
&\quad + (-1)^n \frac{2}{\pi z} \sum_{k=0}^{n-1} \left(\frac{1}{2}-n\right)_k \left(\frac{2}{z}\right)^k {}_2F_3\left(-\frac{k}{2}, \frac{1-k}{2}; -k, \frac{1}{2} \\
&\quad -n, \frac{1}{2}-k+n; -z^2\right).
\end{aligned} \tag{37}$$

$$\begin{aligned}
\mathbf{J}_{-n-1/2}(z) &= (-1)^n \sqrt{\frac{2}{\pi}} \left(\frac{1}{2}\right)_n \left(\frac{2}{z}\right)^{n+1/2} {}_2F_3\left(-\frac{n}{2}, \frac{1-n}{2}; -n, \frac{1}{2}-n, \frac{1}{2}; -z^2\right) \\
&\quad \times \left[\sin\left(z + \frac{\pi}{4}\right) S(z) - \sin\left(z - \frac{\pi}{4}\right) C(z) \right] \\
&\quad + (-1)^n \sqrt{\frac{2}{\pi}} \left(\frac{3}{2}\right)_{n-1} \left(\frac{2}{z}\right)^{n-1/2} {}_2F_3\left(\frac{1-n}{2}, 1-\frac{n}{2}; 1-n, \frac{1}{2}-n, \frac{3}{2}; -z^2\right) \\
&\quad \times \left[\sin\left(z - \frac{\pi}{4}\right) S(z) + \sin\left(z + \frac{\pi}{4}\right) C(z) \right] + (-1)^n \frac{2}{\pi z} \sum_{k=0}^{n-1} \left(\frac{1}{2}-n\right)_k \\
&\quad \times \left(-\frac{2}{z}\right)^k {}_2F_3\left(-\frac{k}{2}, \frac{1-k}{2}; -k, \frac{1}{2}-n, \frac{1}{2}-k+n; -z^2\right).
\end{aligned} \tag{38}$$

Similar formulas for $\mathbf{E}_{\pm n \pm 1/2}(z)$ follow from relations (12) and (13).

5. Derivatives with respect to the order

From (14), we have

$$\begin{aligned} \mathbf{J}_{v-n}(\sqrt{z}) &= 2^n z^{(-v+n)/2} \mathbf{D}^n [z^{v/2} \mathbf{J}_v(\sqrt{z})] \\ &+ (-1)^n \frac{\sin(v\pi)}{\pi} z^{-1/2} \sum_{k=0}^{n-1} \left(\frac{n-k-v+1}{2} \right)_k \left(\frac{2}{\sqrt{z}} \right)^k. \end{aligned}$$

Differentiating with respect to v and posing $v = 0$, we obtain

$$\begin{aligned} \left. \frac{\partial \mathbf{J}_v(z)}{\partial v} \right|_{v=-n} &= -2^{n-1} z^{n/2} \ln z \mathbf{D}^n [J_0(\sqrt{z})] + 2^n z^{n/2} \mathbf{D}^n \left[\ln z J_0(\sqrt{z}) + \frac{\pi}{2} \mathbf{H}_0(\sqrt{z}) \right] \\ &+ (-1)^n z^{-1/2} \sum_{k=0}^{n-1} \left(\frac{n-k+1}{2} \right)_k \left(\frac{2}{\sqrt{z}} \right)^k, \end{aligned}$$

where the relations

$$\mathbf{J}_0(z) = J_0(z)$$

and [1]

$$\left. \frac{\partial \mathbf{J}_v(z)}{\partial v} \right|_{v=0} = \frac{\pi}{2} \mathbf{H}_0(z)$$

were used. Taking into account the differentiation formulas for the Bessel function $J_v(z)$,

$$\mathbf{D}^n [z^{-v/2} J_v(\sqrt{z})] = \left(-\frac{1}{2} \right)^n z^{(-v-n)/2} J_{v+n}(\sqrt{z}),$$

and for the Struve function $\mathbf{H}_v(z)$,

$$\mathbf{D}^n [z^{v/2} \mathbf{H}_v(\sqrt{z})] = \frac{1}{2^n} z^{(v-n)/2} \mathbf{H}_{v-n}(\sqrt{z}),$$

we obtain the equality

$$\begin{aligned} \left. \frac{\partial \mathbf{J}_v(z)}{\partial v} \right|_{v=-n} &= (-1)^{n-1} \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(2/z)^{n-k}}{k!(n-k)} J_k(z) + \frac{\pi}{2} \mathbf{H}_{-n}(z) \\ &+ \frac{(-1)^n}{z} \sum_{k=0}^{n-1} \left(\frac{n-k+1}{2} \right)_k \left(\frac{2}{z} \right)^k. \end{aligned} \quad (39)$$

Similarly, we can derive the formulas for $v = n$ and, using the relation [1]

$$\left. \frac{\partial \mathbf{E}_v(z)}{\partial v} \right|_{v=0} = \frac{\pi}{2} J_0(z),$$

for $\mathbf{E}_v(z)$,

$$\begin{aligned} \left. \frac{\partial \mathbf{J}_v(z)}{\partial v} \right|_{v=n} &= \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(2/z)^{n-k}}{k!(n-k)} J_k(z) + \frac{\pi}{2} \mathbf{H}_n(z) \\ &\quad - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(1/2)_k}{(1/2)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1} + \frac{(-1)^n}{z} \sum_{k=0}^{n-1} (-1)^k \left(\frac{n-k+1}{2}\right)_k \left(\frac{2}{z}\right)^k. \end{aligned} \quad (40)$$

$$\begin{aligned} \left. \frac{\partial \mathbf{E}_v(z)}{\partial v} \right|_{v=n} &= \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(2/z)^{n-k}}{k!(n-k)} \left[-\mathbf{H}_k(z) + \frac{1}{\pi} \sum_{r=0}^{k-1} \frac{(1/2)_r}{(1/2)_{k-r}} \left(\frac{z}{2}\right)^{k-2r-1} \right] + \frac{\pi}{2} J_n(z) \\ &\quad + \frac{1}{2\pi} \sum_{k=0}^{n-1} [(-1)^k + (-1)^n] \left(\frac{n-k+1}{2}\right)_k \left(-\frac{2}{z}\right)^{k+1} \sum_{r=0}^{k-1} \frac{1}{2r+n-k+1}. \end{aligned} \quad (41)$$

$$\begin{aligned} \left. \frac{\partial \mathbf{E}_v(z)}{\partial v} \right|_{v=-n} &= \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(-2/z)^{n-k}}{k!(n-k)} \mathbf{H}_{-k}(z) + (-1)^n \frac{\pi}{2} J_n(z) \\ &\quad + \frac{1}{2\pi} \sum_{k=0}^{n-1} [(-1)^k + (-1)^n] \left(\frac{n-k+1}{2}\right)_k \left(\frac{2}{z}\right)^{k+1} \sum_{r=0}^{k-1} \frac{1}{2r+n-k+1}. \end{aligned} \quad (42)$$

6. Some formulas for $\mathbf{H}_v(z)$, $\mathbf{L}_v(z)$, $\mathbf{J}i_v(z)$, $\mathbf{Y}i_v(z)$, $\mathbf{K}i_v(z)$

In ref. [3], we have derived a number of relations for derivatives of the Struve functions $\mathbf{H}_v(z)$, $\mathbf{L}_v(z)$ and of the integral Bessel functions. Some simpler and corrected formulas are given.

$$\begin{aligned} \left. \frac{\partial \mathbf{H}_v(z)}{\partial v} \right|_{v=n} &= -\frac{\pi}{2} J_n(z) + \frac{2^{n-1} z^{-n}}{\pi} G_{24}^{32} \left(\frac{z^2}{4} \middle| \frac{1/2, 1/2}{1/2, 1/2, n, 0} \right) \\ &\quad + \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(1/2)_k}{(1/2)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1} \left[\log \frac{z}{2} - \psi \left(n - k + \frac{1}{2} \right) \right] \\ &\quad + \frac{n!}{2} \sum_{k=0}^{n-1} (-1)^k \frac{(z/2)^{k-n}}{k!(n-k)} \mathbf{H}_{-k}(z), \end{aligned} \quad (43)$$

$$-\pi/2 < \arg z \leq \pi/2,$$

$$\begin{aligned} \left. \frac{\partial \mathbf{H}_v(z)}{\partial v} \right|_{v=-n} &= (-1)^{n+1} \frac{\pi}{2} J_n(z) + (-1)^n \frac{2^{n-1} z^{-n}}{\pi} G_{24}^{32} \left(\frac{z^2}{4} \middle| \frac{1/2, 1/2}{1/2, 1/2, n, 0} \right) \\ &\quad - \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(-z/2)^{k-n}}{k!(n-k)} \mathbf{H}_{-k}(z), \end{aligned} \quad (44)$$

$$-\pi/2 < \arg z \leq \pi/2,$$

$$\begin{aligned} \left. \frac{\partial \mathbf{H}_v(z)}{\partial v} \right|_{v=n+1/2} &= [\mathbf{Si}(2z) - 2\mathbf{Si}(z)]J_{n+1/2}(z) + (-1)^n [\mathbf{ci}(2z) - 2\mathbf{ci}(z)]J_{-n-1/2}(z) \\ &\quad + \ln \frac{z}{2} [\mathbf{H}_{n+1/2}(z) - Y_{n+1/2}(z)] + \frac{1}{2\sqrt{\pi}} \left(\frac{2}{z}\right)^{n+1/2} \left(\frac{1}{2}\right)_n \\ &\quad \times \left[3\mathbf{C} + 2 \ln 2 + \psi \left(\frac{1}{2} - n\right) \right] - \frac{n!}{2} \left(\frac{2}{z}\right)^n \\ &\quad \times \sum_{k=0}^{n-1} \frac{(-z/2)^k}{k!(n-k)} J_{-k-1/2}(z) - \frac{n!}{2\sqrt{\pi}} \left(\frac{2}{z}\right)^{n+1/2} \sum_{k=0}^{n-1} \frac{(1/2)_k}{k!(n-k)} \\ &\quad - \frac{(z/2)^{n-1/2}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(1/2)_k}{(n-k)!} \left(\frac{2}{z}\right)^{2k} \psi(n-k+1) \\ &\quad - n! \sqrt{\pi} \left(\frac{2}{z}\right)^{1/2-n} \sum_{k=0}^{n-1} \frac{(z/2)^k}{k!(n-k)} \sum_{p=0}^{n-k-1} \frac{(z/2)^p}{p!} \\ &\quad \times \{ (-1)^{p+1} J_{k+1/2}(z) [J_{1/2-p}(z) - 2^{p-1/2} J_{1/2-p}(2z)] \\ &\quad - (-1)^k J_{-k-1/2}(z) [J_{p-1/2}(z) - 2^{p-1/2} J_{p-1/2}(2z)] \}, \end{aligned} \quad (45)$$

$$\begin{aligned} \left. \frac{\partial \mathbf{H}_v(z)}{\partial v} \right|_{v=-n-1/2} &= (-1)^n [2\mathbf{ci}(z) - \mathbf{ci}(2z)]J_{n+1/2}(z) - [2\mathbf{Si}(z) - \mathbf{Si}(2z)]J_{-n-1/2}(z) \\ &\quad - \frac{n!}{2} \left(-\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{(z/2)^k}{k!(n-k)} J_{k+1/2}(z) \\ &\quad + \frac{n!}{2} \sqrt{\pi} z \sum_{k=1}^n \frac{(2/z)^k}{k!(n-k)} \sum_{p=0}^{k-1} \frac{z^p}{p!} \\ &\quad \times \{ (-1)^n J_{n-k+1/2}(z) [J_{p-1/2}(2z) - 2^{1/2-p} J_{p-1/2}(z)] \\ &\quad - (-1)^{k+p} J_{k-n-1/2}(z) [J_{1/2-p}(2z) - 2^{1/2-p} J_{1/2-p}(z)] \}, \end{aligned} \quad (46)$$

$$\begin{aligned} \left. \frac{\partial \mathbf{L}_v(z)}{\partial v} \right|_{v=n} &= (-1)^n K_n(z) + \frac{(-2)^{n-1} z^{-n}}{\pi^2} G_{24}^{42} \left(\frac{z^2}{4} \middle| \begin{matrix} 1/2, 1/2 \\ 0, 1/2, 1/2, n \end{matrix} \right) \\ &\quad - \frac{1}{\pi} \sum_{k=0}^{n-1} (-1)^k \frac{(1/2)_k}{(1/2)_{n-k}} \left(\frac{z}{2}\right)^{n-2k-1} \left[\log \frac{z}{2} \right. \\ &\quad \left. - \psi \left(n - k + \frac{1}{2} \right) \right] + \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(-z/2)^{k-n}}{k!(n-k)} \mathbf{L}_{-k}(z), \end{aligned} \quad (47)$$

$$-\pi/2 < \arg z \leq \pi/2,$$

$$\begin{aligned} \left. \frac{\partial \mathbf{L}_v(z)}{\partial v} \right|_{v=-n} &= (-1)^n K_n(z) + \frac{(-2)^{n-1} z^{-n}}{\pi^2} G_{24}^{42} \left(\frac{z^2}{4} \middle| \begin{matrix} 1/2, 1/2 \\ 0, 1/2, 1/2, n \end{matrix} \right) \\ &\quad - \frac{n!}{2} \sum_{k=0}^{n-1} \frac{(-z/2)^{k-n}}{k!(n-k)} \mathbf{L}_{-k}(z), \end{aligned} \quad (48)$$

$$-\pi/2 < \arg z \leq \pi/2,$$

$$\begin{aligned} \left. \frac{\partial \mathbf{L}_v(z)}{\partial v} \right|_{v=n+1/2} &= [\text{Shi}(2z) - 2 \text{Shi}(z)] I_{n+1/2}(z) - [\text{chi}(2z) - 2 \text{chi}(z)] I_{-n-1/2}(z) \\ &\quad + \ln \frac{z}{2} [I_{-n-1/2}(z) - \mathbf{L}_{n+1/2}(z)] + \frac{(-1)^{n+1}}{2\sqrt{\pi}} \left(\frac{2}{z}\right)^{n+1/2} \left(\frac{1}{2}\right)_n \\ &\quad \times \left[3\mathbf{C} + 2 \ln 2 + \psi\left(\frac{1}{2} - n\right) \right] + \frac{n!}{2} \left(-\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{(-z/2)^k}{k!(n-k)} \\ &\quad \times I_{-k-1/2}(z) + (-1)^n \frac{n!}{2\sqrt{\pi}} \left(\frac{2}{z}\right)^{n+1/2} \sum_{k=0}^{n-1} \frac{(1/2)_k}{k!(n-k)} \\ &\quad + \frac{(z/2)^{n-1/2}}{\sqrt{\pi}} \sum_{k=0}^{n-1} (-1)^k \frac{(1/2)_k}{(n-k)!} \left(\frac{2}{z}\right)^{2k} \psi(n-k+1) \\ &\quad + (-1)^n n! \sqrt{\pi} \left(\frac{z}{2}\right)^{1/2-n} \sum_{k=0}^{n-1} \frac{(-z/2)^k}{k!(n-k)} \sum_{p=0}^{n-k-1} \frac{(-z/2)^p}{p!} \\ &\quad \times \{ I_{k+1/2}(z) [I_{1/2-p}(z) - 2^{p-1/2} I_{1/2-p}(2z)] \\ &\quad - I_{-k-1/2}(z) [I_{p-1/2}(z) - 2^{p-1/2} I_{p-1/2}(2z)] \}, \end{aligned} \quad (49)$$

$$\begin{aligned} \left. \frac{\partial \mathbf{L}_v(z)}{\partial v} \right|_{v=-n-1/2} &= [2 \text{chi}(z) - \text{chi}(2z)] I_{n+1/2}(z) - [2 \text{Shi}(z) - \text{Shi}(2z)] I_{-n-1/2}(z) \\ &\quad - \frac{n!}{2} \left(-\frac{2}{z}\right)^n \sum_{k=0}^{n-1} \frac{(-z/2)^k}{k!(n-k)} I_{k+1/2}(z) + \frac{n!}{2} \sqrt{\pi z} \sum_{k=1}^n \frac{(-2/z)^k}{k!(n-k)} \\ &\quad \times \sum_{p=0}^{k-1} \frac{(-z)^p}{p!} \{ I_{n-k+1/2}(z) [2^{1/2-p} I_{p-1/2}(z) - I_{p-1/2}(2z)] \\ &\quad - I_{-k-n-1/2}(z) [2^{1/2-p} I_{1/2-p}(z) - I_{1/2-p}(2z)] \}, \end{aligned} \quad (50)$$

$$\left. \frac{\partial J i_v(z)}{\partial v} \right|_{v=0} = \frac{\pi}{2} Y i_0(z), \quad \left. \frac{\partial J i_v(z)}{\partial v} \right|_{v=\pm 1} = \pm \frac{\pi}{2} Y i_1(z) - J i_1(z) + \frac{1}{z} J_0(z), \quad (51)$$

$$\left. \frac{\partial Y i_v(z)}{\partial v} \right|_{v=0} = -\frac{\pi}{2} J i_0(z), \quad \left. \frac{\partial Y i_v(z)}{\partial v} \right|_{v=\pm 1} = \mp \frac{\pi}{2} J i_1(z) - Y i_1(z) + \frac{1}{z} Y_0(z), \quad (52)$$

$$\left. \frac{\partial K i_v(z)}{\partial v} \right|_{v=0} = 0, \quad \left. \frac{\partial K i_v(z)}{\partial v} \right|_{v=\pm 1} = \mp K i_1(z) \pm \frac{1}{z} K_0(z). \quad (53)$$

$$\begin{aligned} \left. \frac{\partial J i_v(z)}{\partial v} \right|_{v=\pm n} &= (\pm 1)^n \frac{\pi}{2} Y i_n(z) \pm (\pm 1)^n \frac{n!}{2} \sum_{k=0}^{n-1} \frac{1}{k!(n-k)} \\ &\quad \times \left[\frac{1}{(n-k)!} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} J i_{2p+2k-n}(z) \right. \\ &\quad \left. - \sum_{m=0}^{n-k-1} (-1)^m \frac{2^{n-k-m} z^{k+m-n}}{(k-n)_{m+1}} \sum_{p=0}^m (-1)^p \binom{m}{p} J_{2p+k-m}(z) \right]. \end{aligned} \quad (54)$$

$$\begin{aligned}
\left. \frac{\partial Y i_\nu(z)}{\partial \nu} \right|_{\nu=\pm n} &= -(\pm 1)^n \frac{\pi}{2} J i_n(z) + (\pm 1)^{n+1} \frac{n!}{2} \sum_{k=0}^{n-1} \frac{1}{k!(n-k)} \\
&\times \left[\frac{1}{(n-k)!} \sum_{p=0}^{n-k} (-1)^p \binom{n-k}{p} Y i_{2p+2k-n}(z) \right. \\
&\left. - \sum_{m=0}^{n-k-1} (-1)^m \frac{2^{n-k-m} z^{k+m-n}}{(k-n)_{m+1}} \sum_{p=0}^m (-1)^p \binom{m}{p} Y i_{2p+k-m}(z) \right]. \quad (55)
\end{aligned}$$

$$\begin{aligned}
\left. \frac{\partial K i_\nu(z)}{\partial \nu} \right|_{\nu=\pm n} &= \pm \frac{n!}{2} \sum_{k=0}^{n-1} \frac{1}{k!(n-k)} \left[\frac{(-1)^{n-k}}{(n-k)!} \sum_{p=0}^{n-k} \binom{n-k}{p} K i_{2p+2k-n}(z) \right. \\
&\left. - \sum_{m=0}^{n-k-1} \frac{2^{n-k-m} z^{k+m-n}}{(k-n)_{m+1}} \sum_{p=0}^m \binom{m}{p} K i_{2p+k-m}(z) \right]. \quad (56)
\end{aligned}$$

Here, $I_\nu(z)$ and $K_\nu(z)$ are the Bessel functions, and G_{pq}^{mn} is the Meijer G function.

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On a certain class of spherical harmonic functions and singular pseudo-differential operators

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This article gives some results for the weighted spherical harmonic functions. Some relations are found between the singular integral operators generated by generalized shift operator and singular pseudo-differential operators. Finally, the boundedness of the pseudo-differential operators studied are shown.

Keywords: Spherical harmonic functions; Pseudo-differential operators; Generalized shift operator; Fourier–Bessel transformation

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1. Introduction

It is well known that the special functions play a very important role in the solution of equations of mathematical physics. In this study, we consider a class of special functions weighted spherical harmonic functions (or B_k -spherical functions) on the hemisphere $S_n^{k,+} = \{x \in \mathbb{R}_n : |x| = 1, x_{n-k+1} > 0, \dots, x_n > 0\}$ which satisfy the Laplace–Bessel equation

$$\Delta_{B_k} P(x) = 0, \quad (1)$$

where $P(x)$ is a homogeneous polynomial in $x \in \mathbb{R}_n^{k,+} = \{x \in \mathbb{R}_n : x_{n-k+1} > 0, \dots, x_n > 0\}$, which is even with respect to $x_{n-k+1}, \dots, x_n$, ($1 \leq k \leq n$), and

$$\Delta_{B_k} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \sum_{j=1}^k \frac{2v_j}{x_{n-k+j}} \frac{\partial}{\partial x_{n-k+j}}, \quad (2)$$

$v = (v_1, \dots, v_k)$ is a multi-index consisting of fixed positive numbers. Lyakhov [2, 5] studied this class of spherical functions for the case of $k = 1$ in the equations (1) and (2). Our results are generalizations of the Lyakhov's results.

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The singular integral operators associated with equation (2) Laplace–Bessel differential operator [1, 3] are known to be important differential operators in analysis and its applications have been research areas for many mathematicians such as Kipriyanov and Klyuchantsev [4], Lyakhov [2, 5], Aliev and Gadjiev [6], Ekinçioğlu and Serbetçi [7] and others.

Let $\mathbb{S}(\mathbb{R}_n^{k,+})$ be the space of functions which are the restrictions to $\mathbb{R}_n^{k,+}$ of the test functions of the Schwartz that are even with respect to $x_{n-k+1}, \dots, x_n$, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-k}}}{\partial x_{n-k}^{\alpha_{n-k}}} B_{x_{n-k+1}}^{\alpha_{n-k+1}} \cdots B_{x_n}^{\alpha_n},$$

where $B_{x_i} = \partial^2/\partial x_i^2 + (2\nu/x_i)(\partial/\partial x_i)$. The closure of the space $\mathbb{S}(\mathbb{R}_n^{k,+})$ in the norm

$$\|f\|_{L_{p,v}} \equiv \left(\int_{\mathbb{R}_n^{k,+}} |f(x)|^p \left(\prod_{j=1}^k x_{n-k+j}^{2\nu_j} \right) dx \right)^{1/p}$$

is denoted by $L_{p,v}(\mathbb{R}_n^{k,+})$, $1 \leq p < \infty$. Following ref. [4], we define the singular integral operator generated by the generalized shift operator to be

$$Af(x) = p\nu \int_{\mathbb{R}_n^{k,+}} \left[T^y \frac{g(\theta)}{|x|^{n+2|\nu|}} \right] f(y) \prod_{j=1}^k y_{n-k+j}^{2\nu_j} dy,$$

where $\theta = x/|x|$, $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n$, the characteristic $g(\theta)$ belongs to some function space on the hemisphere $S_n^{k,+}$ and satisfies the cancellation condition $\int_{S_n^{k,+}} g(\theta) \prod_{j=1}^k \theta_{n-k+j}^{2\nu_j} d\theta = 0$, and T^y is the generalized shift operator defined by

$$\begin{aligned} T^y \varphi(x) &= \pi^{-k/2} \prod_{j=1}^k \frac{\Gamma(\nu_j + 1/2)}{\Gamma(\nu_j)} \\ &\times \int_0^\pi \cdots \int_0^\pi \varphi \left(x' - y', \sqrt{x_{n-k+1}^2 + y_{n-k+1}^2 - 2x_{n-k+1}y_{n-k+1} \cos \alpha_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos \alpha_k} \right) \\ &\times \prod_{j=1}^k \left(\sin^{2\nu_j-1} \alpha_j d\alpha_j \right), \quad x', y' \in \mathbb{R}_{n-k}. \end{aligned} \quad (3)$$

The article is organized as follows. In sections 2 and 3 following Lyakhov [2, 5], we introduced a class of spherical functions called spherical B_k -harmonic functions and gave some of their important properties. In section 4, we established some relations between the singular integral operators generated by generalized shift operator and singular pseudo-differential operators. Furthermore, we showed that this type of singular pseudo-differential operators is bounded in the space $L_{2,v}(\mathbb{R}_n^{k,+})$.

2. Weighted spherical harmonic functions

A homogeneous polynomial $P_m(x)$, $x \in \mathbb{R}_n^{k,+}$ of degree m which is even with respect to $x_{n-k+1}, \dots, x_n$, ($1 \leq k \leq n$), and satisfies the Laplace–Bessel equation (1) is called a B_k -spherical polynomial. The space of all B_k -spherical polynomials of degree m is denoted by $\mathcal{H}_m^\nu$. The dimension $\sigma(m)$ of this space is of order m^{n-2} .

The following theorem is an analogy of the Gauss representation of homogeneous polynomials.

THEOREM 2.1 *Any homogeneous polynomial $P_m(x)$ of degree m , even in $x_{n-k+1}, \dots, x_n$, can be uniquely represented in the form*

$$P_m(x) = \sum_{j=0}^{\lfloor m/2 \rfloor} \left(\frac{|x|}{2} \right)^{2j} P_{m-2j,v}(x), \quad (4)$$

where $P_{m-2j,v}(x) \in \mathcal{H}_{m-2j,v}$, $j = 0, 1, \dots$, and are expressed by the formula

$$\begin{aligned} P_{m-2j,v}(x) &= \frac{m-2j + ((n+2|v|)/2) - 1}{(j!) \Gamma(m + ((n+2|v|)/2) - 1)} \\ &\times \sum_{\ell \geq 0} (-1)^\ell \frac{\Gamma(m - \ell - 2j + ((n+2|v|)/2) - 1)}{\Gamma(\ell + 1)} \left(\frac{|x|}{2} \right)^{2\ell} \Delta_{B_k}^\ell P_m(x). \end{aligned}$$

Proof It is not hard to show that for each homogeneous polynomial $P_m(x)$, $x \in \mathbb{R}_n^{k,+}$ of degree m which is even with respect to the k variables $x_{n-k+1}, \dots, x_n$, there exist uniquely defined homogeneous polynomials $P(x)$ and $Q(x)$ such that

$$P_m(x) = P(x) + |x|^2 Q(x),$$

where $\Delta_{B_k} P(x) = 0$ and Q is a homogeneous polynomial of degree $m-2$ which is even with respect to $x_{n-k+1}, \dots, x_n$. By using induction method we can get representation (4). From this it can be calculated that the dimension $\sigma(m)$ of $\mathcal{H}_m^v$ is

$$\sigma(m) = \binom{n+m-2}{m}.$$

A weighted spherical harmonic function, or a spherical B_k -harmonic, of order m is a quotient

$$Y_m^v(\theta) = \frac{P_{m,v}(x)}{|x|^m},$$

where $\theta = x|x|/\overline{S_n^{k,+}} = \{|x| = 1: x_{n-k+1} \geq 0, \dots, x_n \geq 0\}$, $P_{m,v}(x) \in \mathcal{H}_m^v$.

In spherical coordinates

$$\Delta_{B_k} = \frac{\partial^2}{\partial r^2} + \frac{n+2|v|-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{B_k}(\theta), \quad (5)$$

where $r = |x|$ and $\Delta_{B_k}(\theta)$ is the angular coordinate component of Δ_{B_k} . From equation (1), where $P(x) = |x|^m Y_m^v(\theta)$, there follows the differential equation for the weighted spherical functions

$$\Delta_{B_k}(\theta) Y_m^v(\theta) = m(m+n+2|v|-2) Y_m^v(\theta).$$

The operator $\Delta_{B_k}(\theta)$ is symmetric and nonnegative on the space

$$\begin{aligned} L_{2,v}(S_n^{k,+}) &= \left\{ f: \int_{S_n^{k,+}} |f(x)|^2 \left(\prod_{j=1}^k x_{n-k+j}^{2v_j} \right) dx \right. \\ &\quad \left. < \infty \text{ and } f \text{ is even with respect to } x_{n-k+1}, \dots, x_n \right\} \end{aligned}$$

with respect to the product

$$(f, g)_v = \int_{S_n^{k,+}} f(\theta) g(\theta) \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta,$$

i.e., $(\Delta_{B_k}(\theta)f, g)_v = (f, \Delta_{B_k}(\theta)g)_v$ and $(\Delta_B(\theta), f)_v \geq 0$.

These assertions are easily proved by using Green's formula

$$\int_{G^{k,+}} (g \Delta_{B_k} f - f \Delta_{B_k} g) \prod_{j=1}^k x_{n-k+j}^{2v_j} dx = \int_{\Gamma^{k,+}} \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) \prod_{j=1}^k x_{n-k+j}^{2v_j} d\Gamma,$$

where $G^{k,+} \subset \mathbb{R}_n^{k,+}$ is a domain abutting on the hyperplane $x_{n-k+1} = \dots = x_n = 0$, $\Gamma^{k,+}$ is the part of its boundary in the subspace $x_{n-k+1} > 0, \dots, x_n > 0$ and n is the direction of the exterior normal to $\Gamma^{k,+}$. Hence it follows that weighted spherical harmonic functions corresponding to different degrees of homogeneity are orthogonal in $L_{2,v}(S_n^{k,+})$

$$\int_{S_n^{k,+}} Y_m^v(\theta) Y_\ell^v(\theta) \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta = 0 \quad \text{for } l \neq m.$$

From this, for $\ell = 0$ we get

$$\int_{S_n^{k,+}} Y_m^v(\theta) \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta = 0. \quad (6)$$

Then the set of weighted spherical harmonic functions of order m contains $\sigma(m)$ linearly independent elements, which can be taken to be orthogonal in $L_{2,v}(S_n^{k,+})$. Consequently, we have an orthonormal basis

$$\{Y_{m,k}^v(\theta), \quad m = 0, 1, \dots, k; \quad k = 1, 2, \dots, \sigma(m)\}. \quad (7)$$

■

THEOREM 2.2 *The system (7) of weighted spherical harmonic functions is such that*

- (1) *It forms a complete orthonormal system in $L_{2,v}(S_n^{k,+})$;*
- (2) *The set of all linear combinations of functions (7) is dense in $L_{2,v}(S_n^{k,+})$.*

The proof of (1) is similar to the corresponding proof for an orthonormal system of ordinary spherical harmonics [8], while (2) follows directly from Theorem 2.1 and the Weierstrass theorem.

3. The Laplace formula

Let $C_\ell^p(t)$ ($|t| \leq 1$) denote the Gegenbauer polynomial of order ℓ . These polynomials are obtained as coefficients of the expansion in powers of z of the generating function

$$(1 - 2zt - z^2)^{-p} = \sum_{\ell=0}^{\infty} C_\ell^p(t) z^\ell \quad (p > 0). \quad (8)$$

The following analog of the Laplace formula holds for weighted spherical harmonic functions.

THEOREM 3.1 *Let $Y_m^v(\theta)$ be an arbitrary weighted spherical harmonic function of order m . Then it has the integral representation*

$$Y_m^v(\theta) = c_v \int_{S_{n,+}^{k,+}} Y_m^v(\tilde{\theta}) \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle \theta, \tilde{\theta} \rangle) \prod_{j=1}^k \tilde{\theta}_{n-k+j}^{2v_j} d\tilde{\theta}, \quad (9)$$

where $c_v = (n + 2|v| + 2m - 2)/(n + 2|v| - 2)W_{n,v}$,

$$W_{n,v} = \begin{cases} \frac{\pi^{(n-k)/2} \prod_{j=1}^k \Gamma(v_j + 1/2)}{\Gamma((n + 2|v|)/2)}, & \theta_{n-k+1} = \dots = \theta_n = 0; \\ \frac{n}{n + 2|v|} \frac{\pi^{(n-k)/2} \prod_{j=1}^k \Gamma(v_j + 1/2)}{\Gamma((n + 2|v|)/2)}, & \theta_{n-k+1} > 0, \dots, \theta_n > 0; \end{cases} \quad (10)$$

and $\Pi_{\theta_{n,k}}^v$ is the Poisson operator defined by the relation

$$\Pi_{\theta_{n,k}}^v = c'_v \int_0^\pi \dots \int_0^\pi f(\theta_1, \dots, \theta_{n-k}, \theta_{n-k+1} \cos \alpha_1, \dots, \theta_n \cos \alpha_n) \prod_{j=1}^k (\sin^{2v_j-1} \alpha_j d\alpha_j),$$

$c'_v = \pi^{-k/2} \prod_{j=1}^k \frac{\Gamma(v_j+1/2)}{\Gamma(v_j)}$, and $\langle \theta, \tilde{\theta} \rangle$ is the scalar product of the unit vectors θ and $\tilde{\theta}$.

Proof We will use the following Green's formula. Let $u(x)$ be a twice continuously differentiable function on a bounded region $G^{k,+} \subset \mathbb{R}_{n,+}^{k,+}$, abutting the hyperplane $x_{n-k+1} = 0, \dots, x_n = 0$ that is even with respect to $x_{n-k+1}, \dots, x_n$ and vanishes in $\mathbb{R}_n^{k,+} \setminus \overline{G^{k,+}}$. Then we have

$$\begin{aligned} & - \int_{G^{k,+}} \Delta_{B_k} u(y) T^x R^{-(n+2|v|-2)} \prod_{j=1}^k y_{n-k+j}^{2v_j} dy \\ & + \int_{\Gamma^{k,+}} \left[u(y) \frac{\partial}{\partial n} T^x R^{-(n+2|v|-2)} - \frac{\partial u}{\partial n} T^x R^{-(n+2|v|-2)} \right] \prod_{j=1}^k y_{n-k+j}^{2v_j} d\Gamma \\ & = (n + 2|v| - 2) W_{n,v} u(x), \quad x \in G^{k,+} \cup \Gamma^0, \end{aligned} \quad (11)$$

where $R = \sqrt{|y'|^2 + y_{n-k+1}^2 + \dots + y_n^2}$, $\Gamma^{k,+}$ is a part of the boundary of $G^{k,+}$, Γ^0 is the part of the boundary in the hyperplane $x_{n-k+1} = 0, \dots, x_n = 0$ and n is the direction of the exterior normal to $\Gamma^{k,+}$. The coefficient $W_{n,v}$ is given by equation (10). Applying equation (11) to the B_k -spherical polynomial $P_{m,v}(x) = |x|^m Y_m^v(\theta)$ in $G_R^{k,+} = \{|y| < R, y_{n-k+1} > 0, \dots, y_n > 0\}$, we obtain

$$\begin{aligned} (n + 2|v| - 2) W_{n,v} |x|^m Y_m^v(\theta) & = - \int_{S_R^{k,+}} [|y|^m Y_m^v(\theta)] \frac{\partial}{\partial n} T^x R^{-(n+2|v|-2)} \\ & + T^x R^{-(n+2|v|-2)} \frac{\partial}{\partial n} (|y|^m Y_m^v(\theta)) \prod_{j=1}^k y_{n-k+j}^{2v_j} d\Gamma, \end{aligned} \quad (12)$$

where $S_R^{k,+}$ is the surface of the half-ball $G_R^{k,+}$.

Let

$$T^x R^{-(n+2|v|-2)} = \pi^{-k/2} \prod_{j=1}^k \frac{\Gamma(v_j + 1/2)}{\Gamma(v_j)} \times \int_0^\pi \dots \int_0^\pi \prod_{j=1}^k \left[\sin^{2v_j-1} \alpha_j (|x' - y'|^2 \right. \\ \left. + \sum_{j=1}^k (x_{n-k+j}^2 + y_{n-k+j}^2 - 2x_{n-k+j}y_{n-k+j} \cos \alpha_j) \right)^{-(n+2|v|-2)/2} d\alpha_j \Big].$$

The substitutions $z_{n-k+(2j-1)} = y_{n-k+j} \cos \alpha_i$ and $z_{n-k+2j} = y_{n-k+j} \sin \alpha_i$, $j = 1, 2, \dots, k$ transform the last term of the integrand into

$$V = |x' - y'|^2 + \sum_{j=1}^k ((x_{n-k+j} - z_{n-k+(2j+1)})^2 + z_{n-k+2j}^2)^{-(n+2|v|-2)/2}.$$

In the notation $\tilde{y} = (y', \underbrace{z_{n-k+1}, \dots, z_{n+k}}_{2k\text{-times}}), \tilde{x} = (x', x_{n-k+1}, \dots, x_n, \underbrace{0, \dots, 0}_{k\text{-times}}), |\tilde{y}| = |y| = R,$
 $|\tilde{x}| = |x| = r$, we obtain

$$V = |\tilde{x} - \tilde{y}|^{-(n+2|v|-2)} = (R^2 - 2Rr \cos \beta + r^2)^{-(n+2|v|-2)/2} \\ = R^{-(n+2|v|-2)/2} \left(1 + 2 \frac{r}{R} \cos \beta + \frac{r^2}{R^2} \right)^{-(n+2|v|-2)/2},$$

where β is the angle between the vectors $\tilde{x}$ and $\tilde{y}$, *i.e.*,

$$\cos \beta = \frac{\langle \tilde{x}, \tilde{y} \rangle}{Rr} = \sum_{i=1}^{n-k} \theta_i \tilde{\theta}_i + \theta_{n-k+1} \tilde{\theta}_{n-k+1} \cos \alpha_1 + \dots + \theta_n \tilde{\theta}_n \cos \alpha_k,$$

where $\theta_i = x_i/|x|$ and $\tilde{\theta}_i = y_i/|y|$. It follows from equation (8) that

$$(|x' - y'|^2 + \sum_{j=1}^k (x_{n-k+j}^2 + y_{n-k+j}^2 - 2x_{n-k+j}y_{n-k+j} \cos \alpha_j))^{-(n+2|v|-2)/2} \\ = \sum_{l=0}^{\infty} C_l^{((n+2|v|)/2)-1} \left(\sum_{i=1}^{n-k} \theta_i \tilde{\theta}_i + \theta_{n-k+1} \tilde{\theta}_{n-k+1} \cos \alpha_1 + \dots + \theta_n \tilde{\theta}_n \cos \alpha_k \right) \frac{r^l}{R^{n+2|v|-2}};$$

Hence, by virtue of the definition of the Poisson's operator

$$T^x R_y^{-(n+2|v|-2)} = \sum_{l=0}^{\infty} \Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)-1} (\langle \theta, \tilde{\theta} \rangle) \frac{r^l}{R^{n+2|v|-2}}. \quad (13)$$

We now turn to formula (12). As the direction of n coincides with the direction of the radius vector R of the point y on the hemisphere $S_R^{k,+}$; it follows from equation (13) that

$$\frac{\partial}{\partial n} T^x R_y^{-(n+2|v|-2)} = - \sum_{l=0}^{\infty} (n + 2|v| - 2 + l) \Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)-1} (\langle \theta, \tilde{\theta} \rangle) \frac{r^l}{R^{n+2|v|-1}}, \\ \frac{\partial}{\partial n} (R^m Y_m(\theta)) = m R^{m-1} Y_m(\theta).$$

Hence

$$\begin{aligned}
 (n+2|v|-2)W_{n,v}|x|^m Y_m^v(\theta) &= \int_{S_R^{k,+}} \left[R^m Y_m^v(\theta) \sum_{l=0}^{\infty} -(n+2|v|-2+l)\Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)} \right. \\
 &\quad \left. -1(\langle\theta, \tilde{\theta}\rangle) \frac{r^l}{R^{n+2|v|+l-1}} + \sum_{j=0}^{\infty} m Y_m^v(\theta) \frac{r^l}{R^{n+2|v|+l-m-1}} \right. \\
 &\quad \left. \times \Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) Y_m^v(\theta) \right] \prod_{j=1}^k y_{n-k+j}^{2v_j} d\Gamma. \quad (14)
 \end{aligned}$$

We will prove that $\Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)}(\langle\theta, \tilde{\theta}\rangle)$ is a weighted spherical harmonic function of order l . To show this we consider the expression

$$R^{n+2|v|-2} T^x R_y^{-(n+2|v|-2)} = c'_v \int_0^\pi \cdots \int_0^\pi \frac{|\tilde{y}|^{n+2|v|-2} \prod_{j=1}^k (\sin^{2v_j-1} \alpha_j d\alpha_j)}{|\tilde{x} - \tilde{y}|^{n+2|v|-2}}.$$

The Maclaurin expansion of $|\tilde{y}|^{n+2|v|-2}/|\tilde{x} - \tilde{y}|^{n+2|v|-2}$ in powers of $\tilde{x}$ is $\sum_{m=0}^{\infty} \mathcal{P}_m(\tilde{x}/\tilde{y})$, where $\mathcal{P}_m(x)$ is a m th degree homogeneous polynomial of x . It follows that

$$R^{n+2|v|-2} T^x R^{-(n+2|v|-2)} = c'_v \int_0^\pi \cdots \int_0^\pi \sum_{m=0}^{\infty} \mathcal{P}_m\left(\frac{\tilde{x}}{\tilde{y}}\right) \prod_{j=1}^k (\sin^{2v_j-1} \alpha_j d\alpha_j). \quad (15)$$

As $\Delta_{B_k} T^x R^{-(n+2|v|-2)} = 0$ for $x \neq y$, the relation

$$\Delta_{B_k} \int_0^\pi \cdots \int_0^\pi \mathcal{P}_m\left(\frac{\tilde{x}}{\tilde{y}}\right) \prod_{j=1}^k (\sin^{2v_j-1} \alpha_j d\alpha_j) = 0$$

holds for sufficiently small x (we recall that $y \in S_n^{k,+}$). However, comparing equation (15) with equation (13), we find that

$$\Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) \frac{r^l}{R^l} = c'_v \int_0^\pi \cdots \int_0^\pi \mathcal{P}_l\left(\frac{\tilde{x}}{\tilde{y}}\right) \prod_{j=1}^k (\sin^{2v_j-1} \alpha_j d\alpha_j).$$

It follows that $r^l \Pi_{\theta_{n,k}}^v C_l^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle)$ is a B_k -spherical polynomial as a function of θ and as a function of $\tilde{\theta}$ because $T^x R^{-(n+2|v|-2)}$ is symmetric with respect to x and y .

Transforming to polar coordinates and using the orthogonality of the weighted spherical harmonic function of different orders, we obtain

$$\begin{aligned}
 (n+2|v|-2)W_{n,v}Y_m^v(\theta) &= (n+2|v|+m-2) \int_{S_n^{k,+}} Y_m^v(\tilde{\theta}) \\
 &\quad \times \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta,
 \end{aligned}$$

which proves that equation (9) holds and completes the proof. ■

Now we can give the following three corollaries.

COROLLARY 3.2 *If $\{Y_{m,k}^v(\theta)\}$ is an orthonormal system in $L_{2,v}(S_n^{k,+})$ consisting of $\sigma(m)$ weighted spherical harmonic functions of order m , then the following relation holds for an arbitrary fixed unit vector $\theta \in \overline{S_n^{k,+}}$:*

$$\frac{n+2|v|+2m-2}{(n+2|v|-2)W_{n,v}} \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) = \sum_{k=1}^{\sigma(m)} Y_{m,k}^v(\theta) Y_{m,k}^v(\tilde{\theta}). \quad (16)$$

Proof The function $\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle)$, being a spherical weighted-harmonic polynomial of order m , has the representation

$$\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) = \sum_{l=1}^{\sigma(m)} \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \bar{\theta}\rangle, Y_{m,k}^v(\bar{\theta}))_v Y_{m,k}^v(\tilde{\theta}),$$

where $(\cdot)_v$ denotes the scalar product in $L_{2,v}(S_n^{k,+})$. Multiplying this relation to by the constant in Laplace's formula (9) and using this formula, we obtain equation (16). ■

COROLLARY 3.3 *Each spherical weighted-harmonic function $Y_m^v(\theta)$ of order m satisfies the inequality*

$$|Y_m^v(\theta)| \leq C m^{((n+2|v|)/2)-1} \|Y_m^v\|_{L_{2,v}(S_n^{k,+})}. \quad (17)$$

In particular, if $\{Y_{m,k}^v(\theta)\}$ is an orthonormal basis in $L_{2,v}(S_n^{k,+})$, then

$$|Y_{m,k}^v(\theta)| \leq C m^{((n+2|v|)/2)-1}. \quad (18)$$

Proof Applying the Schwartz inequality to the right side of equation (9) we find that

$$\begin{aligned} |Y_m^v(\theta)| &\leq \frac{n+2|v|+2m-2}{(n+2|v|-2)W_{n,v}} \left(\int_{S_n^{k,+}} |Y_m^v(\theta)|^2 \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta \right)^{1/2} \\ &\times \left(\int_{S_n^{k,+}} |\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle)|^2 \prod_{j=1}^k \theta_{n-k+j}^{2v_j} d\theta \right)^{1/2}. \end{aligned}$$

For $\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle)$ we also have the Laplace formula (9):

$$\begin{aligned} \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \tilde{\theta}\rangle) &= \frac{n+2|v|+2m-2}{(n+2|v|-2)W_{n,v}} \int_{S_n^{k,+}} \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\bar{\theta}, \tilde{\theta}\rangle) \\ &\times \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \bar{\theta}\rangle) \prod_{j=1}^k \bar{\theta}_{n-k+j}^{2v_j} d\bar{\theta}. \end{aligned}$$

Putting $\theta = \tilde{\theta}$ in this formula, we obtain

$$\begin{aligned} \Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \theta\rangle) &= \frac{n+2|v|+2m-2}{(n+2|v|-2)W_{n,v}} \\ &\times \int_{S_n^{k,+}} |\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle\theta, \bar{\theta}\rangle)|^2 \prod_{j=1}^k \bar{\theta}_{n-k+j}^{2v_j} d\bar{\theta}. \end{aligned}$$

Hence

$$|Y_m^v(\theta)| \leq \left(\frac{n+2|v|+2m-2}{(n+2|v|-2)W_{n,v}} \right)^{1/2} \left[\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle \theta, \theta \rangle) \right]^{1/2} \|Y_m^v\|_{L_{2,v}(S_n^{k,+})}. \quad (19)$$

The Gegenbauer polynomials [9] of order m satisfy the inequality

$$\max_{|t| \leq 1} |C_m^v(t)| \leq C_1 m^{2v-1},$$

where C_1 is a constant depending only on v . Furthermore it follows from the relation

$$\int_0^\pi \sin^{2v_j-1} \alpha_j \, d\alpha_j = \left[\frac{\Gamma(v_j+1/2)}{\Gamma(v_j)\Gamma(1/2)} \right]^{-1}$$

that

$$\left[\Pi_{\theta_{n,k}}^v C_m^{((n+2|v|)/2)-1}(\langle \theta, \tilde{\theta} \rangle) \right]^{1/2} \leq C_1 m^{(n+2|v|-3)/2}, \quad (20)$$

hence equation (17) follows from equation (19) and the proof is complete. $\blacksquare$

COROLLARY 3.4 *If $\{Y_{m,j}^v(\theta)\}$ is an orthonormal system in $L_{2,v}(S_n^{k,+})$ consisting of $\sigma(m)$ weighted spherical harmonic functions of order m , then*

$$\sum_{k=1}^{\sigma(m)} [Y_{m,k}^v(\theta)]^2 \leq C_2 m^{n+2|v|-2}, \quad (21)$$

where C_2 is a constant depending only on n and v .

Proof We obtain equation (21) by putting $\theta = \tilde{\theta}$ in equation (16) and using equation (20). $\blacksquare$

4. Singular integral operators generated by a generalized shift operator and singular pseudo-differential operators of order zero

On $\mathbb{S}(\mathbb{R}_n^{k,+})$ the direct and inverse Fourier–Bessel transformations are defined as follows:

$$F_{B_k}[f(x)](\xi) = \hat{f}(\xi) = c_{n,v} \int_{\mathbb{R}_n^{k,+}} f(x) e^{-i(x', \xi')} \prod_{l=1}^k [j_{v_{l-1/2}}(x_{n-k+l} \xi_{n-k+l}) x_{n-k+l}^{2v_l}] \, dx,$$

$$F_{B_k}^{-1}[\hat{f}(\xi)](x) = f(x) = \int_{\mathbb{R}_n^{k,+}} \hat{f}(\xi) e^{i(x', \xi')} \prod_{l=1}^k [j_{v_{l-1/2}}(x_{n-k+l} \xi_{n-k+l}) \xi_{n-k+l}^{2v_l}] \, d\xi,$$

where $j_{v_{l-1/2}}(x_{n-k+l} \xi_{n-k+l})$, $l = 1, \dots, k$, is a normalized Bessel function, $\xi = (\xi', \xi_{n-k+1}, \dots, \xi_n) \in \mathbb{R}_n^{k,+}$, and $c_{n,v} = \pi^{(n-k)/2} 2^{(n+2|v|-2k)/2} \prod_{j=1}^k [\Gamma(v_j+1/2)]^{-1}$.

We introduce the singular integral operator A_m^v defined as follows [2, 4]:

$$A_m^v f(x) = \text{pv} \int_{\mathbb{R}_n^{k,+}} \left[T^y \frac{Y_m^v(x)}{|x|^{n+2|v|}} \right] f(y) \prod_{j=1}^k y_{n-k+j}^{2v_j} \, dy, \quad (22)$$

where T^y is the generalized shift operator defined by equation (1), $Y_m^v(x)$ is weighted spherical harmonic function of order m . For brevity we write $Y_m^v(x)$ instead of $Y_m^v(\theta) = Y_m^v(x/|x|)$.

Condition (6) enables us to prove the existence of the principle value (22) for each $f \in L_{2,v}(\mathbb{R}_n^{k,+})$.

Now we will represent the operator A_m^v in terms of Fourier–Bessel transforms. The following result holds.

THEOREM 4.1 *If $f(x) \in L_{2,v}(\mathbb{R}_n^{k,+})$, then*

$$A_m^v f(x) = (-i)^m \frac{\Gamma((n+2|v|)/2)\Gamma(m/2)W_{n,v}}{2^{k+1}\Gamma((n+2|v|+m)/2)} F_{B_k}^{-1}[Y_m^v(\xi)F_{B_k}[f](\xi)](x). \quad (23)$$

The proof of Theorem 4.1 hinges on the following Lemma.

LEMMA 4.2 *For $m > 0$,*

$$\begin{aligned} F_{B_k} \left[\frac{Y_m^v(x)}{|x|^{n+2|v|}} \right] &= c_{n,v} \lim_{\substack{\varepsilon \rightarrow 0 \\ N \rightarrow \infty}} \int_{\varepsilon < |x| < N} e^{-i(x', \xi')} \frac{Y_m^v(x)}{|x|^{n+2|v|}} \times \prod_{l=1}^k [j_{\nu_l-1/2}(x_{n-k+l}\xi_{n-k+l})x_{n-k+l}^{2\nu_l}] dx \\ &= (-i)^m \frac{\Gamma((n+2|v|)/2)\Gamma(m/2)W_{n,v}}{2^{k+1}\Gamma((n+2|v|+m)/2)} Y_m^v(\xi), \end{aligned} \quad (24)$$

where $c_{n,v}$ is the constant in the definition of the Fourier–Bessel transformation.

Proof The proof of the lemma is easily obtained by using the method in ref. [2]. ■

Proof of Theorem 4.1 We use the formula

$$F_{B_k}^{-1}[\hat{g}(\xi)\hat{f}(\xi)](x) = c_{n,v} \int_{\mathbb{R}_n^{k,+}} T^y g(x)f(y) \prod_{j=1}^k y_{n-k+j}^{2\nu_j} dy = c_{n,v}(f * g).$$

The relation

$$\left(\frac{Y_m^v}{|x|^{n+2|v|}} * f(x) \right) = \frac{1}{c_{n,v}} F_{B_k}^{-1} \left[F_{B_k} \left[\frac{Y_m^v}{|x|^{n+2v}} \right] (\xi) \hat{f}(\xi) \right] (x)$$

holds, where the expression on the left is to be understood in the sense of a limit as in equation (22). Formula (23) follows from equation (24), and this proves the theorem. ■

We denote by A a singular pseudo-differential operator which acts on a function $f \in \mathbb{S}(\mathbb{R}_n^{k,+})$ according to the formula

$$Af(x) = \int_{\mathbb{R}_n^{k,+}} e^{i(x', \xi')} a(x; \xi) \hat{f}(\xi) \prod_{l=1}^k j_{\nu_l-1/2}(x_{n-k+l}\xi_{n-k+l}) \xi_{n-k+l} d\xi \quad (25)$$

whose symbol $a(x; \xi)$ is an even function of $x_{n-k+1}, \dots, x_n$ and $\xi_{n-k+1}, \dots, \xi_n$, positively homogeneous in ξ and order zero. From equation (9) it follows that A_m^v is a singular pseudo-differential operator corresponding to the homogeneous symbol of degree zero $a(x; \xi) = Y_m^v(\xi)$. Hence A_m^v is a bounded operator on the weighted spaces of symbols $H_v^s(\mathbb{R}_n^{k,+})$ of the Kipriyanov for arbitrary s . We wish to express the operator (25) in terms of the operators A_m^v . To do this we use the expansion of the symbol in a series of weighted spherical harmonic functions (7). It is sufficient to consider a symbol $a(x; \xi)$ satisfying the following conditions:

(1) $a(x; \xi)$ is a continuous function of x and is independent of x for sufficiently large $|x|$;

- (2) For fixed x , the function $a(x; \xi)$ is in the weighted space $H_\lambda^q(S_n^{k,+})$ [10], where q is fixed so that

$$q > \frac{n + 2|\nu| - 1}{2}. \quad (26)$$

Let $C[H_v^q(S_n^{k,+})]$ denote the space of such symbols. If $a(x; \xi) \in C[H_v^q(S_n^{k,+})]$, then

$$a(x; \xi) = \sum_{m,k} a_{m,k}(x) Y_{m,k}^\nu(\xi), \quad (27)$$

where the coefficient $a_{m,k}(x)$ satisfies the inequality

$$\|a(x; \xi)\| = \left(\max_x |a_{0,0}(x)|^2 + \sum_{m,k} m^{2q} \max_x |a_{m,k}(x)|^2 \right)^{1/2} < \infty. \quad (28)$$

We construct the operator

$$A = \sum_{m,k} a_{m,k}(x) A_{m,k}^\nu, \quad (29)$$

where the $A_{m,k}$ are operators of the form (23) [or (22)].

The following theorem holds [11].

THEOREM 4.3 *If $a(x, \xi) \in C[H_v^q(S_1^{k,+})]$ with q fixed and satisfying equation (26), then the series (29) is convergent in the norm of operators on $H_v^0(\mathbb{R}_n^{k,+}) = L_{2,v}(\mathbb{R}_n^{k,+})$ and*

$$\|Af\|_{0,v} \leq c \|a\| \|f\|_{0,v}, \quad (30)$$

where $\|a\|$ is defined by equation (28) and the constant c is independent of A and f .

Proof Let $f \in H_v^0(\mathbb{R}_n^{k,+})$. Applying the Schwartz inequality to the function

$$S_{\eta,\zeta} f(x) = \sum_{\eta \leq m \leq \zeta, k} a_{m,k}(x) A_{m,k}^\nu f(x)$$

for $\eta > 0$, we obtain

$$\begin{aligned} |S_{\eta,\zeta} f(x)|^2 &\leq \sum_{\eta \leq m \leq \zeta, k} m^{2q} |a_{m,k}(x)|^2 \sum_{\eta \leq m \leq \zeta, k} m^{-2q} |A_{m,k}^\nu f(x)|^2 \\ &\leq \|a\|^2 \sum_{\eta \leq m \leq \zeta, k} m^{-2q} |A_{m,k}^\nu f(x)|^2. \end{aligned}$$

We integrate both sides of this relation with weighted $\prod_{j=1}^k x_{n-k+j}$ with respect to x , apply Parseval's formula [3], and use equation (21) to obtain

$$\begin{aligned} \|S_{\eta,\zeta} f(x)\|_{0,v}^2 &\leq c \|a\|^2 \sum_{m=\eta}^{\zeta} m^{-2q} \int_{\mathbb{R}_n^{k,+}} \sum_{k=1}^{\sigma(m)} (Y_{m,k}^\nu(\xi))^2 |\hat{f}(\xi)|^2 \prod_{j=1}^k \xi_{n-k+j}^{2\nu_j} d\xi \\ &\leq c \|a\|^2 \sum_{m=\eta}^{\zeta} m^{n+2|\nu|-2q-2} \|f\|_{0,v}^2. \end{aligned} \quad (31)$$

As $q > (n + 2|\nu| - 1)/2$, it follows from $\sum_{m=\eta}^{\zeta} m^{-2q+n+2|\nu|-2} \rightarrow 0$ as $\eta \rightarrow \infty$, and so the series (29) is convergent in $H_v^0(\mathbb{R}_n^{k,+})$.

If $a_{00}(x) = 0$, then equation (29) follows from equation (31). In the general case $A = A_0 + A_1$, where A_1 satisfies equation (30) and A_0 is the operator defined by equation (23) corresponding to the symbol $a_{00}(x)$, and also satisfies equation (30). Thus the proof of the theorem is completed. ■

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Relative asymptotics for orthogonal matrix polynomials with unbounded recurrence coefficients

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We study the asymptotic behavior of the ratio of two sequences of orthonormal matrix polynomials P_n (with unbounded three-term recurrence coefficients) and $\tilde{P}_n$, which are associated, respectively, to a matrix of measures and its perturbation by the addition of a Dirac matrix measure.

Keywords: Orthogonal matrix polynomials; Perturbation of varying matrix of measures; Asymptotic behavior

1. Introduction

Orthogonal matrix polynomials appear in a natural way when we consider different kinds of non-standard inner product [see, for instance, ref. 1]. Krein [2] obtained some results about matrix moment problems from the point of view of the operator theory. During the 1980s, they have been connected to the scattering theory by Geronimo [3], and an analog of Favard's theorem has been established for three-term recurrence matrix relation by Aptekarev and Nikishin [4]. Some algebraic results and the results concerning zeros were found by Zhani [5]. Recently, some results concerning zeros and quadrature formulae have been studied by Sinap and Van Assche [6], and finally, some results concerning zeros, quadrature formulae and asymptotic behavior of orthogonal polynomials have been obtained by Durán and coworkers [1, 7, 8]. More recently, a lot of work has been devoted to obtaining asymptotic properties of orthogonal polynomials from the recurrence coefficients. Nevai [9] studied precisely the case when these recurrence coefficients have finite limits. Asymptotically, periodic recurrence coefficients with finite accumulation points have been studied by Geronimo and Van Assche [cf. refs. 10, 11, 12]. Van Assche also studied the case of unbounded recurrence coefficients [cf. ref. 11].

In ref. [13], we have studied a relative asymptotic behavior of two sequences of orthonormal matrix polynomials, with respect to a matrix of measures supported on a real compact set (with convergent three-term matrix recurrence coefficients) and its perturbation by the addition of

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a Dirac delta measure in a point out of the support of the matrix of measures. The support of such a perturbed matrix of measures is also a real compact set, as the perturbed matrix recurrence coefficients are convergent [cf. ref. 14].

In ref. [15], we have studied this problem when the associated matrix of measures is supported on the unit circle.

The aim of this paper is to study the case when the matrix three-term recurrence coefficients are unbounded (Theorem 3.3). We give also a more generalized result (Theorem 2.5) by considering a perturbation of a varying matrix of measures and studying a relative asymptotic for the related matrix orthonormal polynomials. As a consequence, we deduce [13, Theorem 4.1 (with bounded recurrence coefficients)] (see Remark 3.2). We complete this paper giving some examples.

We consider an $N \times N$ positive definite matrix of measures $d\alpha$ (for every Borel set $A \subset \mathbb{R}$ the numerical matrix $d\alpha(A)$ is positive semi-definite) having moments of every order, i.e., the matrix integral $\int t^n d\alpha(t)$ exists for every non-negative integer.

Assuming that $\int P(x)d\alpha(x)P^*(x)$ is non-singular for every matrix polynomial P with non-singular leading coefficient, we introduce an inner product in the linear space of matrix polynomials $\mathbb{C}^{(N,N)}[x]$ by $\langle P, Q \rangle_{d\alpha} \stackrel{\text{def}}{=} \int P(x)d\alpha(x)Q^*(x)$. Using the Gram-Schmidt orthonormalization process for the canonical sequence $\{x^n I_N\}_{n=0}^\infty$, we will obtain many sequences of matrix polynomials (P_n) , which satisfy the orthonormality condition

$$\langle P_n, P_m \rangle_{d\alpha} \stackrel{\text{def}}{=} \int P_n(x)d\alpha(x)P_m^*(x) = \delta_{n,m} I_N \quad n, m \geq 0. \quad (1)$$

$P_n(x)$ is a matrix polynomial with non-singular leading coefficient and is defined up to a multiplication on the left by a unitary matrix.

As in the scalar case, the orthonormal matrix polynomials $(P_n(x; d\alpha))_n$ with respect to the matrix of measures $d\alpha$ are orthogonal to every matrix polynomial of degree less than n and they satisfy a three-term recurrence relation

$$\begin{aligned} xP_n(x; d\alpha) &= A_{n+1}(d\alpha)P_{n+1}(x; d\alpha) \\ &+ B_n(d\alpha)P_n(x; d\alpha) + A_n^*(d\alpha)P_{n-1}(x; d\alpha), \end{aligned} \quad (2)$$

where $P_{-1} = 0$, $P_0 = \int d\alpha = I_N$, $A_n(d\alpha) = \gamma_{n-1}(d\alpha)\gamma_n^{-1}(d\alpha)$ is a positive definite matrix ($\gamma_n(d\alpha)$ is the leading coefficient of $P_n(x; d\alpha)$) and $B_n(d\alpha)$ is an hermitian matrix.

Notice that the polynomial $R_n(x; d\alpha) = U_n P_n(x; d\alpha)$ with $U_n U_n^* = I_N$ is also orthonormal with respect to equation (1) and they satisfy a three-term recurrence relation of type (2) with matrix parameters $U_{n-1} A_n(d\alpha) U_n^*$ and $U_n B_n(d\alpha) U_n^*$, respectively. This three-term recurrence relation characterizes the orthogonality of a sequence of matrix polynomials with respect to a positive definite matrix of measures (see, for instance, refs. [1, 4]).

The corresponding matrix polynomials of the second kind are defined by

$$Q_n(x; d\alpha) \stackrel{\text{def}}{=} \int \frac{P_n(x; d\alpha) - P_n(t; d\alpha)}{x - t} d\alpha(t) \quad (3)$$

and satisfy a recurrence relation of type (2) with initial conditions $Q_0 = 0$ and $Q_1 = A_1^{-1}$. Furthermore, defining

$$K_{n+1}(x, y; d\alpha) \stackrel{\text{def}}{=} \sum_{j=0}^n P_j^*(y; d\alpha) P_j(x; d\alpha),$$

we have

(a) the Christoffel–Darboux formula

$$(x - y)K_{n+1}(x, y; d\alpha) = P_n^*(y; d\alpha)A_{n+1}(d\alpha)P_{n+1}(x; d\alpha) - P_{n+1}^*(y; d\alpha)A_{n+1}^*(d\alpha)P_n(x; d\alpha); \quad (4)$$

(b) the confluent formula

$$K_{n+1}(x, x; d\alpha) = P_{n+1}^*(x; d\alpha)' A_{n+1}^*(d\alpha)P_n(x; d\alpha) - P_n^*(x; d\alpha)' A_{n+1}(d\alpha)P_{n+1}(x; d\alpha); \quad (5)$$

(c) the Green formula

$$I_N + (x - y)K_{n+1}(x, y; d\alpha) = P_n^*(y; d\alpha)A_{n+1}(d\alpha)Q_{n+1}(x; d\alpha) - P_{n+1}^*(y; d\alpha)A_{n+1}^*(d\alpha)Q_n(x; d\alpha); \quad (6)$$

(d) the Liouville–Ostrogradski formula

$$Q_n(x; d\alpha)P_{n-1}^*(x; d\alpha) - P_n(x; d\alpha)Q_{n-1}^*(x; d\alpha) = A_n^{-1}(d\alpha). \quad (7)$$

The matrix $K_n(x, y; d\alpha)$ is called the n th reproducing kernel because of the following property. For every matrix polynomial $\Pi_m(x)$ of degree $m \leq n - 1$, we have

$$\langle \Pi_m(x), K_n(x, y; d\alpha) \rangle_{d\alpha} = \int \Pi_m(x) d\alpha(x) K_n^*(x, y; d\alpha) = \Pi_m(y). \quad (8)$$

We define the support of the matrix of measures $d\alpha$ as the support of the trace of the matrix of measures $d\alpha(\text{supp}(d\alpha) = \text{supp}(d\alpha_{1,1} + d\alpha_{2,2} + \dots + d\alpha_{N,N}))$.

Let Δ_n be the set of zeros of matrix polynomial P_n , i.e.

$$\Delta_n(d\alpha) \stackrel{\text{def}}{=} \{x_{n,j}; j = 1, \dots, nN : \det P_n(x_{n,j}; d\alpha) = 0\}.$$

These zeros are real and have multiplicity at most N [cf. ref. 8, Theorem 1.1]. Setting

$$\Gamma = \bigcap_{N>0} M_N,$$

where

$$M_N = \overline{\bigcup_{n \geq N} \Delta_n(d\alpha)},$$

then for a determinate measure $d\alpha$, we have $\text{supp}(d\alpha) \subset \Gamma$ ($d\alpha$ can be found as a weak limit point of a sequence of discrete positive definite matrices of measures) [cf. ref. 8].

Let $M(A, B)$ be the matrix Nevai class (see, for instance, refs. [16] or [14, Definition 2.1]). When A is non-singular, we associate to the matrix Nevai class $M(A, B)$ the orthonormal matrix polynomials $(U_n^{A,B})_n$ defined by the recurrence formula

$$t U_n^{A,B}(t) = A^* U_{n+1}^{A,B}(t) + B U_n^{A,B}(t) + A U_{n-1}^{A,B}(t), \quad n \geq 0, \quad (9)$$

with initial conditions $U_0^{A,B} = I_N$, $U_{-1}^{A,B} = 0$. This sequence is orthonormal with respect to a positive definite matrix of measures, which we denote by $W_{A,B}$, and it is a matrix analogous to the Chebyshev polynomials of the second kind.

The support of a matrix of measures $d\alpha$ which belongs to a Nevai matrix class $M(A, B)$, i.e. its corresponding three-term recurrence coefficients $A_n(d\alpha)$ and $B_n(d\alpha)$ in equation (2) are convergent to A and B , respectively, is a compact set [16, Lemma 2.1].

In the next and when a matrix of measures belongs to a Nevai class, let $\hat{\Gamma}$ be the smallest closed interval which contains its support.

We recall that if H is a positive definite (resp. positive semi-definite) matrix, then there is a unique square root $H_0 = H^{1/2}$ of H defined as follows. Writing $H = UDU^*$, where $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and $(\lambda_i)_{i=1,n}$ are the positive eigenvalues of H , then $H_0 = UD_0U^*$, where $D_0 = \text{diag}(+\sqrt{\lambda_1}, +\sqrt{\lambda_2}, \dots, +\sqrt{\lambda_N})$.

PROPOSITION 1.1 [16] *Let $(P_n(x; d\alpha))_n$ be a sequence of orthonormal matrix polynomials with recurrence coefficients satisfying equation (47). Then*

$$\lim_{n \rightarrow \infty} P_{n-1}(x; d\alpha) P_n^{-1}(x; d\alpha) A_n^{-1}(d\alpha) = \int \frac{dW_{A,B}(t)}{x-t}; \quad x \in \mathbb{C} \setminus \Gamma. \quad (10)$$

If A is an hermitian positive definite matrix, then

$$\begin{aligned} \int \frac{dW_{A,B}(t)}{x-t} &= \frac{1}{2} A^{-1} (xI_N - B) A^{-1} \\ &\quad - \frac{1}{2} A^{-(1/2)} \left[\sqrt{A^{-(1/2)} (B - xI_N) A^{-1} (B - xI_N) A^{-(1/2)} - 4I_N} \right] A^{-(1/2)}, \end{aligned} \quad (11)$$

where $x \notin \text{supp}(dW_{A,B})$.

Remark 1.2 The Markov matrix function $\int (dW_{A,B}(t))/(z-t)$ is positive or negative definite for each $z \in \mathbb{R} \setminus \hat{\Gamma}$. It is differentiable and its derivative $-\int (dW_{A,B}(t))/(z-t)^2$ is negative definite when z is outside $\text{supp}(dW_{A,B})$.

Proof See ref. [13, Proposition 2.2]. ■

The structure of this paper is as follows. Section 2 is devoted to relative asymptotic of orthogonal matrix polynomials with varying recurrence coefficients. We also formulate the extended markov's theorem. In section 3, we establish the relative asymptotic for orthogonal matrix polynomials with unbounded recurrence coefficients and an example is given.

2. Relative asymptotic for orthogonal matrix polynomials with varying recurrence coefficients

Let $(P_n(x; d\alpha_k) = \gamma_n(d\alpha_k)x^n + b_n(d\alpha_k)x^{n-1} + \dots)_n$ be a sequence of orthonormal matrix polynomials with respect to a varying matrix of measures $d\alpha_k$, $k = 1, 2, \dots$, satisfying the following recurrence relation

$$\begin{aligned} x P_n(x; d\alpha_k) &= A_{n+1}(d\alpha_k) P_{n+1}(x; d\alpha_k) \\ &\quad + B_n(d\alpha_k) P_n(x; d\alpha_k) + A_n^*(d\alpha_k) P_{n-1}(x; d\alpha_k), \end{aligned} \quad (12)$$

where $P_{-1}(x; d\alpha_k) = 0$ and $P_0(x; d\alpha_k) = \int d\alpha_k = I_N$.

Let $(n_m)_m$ and $(k_m)_m$ be two increasing sequences of positive integers. We write $\tilde{\Delta}_m$ for the set of zeros of $P_{n_m}(\cdot; d\alpha_{k_m})$.

$$\tilde{\Delta}_m \stackrel{\text{def}}{=} \{x_{n_m, k_m, j} \mid (1 \leq j \leq n_m N); \det P_{n_m}(x_{n_m, k_m, j}; d\alpha_{k_m}) = 0\}$$

and let $\hat{\Gamma}_1$ be the smallest connected set which contains $\Gamma_1 \stackrel{\text{def}}{=} \bigcap_{N>0} \overline{\bigcup_{m \geq N} \tilde{\Delta}_m}$.

Markov's theorem for orthogonal matrix polynomials has been considered in ref. [7]. Here, we establish in the following result the generalization of Markov's theorem for orthogonal matrix polynomials with respect to a varying matrix of measures, using a generalization of the proof given in ref. [7].

THEOREM 2.1 (*Extended Markov's theorem*) *Let $(n_m)_{m \in \mathbb{R}}$ and $(k_m)_{m \in \mathbb{R}}$ be two subsequences of non-negative integers. Then, there exists a matrix of Borel measures ν such that*

$$\lim_{m \rightarrow \infty} P_{n_m}^{-1}(x; d\alpha_{k_m}) Q_{n_m}(x; d\alpha_{k_m}) = \int \frac{1}{x-t} d\nu(t), \quad \text{for } x \in \mathbb{R} \setminus \Gamma_1. \quad (13)$$

Remark 2.2 When the matrices of measures $(d\alpha_k)_k$ are all equal to a matrix of measures $d\alpha$ which is determinate, then $d\nu$ is unique and is equal to $d\alpha$.

Proof of Theorem 2.1 The decomposition

$$P_{n_m}^{-1}(x; d\alpha_{k_m}) Q_{n_m}(x; d\alpha_{k_m}) = \sum_{j=1}^p \frac{1}{x - x_{n_m, k_m, j}} \Gamma_{n_m, k_m, j} \quad (14)$$

is possible ($\deg(Q_{n_m}) = n_m - 1 < \deg(P_{n_m})$), where $\Gamma_{n, k, j}$ are the matrix weights in the quadrature formula for the polynomials $(P_n(x; d\alpha_k))_n$ associated with $x_{n, k, j}$ [7, Theorem 3.1] and given by

$$\Gamma_{n, k, j} = \frac{l_j}{(\det(P_n(x; d\alpha_k)))^{(l_j)}(x_{n, k, j})} (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)}(x_{n, k, j}) Q_n(x_{n, k, j}; d\alpha_k), \quad (15)$$

with $x_{n, k, j}$, $j = 1, \dots, p$, are the different zeros of the polynomials $P_n(x; d\alpha_k)$ and l_j is the multiplicity of $x_{n, k, j}$. We consider the following matrices of discrete measures

$$\lambda_m(t) \stackrel{\text{def}}{=} \sum_{j=1}^p \delta(t - x_{n_m, k_m, j}) \Gamma_{n_m, k_m, j} \quad (16)$$

and

$$\lambda_{m|[a, b]}(t) = \sum_{\substack{j=1 \\ x_{n_m, k_m, j} \in [a, b]}}^p \delta(t - x_{n_m, k_m, j}) \Gamma_{n_m, k_m, j}.$$

From the quadrature formula [7, Theorem 3.1], we conclude

$$\int_{\mathbb{R}} d\lambda_m(t) = \sum_{j=1}^p \Gamma_{n_m, k_m, j} = \int_{\mathbb{R}} d\alpha_{k_m} = I_N. \quad (17)$$

Using equations (14) and (16), we obtain

$$P_{n_m}^{-1}(x; d\alpha_{k_m}) Q_{n_m}(x; d\alpha_{k_m}) = \int \frac{1}{x-t} d\lambda_m(t), \quad \text{with } x \notin \tilde{\Delta}_m.$$

Now, we will use the following classical result in functional analysis. For $a, b \in \mathbb{R}^+$, we denote by $C([-a, b], \mathbb{C}^{N \times N})$ the space of continuous functions from the interval $[-a, b]$ to the linear space of complex $N \times N$ matrices. The dual space $C'([-a, b], \mathbb{C}^{N \times N})$ will be the space of $N \times N$ matrices whose entries are Borel measures. Then, the unit ball of $C'([-a, b], \mathbb{C}^{N \times N})$ is weakly compact (Banach–Alaoglu). By taking into account equation (17), we have

$$\|\lambda_{m|[-a,b]}\| \stackrel{\text{def}}{=} \sum_{i,j=1}^N |\lambda_{m,i,j}|(\mathbb{R}) \leq \sum_{i,j=1}^N |\lambda_{m,i,j}|(\mathbb{R}) \leq N^2 \tau(\lambda_m(\mathbb{R})) = N^3,$$

where $\tau(A)$ is the trace of the matrix A . Hence, the matrices of measures $\lambda_{m|[-a,b]}$ are in a ball of $C'([-a, b], \mathbb{C}^{N \times N})$ with constant ratio and thus we can apply the Banach–Alaoglu's theorem for $(\lambda_{m|[-a,b]})_m$.

Now to prove equation (13), it is enough to show that there exists some matrix of measures ν such that

$$\lim_{m \rightarrow \infty} \int \frac{1}{x-t} d\lambda_m(t) = \int \frac{1}{x-t} d\nu(t), \quad \text{for } x \in \mathbb{R} \setminus \Gamma_1. \quad (18)$$

If equation (18) is false, then for all positive definite matrix of measures ν , we can find $x \in \mathbb{R} \setminus \Gamma_1$, an increasing sequence of non-negative integers $(m_s)_{s \in \mathbb{N}}$ and a positive constant c such that

$$\left\| \int \frac{d\lambda_{m_s}}{x-t} - \int \frac{d\nu}{x-t} \right\| \geq c > 0; \quad s \geq 0. \quad (19)$$

Given two increasing sequences of positive integers a_k and b_k for which $a_k \rightarrow +\infty$ and $b_k \rightarrow +\infty$, we obtain (using Banach–Alaoglu's theorem) that there exist $(l_s)_{s \in \mathbb{N}}$ a subsequence of $(m_s)_{s \in \mathbb{N}}$ and a positive definite matrix of measures $\nu^{(k)} \in C'([-a_k, b_k], \mathbb{C}^{N \times N})$ such that

$$\lim_{s \rightarrow \infty} \int_{[-a_k, b_k]} g(t) d\lambda_{l_s}(t) = \int_{[-a_k, b_k]} g(t) d\nu^{(k)}(t)$$

for any continuous matrix function g defined in $[a_k, b_k]$. For these matrices of measures $\nu^{(k)}$, we can find a matrix of measures $\nu \in C'(\mathbb{R}, \mathbb{C}^{N \times N})$, which extends all $\nu^{(k)}$, and then

$$\lim_{s \rightarrow \infty} \int_{[-a_k, b_k]} \frac{1}{x-t} d\lambda_{l_s}(t) = \int_{[-a_k, b_k]} \frac{1}{x-t} d\nu(t) \quad (20)$$

for all $k \geq 1$. Taking k and l_s big enough, and from equations (19) and (20), we have

$$0 < \frac{c}{3} < \max \left\{ \frac{1}{x-a_k}, \frac{1}{x-b_k} \right\} \left\| \int_{\mathbb{R}} d\lambda_{l_s} \right\|,$$

which yields $0 < c = 0$. Hence, equation (18) is true and the proof is finished. ■

To establish the main theorem of this section (Theorem 2.5), we need to obtain the following result, which deals with relative asymptotic for the leading coefficients of orthogonal matrix polynomials with respect to a varying matrix of measures.

Let $(n_m)_m$ and $(k_m)_m$ be two increasing sequences of positive integers and assume that there exist two matrices A non-singular and B hermitian such that for all $l \geq 0$,

$$\lim_{m \rightarrow \infty} A_{n_m-l}(d\alpha_{k_m}) = A, \quad \lim_{m \rightarrow \infty} B_{n_m-l}(d\alpha_{k_m}) = B. \quad (21)$$

THEOREM 2.3 *Let $\gamma_n(d\beta_k)$ and $\gamma_n(d\alpha_k)$ be the leading coefficients of the n th orthonormal matrix polynomials $(P_n(x; d\beta_k))_n$ and $(P_n(x; d\alpha_k))_n$ associated, respectively, with $d\beta_k$ and $d\alpha_k$, which are related by $d\beta_k(u) = d\alpha_k(u) + M\delta(u - c)$, $c \in \mathbb{R} \setminus \hat{\Gamma}_1$, where M is a positive definite matrix. Assume that equation (21) holds. Then,*

$$\begin{aligned} & \lim_{m \rightarrow \infty} [\gamma_{n_m}(d\beta_{k_m})\gamma_{n_m}(d\alpha_{k_m})^{-1}]^* [\gamma_{n_m}(d\beta_{k_m})\gamma_{n_m}(d\alpha_{k_m})^{-1}] \\ &= I_N + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{A,B}(t)}{c-t} \right). \end{aligned} \quad (22)$$

We proceed as in the proof of ref. [13, Theorem 3.1] using the generalized result given in Lemma 2.4.

LEMMA 2.4 *Assume that (21) holds, we have*

$$\lim_{m \rightarrow \infty} P_{n_m}^{-1}(x; d\alpha_{k_m}) = 0, \quad \text{for } x \in \mathbb{R} \setminus \hat{\Gamma}_1. \quad (23)$$

Proof From equation (7), we have

$$\begin{aligned} P_{n_m}^{-1}(x; d\alpha_{k_m})A_{n_m}^{-1}(d\alpha_{k_m})P_{n_m-1}^{-*}(x; d\alpha_{k_m}) &= P_{n_m}^{-1}(x; d\alpha_{k_m})Q_{n_m}(x; d\alpha_{k_m}) \\ &\quad - (P_{n_m-1}^{-1}(x; d\alpha_{k_m})Q_{n_m-1}(x; d\alpha_{k_m}))^*. \end{aligned}$$

But

$$\begin{aligned} P_{n_m}^{-1}(x; d\alpha_{k_m})Q_{n_m}(x; d\alpha_{k_m}) &= \int P_{n_m}^{-1}(x; d\alpha_{k_m}) \frac{P_{n_m}(x; d\alpha_{k_m}) - P_{n_m}(t; d\alpha_{k_m})}{x-t} d\alpha_{k_m}(t) \\ &= \int d\alpha_{k_m}(t) \frac{P_{n_m}^*(x; d\alpha_{k_m}) - P_{n_m}^*(t; d\alpha_{k_m})}{x-t} P_{n_m}^{-*}(x; d\alpha_{k_m}) \\ &= (P_{n_m}^{-1}(x; d\alpha_{k_m})Q_{n_m}(x; d\alpha_{k_m}))^*. \end{aligned}$$

Using Theorem 2.1, $(P_{n_m}^{-1}(x, d\alpha_{k_m})Q_{n_m}(x, d\alpha_{k_m}))_m$ is convergent and then

$$\lim_{m \rightarrow \infty} P_{n_m}^{-1}(x; d\alpha_{k_m})A_{n_m}^{-1}(d\alpha_{k_m})P_{n_m-1}^{-*}(x; d\alpha_{k_m}) = 0. \quad (24)$$

Let $\varphi(s)$ be an increasing sequence of non-negative integer numbers such that the limit

$$L(x) = \lim_{s \rightarrow \infty} P_{n_{\varphi(s)}}^{-1}(x; d\alpha_{k_{\varphi(s)}})$$

exists or is ∞ ($n_{\varphi(s)}$ is a subsequence of n_m and $k_{\varphi(s)}$ is a subsequence of k_m). Then,

$$\begin{aligned} P_{n_{\varphi(s)}}^{-1}(x; d\alpha_{k_{\varphi(s)}})A_{n_{\varphi(s)}}^{-1}(d\alpha_{k_{\varphi(s)}})P_{n_{\varphi(s)}-1}^{-*}(x; d\alpha_{k_{\varphi(s)}}) &= P_{n_{\varphi(s)}}^{-1}(x; d\alpha_{k_{\varphi(s)}})A_{n_{\varphi(s)}}^{-1}(d\alpha_{k_{\varphi(s)}}) \\ &\quad \times \left(P_{n_{\varphi(s)}}(x; d\alpha_{k_{\varphi(s)}})P_{n_{\varphi(s)}-1}^{-1}(x; d\alpha_{k_{\varphi(s)}}) \right)^* \\ &\quad \times P_{n_{\varphi(s)}}^{-*}(x; d\alpha_{k_{\varphi(s)}}). \end{aligned} \quad (25)$$

But, from ref. [17, Theorem 2.1]

$$\lim_{s \rightarrow \infty} \left(P_{n_{\varphi(s)}}(x; d\alpha_{k_{\varphi(s)}}) P_{n_{\varphi(s)}-1}^{-1}(x; d\alpha_{k_{\varphi(s)}}) \right)^* = \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-*} A^{-*}$$

for $x \in \mathbb{R} \setminus \Gamma_1$. Using equations (24) and (25), we have

$$\begin{aligned} 0 &= L(x) A^{-1} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-*} A^{-*} L^*(x) \\ &= \left(L(x) A^{-1} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \right) \left(\int \frac{dW_{A,B}(t)}{x-t} \right) \left(L(x) A^{-1} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \right)^*. \end{aligned}$$

The sequence of discrete matrices of measures $(\mu_{n_m}(d\alpha_{k_m}))_m$ defined in equation (32) converges weakly to $W_{A,B}$ (see the proof of Step 1), and as $\text{supp}(\mu_{n_m}(d\alpha_{k_m})) = \tilde{\Delta}_m$ we deduce that $\text{supp}(dW_{A,B}) \subset \Gamma_1 \subset \hat{\Gamma}_1$. Then, for $x \in \mathbb{R} \setminus \hat{\Gamma}_1$, $\int (dW_{A,B}(t))/(x-t)$ is positive or negative definite (Remark 1.2). Hence, we deduce that $L(x) = 0$. Finally, $(P_{n_m}^{-1}(x; d\alpha_{k_m}))_m$ has no a subsequence that converges to a number (or ∞) other than 0. Hence, (23) holds. ■

Proof of Theorem 2.3 Using a similar proof given in ref. [13, pp. 13–15], we obtain

$$\begin{aligned} [\gamma_{n_m}(d\beta_{k_m}) \gamma_{n_m}(d\alpha_{k_m})^{-1}]^* [\gamma_{n_m}(d\beta_{k_m}) \gamma_{n_m}(d\alpha_{k_m})^{-1}] &= I_N - [P_{n_m}^{-*}(c; d\alpha_{k_m}) M^{-1} P_{n_m}^{-1}(c; d\alpha_{k_m}) \\ &\quad + P_{n_m}^{-*}(c; d\alpha_{k_m}) K_{n_m+1}(c, c; d\alpha_{k_m}) \\ &\quad \times P_{n_m}^{-1}(c; d\alpha_{k_m})]^{-1} \end{aligned}$$

and

$$\begin{aligned} P_{n_m}^{-*}(c; d\alpha_{k_m}) K_{n_m+1}(c, c; d\alpha_{k_m}) P_{n_m}^{-1}(c; d\alpha_{k_m}) &= -A_{n_m+1} (P_{n_m}(c; d\alpha_{k_m}) P_{n_m+1}^{-1}(c; d\alpha_{k_m}))^{-1} \\ &\quad \times \left\{ \frac{d}{dx} (P_{n_m}(x; d\alpha_{k_m}) P_{n_m+1}^{-1}(x; d\alpha_{k_m})) (c) \right\} \\ &\quad \times (P_{n_m}(c; d\alpha_{k_m}) P_{n_m+1}^{-1}(c; d\alpha_{k_m}))^{-1}. \end{aligned}$$

Using Lemma 2.4, we get

$$\lim_{m \rightarrow \infty} P_{n_m}^{-*}(c; d\alpha_{k_m}) M^{-1} P_{n_m}^{-1}(c; d\alpha_{k_m}) = 0.$$

Using the theorem of Weierstrass concerning the differentiation of a sequence of holomorphic functions, and Theorem 2.1 of ref. [17], we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} P_{n_m}^{-*}(c; d\alpha_{k_m}) K_{n_m+1}(c, c; d\alpha_{k_m}) P_{n_m}^{-1}(c; d\alpha_{k_m}) \\ = - \left(\int \frac{dW_{D,E}(t)}{c-t} \right)^{-1} \left(\int \frac{dW_{D,E}(t)}{c-t} \right)' \left(\int \frac{dW_{D,E}(t)}{c-t} \right)^{-1}. \end{aligned}$$

Hence, we get equation (22). ■

Now, we are ready to establish the main result given in the following.

THEOREM 2.5 *Let $(P_n(x; d\alpha_k))_n$ be a sequence of orthonormal matrix polynomials with respect to a varying matrix of measures $d\alpha_k$ and satisfying equation (12). Assume that equation (21) holds. Then, there exists a sequence of orthonormal matrix polynomials $(P_n(x; d\beta_k))_n$ with respect to a varying matrix of measures defined by*

$$d\beta_k(u) = d\alpha_k(u) + M\delta(u - c), \quad c \in \mathbb{R} \setminus \hat{\Gamma}_1, \quad (26)$$

where M is a positive definite matrix such that for $x \in \mathbb{R} \setminus \{\hat{\Gamma}_1 \cup \{c\}\}$,

$$\begin{aligned} & \lim_{m \rightarrow \infty} P_{n_m-1}(x; d\beta_{k_m}) P_{n_m}^{-1}(x; d\alpha_{k_m}) A_{n_m}^{-1}(d\alpha_{k_m}) \\ &= \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \right. \\ & \quad \left. \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \right], \end{aligned} \quad (27)$$

with $\Lambda(c) \stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} [\gamma_{n_m-1}(d\beta_{k_m}) \gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^*$.

To prove Theorem 2.5, we need to state the following lemma.

LEMMA 2.6 [13] *Let $d\alpha$ and $d\beta$ be two matrices of measures, and M be a positive definite matrix such that $d\beta(u) = d\alpha(u) + M\delta(u - c)$, where c is a real number. Then,*

$$\begin{aligned} P_n(x; d\beta) &= \gamma_n^{-*}(d\beta) \gamma_n^*(d\alpha) \times [P_n(x; d\alpha) \\ & \quad - P_n(c; d\alpha) (I_N + MK_{n+1}(c, c; d\alpha))^{-1} MK_{n+1}^*(c, x; d\alpha)]. \end{aligned} \quad (28)$$

Proof of Theorem 2.5 The decomposition

$$P_{n-1}(x; d\beta_k) P_n^{-1}(x; d\alpha_k) = \sum_{j=1}^p \frac{1}{x - x_{n,k,j}} C_{n,k,j}$$

is always possible with $C_{n,k,j}$ given, as in ref. [7], by

$$C_{n,k,j} = \frac{l_j}{(\det(P_n(x; d\alpha_k)))^{(l_j)}(x_{n,k,j})} P_{n-1}(x_{n,k,j}; d\beta_k) (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)}(x_{n,k,j})),$$

where $x_{n,k,j}$, $j = 1, \dots, p$, are the different zeros of the polynomials $P_n(x; d\alpha_k)$, l_j the multiplicity of $x_{n,k,j}$ and $\text{Adj}(A)$ the classical adjoint matrix uniquely defined by $A \text{Adj}(A) = \text{Adj}(A)A = \det(A)I_N$. Taking into account the Liouville–Ostrogradski formula (7), we have

$$\begin{aligned} C_{n,k,j} A_n^{-1}(d\alpha_k) &= \frac{l_j}{(\det(P_n(x; d\alpha_k)))^{(l_j)}(x_{n,k,j})} \\ & \times P_{n-1}(x_{n,k,j}; d\beta_k) (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)}(x_{n,k,j})) \\ & \times [Q_n(x_{n,k,j}; d\alpha_k) P_{n-1}^*(x_{n,k,j}; d\alpha_k) - P_n(x_{n,k,j}; d\alpha_k) Q_{n-1}^*(x_{n,k,j}; d\alpha_k)]. \end{aligned}$$

But, from ref. [7, Theorem 2.3], we deduce

$$\begin{aligned} (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)}(x_{n,k,j}) P_n(x_{n,k,j}; d\alpha_k) &= P_n(x_{n,k,j}; d\alpha_k) (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)}(x_{n,k,j})) \\ &= 0. \end{aligned} \quad (29)$$

Then,

$$C_{n,k,j} A_n^{-1}(d\alpha_k) = \frac{l_j}{(\det(P_n(x; d\alpha_k)))^{(l_j)} (x_{n,k,j})} P_{n-1}(x_{n,k,j}; d\beta_k) \\ \times (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)} (x_{n,k,j}) Q_n(x_{n,k,j}; d\alpha_k) P_{n-1}^*(x_{n,k,j}; d\alpha_k).$$

From equation (28), we obtain

$$P_{n-1}(x_{n,k,j}; d\beta_k) = \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k) \times [P_{n-1}(x_{n,k,j}; d\alpha_k) \\ - P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MK_n^*(c, x_{n,k,j}, d\alpha_k)]. \quad (30)$$

Then, using equations (4) and (29)

$$C_{n,k,j} A_n^{-1}(d\alpha_k) = \frac{l_j \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k)}{(\det(P_n(x; d\alpha_k)))^{(l_j)} (x_{n,k,j})} [P_{n-1}(x_{n,k,j}; d\alpha_k) \\ \times (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)} (x_{n,k,j}) Q_n(x_{n,k,j}; d\alpha_k) P_{n-1}^*(x_{n,k,j}; d\alpha_k) \\ - P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MK_n^*(c, x_{n,k,j}, d\alpha_k) \\ \times (\text{Adj}(P_n(x; d\alpha_k))^{(l_j-1)} (x_{n,k,j}) Q_n(x_{n,k,j}; d\alpha_k) P_{n-1}^*(x_{n,k,j}; d\alpha_k))] \\ = \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k) [P_{n-1}(x_{n,k,j}; d\alpha_k) \Gamma_{n,k,j} P_{n-1}^*(x_{n,k,j}; d\alpha_k) \\ - \frac{1}{c - x_{n,k,j}} P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} \\ \times MP_n^*(c; d\alpha_k) A_n^*(d\alpha_k) P_{n-1}(x_{n,k,j}; d\alpha_k) \Gamma_{n,k,j} P_{n-1}^*(x_{n,k,j}; d\alpha_k)],$$

where $\Gamma_{n,k,j}$ is as given in equation (15). Hence

$$P_{n-1}(x; d\beta_k) P_n^{-1}(x; d\alpha_k) A_n^{-1}(d\alpha_k) \\ = \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k) \times \left[\sum_{j=1}^p \frac{1}{x - x_{n,k,j}} P_{n-1}(x_{n,k,j}; d\alpha_k) \Gamma_{n,k,j} P_{n-1}^*(x_{n,k,j}; d\alpha_k) \right. \\ \left. - (P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MP_n^*(c; d\alpha_k) A_n^*(d\alpha_k)) \right. \\ \left. \times \sum_{j=1}^p \frac{1}{(x - x_{n,k,j})(c - x_{n,k,j})} P_{n-1}(x_{n,k,j}; d\alpha_k) \Gamma_{n,k,j} P_{n-1}^*(x_{n,k,j}; d\alpha_k) \right]. \quad (31)$$

Let $(n_m)_m$ and $(k_m)_m$ be two increasing sequences of positive integers for which we assume equation (21) holds. To prove equation (27), we proceed in several steps.

Step 1 Under the hypothesis (21), we have

$$\lim_{m \rightarrow \infty} \sum_{j=1}^p \frac{1}{(x - x_{n_m, k_m, j})(c - x_{n_m, k_m, j})} P_{n_m-1}(x_{n_m, k_m, j}; d\alpha_{k_m}) \Gamma_{n_m, k_m, j} P_{n_m-1}^*(x_{n_m, k_m, j}; d\alpha_{k_m}) \\ = \int \frac{dW_{A,B}(t)}{(x-t)(c-t)}$$

for $x \in \mathbb{C} \setminus \Gamma_1$.

Proof of Step 1 We consider the following matrices of measures

$$\mu_n(d\alpha_k)(t) = \sum_{j=1}^p \delta(t - x_{n,k,j}) P_{n-1}(x_{n,k,j}; d\alpha_k) \Gamma_{n,k,j} P_{n-1}^*(x_{n,k,j}; d\alpha_k). \quad (32)$$

Then

$$\begin{aligned} & \sum_{j=1}^p \frac{1}{(x - x_{n_m,k_m,j})(c - x_{n_m,k_m,j})} P_{n_m-1}(x_{n_m,k_m,j}; d\alpha_{k_m}) \Gamma_{n_m,k_m,j} P_{n_m-1}^*(x_{n_m,k_m,j}; d\alpha_{k_m}) \\ &= \int \frac{1}{(x - t)(c - t)} d\mu_{n_m}(d\alpha_{k_m})(t). \end{aligned} \quad (33)$$

To prove Step 1, we will use the so-called method of moments: let μ_m and μ be two matrices of measures such that $\int_{\mathbb{R}} d\mu_m = \int_{\mathbb{R}} d\mu = I_N$, with moment of every order and that μ has compact support. If r_n is a sequence of matrix polynomials with degree n , and $\lim_{m \rightarrow \infty} \int r_k(t) d\mu_m = \int r_k(t) d\mu$ for $k = 0, 1, 2, \dots$, then μ_m converges weakly to μ .

From the quadrature formula [7, Theorem 3.1], we conclude that

$$\begin{aligned} \int_{\mathbb{R}} d\mu_{n_m}(d\alpha_{k_m})(t) &= \sum_{j=1}^p P_{n_m-1}(x_{n_m,k_m,j}; d\alpha_{k_m}) \Gamma_{n_m,k_m,j} P_{n_m-1}^*(x_{n_m,k_m,j}; d\alpha_{k_m}) \\ &= \int_{\mathbb{R}} P_{n_m-1}(t; d\alpha_{k_m}) d\alpha_{k_m}(t) P_{n_m-1}^*(t; d\alpha_{k_m}) \\ &= I_N. \end{aligned} \quad (34)$$

Let us consider the Chebyshev orthonormal matrix polynomials of the second kind $(U_n^{A,B})_n$, which satisfy the recurrence relation (9),

$$\int U_l^{A,B}(t) dW_{A,B}(t) = \begin{cases} I_N & \text{for } l = 0 \\ 0 & \text{for } l \neq 0, \end{cases} \quad (35)$$

where $W_{A,B}$ is the corresponding matrix weight which has a compact support.

Under the hypothesis (21), and by a similar proof given in ref. [17, pp. 8–9], we get

$$\lim_{m \rightarrow \infty} \int U_l^{A,B}(t) d\mu_{n_m}(d\alpha_{k_m})(t) = \begin{cases} I_N & \text{for } l = 0 \\ 0 & \text{for } l \neq 0. \end{cases} \quad (36)$$

Taking into account equations (35) and (36), we deduce that $(\mu_{n_m}(d\alpha_{k_m}))_m$ converges weakly to $W_{A,B}$. Hence,

$$\lim_{m \rightarrow \infty} \int \frac{1}{(x - t)(c - t)} d\mu_{n_m}(d\alpha_{k_m})(t) = \int \frac{1}{(x - t)(c - t)} dW_{A,B}(t) \quad (37)$$

for $x, c \in \mathbb{C} \setminus \Gamma_1$. By equations (33) and (37), we get the result in Step 1. ■

Step 2 Under the hypothesis (21), we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{j=1}^p \frac{1}{x - x_{n_m,k_m,j}} P_{n_m-1}(x_{n_m,k_m,j}; d\alpha_{k_m}) \Gamma_{n_m,k_m,j} P_{n_m-1}^*(x_{n_m,k_m,j}; d\alpha_{k_m}) \\ &= \int \frac{1}{x - t} dW_{A,B}(t) \end{aligned}$$

for $x \in \mathbb{C} \setminus \Gamma_1$.

Proof of Step 2 Similar to the proof given for the Step 1, if we consider the bounded function $1/(x - t)$ instead of $1/((x - t)(c - t))$ in equation (37). ■

Step 3 Under the hypothesis (21), we have

$$\begin{aligned} & \lim_{m \rightarrow \infty} P_{n_m-1}(c; d\alpha_{k_m}) (I_N + MK_{n_m}(c, c; d\alpha_{k_m}))^{-1} MP_{n_m}^*(c; d\alpha_{k_m}) A_{n_m}^*(d\alpha_{k_m}) \\ &= - \left(\int \frac{dW_{A,B}(t)}{c - t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x - t} \right) (c) \right\}^{-1} \\ & \quad \times \left(\int \frac{dW_{A,B}(t)}{c - t} \right) \left(\int \frac{dW_{A,B}(t)}{c - t} \right)^{-*} \end{aligned}$$

for $c \in \mathbb{R} \setminus \hat{\Gamma}_1$.

Proof of Step 3 We observe that

$$\begin{aligned} & P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MP_n^*(c; d\alpha_k) A_n^*(d\alpha_k) \\ &= P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} \\ & \quad \times MP_{n-1}^*(c; d\alpha_k) (P_{n-1}(c; d\alpha_k) P_n^{-1}(c; d\alpha_k) A_n^{-1}(d\alpha_k))^{-*}. \end{aligned} \quad (38)$$

From ref. [17, Theorem 2.1] and under equation (21), we have

$$\lim_{m \rightarrow \infty} P_{n_m-1}(c; d\alpha_{k_m}) P_{n_m}^{-1}(c; d\alpha_{k_m}) A_{n_m}^{-1}(d\alpha_{k_m}) = \int \frac{dW_{A,B}(t)}{c - t}. \quad (39)$$

From equation (28) and using equation (8), we obtain

$$\begin{aligned} & \int P_{n-1}(t; d\beta_k) d\alpha_k(t) P_{n-1}^*(t; d\alpha_k) = \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k) \\ & \quad \left\{ I_N - P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} M \right. \\ & \quad \left. \times \int K_n^*(c, t; d\alpha_k) d\alpha_k(t) P_{n-1}^*(t; d\alpha_k) \right\}. \end{aligned}$$

This means that

$$\begin{aligned} & \gamma_{n-1}(d\beta_k) \gamma_{n-1}^{-1}(d\alpha_k) = \gamma_{n-1}^{-*}(d\beta_k) \gamma_{n-1}^*(d\alpha_k) \\ & \quad \left\{ I_N - P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MP_{n-1}^*(c; d\alpha_k) \right\}, \end{aligned}$$

that is,

$$\begin{aligned} & [\gamma_{n-1}(d\beta_k) \gamma_{n-1}^{-1}(d\alpha_k)]^* [\gamma_{n-1}(d\beta_k) \gamma_{n-1}^{-1}(d\alpha_k)] \\ &= I_N - P_{n-1}(c; d\alpha_k) (I_N + MK_n(c, c; d\alpha_k))^{-1} MP_{n-1}^*(c; d\alpha_k). \end{aligned}$$

Application of Theorem 2.3 yields

$$\begin{aligned} & \lim_{m \rightarrow \infty} P_{n_m-1}(c; d\alpha_{k_m}) (I_N + MK_{n_m}(c, c; d\alpha_{k_m}))^{-1} MP_{n_m-1}^*(c; d\alpha_{k_m}) \\ &= - \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{A,B}(t)}{c-t} \right). \end{aligned} \quad (40)$$

From equation (38)–(40), we get the result in Step 3. ■

Now, using Theorem 2.3, we get

$$\begin{aligned} & \lim_{m \rightarrow \infty} [\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^* [\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}] \\ &= I_N + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \\ &= I_N - \int \frac{dW_{A,B}(t)}{c-t} \left\{ \int \frac{dW_{A,B}(t)}{(c-t)^2} \right\}^{-1} \int \frac{dW_{A,B}(t)}{c-t}, \end{aligned} \quad (41)$$

and the right-hand side of equation (41) is positive definite [13, pp. 17–18]. Let $\Lambda(c)$ be a limit of any subsequence of

$$\left([\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^* \right)_m. \quad (42)$$

To guarantee the uniqueness of the limit $\Lambda(c)$, we can choose $(P_{n_m-1}(x; d\beta_{k_m}))_m$ such that $[\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^*$ are, respectively, lower or upper triangular matrices with positive diagonal elements (we use either the QR or QL factorization of Francis and Kublanovskaja as in ref. [13, pp. 15–16]), and we use the Cholesky factorization of the right-hand side of equation (41), which get us the unique expression of $\Lambda(c)$ with positive diagonal elements. Hence, every subsequence of equation (42) converges to $\Lambda(c)$ and then

$$\lim_{m \rightarrow \infty} [\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^* = \Lambda(c). \quad (43)$$

Now, we are ready to conclude the proof of Theorem 2.5. First, we use equation (37), and we take into account Step 1, Step 2, Step 3 and equation (43). Then, there exists a sequence of orthonormal matrix polynomials $(P_{n_m-1}(x; d\beta_{k_m}))_m$ such that for $x \in \mathbb{R} \setminus \{\hat{\Gamma}_1 \cup \{c\}\}$

$$\begin{aligned} & \lim_{m \rightarrow \infty} P_{n_m-1}(x; d\beta_{k_m}) P_{n_m}^{-1}(x; d\alpha_{k_m}) A_{n_m}^{-1}(d\alpha_{k_m}) \\ &= \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \right. \\ & \quad \left. \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \right]. \end{aligned} \quad \blacksquare$$

Remark 2.7 We select $(P_{n_m-1}(x; d\beta_{k_m}))_m$ orthonormal with respect to $d\beta_{k_m}$ such that $[\gamma_{n_m-1}(d\beta_{k_m})\gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^*$ are lower triangular matrices (or upper triangular matrices) with positive diagonal elements.

Proof See ref. [13, pp. 15–16]. ■

Remark 2.8 Under the hypothesis of Theorem 2.5, we obtain

$$\begin{aligned}
 \lim_{m \rightarrow \infty} P_{n_m-1}(x; d\beta_{k_m}) P_{n_m-1}^{-1}(x; d\alpha_{k_m}) &= \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \right. \\
 &\quad \times \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \\
 &\quad \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \\
 &\quad \left. \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-1} \right], \tag{44}
 \end{aligned}$$

with $\Lambda(c) \stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} [\gamma_{n_m-1}(d\beta_{k_m}) \gamma_{n_m-1}(d\alpha_{k_m})^{-1}]^*$.

Proof We write

$$\begin{aligned}
 P_{n_m-1}(x; d\beta_{k_m}) P_{n_m-1}^{-1}(x; d\alpha_{k_m}) &= [P_{n_m-1}(x; d\beta_{k_m}) P_{n_m}^{-1}(x; d\alpha_{k_m}) A_{n_m}^{-1}(d\alpha_{k_m})] \\
 &\quad \times [P_{n_m-1}(x; d\alpha_{k_m}) P_{n_m}^{-1}(x; d\alpha_{k_m}) A_{n_m}^{-1}(d\alpha_{k_m})]^{-1}
 \end{aligned}$$

and we use Theorem 2.5 and Theorem 2.1 of ref. [17], we obtain equation (44). ■

Example 2.9 We consider

$$A_{n,k} = \begin{pmatrix} a_{n,k} & 0 \\ 0 & a_{n,k} \end{pmatrix} \quad \text{and} \quad B_{n,k} = \begin{pmatrix} b_{n,k} & b_{n,k} \\ b_{n,k} & b_{n,k} \end{pmatrix},$$

where

$$\begin{aligned}
 a_{n,k} &= 2 \sqrt{\frac{n(n + \alpha_k + \beta_k)(n + \alpha_k)(n + \beta_k)}{(2n + \alpha_k + \beta_k - 1)(2n + \alpha_k + \beta_k)^2(2n + \alpha_k + \beta_k + 1)}} \\
 b_{n,k} &= \frac{\beta_k^2 - \alpha_k^2}{(2n + \alpha_k + \beta_k)(2n + \alpha_k + \beta_k + 2)}
 \end{aligned}$$

are the recurrence coefficients of orthonormal varying Jacobi polynomials $p_n^{(\alpha_k, \beta_k)}$, with $\alpha_k, \beta_k > -1$. We take $\alpha_k = (-1/2)^{k+1}$, $\beta_k = (-1)^k/3^k$, $n_m = m$ and $k_m = 2m$. A straightforward computations yields

$$\begin{aligned}
 a_{n_m, k_m} &= 2 \sqrt{\frac{n_m ((-1/2)^{1+k_m} + n_m)}{-1 + (-1/2)^{1+k_m} + (-1/3)^{k_m} + 2n_m}} \\
 &\quad \times \sqrt{\frac{((-1/3)^{k_m} + n_m) ((-1/2)^{1+k_m} + (-1/3)^{k_m} + n_m)}{((-1/2)^{1+k_m} + (-1/3)^{k_m} + 2n_m)^2 (1 + (-1/2)^{1+k_m} + (-1/3)^{k_m} + 2n_m)}}, \\
 b_{n_m, k_m} &= \frac{-((-1/2)^{2+2k_m} + (-1/3)^{2k_m})}{((-1/2)^{1+k_m} + (-1/3)^{k_m} + 2n_m) (2 + (-1/2)^{1+k_m} + (-1/3)^{k_m} + 2n_m)}
 \end{aligned}$$

and

$$\lim_{m \rightarrow \infty} a_{n_m-l, k_m} = 1/2 \quad \text{and} \quad \lim_{m \rightarrow \infty} b_{n_m-l, k_m} = 0, \quad \text{for all } l > 0.$$

Hence, $A \stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} A_{n_m - l, k_m} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ and $B \stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} B_{n_m - l, k_m} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, for all $l > 0$, then computing the terms given in equation (11), we obtain

$$\int \frac{dW_{A,B}}{c-t} = \begin{pmatrix} 2\phi(c) & 0 \\ 0 & 2\phi(c) \end{pmatrix} \quad (45)$$

and

$$\left(\frac{d}{dx} \left(\int \frac{dW_{A,B}}{x-t} \right) (c) \right)^{-1} = 1/2 \begin{pmatrix} -\frac{\sqrt{-1+c^2}}{\phi(c)} & 0 \\ 0 & -\frac{\sqrt{-1+c^2}}{\phi(c)} \end{pmatrix},$$

with $\phi(c) = c - \sqrt{c^2 - 1}$ and where $\text{supp}(dW_{A,B}) = [-1, 1]$.

Taking into account Theorem 2.3 and equation (43) and using the Cholesky decomposition, we obtain

$$\Lambda(c) = \begin{pmatrix} \sqrt{-1+2c\phi(c)} & 0 \\ 0 & \sqrt{-1+2c\phi(c)} \end{pmatrix}.$$

We can compute equation (27) in terms of equation (45), and then

$$\lim_{n \rightarrow \infty} P_{n_m-1}(x; d\beta_{k_m}) P_{n_m}^{-1}(x; d\alpha_{k_m}) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

where

$$u(c, x) = \frac{\sqrt{-1+2c\phi(c)} (c-x-\phi(c)+\phi(x)+c(1-(2c-\phi(c))\phi(x)))}{c-x}. \quad (46)$$

3. Relative asymptotic for orthogonal matrix polynomials with unbounded recurrence coefficients

In a previous paper, we have considered a perturbation of $d\alpha$ by the addition of a Dirac matrix measure $d\beta(u) = d\alpha(u) + M\delta(u-c)$, where M is a positive definite matrix and $c \in \mathbb{R} \setminus \hat{\Gamma}$; when the three-term recurrence coefficients in equation (2) are convergent (bounded), we have established the following result.

THEOREM 3.1 [13] *Let $(P_n(x, d\alpha) = \gamma_n(d\alpha)x^n + \text{lower degree terms})_n$ be a sequence of orthonormal matrix polynomials with respect to the matrix of measures $d\alpha$. Assume that the matrix coefficients in equation (2) satisfy*

$$\lim_{n \rightarrow \infty} A_n(d\alpha) = A, \quad \lim_{n \rightarrow \infty} B_n(d\alpha) = B \quad \text{and } A \text{ is nonsingular.} \quad (47)$$

Then, there exists a sequence of orthonormal matrix polynomials $(P_n(x, d\beta))_n$ with respect to $d\beta$ such that for $x \in \mathbb{R} \setminus \{\hat{\Gamma} \cup \{c\}\}$

$$\begin{aligned} \lim_{n \rightarrow \infty} P_n(x; d\beta) P_n^{-1}(x; d\alpha) &= \Lambda(c)^{-1} + \frac{1}{c-x} [\Lambda(c)^* - \Lambda(c)^{-1}] \\ &\quad \times \left[\left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} - \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \right], \end{aligned} \quad (48)$$

where $\Lambda(c) = \lim_{n \rightarrow \infty} [\gamma_n(d\beta) \gamma_n^{-1}(d\alpha)]^*$.

Remark 3.2 Theorem 3.1 can be deduced from Theorem 2.5.

In fact, let us consider $(P_n(x; d\alpha_k))_n$ with $d\alpha = d\alpha_1 = d\alpha_2 = \dots$

From equation (47), if we take $n_m = n$, we get equation (21) (all subsequences of a convergent sequence are convergent). Taking into account that

$$P_{n-1}(x; d\beta)P_{n-1}^{-1}(x; d\alpha) = P_{n-1}(x; d\beta)P_n^{-1}(x; d\alpha)A_n^{-1}(d\alpha) \\ \times [P_{n-1}(x; d\alpha)P_n^{-1}(x; d\alpha)A_n^{-1}(d\alpha)]^{-1}$$

and, using Theorem 2.5, equation (10) and Theorem 2.3, there exists $(P_n(x; d\beta))_n$ such that

$$\lim_{n \rightarrow \infty} P_n(x; d\beta)P_n^{-1}(x; d\alpha) = \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \right. \\ \times \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \\ \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \left. \right] \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \\ = \Lambda(c)^{-1} + [(\Lambda(c)^* - \Lambda(c)^{-1}) \\ \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1}] ,$$

for $x \in \mathbb{R} \setminus \{\hat{\Gamma}_1 \cup \{c\}\}$. As Γ_1 coincide with Γ and taking into account (47), then Γ is bounded (see [ref. 16, Lemma 2.1]), then $\hat{\Gamma}_1 = \hat{\Gamma}$. But $1/(x-t)(c-t) = 1/(c-x) \times (1/(x-t) - 1/(c-t))$, then

$$\left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \\ = \frac{1}{c-x} \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left\{ \int \frac{dW_{A,B}(t)}{x-t} - \int \frac{dW_{A,B}(t)}{c-t} \right\} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \\ = \frac{1}{c-x} \left[\left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} - \left(\int \frac{dW_{A,B}(t)}{x-t} \right)^{-1} \right]. \quad (49)$$

Hence, we get equation (48).

Now, we assume that the matrix recurrence coefficients in equation (2) are diverging in a particular way: we will suppose that there exists a sequence of a positive definite $(C_n)_n$ matrices $(C_n)_n$ such that

$$\lim_{n \rightarrow \infty} C_n^{-1/2} A_n(d\alpha) C_n^{-1/2} = A, \quad \lim_{n \rightarrow \infty} C_n^{-1/2} B_n(d\alpha) C_n^{-1/2} = B, \quad \lim_{n \rightarrow \infty} C_n^{-1/2} C_{n-1}^{1/2} = I_N. \quad (50)$$

When unbounded coefficients are considered in the scalar case, the relative asymptotic behavior is then obtained for the scaled polynomials $p_n(c_n x)$. In the matrix case, we define the sequence of the scaled matrix polynomials $P_n(C_k; x)$ as in ref. [17]. We consider a matrix polynomial P of a matrix variable T .

$$P(T) \stackrel{\text{def}}{=} \sum_{finite} \alpha_{n_1} T^{n_1} \alpha_{n_2} T^{n_2} \dots \alpha_{n_k} T^{n_k} \alpha_{n_{k+1}},$$

with $T = [t_{i,j}]$ is the matrix as a variable and the α 's are matrices of size $N \times N$. The matrix polynomial $P(x)$ can be obtained from many different matrix polynomials P of matrix variable

just by substituting T by xI_N in $P(T)$ ($P(x) \stackrel{\text{def}}{=} P(xI_N)$). We use the recurrence coefficients $(A_n)_n$ and $(B_n)_n$, and we consider a sequence of matrix polynomials of one matrix variable by

$$TP_n(T) = A_{n+1}P_{n+1}(T) + B_nP_n(T) + A_n^*P_{n-1}(T), \quad (51)$$

with initial conditions $P_{-1}(T) = 0$ and $P_0(T) = P_0$. It is clear that $P_n(xI_N) = P_n(x)$. Then, we define $P_n(C; x) \stackrel{\text{def}}{=} P_n(Cx)$. The scaled matrix polynomials $(P_n(C_k; x))_n$ are orthonormal with respect to some varying matrix of measures $d\alpha_k$ (for $k = 1, 2, \dots$).

Now, we get the relative asymptotic for matrix orthonormal polynomials with unbounded matrix recurrence coefficients. That is, a sequence of matrix polynomials satisfying a relation (12) under the condition (50).

Let $\tilde{\Delta}_n$ be the set of zeros of the scaled matrix polynomials $(P_n(C_n; x))_n$ associated to a varying matrix of measures $d\alpha_n$, and let $\hat{\Gamma}_1$ be the smallest connected set which contains $\Gamma_1 \stackrel{\text{def}}{=} \bigcap_{m>0} \overline{\bigcup_{n \geq m} \tilde{\Delta}_n}$.

THEOREM 3.3 *Let $(P_n)_n$ be a sequence of orthonormal matrix polynomials which satisfies equation (2). We suppose that there exists a sequence of a positive definite matrices $(C_n)_n$ such that equation (50) holds with A non-singular and B hermitian. Then, there exists a sequence of orthonormal matrix polynomials $(P_{n-1}(x; d\beta_n))_n$ with respect to $d\beta_n(u) \stackrel{\text{def}}{=} d\alpha_n(u) + M\delta(u - c)$, $c \in \mathbb{R} \setminus \hat{\Gamma}_1$, where M is a positive definite matrix, such that*

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_{n-1}(x; d\beta_n) P_n^{-1}(C_n; x) C_n^{-1/2} \\ &= \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right) (c) \right\}^{-1} \right. \\ & \quad \times \left. \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \right] A, \end{aligned} \quad (52)$$

for $x \in \mathbb{R} \setminus \{\hat{\Gamma}_1 \cup \{c\}\}$.

Proof Let $(C_n)_n$ be a sequence of $N \times N$ positive definite matrices for which equation (50) holds. From equation (51), we have

$$C_k x P_n(C_k; x) = A_{n+1} P_{n+1}(C_k; x) + B_n P_n(C_k; x) + A_n^* P_{n-1}(C_k; x) \quad n \geq 0.$$

Then,

$$\begin{aligned} x C_k^{1/2} P_n(C_k; x) &= C_k^{-1/2} A_{n+1} C_k^{-1/2} C_k^{1/2} P_{n+1}(C_k; x) + C_k^{-1/2} B_n C_k^{-1/2} C_k^{1/2} P_n(C_k; x) \\ & \quad + C_k^{-1/2} A_n^* C_k^{-1/2} C_k^{1/2} P_{n-1}(C_k; x), \quad n \geq 0. \end{aligned}$$

The matrix polynomials $(P_n(C_k; x))_n$ are orthonormal with respect to some varying matrix of measures, which we denote by $d\alpha_k$.

Now, we are ready to use Theorem 2.5 with $P_n(x; d\alpha_k) = C_k^{1/2} P_n(C_k; x)$, $A_n(d\alpha_k) = C_k^{-1/2} A_n C_k^{-1/2}$ and $B_n(d\alpha_k) = C_k^{-1/2} B_n C_k^{-1/2}$. For $n_m = k_m = n$ and under the hypothesis (50), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} A_{n-l}(d\alpha_n) &= \lim_{n \rightarrow \infty} C_n^{-1/2} A_{n-l} C_n^{-1/2} = A \\ \lim_{n \rightarrow \infty} B_{n-l}(d\alpha_n) &= B \quad \text{for all } l \geq 0. \end{aligned}$$

Hence,

$$\begin{aligned} &\lim_{n \rightarrow \infty} P_{n-1}(x; d\beta_n) P_n^{-1}(C_n; x) C_n^{-1/2} \\ &= \lim_{n \rightarrow \infty} P_{n-1}(x; d\beta_n) P_n^{-1}(x; d\alpha_n) A_n^{-1}(d\alpha_n) \times \lim_{n \rightarrow \infty} C_n^{-1/2} A_n C_n^{-1/2} \\ &= \Lambda(c)^{-1} \left[\left(\int \frac{dW_{A,B}(t)}{x-t} \right) + \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left\{ \frac{d}{dx} \left(\int \frac{dW_{A,B}(t)}{x-t} \right)_{(c)} \right\}^{-1} \right. \\ &\quad \left. \times \left(\int \frac{dW_{A,B}(t)}{c-t} \right) \left(\int \frac{dW_{A,B}(t)}{c-t} \right)^{-*} \left(\int \frac{dW_{A,B}(t)}{(x-t)(c-t)} \right) \right] A, \end{aligned}$$

for $x \in \mathbb{R} \setminus \{\hat{\Gamma}_1 \cup \{c\}\}$. ■

Example 3.4 We consider (P_n) a sequence of orthonormal matrix polynomials which satisfies equation (2) with

$$A_n = \frac{1}{2} \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \quad \text{and} \quad B_n = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n} \end{pmatrix}.$$

We take $C_n = nI_2$, and using equation (51), we define the scaled polynomials $(P_n(C_n; x))_n$. The sequence $(C_n)_n$ satisfies equation (50) with $A = 1/2 I_2$ and $B = 0$.

We compute the terms given in equation (52) as in example 1. Then, there exists $(P_n(x; d\beta_k))_n$ such that

$$\lim_{n \rightarrow \infty} P_{n-1}(x; d\beta_n) P_n^{-1}(C_n; x) C_n^{-1/2} = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix},$$

where u is given in equation (46).

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Euclidean motion group representations and the singular value decomposition of the Radon transform

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The matrix elements of the unitary irreducible representations of the Euclidean motion group are closely related to the Bessel and Gegenbauer functions. These special functions also arise in the singular functions of the singular value decomposition (SVD) of the Radon transform. In this paper, our objective is to study the Radon transform using the harmonic analysis over the Euclidean motion group and explain the origin of the special functions present in the SVD of the Radon transform from the perspective of group representation theory. Starting with a convolution representation of the Radon transform over the Euclidean motion group, we derive a method of inversion for the Radon transform using the harmonic analysis over the Euclidean motion group. We show that this inversion formula leads to an alternative derivation of the SVD of the Radon transform. This derivation reveals the origin of the special functions present in the SVD of the Radon transform. The derivation of the SVD of the Radon transform is a special case of a general result developed in this paper. This result shows that an integral transform with a convolution kernel is separable if the matrix elements of the irreducible unitary representations of the underlying group are separable.

Keywords: Radon transform; Singular value decomposition; Special functions; Euclidean motion group; Harmonic analysis

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1. Introduction

Since the seminal work of Radon in 1917 [1], the Radon transform has become a major area of research in both pure and applied mathematics (see [2–13] and references therein). The importance of the Radon transform for today's imaging technologies is another motivation for investigating the properties of the Radon transform [14–19].

In this paper, we present a new derivation of the singular value decomposition (SVD) of the Radon transform using the harmonic analysis over the Euclidean motion group, $M(N)$. The SVD of the Radon transform has been studied in ref. [20–22]. However, the origin of the specific special functions that arise in the SVD of the Radon transform was not discussed. Our derivation explains the presence of the special functions in the SVD of the Radon transform by

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using the relationship between the irreducible unitary representations of the Euclidean motion group and the special functions.

It is well known that the Radon transform exhibits symmetries with respect to the Euclidean motion group [7, 10]. These symmetries are represented by the convolution representation of the Radon transform. Starting with a convolution representation of the Radon transform over the Euclidean motion group, we diagonalize the Radon transform in the Euclidean motion group Fourier ($M(N)$ -Fourier) domain and obtain the spherical harmonic decomposition of the projection slice theorem. We show that an integral transform with a convolution kernel is separable if the matrix elements of the unitary representations of the underlying group are separable. We next derive the singular functions of the inverse Radon transform, which leads to the SVD of the Radon transform. It is the separability of the matrix elements of the irreducible unitary representations of the Euclidean motion group that gives rise to the SVD of the Radon transform and the special functions present in this decomposition.

The rest of the paper is organized as follows: Section 2 provides a general overview of harmonic analysis over groups. Section 3 introduces harmonic analysis over the Euclidean motion group and related special functions. Section 4 introduces the Radon transform and spherical harmonic decomposition of the projection slice theorem by using Fourier analysis over $M(N)$. Section 5 presents the main results of the paper. Finally, section 6 concludes our discussion. Appendices A and B provide the computation of the $M(N)$ -Fourier coefficients of the projections and the convolution kernel of the Radon transform.

2. Harmonic analysis over groups

Harmonic analysis over groups is directly related to the decomposition of unitary representations of a group as a direct sum of its irreducible unitary representations. This decomposition enables spectral or Fourier decomposition of functions defined over the group. Fourier transform over groups has the important property of mapping convolution into multiplication in the Fourier transform domain.

Representation of a group G over a vector space V is a homomorphism $T: g \rightarrow T(g)$ of G into the linear transformations of V , $GL(V)$. It follows from this definition that,

$$\begin{aligned} T(g_1)T(g_2) &= T(g_1g_2), \quad T(g)^{-1} = T(g^{-1}), \\ T(e) &= I, \quad g_1, g_2 \in G, \end{aligned}$$

where $e \in G$ is the identity element and I is the identity transformation in $GL(V)$.

Given an orthonormal basis $\{\mathbf{e}_i\}$ of V , for each $T(g)$, we can associate the matrix $(t_{ij}(g))$, $1 \leq i, j \leq n$ given by

$$T(g)\mathbf{e}_j = \sum_{i=1}^n t_{ij}(g)\mathbf{e}_i,$$

where n is the dimension of the vector space V . The matrix $(t_{ij}(g))$ is called ‘matrix representation’ for the representation $T(g)$ with respect to the basis $\{\mathbf{e}_i\}$. The n^2 continuous functionals $\{t_{ij}(g)\}$ also defines the n -dimensional representation of the group G . If V is an infinite-dimensional vector space, then $(t_{ij}(g))$ is an infinite matrix with elements given by

$$t_{ij}(g) = (\mathbf{e}_i, T(g)\mathbf{e}_j), \quad 1 \leq i, j < \infty.$$

By the homomorphism property of $T(g)$, the matrices $(t_{ij}(g))$, $g \in G$ satisfy the usual matrix multiplication rule, that is

$$t_{ij}(g_1 g_2) = \sum_k t_{ik}(g_1) t_{kj}(g_2).$$

If $T(g)$ and $Q(g)$ are two representations of a group G over the spaces V and W , respectively, then $T(g)$ and $Q(g)$ are said to be equivalent if there is an invertible linear mapping between V and W such that

$$Q(g) = AT(g)A^{-1}.$$

Equivalent representations have the same matrix representations, that is

$$Q(g)Ae_k = AT(g)Ae_k = A \sum_j t_{jk}(g) e_j = \sum_j t_{jk}(g) Ae_j$$

and form an equivalence class. Therefore, it is sufficient to find one representation $T(g)$ in each equivalence class.

A representation $T(g)$ of a group G in an inner-product space V is called unitary if it preserves the inner product over V , that is

$$(T(g)x, T(g)y) = (x, y).$$

A measure $d(g)$ on the group G is called left-invariant if for any measurable subset H of G , and for any $g \in G$,

$$\int_H d(h) = \int_{gH} d(gh),$$

and right-invariant if

$$\int_H d(h) = \int_{Hg} d(hg).$$

A measure $d(g)$ on the group G is called invariant if it is both left- and right-invariant. A group having an invariant measure is called a unimodular group. Otherwise it is called a non-unimodular group. For any given representation $T(g)$ of the group G on the inner-product space V , one can define a new inner product, $(\cdot, \cdot)_M$, on V

$$(x, y)_M = \int_G (T(g)x, T(g)y) d(g),$$

such that $T(g)$ becomes a unitary representation over V with respect to this new inner product $(\cdot, \cdot)_M$, where $d(g)$ is the invariant measure over G . Therefore, without loss of generality, given any representation, we may assume that it is unitary, and denote it by $U(g)$.

A subspace $V^{(0)}$ of V is called invariant if for any $x \in V^{(0)}$, $U(g)x \in V^{(0)}$, for all $g \in G$. The null-subspace and the whole space V are the trivial invariant subspaces. A representation is called irreducible if the only invariant subspaces are the trivial ones. Otherwise it is called reducible. Decomposing V as a direct sum of mutually orthogonal invariant subspaces $\{V^{(\lambda)}\}$

such that $U(g)$ is irreducible on each $V^{(\lambda)}$,

$$V = \sum_{\lambda} \oplus V^{(\lambda)}$$

enables decomposition of $U(g)$ as an orthogonal direct sum

$$U(g) = \sum_{\lambda} \oplus U^{(\lambda)}(g),$$

where $U^{(\lambda)}(g)$ is the restriction of $U(g)$ to the subspace $V^{(\lambda)}$. A representation is called completely reducible if it is the direct sum of irreducible representations.

Let G be a unimodular, separable, locally compact group of Type I [23, 24]. Then, the Fourier transform of $f \in L^2(G)$ is defined as,

$$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_G d(g) f(g) U^{(\lambda)}(g^{-1})$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}(\hat{f})(g) = f(g) = \int_{\hat{G}} \text{Tr}[\hat{f}(\lambda) U^{(\lambda)}(g)] d(\lambda),$$

where $\hat{G}$ is the collection of all $\{\lambda\}$ values and is called the dual of G [25]. Using the matrix representations, the matrix elements of the Fourier transform and the corresponding inverse Fourier transform are given by

$$\begin{aligned} \hat{f}_{ij}(\lambda) &= \int_G f(g) u_{ij}^{(\lambda)}(g^{-1}) d(g), \\ f(g) &= \int_{\hat{G}} \sum_{i,j} \hat{f}_{ij}(\lambda) u_{ji}^{(\lambda)}(g) d(\lambda). \end{aligned}$$

In the next section, we will look into the case where G is the group of motions of the N -dimensional Euclidean space.

3. Harmonic analysis over the Euclidean motion group

In this section, we will provide a review of the irreducible representations and the Fourier transform of the Euclidean motion group and some related properties. A more detailed discussion on these topics can be found in [25, 26].

3.1 Euclidean motion group

The isometries of $\mathbb{R}^N$ that preserve orientation form a group called the ‘Euclidean motion group’ or motion group of N dimensions, denoted by $M(N)$. The elements of $M(N)$ are formed by tuples $(R_{\theta}, \mathbf{r})$, where $\mathbf{r} \in \mathbb{R}^N$ is the translation component, and $R_{\theta} \in SO(N)$ is the rotation component parameterized by θ . For $N = 2$, θ can be parameterized as an element of the unit circle. For $N = 3$, θ can be parameterized by, for example, axis-angle description of the rotation, Euler angles, Caley–Klein parameters or any other parameterization for three-dimensional rotation.

The group operation of $M(N)$ is given by

$$(R_\theta, \mathbf{r})(R_\phi, \mathbf{x}) = (R_\theta R_\phi, R_\theta \mathbf{x} + \mathbf{r}). \quad (1)$$

This defines $M(N)$ as the semi-direct product of the additive group $\mathbb{R}^N$ and the rotation group $SO(N)$. The columns of $R_\theta \in SO(N)$ form an orthonormal basis for $\mathbb{R}^N$, meaning that each column is an element of $(N - 1)$ -dimensional sphere, S^{N-1} . Consequently, rows of R_θ also form an orthonormal basis.

The identity element of the group is given by $(I, \mathbf{0})$, I being the identity rotation, and the inverse element of $g = (R_\theta, \mathbf{r})$ is given by $g^{-1} = (R_\theta^{-1}, -R_\theta^{-1}\mathbf{r})$.

Alternatively, the elements of $M(N)$, can be represented as $(N + 1) \times (N + 1)$ dimensional matrices of the form

$$A(g) = \begin{pmatrix} R_\theta & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix}. \quad (2)$$

Then, the group operation becomes the usual matrix multiplication. The identity element is the identity matrix and the inverse of each element can be obtained by matrix inversion.

3.2 Irreducible, unitary representations of $M(N)$

Irreducible, unitary representations $U^{(\lambda)}(g)$, $\lambda > 0$, of $M(N)$ on $L^2(S^{N-1})$ is given by

$$(U^{(\lambda)}(g)F)(\mathbf{s}) = e^{-i\lambda(\mathbf{r} \cdot \mathbf{s})} F(R_\theta^{-1}\mathbf{s}), \quad F \in L^2(S^{N-1}), \quad (3)$$

where $g = (R_\theta, \mathbf{r})$ is an element of $M(N)$, $\mathbf{s}$ is a point on the unit sphere S^{N-1} , and $(\cdot)$ is the standard inner product over $\mathbb{R}^N$ [26].

Because the spherical harmonics form an orthonormal basis for $L^2(S^{N-1})$, matrix elements for the unitary representation $U^{(\lambda)}(g)$ of $M(N)$ is expressed by

$$u_{mn}^{(\lambda)}(g) = (S_m, U^{(\lambda)}(g)S_n) = \int_{S^{N-1}} \overline{S_m(\mathbf{s})} e^{-i\lambda(\mathbf{r} \cdot \mathbf{s})} S_n(R_\theta^{-1}\mathbf{s}) d(\mathbf{s}), \quad (4)$$

where m and n are multi-indices for the spherical harmonics, with the first element denoting the degree of the spherical harmonics, and $d(\mathbf{s})$ is the invariant normalized measure on S^{N-1} .

The matrix elements of $U^{(\lambda)}(g)$ satisfy the following properties:

1. Adjoint property:

$$(u_{mn}^{(\lambda)}(g^{-1})) = \overline{(u_{nm}^{(\lambda)}(g))}. \quad (5)$$

2. Homomorphism property:

$$u_{mn}^{(\lambda)}(g_1 g_2) = \sum_k u_{mk}^{(\lambda)}(g_1) u_{kn}^{(\lambda)}(g_2). \quad (6)$$

3. Orthogonality property:

$$\int_G u_{mk}^{(\lambda)}(g) \overline{u_{ls}^{(\mu)}(g)} d(g) = \delta_{\mu\lambda} \delta_{ml} \delta_{ks}, \quad (7)$$

and $\{u_{mn}^{(\lambda)}(g)\}$ form a complete orthonormal system in $L^2(M(N), d(g))$.

3.3 Associated spherical and zonal spherical functions

Let $\mathfrak{S}^l$ denote the space of spherical harmonics of degree l over S^{N-1} . $\{\mathfrak{S}^l\}_{l=0}^\infty$ are orthogonal and span $L^2(S^{N-1})$. A basis for $\mathfrak{S}^l$ is given by

$$\Xi_m(s) = A_m^l \prod_{j=0}^{N-3} r_{N-j}^{m_j - m_{j+1}} C_{m_j - m_{j+1}}^{(N-j-2)/2 + m_{j+1}} \left(\frac{s_{N-j}}{r_{N-j}} \right) (s_1 + i s_2)^{m_{N-2}},$$

where $C_n^v(r)$ is the Gegenbauer polynomial with associated weight function given by $(1-r^2)^{v-1/2}$ for $v > 1/2$, $m = (m_0, m_1, \dots, \pm m_{N-2})$ such that $l = m_0 \geq m_1 \geq \dots \geq m_{N-2}$, $r_{N-j} = (\sum_{k=1}^{N-j} s_k^2)^{1/2}$, and A_m^l is the normalizing factor such that

$$\int_{S^{N-1}} |\Xi_m(s)|^2 d(s) = 1.$$

If $l = 0$, $m = (0, \dots, 0) = 0$. Then, $\Xi_0(s)$ is equal to 1, and invariant under the rotations. For $h_1, h_2 \in SO(n-1) \times 0$ and $g = (R_\theta, \mathbf{r}) \in M(N)$, the functions $u_{m0}^{(\lambda)}(g)$ are invariant under right rotations,

$$u_{m0}^{(\lambda)}(gh_2) = u_{m0}^{(\lambda)}(g) = u_{m0}^{(\lambda)}((I, \mathbf{r})),$$

and the functions $u_{00}^{(\lambda)}(g)$ are invariant under left and right rotations,

$$u_{00}^{(\lambda)}(h_1gh_2) = u_{00}^{(\lambda)}(g) = u_{00}^{(\lambda)}((I, |\mathbf{r}|e_n)).$$

The matrix elements $\{u_{m0}^{(\lambda)}(g)\}_m$ depend only on the translation component of g and are called the associated spherical functions of the representation $U^{(\lambda)}(g)$. The matrix element $u_{00}^{(\lambda)}(g)$ depends only on the radius of the translation component and is called the zonal spherical function of the representation $U^{(\lambda)}(g)$.

3.3.1 Zonal spherical functions. The zonal spherical functions are given by

$$\begin{aligned} u_{00}^{(\lambda)}(g) &= (\Xi_0, U^{(\lambda)}(g)\Xi_0) = \int_{S^{N-1}} e^{-i\lambda|\mathbf{r}|s_N} d(s) \\ &= \frac{\Gamma(N/2)}{\sqrt{\pi} \Gamma((N-1)/2)} \int_0^\pi e^{-i\lambda|\mathbf{r}|\cos(\varphi)} \sin^{N-2}(\varphi) d\varphi. \end{aligned} \quad (8)$$

This integral representation can be rewritten in terms of Bessel functions by

$$u_{00}^{(\lambda)}(g) = \Gamma\left(\frac{N}{2}\right) \frac{J_{(N-2)/2}(\lambda|\mathbf{r}|)}{(\lambda|\mathbf{r}|/2)^{(N-2)/2}}. \quad (9)$$

Equating (8) and (9), one obtains the identity

$$\int_{-1}^1 e^{ixt} (1-t^2)^{(n-1)/2} dt = \frac{\sqrt{\pi} \Gamma((n+1)/2) J_{n/2}(x)}{(x/2)^{n/2}}. \quad (10)$$

We will use identity (10) in the derivation of the identity (16), which is a key property towards the derivation of the SVD of the Radon transform.

3.3.2 Associated spherical functions. The associated spherical functions are given by

$$u_{m0}^{(\lambda)}(g) = (\Xi_m, U^{(\lambda)}(g)\Xi_0) = \int_{S^{N-1}} e^{-i\lambda(r \cdot s)} \overline{\Xi_m(s)} d(s). \quad (11)$$

For $r = r e_n$, $r \in \mathbb{R}$, equation (11) is non-zero if and only if $m = (l, 0, \dots, 0)$ and takes the form

$$u_{m0}^{(\lambda)}(g) = \frac{\Gamma(N/2)}{\sqrt{\pi} \Gamma((N-1)/2)} \sqrt{\frac{l! \Gamma(N-2)(2l+N-2)}{\Gamma(N+l-2)(N-2)}} \\ \times \int_{-1}^1 e^{-i\lambda r x} C_l^{(N-2)/2}(x) (1-x^2)^{(n-3)/2} dx \quad (12)$$

$$= \frac{\Gamma(N/2)}{2^l \sqrt{\pi} \Gamma(l + (N-1)/2)} \sqrt{\frac{\Gamma(N+l-2)(2l+N-2)}{l! \Gamma(N-1)}} \\ \times (-i\lambda r)^l \int_{-1}^1 e^{-i\lambda r x} (1-x^2)^{l+(n-3)/2} dx. \quad (13)$$

Using the equality (10), (13) can be written as

$$u_{m0}^{(\lambda)}(g) = i^{-l} \Gamma\left(\frac{N}{2}\right) \sqrt{\frac{\Gamma(N+l-2)(2l+N-2)}{l! \Gamma(N-1)}} \frac{J_{l+(N-2)/2}(\lambda r)}{(\lambda r/2)^{(n-2)/2}}. \quad (14)$$

Equating (12) and (14), one obtains the identity

$$\int_{-1}^1 e^{i\lambda r} C_l^{(N-2)/2}(r) (1-r^2)^{(N-3)/2} dr = \frac{i^l \sqrt{\pi} \Gamma((N-1)/2) \Gamma(N+l-2)}{l! \Gamma(N-2)} \frac{J_{l+(N-2)/2}(\lambda)}{(\lambda/2)^{(N-2)/2}}. \quad (15)$$

By the orthogonality relationship of the Gegenbauer polynomials,

$$e^{i\lambda r} = \Gamma\left(\frac{N-2}{2}\right) \sum_{l=0}^{\infty} i^l \left(l + \frac{N-2}{2}\right) \frac{J_{l+(N-2)/2}(\lambda)}{(\lambda/2)^{(N-2)/2}} C_l^{(N-2)/2}(r).$$

More generally, it can be verified that (see [34] 9.3.8 equation (8))

$$e^{i\lambda r} = \Gamma(\nu)(\lambda/2)^{-\nu} \sum_{l=0}^{\infty} i^l (l+\nu) J_{l+\nu}(\lambda) C_l^{\nu}(r), \quad -1 < x < 1, \quad \nu > -1/2. \quad (16)$$

The associated spherical functions and the zonal spherical functions of the irreducible unitary representation of the group $M(N)$ are closely related to the Gegenbauer polynomials and Bessel functions. By the orthogonality of Gegenbauer polynomials and Bessel functions, equation (16) provides a decomposition of $e^{i\lambda r}$. This property plays an important role in the derivation of the SVD of the Radon transform.

Further properties involving Bessel function and Gegenbauer polynomials can be derived through the properties of the unitary representations, which is beyond the scope of our study. For an extensive study on the group representations and their relationship with the special functions, we refer the reader to [26–29].

3.4 Fourier transform over $M(N)$

Let $f \in L^2(M(N))$, then its Fourier transform is defined as

$$\mathcal{F}(f)(\lambda) = \hat{f}(\lambda) = \int_{M(N)} d(g) f(g) U^{(\lambda)}(g^{-1}), \quad \lambda > 0 \quad (17)$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}(\hat{f})(g) = f(g) = \int_0^\infty \text{Tr}[\hat{f}(\lambda) U^{(\lambda)}(g)] \lambda^{N-1} d\lambda. \quad (18)$$

Using the matrix elements of the unitary representation, the matrix elements of the Fourier and the inverse Fourier transforms over $M(N)$ can be expressed as

$$\hat{f}_{mn}(\lambda) = \int_{M(N)} f(g) u_{mn}^{(\lambda)}(g^{-1}) d(g), \quad (19)$$

$$f(g) = \int_0^\infty \sum_{m,n} \hat{f}_{mn}(\lambda) u_{nm}^{(\lambda)}(g) \lambda^{N-1} d\lambda. \quad (20)$$

3.4.1 Properties of $M(N)$ -Fourier transform. Fourier transform over $M(N)$ satisfies the following properties:

1. Plancherel equality:

$$\int_{M(N)} f_1(g) \overline{f_2(g)} d(g) = \int_0^\infty \text{Tr}[\hat{f}_2^\dagger(\lambda) \hat{f}_1(\lambda)] \lambda^{N-1} d\lambda. \quad (21)$$

For $f_1 = f_2 = f$,

$$\int_{M(N)} |f(g)|^2 d(g) = \int_0^\infty \|\hat{f}(\lambda)\|_2^2 \lambda^{N-1} d\lambda, \quad (22)$$

This property is analogous to the Parsevals equality of the standard Fourier transform.

2. Convolution property:

$$\mathcal{F}(f_1 * f_2)(\lambda) = \mathcal{F}(f_2)(\lambda) \mathcal{F}(f_1)(\lambda). \quad (23)$$

Equivalently,

$$\mathcal{F}(f_1 * f_2)_{mn}(\lambda) = \sum_q \mathcal{F}(f_2)_{mq}(\lambda) \mathcal{F}(f_1)_{qn}(\lambda). \quad (24)$$

Note that since $M(N)$ is not commutative, $\mathcal{F}(f_1)(\lambda) \mathcal{F}(f_2)(\lambda)$ is not necessarily equal to $\mathcal{F}(f_2)(\lambda) \mathcal{F}(f_1)(\lambda)$.

3. Adjoint property:

$$\mathcal{F}(f^*)(\lambda) = [\mathcal{F}(f)(\lambda)]^\dagger, \quad (25)$$

where $f^*(g) = \overline{f(g^{-1})}$ and $U^\dagger$ denotes the adjoint of the operator U . Equivalently,

$$\widehat{f^*}_{mn}(\lambda) = \overline{\hat{f}_{nm}(\lambda)}. \quad (26)$$

3.5 $M(N)$ -Fourier transform over the homogeneous space $\mathbb{R}^N$

Any function $f \in L^2(\mathbb{R}^N)$ can be treated as a rotation-invariant function over $M(N)$, by $f(g) = f(R_\theta, \mathbf{r}) = f(\mathbf{r})$. This extension is not only well-defined, but also treats f as an $L^2(M(N))$ function, because $SO(N)$ is a compact subgroup of $M(N)$, and the measure on $\mathbb{R}^N$ is invariant under the action of $M(N)$. The matrix coefficients of $M(N)$ -Fourier transform of f is given by

$$\begin{aligned}\hat{f}_{mn}(\lambda) &= \int_{M(N)} f(g) u_{mn}^{(\lambda)}(g^{-1}) d(g) \\ &= \delta_m \int_{\mathbb{R}^N} f(\mathbf{r}) \int_{S^{N-1}} e^{i\lambda \mathbf{r} \cdot \mathbf{s}} S_n(\mathbf{s}) d(\mathbf{s}) d\mathbf{r} \end{aligned} \quad (27)$$

$$= \delta_m \tilde{f}_n(-\lambda), \quad (28)$$

where $\tilde{f}_n(\lambda)$ is the spherical harmonic decomposition of the standard Fourier transform of f . Observe from equation (27) that taking Fourier transform of f is equivalent to taking an N -dimensional standard Fourier transform followed by spherical harmonic decomposition. This decomposes f in terms of the associated spherical functions $\{u_{n0}(g)\}$ in $L^2(\mathbb{R}^N)$, that is

$$\int_{\mathbb{R}^N} f(\mathbf{r}) \int_{S^{N-1}} e^{i\lambda \mathbf{r} \cdot \mathbf{s}} S_n(\mathbf{s}) d(\mathbf{s}) d\mathbf{r} = (f, u_{n0}^{(\lambda)}(g)),$$

where $(f_1, f_2) = \int_{\mathbb{R}^N} f_1(\mathbf{r}) \overline{f_2(\mathbf{r})} d\mathbf{r}$ is the inner product over $L^2(\mathbb{R}^N)$. The inverse $M(N)$ -Fourier transform then becomes

$$\begin{aligned}f(g) &= \int_0^\infty \sum_{m,n} \hat{f}_{mn}(\lambda) u_{nm}^{(\lambda)}(g) \lambda^{N-1} d\lambda \\ &= \int_0^\infty \sum_n \tilde{f}_n(-\lambda) u_{n0}^{(\lambda)}(g) \lambda^{N-1} d\lambda. \end{aligned}$$

It is straightforward to show that $u_{m0}^{(\lambda)}(g) = u_{0m}^{(\lambda)}(g)$, when $u_{m0}^{(\lambda)}(g)$ is calculated with respect to Ξ_m [26].

3.6 Distributions and $M(N)$ -Fourier transform

Let $\mathcal{S}(M(N))$ denote the space of rapidly decreasing functions on $M(N)$. The $M(N)$ -Fourier transform can be extended to $\mathcal{S}(M(N))$, and is injective [25, 26]. Let $\mathcal{S}'(M(N))$ denote the space of linear functionals over $\mathcal{S}(M(N))$. $\mathcal{S}'(M(N))$ is called the space of tempered distributions over $M(N)$. Let $u \in \mathcal{S}'(M(N))$ and $\varphi \in \mathcal{S}(M(N))$. The value $u(\varphi)$ is denoted by $\langle u, \varphi \rangle$ or $\int_{M(N)} u(g) \varphi(g) d(g)$.

Let $\varphi \in \mathcal{S}(M(N))$ and $u \in \mathcal{S}'(M(N))$. The $M(N)$ -Fourier transform $\hat{u}$ of u is defined by

$$\langle \hat{u}, \hat{\varphi} \rangle = \langle u, \varphi \rangle. \quad (29)$$

Let u and v be two distributions, at least one of which has compact support. Then the convolution of u and v is a distribution that can be computed using either of the following

equations

$$\langle u *_{M(N)} v, \varphi \rangle = \langle u(h), \langle v(g), \varphi(hg) \rangle \rangle \quad (30)$$

$$= \langle v(g), \langle u(h), \varphi(hg) \rangle \rangle. \quad (31)$$

If either u or v is a tempered distribution, and the other is compactly supported, then $u *_{M(N)} v$ is a tempered distribution. Without loss of generality, assume that u is compactly supported and $v \in \mathcal{S}'(M(N))$. Then $\hat{u}$ can be computed using equation (29) by

$$\hat{u} = \langle u(g), u_{mn}^{(\lambda)}(g^{-1}) \rangle. \quad (32)$$

Using equations (30) and (32), the $M(N)$ -Fourier transform of the convolution $u *_{M(N)} v$ is obtained to be

$$\mathcal{F}_{M(N)}(u *_{M(N)} v)_{mn}(\lambda) = \sum_k \hat{v}_{mk}(\lambda) \hat{u}_{kn}(\lambda). \quad (33)$$

For the rest of the paper, we will use the integral representations for distributions.

4. Spherical harmonic decomposition of the Radon transform and $M(N)$

4.1 Radon transform

The Radon transform of a compactly supported function [1, 10] over $\mathbb{R}^N$ is defined as

$$\mathcal{R}f(\boldsymbol{\vartheta}, t) = \int_{\mathbb{R}^N} f(\mathbf{x}) \delta(\mathbf{x} \cdot \boldsymbol{\vartheta} - t) d\mathbf{x}, \quad (34)$$

where $\boldsymbol{\vartheta} \in S^{N-1}$, $t \in \mathbb{R}$ and δ is the generalized Dirac delta distribution. Without loss of generality, we assume that the support of f is within the unit ball.

Let $\tilde{f}(\boldsymbol{\xi})$ be the N -dimensional standard Fourier transform of a real-valued function $f \in L^2(\mathbb{R}^N)$,

$$\tilde{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} f(\mathbf{x}) e^{-i\mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x},$$

and $\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma)$ be the one-dimensional standard Fourier transform of $\mathcal{R}f(\boldsymbol{\vartheta}, t)$ with respect to t ,

$$\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma) = \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, t) e^{-it\sigma} dt.$$

The Fourier slice theorem states that $\tilde{f}(\sigma \boldsymbol{\vartheta}) = \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \sigma)$ for $\sigma \in \mathbb{R}$, $\boldsymbol{\vartheta} \in S^{N-1}$ [10]. Let $\tilde{f}_l(\sigma)$ and $\widetilde{\mathcal{R}f}_l(\sigma)$ denote the spherical harmonic decomposition of the Fourier transforms of the function f and its projections $\mathcal{R}f$. Then, by the Fourier slice theorem, we can show that $\tilde{f}_l(\sigma) = \widetilde{\mathcal{R}f}_l(\sigma)$. This relationship leads to the spherical harmonic decomposition of Radon transform in terms of f [30, 31]. In the following section, we will derive this relationship from a convolution-type representation of the Radon transform over $M(N)$.

4.2 Radon transform as a convolution over $M(N)$

In ref. [32], Radon transform over $\mathbb{R}^2$ was recognized as a cross-correlation over $M(2)$. The cross-correlation representation was restated and generalized as a convolution over $M(N)$ in

Chapter 13 of [33]. The convolution representation of the Radon transform over $M(N)$ is an alternative representation of the Radon transform for the double fibration ([17, section 2.2]) using distributions. We also adapt a convolution formulation similar to the one introduced in ref. [33], and derive the SVD of the Radon transform from the convolution representation using the Fourier analysis over $M(N)$.

In order to express the Radon transform as a convolution over $M(N)$, let us revisit the convolution over $M(N)$. Let f_1 and f_2 be real-valued functions over $M(N)$. Then, convolution of f_1 and f_2^* over $M(N)$ is given by

$$\begin{aligned} (f_1 *_{M(N)} f_2^*)(g) &= \int_{M(N)} f_1(h) f_2(g^{-1}h) d(h) \\ &= \int_{SO(N)} \int_{\mathbb{R}^N} f_1(R_\phi, \mathbf{x}) f_2(R_\phi R_\theta^{-1}, R_\theta^{-1}\mathbf{x} - R_\theta^{-1}\mathbf{r}) d(\phi) d\mathbf{x}, \end{aligned} \quad (35)$$

where $g = (R_\theta, \mathbf{r})$ and $h = (R_\phi, \mathbf{x})$.

Let $\delta(R_\phi)$ denote the distribution over $SO(N)$ defined as follows

$$\int_{SO(N)} \delta(R_\phi) \varphi(R_\phi) d(R_\phi) = \varphi(I), \quad (36)$$

where I is the identity rotation. Then, setting $f_\delta(R_\phi, \mathbf{x}) = f(\mathbf{x})\delta(R_\phi)$ and $\Lambda(R_\phi, \mathbf{x}) = \delta(\mathbf{x} \cdot \mathbf{e}_1)$, we can write the Radon transform of f by

$$\mathcal{R}f(\boldsymbol{\vartheta}, t) = (f_\delta *_{M(N)} \Lambda^*)(g), \quad (37)$$

where $\boldsymbol{\vartheta} = R_\theta \mathbf{e}_1 \in S^{N-1}$, the first column of R_θ , and $t = (R_\theta^{-1}\mathbf{r}) \cdot \mathbf{e}_1$.

Using an equivalent formulation,

$$(f_1 *_{M(N)} f_2^*)(g) = \int_{M(N)} f_1(R_\theta R_\phi, R_\theta \mathbf{x} + \mathbf{r}) f_2(\phi, \mathbf{x}) d(h). \quad (38)$$

In this case, the Radon transform of f is given by,

$$\mathcal{R}f(\boldsymbol{\vartheta}, r_1) = (\Lambda *_{M(N)} f^*)(g), \quad (39)$$

where $\Lambda(R_\phi, \mathbf{x}) = \delta(\mathbf{x} \cdot \mathbf{e}_1)$, $f(R_\phi, \mathbf{x}) = f(\mathbf{x})$, and $\boldsymbol{\vartheta} = -R_\theta^{-1}\mathbf{e}_1$.

The formulation in equation (39) leads to a simpler algorithm for the inversion, because, unlike the formulation in equation (37), the projection data can be used without any pre-processing. The formulation in equations (39) and (37) carries the tomographic inversion problem over $M(N)$. Now, we wish to determine f given $\mathcal{R}f$ and Λ over $M(N)$.

Equation (39) should be understood in the distribution sense where $\mathcal{R}f$ is a tempered distribution given by the convolution of the tempered distribution Λ and the compactly supported distribution f over $M(N)$.

4.3 Spherical harmonic decomposition of the radon transform

$M(N)$ -Fourier transform maps the convolution into multiplication in the transform domain. Hence, taking $M(N)$ -Fourier transform of both sides of equation (39) gives the relationship

between the $M(N)$ -Fourier coefficients of $\mathcal{R}f$ and f as

$$\widehat{\mathcal{R}f}(\lambda) = \hat{f}(\lambda)^\dagger \hat{\Lambda}(\lambda), \quad (40)$$

or in matrix form

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \sum_q \overline{\hat{f}_{qm}(\lambda)} \hat{\Lambda}_{qn}(\lambda). \quad (41)$$

As $f \in L^2(\mathbb{R}^N)$, $M(N)$ -Fourier coefficients of f are non-zero if and only if $q = 0$. Equation (41) becomes

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \overline{\hat{f}_{0m}(\lambda)} \hat{\Lambda}_{0n}(\lambda). \quad (42)$$

From equation (42), $\hat{f}_{0m}(\lambda)$ is independent of the choice of n . Therefore, as long as $\hat{\Lambda}_{0n}(\lambda)$ is not equal to zero, $M(N)$ -Fourier coefficients of f are computed as

$$\overline{\hat{f}_{0m}(\lambda)} = \frac{\widehat{\mathcal{R}f}_{mn}(\lambda)}{\hat{\Lambda}_{0n}(\lambda)}. \quad (43)$$

The $M(N)$ -Fourier coefficients, $\widehat{\mathcal{R}f}_{mn}(\lambda)$ and $\hat{\Lambda}_{0n}(\lambda)$ are (see sections A and B) given by

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta}), \quad (44)$$

$$\hat{\Lambda}_{mn}(\lambda) = \delta_m C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}}. \quad (45)$$

Note that, whenever $\hat{\Lambda}_{0n}(\lambda)$ is zero, so is $\widehat{\mathcal{R}f}_{mn}(\lambda)$. Substituting equations (44) and (45) into the right-hand side of equation (43), the $M(N)$ -Fourier coefficients of f becomes

$$\hat{f}_{0m}(\lambda) = \int_{S^{N-1}} \overline{\widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda)} S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}). \quad (46)$$

Using the $M(N)$ -Fourier transform over $\mathbb{R}^N$, and the property of the standard Fourier transform under conjugation, one obtains

$$\tilde{f}_m(-\lambda) = \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}).$$

This is nothing but the spherical harmonic decomposition of the Radon transform

$$\tilde{f}_m(\lambda) = \widetilde{\mathcal{R}f}_m(\lambda). \quad (47)$$

Equation (47) shows that $M(N)$ -Fourier analysis of the Radon transform implicitly combines the spherical harmonic decomposition and the Fourier slice theorem.

While the harmonic analysis over $M(N)$ provides a framework to analyze the Radon transform, the relationship between the representations of $M(N)$ and the special functions provides a natural approach in understanding the presence of the special functions in the SVD of the Radon transform.

5. Inversion of the radon transform

Equation (46) provides an inversion formula for the Radon transform based on the $M(N)$ -Fourier transform. This inversion method facilitates the definition of the inverse Radon transform, $\mathcal{R}^{-1}$, as an integral operator over $M(N)$.

In the following sections, we will first derive the sufficient conditions for the separability of the convolution kernels. This will be followed by the decomposition of the kernel of $\mathcal{R}^{-1}$, leading to the SVD of the Radon transform.

5.1 Linear integral operators over groups

Let G be a unimodular, separable, locally compact group of Type I, and A be a linear integral operator acting on the functions defined over a homogeneous space Z of G with kernel $\tilde{K}_A(g, h)$:

$$Af(h) = \int_Z \tilde{K}_A(g, h) f(g) d(g), \quad (48)$$

where h is an element of another homogeneous space H of G . If $\tilde{K}_A(g, h)$ can be decomposed as

$$\tilde{K}_A(g, h) = \sum_i c_i a_i(g) b_i(h), \quad (49)$$

where $\{c_i\}$ are constants and $\{a_i(g)\}$ and $\{b_i(h)\}$ are linearly independent normalized functions, then we shall call $\tilde{K}_A(g, h)$ a separable kernel. For linear integral operators with separable kernels, equation (48) becomes

$$Af(h) = \sum_i c_i b_i(h) \int_Z a_i(g) f(g) d(g). \quad (50)$$

If $\{a_i(g)\}$ and $\{b_i(h)\}$ are orthonormal and complete with respect to some weight functions $w_a(g)$ and $w_b(h)$,

$$\begin{aligned} Aa_j w_a(h) &= \sum_i c_i b_i(h) \int_Z a_i(g) a_j(g) w_a(g) d(g) \\ &= \sum_i c_i b_i(h) \delta_{ij} = c_j b_j(h), \end{aligned} \quad (51)$$

where

$$c_i = \int_H Aa_j w_a(h) b_j(h) w_b(h) d(h),$$

then $\{a_i w_a, b_i, c_i\}$ becomes the SVD of A . Furthermore, if A is invertible, the SVD of A^{-1} is given by $\{b_i, a_i w_a, c_i^{-1}\}$.

Let $\tilde{K}_A(g, h) = K_A(g^{-1}h)$, and K_A has a Fourier transform over G . Then we can write K_A as

$$\begin{aligned}
 K_A(g^{-1}h) &= \int_G \text{Tr} \left[\left[\int_H K_A(g^{-1}h_0) U^{(\lambda)}(h_0^{-1}) d(h_0) \right] U^{(\lambda)}(h) \right] d(\lambda) \\
 &= \int_G \text{Tr} \left[\widehat{K}_A(\lambda) U^{(\lambda)}(g^{-1}) U^{(\lambda)}(h) \right] d(\lambda) \\
 &= \sum_{m,n,k} \int_G \widehat{K}_{Amk}(\lambda) u_{kn}^{(\lambda)}(g^{-1}h) d(\lambda) \\
 &= \sum_{m,n,k,l} \int_G \widehat{K}_{Amk}(\lambda) \overline{u_{lk}^{(\lambda)}(g)} u_{ln}^{(\lambda)}(h) d(\lambda). \tag{52}
 \end{aligned}$$

Equation (52) implies that if $u_{mn}^{(\lambda)}(g)$ is separable in λ and g , then K_A is separable. Let $u_{mn}^{(\lambda)}(g) = \sum_p u_{mnp}^{(1)}(\lambda) u_{mnp}^{(2)}(g)$. Then,

$$K_A(g^{-1}h) = \sum_{m,n,k,l,p,q} \left(\int_G \widehat{K}_{Amk}(\lambda) \overline{u_{lkp}^{(1)}(\lambda)} u_{lnq}^{(1)}(\lambda) d(\lambda) \right) \overline{u_{lkp}^{(2)}(g)} u_{lnq}^{(2)}(h). \tag{53}$$

Furthermore, if $u_{lkp}^{(2)}(g)$ and $u_{lnq}^{(2)}(h)$ are orthogonal and complete over Z and H , respectively, equation (53) leads to the SVD of A .

This is indeed the case for the unitary representations of $M(N)$, and can be seen by substituting the identity (16) into the matrix elements $u_{mn}^{(\lambda)}(g)$ of the unitary representations given in equation (4). In the next section, we will derive the singular functions arising in the SVD of the inverse Radon transform by using the identity (16).

5.2 Inverse Radon transform as a linear integral operator

By equation (46), the $M(N)$ -Fourier coefficients of f are given by

$$\begin{aligned}
 \hat{f}_{0m}(\lambda) &= \int_{S^{N-1}} \widetilde{\mathcal{R}}f(\boldsymbol{\vartheta}, -\lambda) S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}) \\
 &= \int_{S^{N-1}} \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, r) e^{ir\lambda} dr S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}). \tag{54}
 \end{aligned}$$

Using the inverse $M(N)$ -Fourier transform, f can be obtained as

$$f(h) = \int_0^\infty \int_{S^{N-1}} \int_{\mathbb{R}} \mathcal{R}f(\boldsymbol{\vartheta}, r_1) e^{ir_1\lambda} dr S_m(\boldsymbol{\vartheta}) d(\boldsymbol{\vartheta}) u_{m0}^{(\lambda)}(h) \lambda^{N-1} d\lambda, \tag{55}$$

where $h = (R_\phi, \mathbf{x})$. As $u_{m0}^{(\lambda)}(h)$ is equal to $u_{m0}^{(\lambda)}(h)$, f does not depend on the rotation component of h . Hence, equation (55) is an exact inversion formula for the Radon transform.

Furthermore, equation (55) defines the inverse Radon transform as an integral operator with the following kernel:

$$K_{\mathcal{R}^{-1}}(g, h) = \int_0^\infty e^{ir_1\lambda} S_m(\boldsymbol{\vartheta}) u_{m0}^{(\lambda)}(h) \lambda^{N-1} d\lambda, \tag{56}$$

where $g = (\boldsymbol{\vartheta}, r_1) \in S^{N-1} \times \mathbb{Z}$. Now, our objective is to decompose the kernel $K_{\mathcal{R}^{-1}}(g, h)$ as in equation (46). To do so, we first decompose $e^{i\lambda r_1}$ by using the equality (16), that is

$$e^{i\lambda r_1} = \Gamma(\nu) \left(\frac{\lambda}{2}\right)^{-\nu} \sum_{k=0}^{\infty} i^k (k + \nu) J_{k+\nu}(\lambda) C_k^\nu(r_1), \quad -1 < x < 1, \quad \nu > -1/2. \quad (57)$$

Secondly, we compute the associated spherical functions $u_{m0}^{(\lambda)}(h)$. $u_{m0}^{(\lambda)}(h)$ is given by

$$u_{m0}^{(\lambda)}(h) = \int_{S^{N-1}} S_m(\boldsymbol{\omega}) e^{-i\lambda(\mathbf{x} \cdot \boldsymbol{\omega})} d(\boldsymbol{\omega}) = \int_{S^{N-1}} S_m(\boldsymbol{\omega}) e^{-i\lambda|\mathbf{x}|((\mathbf{x}/|\mathbf{x}|) \cdot \boldsymbol{\omega})} d(\boldsymbol{\omega}), \quad (58)$$

showing that $u_{m0}^{(\lambda)}(h)$ is independent of the rotation component of h . Therefore, we will use h and $\mathbf{x}$ interchangeably. Furthermore, by Funk–Hecke theorem [34],

$$u_{m0}^{(\lambda)}(\mathbf{x}) = c_{N,l} \left[\int_{-1}^1 e^{-i\lambda|\mathbf{x}|t} C_l^{(N-2)/2}(t) (1-t^2)^{(N-3)/2} dt \right] S_m\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad (59)$$

where l is the first component of the multi-index m representing the degree of the spherical harmonics. Using the identity (10), $u_{m0}^{(\lambda)}(\mathbf{x})$ becomes

$$u_{m0}^{(\lambda)}(\mathbf{x}) = c_{N,l} \frac{(-i)^l \pi \Gamma(N+l-2)}{2^{(N-4)/2} l! \Gamma((N-2)/2)} \frac{J_{l+(N-2)/2}(\lambda|\mathbf{x}|)}{(\lambda|\mathbf{x}|)^{(N-2)/2}} S_m\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right). \quad (60)$$

Note that equation (14) is a special case for $u_{m0}^{(\lambda)}(g)$, where $S_m = \Xi_m$.

Substituting equations (60) and (57) into (56) and collecting all the constants under a single expression $C_1(N, k, l, \nu)$, and reorganizing the terms, $K_{\mathcal{R}^{-1}}(g, h)$ can be expressed as

$$K_{\mathcal{R}^{-1}}(g, \mathbf{x}) = \sum_{k,m} C_1(N, k, l, \nu) C_k^\nu(r_1) S_m(\boldsymbol{\vartheta}) \quad (61)$$

$$\times |\mathbf{x}|^{1-N/2} \left(\int_0^\infty \lambda^{-\nu} J_{k+\nu}(\lambda) \frac{J_{l+(N-2)/2}(\lambda|\mathbf{x}|)}{\lambda^{(N-2)/2}} \lambda^{N-1} d\lambda \right) S_m\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right). \quad (62)$$

From equation (61), we obtain the first set of functions, $\{a_{km}\}$, of the kernel:

$$a_{km}(g) = C_k^\nu(r_1) S_m(\boldsymbol{\vartheta}). \quad (63)$$

The integral inside the brackets in equation (62) when constants are neglected gives the second set of functions, $\{b_{km}\}$, of the kernel: (See [20], equation (3.6))

$$b_{km}(\mathbf{x}) = (1 - |\mathbf{x}|^2)^{\nu-N/2} Q_{k,l}^{\nu,N}(|\mathbf{x}|) S_m\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad (64)$$

where

$$Q_{k,l}^{\nu,N}(|\mathbf{x}|) = |\mathbf{x}|^l P_{(k-l)/2}^{(\nu-N/2, l+N/2-1)}(2|\mathbf{x}|^2 - 1).$$

Using $\{a_{km}\}$ and $\{b_{km}\}$, $K_{\mathcal{R}^{-1}}(g, \mathbf{x})$ is rewritten as

$$K_{\mathcal{R}^{-1}}(g, h) = \sum_{k,m} c(N, k, l, \nu) a_{km}(g) b_{km}\left(|\mathbf{x}|, \frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad (65)$$

where

$$c(N, k, l, v) = C_1(N, k, l, v) \times \left[\int_0^\infty |\mathbf{x}| \left(\int_0^\infty \lambda^{-v} J_{k+v}(\lambda) \frac{J_{l+(N-2)/2}(\lambda|\mathbf{x}|)}{\lambda^{(N-2)/2}} \lambda^{N-1} d\lambda \right)^2 d|\mathbf{x}| \right]^{1/2}. \quad (66)$$

An explicit computation of $c(N, k, l, v)$ is given in equation [21].

5.3 SVD of the Radon transform

By equation (50),

$$f(\mathbf{x}) = \sum_{m,k} c(N, k, l, v) b_{km}(\mathbf{x}) \int_{S^{N-1} \times \mathbb{R}} a_{km}(g) \mathcal{R}f(g) dg.$$

$\{a_{km}\}$ are orthogonal with respect to the weight function $w_{a,v}(r_1) = (1 - r_1^2)^{v-1/2}$; and $\{b_{km}\}$ are orthogonal with respect to the weight function $w_{b,v}(|\mathbf{x}|) = (1 - |\mathbf{x}|^2)^{v-N/2}$; and both bases are complete. Both the orthogonality and completeness are direct consequences of the Gegenbauer polynomials, Jacobi polynomials and spherical harmonics [21, 26, 35].

Therefore, the SVD of the inverse Radon transform is given by $\{a_{km} w_{a,v}, b_{km}, c(N, k, l, v)\}$. Hence, the SVD of the Radon transform is $\{b_{km}, a_{km} w_{a,v}, c(N, k, l, v)^{-1}\}$.

6. Conclusions

We presented a new derivation of the SVD of the Radon transform using harmonic analysis over the Euclidean motion group. We showed that the separability of the unitary representations of the underlying group leads to the decomposition of the integral transforms with convolution kernels. The separability of the unitary representations also gives rise to the special functions present in the SVD of the Radon transform. Additionally, the decomposition method presented here can be used to show the separability of the kernel of the convolution representation of the weighted Radon transform presented in [36].

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Appendix A: computation of $\widehat{\mathcal{R}f}_{mn}(\lambda)$

Let $S_n(s)$ and $S_m(s)$ be any two n th and m th order spherical harmonics. Then, the $M(N)$ Fourier transform of the projections is computed as

$$\widehat{\mathcal{R}f}_{mn}(\lambda) = \int_{M(N)} \mathcal{R}f(\vartheta, r_1) \int_{S^{N-1}} S_n(s) e^{i\lambda(r \cdot s)} \overline{S_m(R_\theta^{-1}s)} d(s) d(g)$$

$$\begin{aligned}
&= \int_{SO(N)} \int_{\mathbb{R}^N} \mathcal{R}f(\boldsymbol{\vartheta}, r_1) \int_{S^{N-1}} S_n(\mathbf{s}) e^{i\lambda(r \cdot \mathbf{s})} \overline{S_m(R_\theta^{-1} \mathbf{s})} d(\mathbf{s}) d\mathbf{r} d(\theta) \\
&= (2\pi)^{N-1} \int_{SO(N)} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda s_1) \delta(\lambda \mathbf{s}_2) S_n(\mathbf{s}) \overline{S_m(R_\theta^{-1} \mathbf{s})} d(\mathbf{s}) d(\theta)
\end{aligned}$$

where $\mathbf{s}_2 = (s_2, \dots, s_N)$, and $\|\mathbf{s}_2\| = 1 - s_1^2 \implies s_1 = \pm 1$ when $\mathbf{s}_2 = \mathbf{0}$

$$\begin{aligned}
&= \frac{C_1}{\lambda^{N-1}} \left(\int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_n(\mathbf{e}_1) \overline{S_m(R_\theta^{-1} \mathbf{e}_1)} d(\theta) \right. \\
&\quad \left. + \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) S_n(-\mathbf{e}_1) \overline{S_m(-R_\theta^{-1} \mathbf{e}_1)} d(\theta) \right)
\end{aligned}$$

where $C_1 = \frac{2\pi^{N-1}}{|S^{N-1}|}$

$$\begin{aligned}
&= \frac{C_1}{\lambda^{N-1}} \left(\int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, -\lambda) S_n(\mathbf{e}_1) \overline{S_m(-\boldsymbol{\vartheta})} d(\theta) \right. \\
&\quad \left. + \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) S_n(-\mathbf{e}_1) \overline{S_m(\boldsymbol{\vartheta})} d(\theta) \right) \\
&= C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}} \int_{SO(N)} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\theta) \\
&= C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}} \int_{SO(N-1)} d(\theta) \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta}) \\
&= C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}} \int_{S^{N-1}} \widetilde{\mathcal{R}f}(\boldsymbol{\vartheta}, \lambda) \overline{S_m(\boldsymbol{\vartheta})} d(\boldsymbol{\vartheta}).
\end{aligned}$$

Appendix B: computation of $\hat{\Lambda}_{0n}(\lambda)$

Similar to $\widehat{\mathcal{R}f}_{mn}(\lambda)$, we compute $\hat{\Lambda}_{mn}(\lambda)$,

$$\begin{aligned}
\hat{\Lambda}_{mn}(\lambda) &= \int_{M(N)} \delta(r_1) \int_{S^{N-1}} S_n(\mathbf{s}) e^{i\lambda(r \cdot \mathbf{s})} \overline{S_m(R_\theta^{-1} \mathbf{s})} d(\mathbf{s}) d(g) \\
&= \int_{SO(N)} \int_{\mathbb{R}^N} \delta(r_1) \int_{S^{N-1}} S_n(\mathbf{s}) e^{i\lambda(r \cdot \mathbf{s})} \overline{S_m(R_\theta^{-1} \mathbf{s})} d(\mathbf{s}) d\mathbf{r} d(\theta) \\
&= \delta_m (2\pi)^{N-1} \int_{S^{N-1}} \delta(\lambda \mathbf{s}_2) S_n(\mathbf{s}) d(\mathbf{s}) \\
&= \delta_m C_1 \frac{S_n(\mathbf{e}_1) + S_n(-\mathbf{e}_1)}{\lambda^{N-1}}.
\end{aligned}$$

A generalization of k th derivative of Dirac delta in a hypercone

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In this article, we obtain relations between the distributions $\delta^{(r-1)}(G_+)$, $\delta^{(r-1)}(G_-)$, $\delta_1^{(r-1)}(G)$ and $\delta_2^{(r-1)}(G)$ in terms of the operator L_m iterated $(r - n/2m)$ times under the conditions $r - 1 \geq n/2m - 1$, $\mu = (2s)m$ and $\nu = (2l)m$; $r - 1 \geq n/2m - 1$, $\mu = (2s)m + m$ and $\nu = (2l)m + m$ with r non-negative integer, where L_m^j is defined by equation (59).

Keywords: Property of distributions theory

1. Introduction

Let $c^2 + G$ be defined by

$$c^2 + G = c^2 + G(x, m) = c^2 + (x_1^2 + \cdots + x_\mu^2)^m - (x_1^2 + \cdots + x_\nu^2)^m, \quad (1)$$

where c^2 is a real number, m is a positive integer and $\mu + \nu = n$ dimension of the space.

The distributions $(c^2 + G)_+^\lambda$ and $(c^2 + G)_-^\lambda$ are defined by

$$(c^2 + G)_+^\lambda = \int_{c^2 + G > 0} (c^2 + G(x, m))^\lambda \varphi(x) \, dx \quad (2)$$

and

$$(c^2 + G)_-^\lambda = \int_{c^2 + G < 0} (-(c^2 + G(x, m)))^\lambda \varphi(x) \, dx, \quad (3)$$

where λ is a complex number.

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Aguirre in ref. [1] showed the following formula:

$$\operatorname{Re} s_{\lambda=-k}(c^2 + G)_+^\lambda = \frac{(-1)^{k-1}}{k!} \delta^{(k)}(c^2 + G) \quad (4)$$

and

$$\operatorname{Re} s_{\lambda=-k}(c^2 + G)_-^\lambda = \frac{(-1)^{k-1}}{k!} \delta^{(k)}(-c^2 - G), \quad (5)$$

which are generalization of Bresters' formulae [2, p. 566].

On the other hand, let ϕ_s be a distribution of variable s , and let $u(x) \in C^\infty(R^n)$ be such that the $(n-1)$ dimension manifold $u(x_1, x_2, \dots, x_n) = 0$ has no critical points; $\phi_{u(x)}$ denotes the distribution defined on R^n by the formula (called the Leray formula [3, p. 102])

$$\int_{R^n} \phi_{u(x)} f(x) dx_1 \cdots dx_n = \int_{-\infty}^{+\infty} \phi_s ds \int_{u(x)=s} f(x) w_u(x, dx), \quad (6)$$

here w_u is an $(n-1)$ -dimensional form on u defined as follows:

$$du \wedge dw = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n; \quad (7)$$

the manifold $u(x) = s$ has the orientation such that $w(x, dx) > 0$.

Using (6) (the Leray formula) for c^2 , $\delta^{(k)}(c^2 + G)$, also can be defined in the following form,

$$\langle \delta^{(k)}(c^2 + G), \varphi \rangle = \langle \delta^{(k)}(s), \gamma(s) \rangle = (-1)^k \psi^{(k)}(0), \quad (8)$$

where

$$\gamma(s) = \int_{c^2+G=s} \varphi w_{c^2+G}(x, dx). \quad (9)$$

If

$$G = G(m, x) = (x_1^2 + \cdots + x_p^2)^m - (x_{p+1}^2 + \cdots + x_{p+q}^2)^m, \quad (10)$$

we know that distribution

$$\delta^{(k-1)}(G) \text{ exists if } n \text{ is odd and } k-1 < \frac{n}{2m} - 1 \text{ if } n \text{ even [4, p. 52]}. \quad (11)$$

If, on the other hand

$$k-1 \geq \frac{n}{2m} - 1, \quad (12)$$

$\langle \delta_1^{(k-1)}(G), \varphi \rangle$ and $\langle \delta_2^{(k-1)}(G), \varphi \rangle$ are defined by [4, p. 53],

$$\langle \delta_1^{(k-1)}(G), \varphi \rangle = \int_0^\infty \left[\left(\frac{\partial}{2ms^{2m-1}\partial s} \right)^{k-1} \left\{ s^{q-2m} \cdot \frac{\psi(r, s)}{2m} \right\} \right]_{s=r} r^{p-1} dr \quad (13)$$

and

$$\langle \delta_2^{(k-1)}(G), \varphi \rangle = (-1)^{k-1} \int_0^\infty \left[\left(\frac{\partial}{2mr^{2m-1}\partial r} \right)^{k-1} \left\{ r^{p-2m} \cdot \frac{\psi(r, s)}{2m} \right\} \right]_{r=s} s^{q-1} ds \quad (14)$$

are regularizations of $\langle \delta^{(k-1)}(G), \varphi \rangle$.

On the other hand, from ref. [5] and considering all the above G_+^λ remains true also for G_-^λ except that μ and ν must be interchanges, and in all the formulae $\delta_1^{(k-1)}(G)$ must be replaced by $\delta_2^{(k-1)}(-G) = (-1)^{k-1}\delta_1^{(k-1)}(G)$ and L_m by $(-L_m)$, where L_m is defined by (59), we have

$$\delta^{(k-1)}(G_+) = (-1)^{k-1}(k-1)!\operatorname{Re} s_{\lambda=-k} G_+^\lambda \quad (15)$$

and

$$\delta^{(k-1)}(G_-) = (-1)^{k-1}(k-1)!\operatorname{Re} s_{\lambda=-k} G_-^\lambda, \quad (16)$$

then

$$\delta^{(k-1)}(G_+) = \delta_1^{(k-1)}(G) \quad (17)$$

and

$$\delta^{(k-1)}(G_-) = \delta_1^{(k-1)}(-G) \quad (18)$$

if n is odd and $k-1 < n/2m-1$ if n is even.

For the case $k-1 \geq n/2m-1$ the relations (17) and (18) are not valid, for example $\delta^{(k-1)}(G_+) - \delta_1^{(k-1)}(G)$ and $\delta^{(k-1)}(G_-) - \delta_1^{(k-1)}(-G)$ are generalized functions concentrated in $G = 0$.

It is important to observe that for (4), (5), (8) and (9) to hold, c must be different from zero. Indeed, formulae (4), (5), (8) and (9) may not hold if we put in it $u(x) = G(x)$ where $G(x)$ is defined in (10). This is due to the fact that $G(x) = 0$ has a critical point (namely the origin).

For going from $\delta^{(k-1)}(c^2 + G)$ to $\delta^{(k-1)}(G)$ taking $c^2 = 0$ we consider the conditions (11) and (12) and formulae (13)–(18), therefore the following relations are valid,

$$\delta^{(k-1)}(c^2 + G) = \delta_1^{(k-1)}(G) = \delta^{(k-1)}(G_+) \quad (19)$$

if $c^2 = 0$ for $n = (2s)m + 1$ and for even n such that $n = (2s)m$ and $k-1 < n/2m-1$, whereas taking into account (4), (5) and (12)–(14) the following relations are valid,

$$\delta^{(k-1)}(c^2 + G) = \delta_1^{(k)}(G) \quad (20)$$

or

$$(-1)^{k-1}\delta^{(k-1)}(-c^2 - G) = \delta_2^{(k-1)}(G) \quad (21)$$

if $c^2 = 0$ for $n = (2s)m$ and $k-1 \geq n/2m-1$, where $\delta_2^{(k-1)}(G) = (-1)^{k-1}\delta_1^{(k-1)}(-G)$ [4, p. 53].

In this article, we obtain relations between the distributions $\delta^{(r-1)}(G_+)$, $\delta^{(r-1)}(G_-)$, $\delta_1^{(r-1)}(G)$ and $\delta_2^{(r-1)}(G)$ in terms of the operator L_m iterated $(r - n/2m)$ times under the conditions $r-1 \geq n/2m-1$, $\mu = (2s)m$ and $\nu = (2l)m$; $r-1 \geq n/2m-1$, $\mu = (2s)m + m$ and $\nu = (2l)m + m$ with r non-negative integer, where L_m^j is defined by (59).

2. A generalization of k th derivative of Dirac delta in a hypercone

In this paragraph, we give a sense to relations between the distributions $\delta^{(r-1)}(G_+)$, $\delta^{(r-1)}(G_-)$, $\delta_1^{(r-1)}(G)$ and $\delta_2^{(r-1)}(G)$ in terms of the operator L_m iterated $(r - n/2m)$ times under the conditions $r-1 \geq n/2m-1$, $\mu = (2s)m$ and $\nu = (2l)m$; $r-1 \geq n/2m-1$, $\mu = (2s)m + m$ and $\nu = (2l)m + m$ with r non-negative integer, where L_m is defined by (59).

To obtain our result, we need the following formulae

$$\begin{aligned}
 & \text{Re } s_{\lambda=-\frac{n}{2m}-k, k=0,1,2,\dots} G_+^\lambda \\
 &= \frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)} \delta_1^{(\frac{n}{2m}+k-1)}(G) + \frac{\pi^{\frac{n}{2}} \Gamma(\frac{\nu}{2m})(-1)^{\frac{\nu}{2m}} \Gamma(\frac{\mu}{2m})}{m^2 \Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu+\nu}{2m})} \\
 & \quad \cdot \frac{1}{k!(2m)^{2k} (\frac{n}{2m} + \frac{2(m-1)}{m})(\frac{n}{2m} + \frac{2(m-1)}{m} + 1) \cdots (\frac{n}{2m} + \frac{2(m-1)}{m} + k-1)} L_m^k \{\delta\} \quad (22)
 \end{aligned}$$

if n is even, $\mu = 2ms$, $\nu = 2ml$; $s, l = 0, 1, 2, \dots$ ([5], formula (106)) and

$$\begin{aligned}
 & \text{Re } s_{\lambda=-\frac{n}{2m}-k, k=0,1,2,\dots} G_+^\lambda \\
 &= \frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)} \delta_1^{(\frac{n}{2m}+k-1)}(G) + \frac{\pi^{\frac{n}{2}-1} \Gamma(\frac{\nu}{2m})(-1)^{\frac{\nu-m}{2m}} \Gamma(\frac{\nu}{2m})}{m^2 \Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu+\nu}{2m})} \\
 & \quad \cdot \left\{ \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] \right\} \\
 & \quad \cdot \frac{1}{k!(2m)^{2k} (\frac{n}{2m} + \frac{2(m-1)}{2m})(\frac{n}{2m} + \frac{2(m-1)}{m} + 1) \cdots (\frac{n}{2m} + \frac{2(m-1)}{m} + k-1)} L_m^k \{\delta\} \quad (23)
 \end{aligned}$$

if n is even, $\mu = 2ms + m$, $\nu = 2ml + m$; $s, l = 0, 1, 2, \dots$ ([5], formula (108)). Now considering all the above G_+^λ remains true also for G_-^λ except that μ and ν must be interchanged, and in all the formulae $\delta_1^{(k-1)}(G)$ must be replaced by $\delta_2^{(k-1)}(-G) = (-1)^{k-1} \delta_1^{(k-1)}(G)$ and L_m by $(-L_m)$, from (22) and (23) we have

$$\begin{aligned}
 & \text{Re } s_{\lambda=-\frac{n}{2m}-k, k=0,1,2,\dots} G_-^\lambda \\
 &= \frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)} \delta_1^{(\frac{n}{2m}+k-1)}(-G) + \frac{\pi^{\frac{n}{2}} \Gamma(\frac{\mu}{2m})(-1)^{\frac{\mu}{2m}} \Gamma(\frac{\nu}{2m})}{m^2 \pi \Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu+\mu}{2m})} \\
 & \quad \cdot \frac{1}{k!(2m)^{2k} (\frac{n}{2m} + \frac{2(m-1)}{m})(\frac{n}{2m} + \frac{2(m-1)}{m} + 1) \cdots (\frac{n}{2m} + \frac{2(m-1)}{m} + k-1)} (-L_m)^k \{\delta\} \quad (24)
 \end{aligned}$$

if n is even, $\mu = 2ms$, $\nu = 2ml$; $s, l = 0, 1, 2, \dots$ and

$$\begin{aligned}
 & \text{Re } s_{\lambda=-\frac{n}{2m}-k, k=0,1,2,\dots} G_-^\lambda \\
 &= \frac{(-1)^{\frac{n}{2m}+k-1}}{\Gamma(\frac{n}{2m}+k)} \delta_1^{(\frac{n}{2m}+k-1)}(-G) + \frac{\pi^{\frac{n}{2}-1} \Gamma(\frac{\mu}{2m})(-1)^{\frac{\mu-m}{2m}} \Gamma(\frac{\mu}{2m})}{m^2 \Gamma(\frac{\nu}{2}) \Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu+\mu}{2m})} \\
 & \quad \cdot \left\{ \left[\psi\left(\frac{\nu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] \right\} \\
 & \quad \cdot \frac{1}{k!(2m)^{2k} (\frac{n}{2m} + \frac{2(m-1)}{m})(\frac{n}{2m} + \frac{2(m-1)}{m} + 1) \cdots (\frac{n}{2m} + \frac{2(m-1)}{m} + k-1)} \\
 & \quad \times (-L_m)^k \{\delta\} \quad (25)
 \end{aligned}$$

if n is even, $\mu = 2ms + 1$, $\nu = 2ml + 1$; $s, l = 0, 1, 2, \dots$

LEMMA 1 *The following formulae are valid*

$$\delta^{(r-1)}(G_+) - \delta_1^{(r-1)}(G) = \frac{(r-1)!}{(-1)^{r-1}} a_{m,n,\mu,v} (-1)^{v/2m} b_{r-(n/2)m,n,m} L_m^{r-(n/2)} \{\delta(x)\} \quad (26)$$

and

$$\delta^{(r-1)}(G_-) - \delta_1^{(r-1)}(-G) = (r-1)! a_{m,n,\mu,v} (-1)^{-v/2m} (-1) b_{r-(n/2)m,n,m} L_m^{r-(n/2)} \{\delta(x)\} \quad (27)$$

if $\mu = (2s)m$, $v = (2l)m$, n even, $s, l = 0, 1, 2, \dots$ and $r \geq n/2m - 1$. Where

$$a_{m,n,\mu,v} = \frac{\pi^{n/2} \Gamma(v/2m) \Gamma(\mu/2m)}{m^2 \Gamma(\mu/2) \Gamma(v/2) \Gamma((\mu+v)/2m)} \quad (28)$$

and

$$b_{r,n,m} = \frac{1}{r! (2m)^{2r} \left(\frac{n}{2m} + \frac{2(m-1)}{m}\right) \left(\frac{n}{2m} + \frac{2(m-1)}{m} + 1\right) \dots \left(\frac{n}{2m} + \frac{2(m-1)}{m} + r - 1\right)}. \quad (29)$$

Proof The proof of Lemma 1 is consequence of formulae (15)–(18) and (22) and (24). ■

LEMMA 2 *The following formulae are valid*

$$\begin{aligned} \delta^{(r-1)}(G_+) - \delta_1^{(r-1)}(G) &= \frac{(r-1)!}{(-1)^{r-1}} a_{m,n,\mu,v} \pi^{-1} (-1)^{-v-m/2m} b_{r-n/2m,n,m} \\ &\times \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{r-(n/2)} \{\delta(x)\} \end{aligned} \quad (30)$$

if $\mu = (2s)m + m$, $v = (2l)m + m$; $s, l = 0, 1, 2, \dots$ and $r \geq n/2m$ and

$$\begin{aligned} \delta^{(r-1)}(G_-) - \delta_1^{(r-1)}(-G) &= (r-1)! a_{m,n,\mu,v} (-1)^{v-m-n/2m} (-1) b_{r-(n/2m),n,m} L_m^{k-(n/2)} \{\delta(x)\} \\ &\times \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{k-(n/2)} \{\delta(x)\} \end{aligned} \quad (31)$$

if $\mu = (2s)m + m$, $v = (2l)m + m$; $s, l = 0, 1, 2, \dots$ and $k \geq n/2m$. Where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and for integral and half-integral values of the argument, $\psi(x)$ is given by $\psi(s) = -C + 1 + 1/2 + \dots + 1/s - 1$, $s = 2, 3, \dots$ $\psi(s + 1/2) = -C - 2 \ln(2) + 2(1 + 1/3 + \dots + 1/2s - 1)$, $s = 1, 2, \dots$ and C is the Euler's constant and

$$L_m^j = \left\{ \left(\sum_{i=1}^p x_i^2 \right)^{1-m} \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^{1-m} \left(\sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right) \right\}^j.$$

Proof The proof of Lemma 2 is consequence of formulae (15)–(18) and (23) and (25).

Formulae (26), (27), (30) and (31) are generalized functions concentrated in $G = 0$. ■

On the other hand, from [1] we have the following formulae:

$$\operatorname{Re} s_{\lambda=-k}(c^2 + G)_+^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta^{(k-1)}(c^2 + G), \quad (32)$$

$$\delta^{(k-1)}(-c^2 - G) = \sum_{l \geq 0} \frac{1}{l!} \delta^{(k+l-1)}(-G)(c^2)^l, \quad (33)$$

$$\begin{aligned} \operatorname{Re} s_{\lambda=-k}(c^2 + G)_+^\lambda &= \sum_{l \geq 0} (c^2)^l \left(\frac{\lambda}{l} \right) \Big|_{\lambda=-k} \operatorname{Re} s_{\lambda=-k} G_+^{\lambda-l} \\ &= \sum_{l \geq 0} \frac{(-1)^l (c^2)^l (k+l-1)!}{l!(k-1)!} \operatorname{Re} s_{\alpha=-(k+l)} G_+^\alpha \end{aligned} \quad (34)$$

and

$$\begin{aligned} \operatorname{Re} s_{\lambda=-k}(c^2 + G)_-^\lambda &= \sum_{l \geq 0} (c^2)^l \frac{\Gamma(-\lambda + l)}{\Gamma(-\lambda)} \Big|_{\lambda=-k} \operatorname{Re} s_{\lambda=-k} G_-^{\lambda-l} \\ &= \sum_{l \geq 0} \frac{(c^2)^l (k+l-1)!}{l!(k-1)!} \operatorname{Re} s_{\alpha=-(k+l)} G_-^\alpha. \end{aligned} \quad (35)$$

From equations (32), (34), (35) and using equations (26) and (27) we have

1. If $\mu = (2s)m$, $\nu = (2l)m$, n even, $s, l = 0, 1, 2, \dots$

$$\begin{aligned} &\delta^{(k-1)}(c^2 + G) \\ &= \delta^{(k-1)}(G_+) + \sum_{\gamma=1}^{\frac{n}{2m}-k-1} \frac{(c^2)^\gamma}{\gamma!} \delta^{(k+\gamma-1)}(G_+) + \sum_{l \geq 0} \frac{(c^2)^{l+\frac{n}{2}-k} (l + \frac{n}{2m} - 1)!}{(l + \frac{n}{2} - k)!} \\ &\quad \times \left\{ \frac{(-1)^{l+\frac{n}{2m}-1}}{\Gamma((l + \frac{n}{2m}))} \delta_1^{(l+\frac{n}{2m}-1)}(G) + a_{m,n,\mu,\nu}(-1)^{\frac{\nu}{2m}} \cdot b_{l,n,m} L_m^l \{\delta(x)\} \right\} \end{aligned} \quad (36)$$

and

$$\begin{aligned} &\delta^{(k-1)}(-c^2 - G) \\ &= \delta^{(k-1)}(G_-) + \sum_{l=1}^{\frac{n}{2m}-k-1} \frac{(c^2)^l}{l!} \delta^{(k+l-1)}(G_-) + \sum_{l \geq 0} \frac{(c^2)^{l+\frac{n}{2m}-1} (l + \frac{n}{2m} - 1)!}{(l + \frac{n}{2m} - k)! (-1)^{(l+\frac{n}{2m}-1)}} \\ &\quad \times \left\{ \frac{(-1)^{(l+\frac{n}{2m}-1)!}}{\Gamma((l + \frac{n}{2m}))} \delta_1^{(l+\frac{n}{2m}-1)}(-G) + a_{m,n,\mu,\nu}(-1)^{\frac{\mu}{2m}} \cdot b_{l,n,m} (-L_m)^l \{\delta(x)\} \right\}. \end{aligned} \quad (37)$$

2. If $\mu = (2s)m + m$, $\nu = (2l)m + m$, n even, $s, l = 0, 1, 2, \dots$

$$\begin{aligned}
 & \delta^{(k-1)}(c^2 + P) \\
 &= \delta^{(k-1)}(G_+) + \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(c^2)^\nu}{\nu!} \delta^{(k+\nu-1)}(G_+) + \sum_{l \geq 0} \frac{(m^2)^{l+\frac{n}{2}-k} (l + \frac{n}{2m} - 1)!}{(l + \frac{n}{2} - k)! (-1)^{l+\frac{n}{2m}-1}} \\
 & \quad \times \left\{ \frac{(-1)^{(l+\frac{n}{2m}-1)}}{\Gamma((l + \frac{n}{2m}))} \delta_1^{(l+\frac{n}{2}-1)}(P) + a_{m,n,\mu,\nu} (-1)^{\nu-\frac{m}{2m}} \cdot b_{l,n,m} \right. \\
 & \quad \times \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] (L_m)^l \{\delta(x)\} \Big\} \quad (38)
 \end{aligned}$$

and

$$\begin{aligned}
 & \delta^{(k-1)}(-c^2 - G) \\
 &= \delta^{(k-1)}(G_-) + (-1)^{k-1} \sum_{\nu=1}^{\frac{n}{2}-k-1} \frac{(c^2)^\nu}{\nu!} \delta^{(k+\nu-1)}(G_-) \\
 & \quad + (-1)^{k-1} \sum_{l \geq 0} \frac{(m^2)^{l+\frac{n}{2}-k} (l + \frac{n}{2m} - 1)!}{(l + \frac{n}{2} - k)!} \left\{ \frac{(-1)^{l+\frac{n}{2m}-1}}{\Gamma((l + \frac{n}{2m}))} \delta_1^{(l+\frac{n}{2}-1)}(-G) \right. \\
 & \quad \left. + a_{m,n,\mu,\nu} \pi^{-1} (-1)^{\mu-\frac{m}{2m}} b_{l,n,m} \left[\psi\left(\frac{\nu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] (-L_m)^l \{\delta(x)\} \right\}. \quad (39)
 \end{aligned}$$

Remark 3 We observe that in the first and second term of above formulae (36)–(39) $n/2m - k - 1 \geq 0$ or equivalently $k \leq n/2m - 1$ while that in the third $k \geq n/2m$.

Putting $c^2 = 0$ in formulae (36)–(39) and taking into account formulae (19)–(21) we obtain the following formulae,

$$\delta_1^{(k-1)}(G) = \delta^{(k-1)}(G_+) + \delta_1^{(k-1)}(G) + \frac{(k-1)!}{(-1)^{k-1}} a_{m,n,\mu,\nu} (-1)^{\nu/2m} b_{k-n/2m,n,m} L_m^{k-n/2m} \{\delta(x)\} \quad (40)$$

and

$$\begin{aligned}
 \delta_1^{(k-1)}(-G) &= \delta^{(k-1)}(G_-) + \delta_1^{(k-1)}(-G) + (k-1)! a_{m,n,\mu,\nu} (-1)^{\mu-n/2m} (-1) \\
 & \quad \cdot b_{k-n/2m,n,m} L_m^{k-n/2m} \{\delta(x)\} \quad (41)
 \end{aligned}$$

if $\mu = (2s)m$, $\nu = (2l)m$; $s, l = 0, 1, \dots$ and $k \geq n/2m$, and

$$\begin{aligned}
 \delta_1^{(k-1)}(G) &= \delta^{(k-1)}(G_+) + \delta_1^{(k-1)}(G) + \frac{(k-1)!}{(-1)^{k-1}} a_{m,n,\mu,\nu} \pi^{-1} (-1)^{\nu-m/2m} \\
 & \quad \cdot b_{k-n/2m,n,m} \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{k-n/2m} \{\delta(x)\} \quad (42)
 \end{aligned}$$

and

$$\begin{aligned}
 \delta_1^{(k-1)}(G) &= \delta^{(k-1)}(G_-) + \delta_1^{(k-1)}(-G) + (k-1)! a_{m,n,\mu,\nu} \pi^{-1} (-1)^{\mu-m-n/2m} (-1) \\
 & \quad \cdot b_{k-n/2m,n,m} \left[\psi\left(\frac{\nu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{k-n/2m} \{\delta(x)\} \quad (43)
 \end{aligned}$$

if $\mu = (2s)m + m$, $\nu = (2l)m + m$; $s, l = 0, 1, \dots$ and $k \geq n/2m$. Where $a_{m,n,\mu,\nu}$ is defined by (28), L_m by (59) and from (29)

$$b_{k-\frac{n}{2m},n,m} = \frac{1}{(k-\frac{n}{2m})!(2m)^{2(k-\frac{n}{2m})}(\frac{n}{2m}+\frac{2(m-1)}{m})(\frac{n}{2m}+\frac{2(m-1)}{m}+1)\dots(\frac{n}{2m}+\frac{2(m-1)}{m}+k-\frac{n}{2m}-1)} \\ = \frac{\Gamma(\frac{n}{2m}+\frac{2(m-1)}{2m})}{(k-\frac{n}{2m})!(2m)^{2(k-\frac{n}{2m})}\Gamma(\frac{2(m-1)}{m}+k)} \quad (44)$$

From (40)–(43) we arrive at the following formulae:

$$\delta^{(r-1)}(G_+) = -\frac{(r-1)!}{(-1)^{r-1}}a_{m,n,\mu,\nu}(-1)^{\nu/2m} \cdot b_{r-n/2m,n,m}L_m^{r-n/2m}\{\delta(x)\} \quad (45)$$

and

$$\delta^{(r-1)}(G_-) = (r-1)!a_{m,n,\mu,\nu}(-1)^{\mu-n/2m} \cdot b_{r-n/2m,n,m}L_m^{r-n/2m}\{\delta(x)\} \quad (46)$$

if $\mu = (2s)m$, $\nu = (2l)m$; $s, l = 0, 1, \dots$ and $r \geq n/2m$, and

$$\delta^{(r-1)}(G_+) = -\frac{(r-1)!}{(-1)^{r-1}}a_{m,n,\mu,\nu}\pi^{-1}(-1)^{\nu-m/2m}b_{r-n/2m,n,m} \\ \cdot \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{r-n/2m}\{\delta(x)\} \quad (47)$$

and

$$\delta^{(r-1)}(G_-) = (r-1)!a_{m,n,\mu,\nu}\pi^{-1}(-1)^{\mu-m-n/2m} \\ \cdot b_{r-n/2m,n,m} \left[\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{n}{2}\right) \right] L_m^{r-n/2m}\{\delta(x)\} \quad (48)$$

if $\mu = (2s)m + m$, $\nu = (2l)m + m$; $s, l = 0, 1, \dots$ and $r \geq n/2m$.

Now taking into account formulae (26), (27), (30) and (31), from (45)–(48) we obtain the following formulae:

$$\delta_1^{(r-1)}(G) = -2\frac{(r-1)!}{(-1)^{r-1}}(-1)^{\nu/2m}a_{m,n,\mu,\nu}b_{r-n/2m,n,m}L_m^{r-n/2m}\{\delta(x)\} \quad (49)$$

and

$$(-1)^{r-1}\delta_2^{(r-1)}(G) = \delta_1^{(r-1)}(-G) = 2(r-1)!(-1)^{\nu/2m}a_{m,n,\mu,\nu} \cdot b_{r-n/2m,n,m}L_m^{r-n/2m}\{\delta(x)\} \quad (50)$$

if $\mu = (2s)m$, $\nu = (2l)m$; $s, l = 0, 1, \dots$ and $r \geq n/2m$. And

$$\delta_1^{(r-1)}(G) = -2(-1)^{r-1}(r-1)!(-1)^{\nu-m/2m}a_{m,n,\mu,\nu}\pi^{-1} \\ \cdot b_{r-n/2m,n,m} \left[\psi\left(\frac{\mu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{r-n/2m}\{\delta(x)\} \quad (51)$$

and

$$(-1)^{r-1}\delta_2^{(r-1)}(G) = \delta_1^{(r-1)}(-G) = 2(r-1)!(-1)^{\nu-m/2m}a_{m,n,\mu,\nu}\pi^{-1}b_{r-n/2m,n,m} \\ \times \left[\psi\left(\frac{\nu}{2m}\right) - \psi\left(\frac{n}{2m}\right) \right] L_m^{r-n/2m}\{\delta(x)\} \quad (52)$$

if $\mu = (2s)m + m$, $\nu = (2l)m + m$; $s, l = 0, 1, \dots$ and $r \geq n/2m$.

Formulae (45)–(52) are generalizations of formulae that appear in [6, p. 260–262]. For example getting $m = 1$ in (45) and considering (10), (28) and (44) we have

$$\begin{aligned}\delta^{(r-1)}(P_+) &= -\frac{(r-1)!}{(-1)^{r-1}} a_{1,n,\mu,\nu} (-1)^{\nu/2} \cdot b_{r-n/2,n,1} L_1^{r-n/2} \{\delta(x)\} \\ &= -\frac{(r-1)!}{(-1)^{r-1}} \frac{(-1)^{\nu/2} \pi^{n/2}}{2^{2(r-n/2)} (r-n/2)!} L_1^{r-n/2} \{\delta(x)\}\end{aligned}\quad (53)$$

if $\mu = (2s)$, $\nu = (2l)$; $s, l = 0, 1, \dots$ and $k \geq n/2$. Where

$$P = G(1, x) = (x_1^2 + \dots + x_p^2) - (x_{p+1}^2 + \dots + x_{p+q}^2) \quad (54)$$

and

$$L_1 = L = \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) - \left(\sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right). \quad (55)$$

Formula (53) appears in [6, p. 260].

Formulae (45)–(52) represent the proportionality between $\delta^{(k)}(G_+)$, $\delta^{(k)}(G_-)$, $\delta_1^{(k)}(G)$ and $\delta_2^{(k)}(G)$ and the operator L_m iterated $(k - n/2m)$ times.

3. Application of the basic formulae ((47) and (48))

We know from ref. [7] that the following formula is valid:

$$\delta^{(k-1)}(G_+) \cdot \delta^{(l-1)}(G_+) = A_{l,k,\nu,n} L_m^{k+l-n/2m} \delta(x) \quad (56)$$

under the following conditions,

$$(a) \quad 0 \leq k + l - \frac{n}{2m} < \frac{n}{2m} \quad (57)$$

and

$$(b) \quad \mu = 2ms + 1, \nu = 2ml + 1, \quad (58)$$

$s, l, = 0, 1, 2, \dots$ and $m = 1, 2, \dots$. Where L_m^j is defined by

$$L_m^j = \left\{ \left(\sum_{i=1}^p x_i^2 \right)^{1-m} \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^{1-m} \left(\sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right) \right\}^j \quad (59)$$

and

$$\begin{aligned}A_{l,k,\nu,n} &= -\frac{1}{2} (k-1)! (l-1)! \frac{(-1)^{k+l-1-1} (-1)^{k+l-n/2m} (-1)^{n+\mu-m/2m}}{(2m)^{2(k+l-n/2m)}} \\ &\quad \cdot \frac{\pi^{n/2-1} \Gamma(\mu/2m) \Gamma(\nu/2m)}{\Gamma(k+l) \Gamma(\mu/2) \Gamma(\nu/2) (-1)^{k+l-n/2m} (k+l-n/2m)!}\end{aligned}\quad (60)$$

From (56) and using (47) we obtain the following formula.

$$\begin{aligned}
 \delta^{(k-1)}(G_+) \cdot \delta^{(l-1)}(G_+) &= A_{l,k,v,n} L_m^{k+l-n/2m} \delta(x) = A_{l,k,v,n} \\
 &\cdot \frac{1}{-(k+l-1)!/(-1)^{k+l-1}} a_{m,n,\mu,v} \pi^{-1} (-1)^{v-m/2m} b_{k+l-n/2m,n,m} \\
 &\cdot \frac{1}{[\psi(\mu/2m) - \psi(n/2m)]} \cdot \delta^{(k+l-1)}(G_+) \\
 &= -\frac{1}{2} \frac{(-1)^{\mu-v/2m} (k-1)!(l-1)! \Gamma(2(m-1)/m + k+l)}{\Gamma(k+l)(k+l-1)! \Gamma(n/2m + 2(m-1)/m)} \\
 &\cdot \frac{1}{[\psi(\mu/2m) - \psi(n/2m)]} \delta^{(k+l-1)}(G_+) \tag{61}
 \end{aligned}$$

under the conditions $\mu = 2ms + m$, $v = 2ml + m$; $s, l = 0, 1, \dots$ and $0 \leq k+l-n/2m < n/2m$, $m = 1, 2, \dots$

Similarly from (61) and considering the formulae (15)–(18) we obtain

$$\begin{aligned}
 &\delta^{(k-1)}(G_-) \cdot \delta^{(l-1)}(G_-) \\
 &= (-1)^{k+l-2} \delta^{(k-1)}(G_+) \cdot \delta^{(l-1)}(G_+) \\
 &= (-1)^{k+l} \cdot \left(-\frac{1}{2}\right) \frac{(-1)^{\mu-\frac{v}{2m}} (k-1)!(l-1)! \Gamma(\frac{n}{2m} + \frac{2(m-1)}{2m} + k+l)}{\Gamma(k+l)(k+l-1)! \Gamma(\frac{n}{2m} + \frac{2(m-1)}{m})} \\
 &\quad \times \frac{1}{[\psi(\frac{\mu}{2m}) - \psi(\frac{n}{2m})]} \delta^{(k+l-1)}(G_+) = \frac{(-1)^{\mu-\frac{v}{2m}} (k-1)!(l-1)! \Gamma(\frac{2(m-1)}{m} + k+l)}{\Gamma(k+l)(k+l-1)! \Gamma(\frac{n}{2m} + \frac{2(m-1)}{m})} \\
 &\quad \cdot \frac{1}{[\psi(\frac{\mu}{2m}) - \psi(\frac{n}{2m})]} \delta^{(k+l-1)}(G_-) \tag{62}
 \end{aligned}$$

under conditions $\mu = 2ms + 1$, $v = 2ml + 1$; $s, l = 0, 1, \dots$ and $0 \leq k+l-n/2m < n/2m$, $m = 1, 2, \dots$

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Norm estimates of a class of Calderon–Zygmund type strongly singular integral operators

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In this article, we prove the L^p boundedness of a class of Calderon–Zygmund type strongly singular operators. In particular, we give an estimate for the L^2 norm of these operators in the unit disc of the complex plane.

Keywords: Singular integral operator; Calderon–Zygmund type operator; Forelli–Rudin estimates; Pompeiu operator

Mathematics Subject Classification: 47G10; 47B38; 30G20

1. Introduction

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$. We consider the following class of operators in $\mathbb{D}$ for $f \in L^1(\mathbb{D})$ and $k \in \mathbb{N}$,

$$\begin{aligned} \Pi_k f(z) = & \frac{(-1)^k k}{\pi} \int \int_{\mathbb{D}} \left[\left(\frac{\bar{\zeta} - \bar{z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \right. \\ & \left. + \left(\frac{\bar{\zeta} - \bar{z} + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \right] d\zeta d\bar{\zeta} \quad (z \in \mathbb{D}), \end{aligned}$$

which are strongly singular integral operators. It is known that $[1, 2]$, $\|\Pi_1\|_{L^2(\mathbb{D})} = 1$. The operators Π_k that are given in refs. [3–7], are k th order complex partial derivatives with respect to z of the operators

$$\tilde{T}_k f(z) = \frac{(-1)^k}{2\pi(k-1)!} \int \int_{\mathbb{D}} (\bar{\zeta} - \bar{z} + \zeta - z)^{k-1} \left[\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right] d\zeta d\bar{\zeta},$$

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where $f \in L^1(\mathbb{D})$ and $z \in \mathbb{D}$, i.e., $\Pi_k f(z) = \partial_z^k \tilde{T}_k$. The operator $\tilde{T}_k$, which have been introduced and investigated extensively in refs. [3–6], provides a particular solution to the equation $\partial_z^k w = f$ subject to the higher order homogeneous Schwarz conditions

$$\operatorname{Re} \partial_z^l \tilde{T}_k f = 0 \text{ on } \partial\mathbb{D}, \quad \operatorname{Im} \partial_z^l \tilde{T}_k f(0) = 0, \quad 0 \leq l \leq k-1.$$

$\tilde{T}_k$ is obtained by iterating

$$\tilde{T}_1 f(z) := -\frac{1}{2\pi} \int \int_{\mathbb{D}} \left(\frac{f(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \right) d\xi d\eta,$$

i.e., $\tilde{T}_k f(z) = \tilde{T}_{k-1}(\tilde{T}_1 f)(z)$ for $f \in L^1(\overline{\mathbb{D}})$. $\tilde{T}_1$ is a modification of the Pompeiu operator

$$Tf(z) := -\frac{1}{\pi} \int \int_{\mathbb{D}} f(\zeta) \frac{d\xi d\eta}{\zeta - z}$$

in $\mathbb{D}$, which is extensively studied by Vekua in ref. [1]. $\tilde{T}_1$ -operator is important for treating complex first-order equations, see refs. [1, 2, 5, 8].

Iterating the operator T with itself and with the $\bar{T}$ -operator

$$\bar{T}f(z) := -\frac{1}{\pi} \int \int_{\mathbb{D}} f(\zeta) \frac{d\xi d\eta}{\bar{\zeta} - \bar{z}}$$

leads to an operator $T_{m,n}$ for $m, n \in \mathbb{Z}$, $m + n \geq 0$, defined by

$$T_{0,0}f(z) = f(z),$$

$$T_{m,n}f(z) = \int \int_D K_{m,n}(z, \zeta) f(\zeta) d\xi d\eta \quad \text{for } (m, n) \neq 0,$$

where

$$K_{m,n} = \begin{cases} \frac{(-1)^m (-m)!}{(n-1)! \pi} (\zeta - z)^{m-1} (\bar{\zeta} - \bar{z})^{n-1} & \text{if } m \leq 0, \\ \frac{(-1)^n (-n)!}{(m-1)! \pi} (\zeta - z)^{m-1} (\bar{\zeta} - \bar{z})^{n-1} & \text{if } n \leq 0, \\ \frac{(\zeta - z)^{m-1} (\bar{\zeta} - \bar{z})^{n-1}}{(m-1)! (n-1)! \pi} \left[\log |\zeta - z|^2 - \sum_{\mu=1}^{m-1} \frac{1}{\mu} - \sum_{\nu=1}^{n-1} \frac{1}{\nu} \right] & \text{if } 1 \leq m, n \end{cases}$$

for $f \in L^1(D)$, $D \subset \mathbb{C}$ a domain, [6]. $T_{m,n}f$ is a particular solution of the equation $\partial_z^m \partial_{\bar{z}}^n w = f$. The operator $T_{m,n}$ is weakly singular for $m + n > 0$ and is strongly singular of Calderon–Zygmund type for $m + n = 0 < m^2 + n^2$. In particular, for strongly singular operator, it is known that [5–7],

$$\|T_{-m,m}f\|_{L^p(\mathbb{D})} \leq \Lambda_p \|f\|_{L^p(\mathbb{D})},$$

where $\Lambda_p = \|T_{-m,m}\|_{L^p(\mathbb{D})}$, $\Lambda_p \geq \Lambda_2 = 1$, for $1 < p < \infty$.

The purpose of this article is to give an estimate of the L^2 -norm of Π_k . In the following section, we will give some preliminary results and notations. The proof of the main results are in the last section.

2. Preliminaries

In the proof of our norm estimates, one of the two results we will use is given by the following theorem, usually known as Schur's test, which is a very effective tool in proving the L^p boundedness of integral operators.

THEOREM 2.1 (Schur) *Let μ be a positive measure on a measure space X . Let $K(x, y)$ be a positive measurable function on $X \times X$, and let $p > 1$ with $1/p + 1/q = 1$. If there exists a positive constant C and a positive measurable function h on X such that*

$$\int_X K(x, y)h(y)^q d\mu(y) \leq Ch(x)^q$$

and

$$\int_X K(x, y)h(x)^p d\mu(x) \leq Ch(y)^p$$

for all x and y in X , then the integral operator

$$Tf(x) = \int_X K(x, y)f(y) d\mu(y)$$

is bounded on $L^p(X, d\mu)$ with $\|T\|_{L^p(X)} \leq C$.

For the proof, see ref. [9] for example.

The second result we will use is due to Forelli–Rudin [10]. Let us start with the inner product

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$$

in $\mathbb{C}^n$. $\mathbb{B} = \{z \in \mathbb{C}^n : |z| = \langle z, z \rangle^{1/2} < 1\}$ is the unit ball of $\mathbb{C}^n$. The letter ν shall denote Lebesgue measure on $\mathbb{C}^n$, normalized so that $\nu(\mathbb{B}) = 1$. Let us associate to each complex number $s = \sigma + it$ ($\sigma > -1$, $-\infty < t < \infty$) a kernel K_s defined by

$$K_s(z, w) = \frac{(1 - |w|^2)^s}{(1 - \langle z, w \rangle)^{n+1+s}},$$

where $z, w \in \mathbb{B}$. For $z \in \mathbb{C}^n$, the letter h will be used for the function $h(z) = 1 - |z|^2$. The following proposition gives estimates for certain integrals on $\mathbb{B}$.

PROPOSITION 2.2 (a) *If $0 < c < 1 + \sigma$, then*

$$\sup_{z \in \mathbb{B}} h^c(z) \int_{\mathbb{B}} |K_s(z, w)| h^{-c}(w) d\nu(w) = A(c, s, n),$$

where

$$A(c, s, n) = \frac{n! \Gamma(1 + \sigma - c) \Gamma(c)}{|\Gamma((n + 1 + s)/2)|^2}.$$

(b) *If $0 < c + \sigma < 1 + \sigma$, then*

$$\sup_{w \in \mathbb{B}} h^c(w) \int_{\mathbb{B}} |K_s(z, w)| h^{-c}(z) d\nu(z) = A(c + \sigma, s, n).$$

In this article, Forelli and Rudin investigate a class of operators T_s , defined by

$$T_s f(z) = \binom{n+s}{n} \int_{\mathbb{B}} K_s(z, w) f(w) d\nu(w) \quad (z \in \mathbb{B}),$$

which are induced by K_s . For $1 < p < \infty$, these operators are bounded on $L^p(\mathbb{B})$ if and only if $\operatorname{Re}(1 + s) > 1/p$. This property is also observed by Begehr and Dzhravayev [11].

3. L^p norms of the operators Π_k

Our main results are the following theorems.

THEOREM 3.1 *Let f be a complex valued function in $L^p(\mathbb{D})$ where $1 < p < \infty$. Then $\Pi_k f$ also belongs to $L^p(\mathbb{D})$, and there exists a constant $C(p, k) > 0$ depending on p and k such that*

$$\|\Pi_k f\|_{L^p(\mathbb{D})} \leq C(p, k) \|f\|_{L^p(\mathbb{D})}$$

holds.

THEOREM 3.2 Π_k is a bounded operator on $L^2(\mathbb{D})$ with norm less than or equal to

$$1 + k\Gamma\left(\frac{1}{2}\right) \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{\Gamma(m + (1/2))}{\Gamma^2((m+2)/2)}.$$

Before giving the proofs of the theorems, we start with decomposing the operator Π_k

$$\Pi_k f(z) = T_{-k,k} f(z) + P_k f(z),$$

where

$$T_{-k,k} f(z) = \frac{(-1)^k k}{\pi} \int \int_{\mathbb{D}} \left(\frac{\bar{\zeta} - z}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta$$

and

$$P_k f(z) = \frac{(-1)^k k}{\pi} \int \int_{\mathbb{D}} \left(\frac{\bar{\zeta} - z + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta.$$

Proof of Theorem 3.1 Let us start with

$$\begin{aligned} P_k f(z) &= \frac{(-1)^k k}{\pi} \int \int_{\mathbb{D}} \left(\frac{\bar{\zeta} - z + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta \\ &= \frac{(-1)^k k}{\pi} \int \int_{\mathbb{D}} \left(\frac{|\zeta|^2 - 1 + \bar{\zeta}(\bar{\zeta} - z)}{1 - z\bar{\zeta}} \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta. \end{aligned}$$

Then

$$|P_k f(z)| \leq \frac{k}{\pi} \int \int_{\mathbb{D}} \frac{||\zeta|^2 - 1 + \bar{\zeta}(\bar{\zeta} - z)|^{k-1}}{|1 - z\bar{\zeta}|^{k+1}} |f(\zeta)| d\xi d\eta.$$

Using the binomial theorem and by the fact that $(\zeta - z)/(1 - z\bar{\zeta}) \leq 1$ for $|z| \leq 1$, $|\zeta| < 1$, we obtain

$$|P_k f(z)| \leq \frac{k}{\pi} \sum_{m=0}^{k-1} \binom{k-1}{m} \int \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m}{|1 - z\bar{\zeta}|^{m+2}} |f(\zeta)| d\xi d\eta.$$

Letting $0 < a < \min\{(1+m), 1/(p-1)\}$, $m = 0, 1, \dots, k-1$ we have

$$|P_k f(z)| \leq \frac{k}{\pi} \sum_{m=0}^{k-1} \binom{k-1}{m} \int \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m (1 - |\zeta|^2)^{-a/q} (1 - |\zeta|^2)^{a/q}}{|1 - z\bar{\zeta}|^{m+2}} |f(\zeta)| d\xi d\eta,$$

where $(1/p) + (1/q) = 1$. Hölder's inequality gives

$$|P_k f(z)| \leq k \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{-a}}{|1-z\bar{\zeta}|^{m+2}} d\xi d\eta \right)^{1/q} \\ \times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^m (1-|\zeta|^2)^{pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p d\xi d\eta \right)^{1/p}$$

or

$$|P_k f(z)| = k \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m-a}}{|1-z\bar{\zeta}|^{m+2}} d\xi d\eta \right)^{1/q} \\ \times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p d\xi d\eta \right)^{1/p}.$$

Since $(1/\pi) dx dy$ is the normalized measure in $\mathbb{D}$, using Proposition 2.2 part (a) with

$$0 < a < 1 + m, \quad m = 0, 1, \dots, k-1,$$

we get

$$|P_k f(z)|^p \leq k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) (1-|z|^2)^{-a} \right)^{p/q} \\ \times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p d\xi d\eta \right),$$

where

$$C_1(m, a) = \frac{\Gamma(1+m-a)\Gamma(a)}{\Gamma^2((m+2)/2)}.$$

So,

$$\int \int_{\mathbb{D}} |P_k f(z)|^p dx dy \leq k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) \right)^{p/q} \\ \times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \int \int_{\mathbb{D}} (1-|z|^2)^{-pa/q} \right. \\ \left. \times \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1-|\zeta|^2)^{m+pa/q}}{|1-z\bar{\zeta}|^{m+2}} |f(\zeta)|^p d\xi d\eta dx dy \right)$$

and utilizing Fubini theorem we obtain

$$\begin{aligned} \int \int_{\mathbb{D}} |P_k f(z)|^p \, dx \, dy &= k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) \right)^{p/q} \\ &\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} \int \int_{\mathbb{D}} (1 - |\zeta|^2)^{pa/q} |f(\zeta)|^p \right. \\ &\times \left. \frac{1}{\pi} \int \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^m (1 - |z|^2)^{-pa/q}}{|1 - z\bar{\zeta}|^{m+2}} \, dx \, dy \, d\xi \, d\eta \right). \end{aligned}$$

Applying Proposition 2.2 part (b) for

$$0 < \frac{pa}{q} + m < 1 + m, \quad m = 0, 1, \dots, k-1,$$

we get

$$\begin{aligned} \int \int_{\mathbb{D}} |P_k f(z)|^p \, dx \, dy &\leq k^p \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) \right)^{p/q} \\ &\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_2(m, a) \right) \int \int_{\mathbb{D}} |f(\zeta)|^p \, d\xi \, d\eta, \end{aligned}$$

where

$$C_2(m, a) = \Gamma(1 - (pa/q)) \frac{\Gamma((pa/q) + m)}{\Gamma^2((m+2)/2)}.$$

Thus

$$\begin{aligned} \left(\int \int_{\mathbb{D}} |P_k f(z)|^p \, dx \, dy \right)^{1/p} &\leq k \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) \right)^{1/q} \\ &\times \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_2(m, a) \right)^{1/p} \left(\int \int_{\mathbb{D}} |f(z)|^p \, d\xi \, d\eta \right)^{1/p}. \end{aligned}$$

Now, for

$$N(k, p) = k \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_1(m, a) \right)^{1/q} \left(\sum_{m=0}^{k-1} \binom{k-1}{m} C_2(m, a) \right)^{1/p},$$

we have

$$\|P_k f\|_{L^p(\mathbb{D})} \leq N(k, p) \|f\|_{L^p(\mathbb{D})}.$$

Therefore,

$$\begin{aligned} \|\Pi_k f\|_{L^p(\mathbb{D})} &\leq \|T_{-k,k} f\|_{L^p(\mathbb{D})} + \|P_k f\|_{L^p(\mathbb{D})} \\ &\leq (\Lambda_p + N(k, p)) \|f\|_{L^p(\mathbb{D})} \\ &= C(k, p) \|f\|_{L^p(\mathbb{D})}. \end{aligned}$$

■

Proof of Theorem 3.2 For $p = 2$, we have $0 < a < 1$, $m = 0, 1, \dots, k - 1$. Repeating the computations above, we find

$$\begin{aligned} & \int \int_{\mathbb{D}} |P_k f(z)|^2 \, dx \, dy \\ &= \left(k \sum_{m=0}^{k-1} \binom{k-1}{m} A(m, a) \right) \left(k \sum_{m=0}^{k-1} \binom{k-1}{m} B(m, a) \right) \int \int_{\mathbb{D}} |f(\zeta)|^2 \, d\xi \, d\eta, \end{aligned}$$

where

$$A(m, a) = \frac{\Gamma(a)\Gamma(1+m-a)}{\Gamma^2((m+2)/2)} \quad \text{and} \quad B(m, a) = \frac{\Gamma(1-a)\Gamma(m+a)}{\Gamma^2((m+2)/2)}.$$

If $a = 1/2$, then $A(m, 1/2) = B(m, 1/2)$ and

$$(C(k))^2 = \left(k \sum_{m=0}^{k-1} \binom{k-1}{m} A\left(m, \frac{1}{2}\right) \right)^2 = \left(k \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{\Gamma(1/2)\Gamma(m+(1/2))}{\Gamma^2((m+2)/2)} \right)^2$$

assumes its minimum value. Thus,

$$\|P_k f\|_{L^2(\mathbb{D})} \leq C(k) \|f\|_{L^2(\mathbb{D})}$$

which completes the proof of Theorem 3.2. ■

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A class of nonlinear integral operators preserving subordination and superordination

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In the present article, we obtain the subordination- and superordination-preserving properties of certain nonlinear integral operators defined on the space of normalized analytic functions in the open unit disk. A differential sandwich-type theorem for these integral operators is also presented.

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1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] := \{f : f \in \mathcal{H} \text{ and } f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F , or F is said to be superordinate to f , if there exists a function w , analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1 \text{ } (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \text{ } (z \in \mathbb{U}).$$

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In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [1, 2])

$$f \prec F \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

DEFINITION 1 (Miller and Mocanu [1]) *Let $\phi : \mathbb{C}^2 \rightarrow \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination:*

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \quad (1)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination or, more simply, a dominant if $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant.

DEFINITION 2 (Miller and Mocanu [3]) *Let $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the following differential superordination:*

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \quad (2)$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinated of the solutions of the differential superordination or, more simply, a subordinated if $q \prec p$ for all p satisfying (2). A univalent subordinated $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinated.

DEFINITION 3 (Miller and Mocanu [3]) *We denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where*

$$E(f) := \left\{ \zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that

$$f'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \setminus E(f)).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ with the usual normalization given by

$$f(0) = f'(0) - 1 = 0.$$

We also denote by $\mathcal{S}^*$ and $\mathcal{K}$ the well-known subclasses of $\mathcal{A}$ consisting of all functions which are, respectively, starlike and convex in $\mathbb{U}$ (see, for details, ref. [2]).

For a function $f \in \mathcal{A}$, we now introduce the following family of nonlinear integral operators $I_{\alpha, \beta}$ defined by

$$I_{\alpha, \beta}(f)(z) := \left(\frac{\alpha + \beta}{z^\beta} \int_0^z t^{\beta-1} \{f(t)\}^\alpha dt \right)^{1/\alpha} \quad (3)$$

$$(f \in \mathcal{A}; \alpha \in \mathbb{C} \setminus \{0\}; \beta \in \mathbb{C}; \Re(\alpha + \beta) > 0).$$

Various cases of the nonlinear integral operators defined by equation (3) have been extensively studied by many authors [4–10] with suitable restriction on the parameters α and β , and for f belonging to some favored classes of analytic functions. In particular, Miller *et al.* [9]

showed that the integral operator $I_{\alpha,\beta}(f)$ belongs to the classes $\mathcal{S}^*$ and $\mathcal{K}$, respectively, under certain restrictions on the parameters α and β , whenever f belongs to the classes $\mathcal{S}^*$ and $\mathcal{K}$, respectively. Their results include the results given earlier by Bernardi [4] and Libera [7].

Making use of the principle of subordination between analytic functions, Miller *et al.* [11] obtained some subordination-preserving properties for certain integral operators. Recently, Owa and Srivastava [10] investigated some subordination theorems involving the integral operators defined by equation (3). Moreover, Miller and Mocanu [3] considered differential superordinations as the dual concept of differential subordinations (see also ref. [5]). In the present article, we investigate the subordination- and superordination-preserving properties of the integral operator $I_{\alpha,\beta}$ defined by equation (3), together with a certain differential sandwich-type theorem as an interesting consequence of the results proven here.

The following lemmas will be required in our present investigation.

LEMMA 1 (Miller and Mocanu [12]) *Suppose that the function $H : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition:*

$$\Re(H(is, t)) \leq 0,$$

for all real s and $t \geq -n(1 + s^2)/2$, where n is a positive integer. If the function

$$p(z) = 1 + p_n z^n + \dots$$

is analytic in $\mathbb{U}$ and

$$\Re(H(p(z), zp'(z))) > 0 \quad (z \in \mathbb{U}),$$

then

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

LEMMA 2 (Miller and Mocanu [13]) *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If*

$$\Re(\beta h(z) + \gamma) > 0 \quad (z \in \mathbb{U}),$$

then the solution of the differential equation:

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}; q(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies the following inequality:

$$\Re(\beta q(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

LEMMA 3 (Miller and Mocanu [1]) *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let*

$$q(z) = a + a_n z^n + \dots$$

be analytic in $\mathbb{U}$ with

$$q(z) \neq a \quad \text{and} \quad n \geq 1.$$

If q is not subordinate to p , then there exist points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U} \setminus E(f)$$

for which

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \quad q(z_0) = p(\zeta_0) \quad \text{and} \quad z_0 q'(z_0) = m \zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Let $c \in \mathbb{C}$ with $\Re(c) > 0$ and let

$$N = N(c) := \frac{|c|\sqrt{1+2\Re(c)} + \Im(c)}{\Re(c)}.$$

If $R(z)$ is the univalent function defined in $\mathbb{U}$ by $R(z) = 2Nz/(1-z^2)$, then the open-door function is defined by

$$R_c(z) := R\left(\frac{z+b}{1+\bar{b}z}\right) \quad (z \in \mathbb{U}), \quad (4)$$

where $b = R^{-1}(c)$ (see, for details, Miller and Mocanu [1]).

Remark 1 The function R_c defined by equation (4) is univalent in $\mathbb{U}$,

$$R_c(0) = c \quad \text{and} \quad R_c(\mathbb{U}) = R(\mathbb{U})$$

in the complex w -plane with slits along the half-lines

$$\Re(w) = 0 \quad \text{and} \quad \Im(|w|) \geq N.$$

LEMMA 4 (Miller and Mocanu [1]) *Let*

$$\alpha, \beta \in \mathbb{C} \quad (\alpha \neq 0; \Re(\alpha + \beta) > 0).$$

If

$$f \in \mathcal{A}_{\alpha, \beta} := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \alpha \frac{zf'(z)}{f(z)} + \beta \prec R_{\alpha+\beta}(z) \quad (z \in \mathbb{U}) \right\}$$

and $R_{\alpha+\beta}(z)$ is defined by (4) with $c = \alpha + \beta$, then

$$I_{\alpha, \beta} \in \mathcal{A}, \quad \frac{I_{\alpha, \beta}(f)(z)}{z} \neq 0 \quad \text{and} \quad \Re\left(\alpha \frac{z(I_{\alpha, \beta}(f)(z))'}{I_{\alpha, \beta}(f)(z)} + \beta\right) > 0 \quad (z \in \mathbb{U}),$$

where $I_{\alpha, \beta}$ is the nonlinear integral operator defined by equation (3).

Our next two lemmas deal with the notion of subordination chain. A function $L(z, t)$ defined on $\mathbb{U} \times [0, \infty)$ is the subordination chain (or Löwner chain) if $L(\cdot, t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0, \infty)$, $L(z, \cdot)$ is continuously differentiable on $[0, \infty)$ for all $z \in \mathbb{U}$ and $L(z, s) \prec L(z, t)$ for $z \in \mathbb{U}$ and $0 \leq s < t$.

LEMMA 5 (Miller and Mocanu [3]) *Let $q \in \mathcal{H}[a, 1]$ and $\varphi : \mathbb{C}^2 \rightarrow \mathbb{C}$. Also set*

$$\varphi(q(z), zq'(z)) \equiv h(z).$$

If $L(z, t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a, 1] \cap \mathcal{Q}$, then

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U})$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

Furthermore, if $\varphi(q(z), zp'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

LEMMA 6 (Pommerenke [14]) *The function*

$$L(z, t) = a_1(t)z + \cdots$$

with

$$a_1(t) \neq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} |a_1(t)| = \infty$$

is a subordination chain if and only if

$$\Re \left(\frac{z(\partial L(z, t)/\partial z)}{\partial L(z, t)/\partial t} \right) > 0 \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

2. A set of main results

Our first subordination result involving the integral operator $I_{\alpha, \beta}$ defined by equation (3) is contained in Theorem 1 below.

THEOREM 1 *Let $f, g \in \mathcal{A}_{\alpha, \beta}$. Suppose also that*

$$\Re \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) > -\delta \quad \left(z \in \mathbb{U}; \phi(z) := \left(\frac{g(z)}{z} \right)^\alpha \right), \quad (5)$$

where

$$\delta = \frac{1 + |\alpha + \beta|^2 - |1 - (\alpha + \beta)^2|}{4\Re(\alpha + \beta)} \quad (\Re(\alpha + \beta) > 0). \quad (6)$$

Then the subordination:

$$\left(\frac{f(z)}{z} \right)^\alpha \prec \left(\frac{g(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}) \quad (7)$$

implies that

$$\left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, \beta}(g)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}), \quad (8)$$

where the integral operator $I_{\alpha, \beta}$ is defined by equation (3). Moreover, the function

$$\left(\frac{I_{\alpha, \beta}(g)(z)}{z} \right)^\alpha$$

is the best dominant.

Proof Let us define the functions F and G by

$$F(z) := \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \quad \text{and} \quad G(z) := \left(\frac{I_{\alpha, \beta}(g)(z)}{z} \right)^\alpha, \quad (9)$$

respectively. We note that F and G are well defined by Lemma 4. Without loss of generality, we can assume that G is analytic and univalent in $\overline{\mathbb{U}}$ and that

$$G'(\zeta) \neq 0 \quad (|\zeta| = 1).$$

Otherwise, we replace F and G by

$$F_r(z) = F(rz) \quad \text{and} \quad G_r(z) = G(rz) \quad (0 < r < 1),$$

respectively. Then these functions satisfy the conditions of Theorem 1 on $\overline{\mathbb{U}}$. We can thus prove that

$$F_r(z) < G_r(z) \quad (z \in \mathbb{U}),$$

which enables us to obtain (8) on letting $r \rightarrow 1$.

We first show that, if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \quad (10)$$

then

$$\Re(q(z)) > 0 \quad (z \in \mathbb{U}).$$

From the definition of (3), we obtain

$$(I_{\alpha, \beta}g(z))^\alpha \left(\alpha \frac{z(I_{\alpha, \beta}(g)(z))'}{I_{\alpha, \beta}(g)(z)} + \beta \right) \frac{1}{\alpha + \beta} = \{g(z)\}^\alpha. \quad (11)$$

We also have

$$\alpha \frac{z(I_{\alpha, \beta}(g)(z))'}{I_{\alpha, \beta}(g)(z)} = \alpha + \frac{zG'(z)}{G(z)}. \quad (12)$$

It follows from equations (11) and (12) that

$$(\alpha + \beta)\phi(z) = (\alpha + \beta)G(z) + zG'(z). \quad (13)$$

Now, by differentiating both sides of equation (13), we obtain

$$(\alpha + \beta)z\phi'(z) = zG'(z)(q(z) + \alpha + \beta),$$

which, in conjunction with equation (13), yields the following relationship:

$$\begin{aligned} 1 + \frac{z\phi''(z)}{\phi'(z)} &= 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \alpha + \beta} \\ &= q(z) + \frac{zq'(z)}{q(z) + \alpha + \beta} =: h(z). \end{aligned} \quad (14)$$

From equation (5), we have

$$\Re(h(z) + \alpha + \beta) > 0 \quad (z \in \mathbb{U}),$$

and (by using Lemma 2) we conclude that the differential equation (14) has a solution $q \in \mathcal{H}(\mathbb{U})$ with

$$q(0) = h(0) = 1.$$

Let us put

$$H(u, v) = u + \frac{v}{u + \alpha + \beta} + \delta, \quad (15)$$

where δ is given by equation (6). From equations (5), (14) and (15), we have

$$\Re(H(q(z), zq'(z))) > 0 \quad (z \in \mathbb{U}).$$

Now we proceed to show that $\Re(H(is, t)) \leq 0$ for all real s and $t \leq -(1 + s^2)/2$. From equation (15), we have

$$\begin{aligned} \Re(H(is, t)) &= \Re\left(is + \frac{t}{is + \alpha + \beta} + \delta\right) \\ &= \Re\left(\frac{t[\Re(\alpha + \beta) - (\Im(\alpha + \beta) + s)i]}{|\alpha + \beta + is|^2}\right) + \delta \\ &= \frac{t\Re(\alpha + \beta)}{|\alpha + \beta + is|^2} + \delta. \\ &\leq -\frac{E_\delta(s)}{2|\alpha + \beta + is|^2}, \end{aligned} \quad (16)$$

where

$$E_\delta(s) := (\Re(\alpha + \beta) - 2\delta)s^2 - 4\delta(\Im(\alpha + \beta))s - 2\delta|\alpha + \beta|^2 + \Re(\alpha + \beta). \quad (17)$$

For δ given by equation (6), the coefficient of s^2 in the quadratic expression $E_\delta(s)$ given by equation (17) is positive or equal to zero. To check this, put $\alpha + \beta = c$, so that

$$\Re(\alpha + \beta) = c_1 \quad \text{and} \quad \Im(\alpha + \beta) = c_2.$$

We thus have to verify that

$$c_1 \geq 2\delta = \frac{1 + |c|^2 - |1 - c^2|}{2c_1}. \quad (18)$$

The inequality in equation (18) will hold true if

$$2c_1^2 + |1 - c^2| \geq 1 + |c|^2 = 1 + c_1^2 + c_2^2,$$

that is, if

$$|1 - c^2| \geq 1 - \Re(c^2),$$

which is obviously true. Moreover, the quadratic expression $E_\delta(s)$ in s in equation (17) is a perfect square for the assumed value of δ . Hence, from equation (16), we see that $\Re(H(is, t)) \leq 0$ for all real s and $t \leq -(1 + s^2)/2$. Thus, by using Lemma 1, we conclude that

$$\Re(q(z)) > 0 \quad (z \in \mathbb{U}),$$

that is, that G defined by equation (10) is convex (univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (7) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \quad (19)$$

for the functions F and G defined by equation (9). For this purpose, we consider the function $L(z, t)$ given by

$$L(z, t) := G(z) + \frac{1+t}{\alpha+\beta} z G'(z) \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

We note that

$$\left. \frac{\partial L(z, t)}{\partial z} \right|_{z=0} = G'(0) \left(\frac{\alpha + \beta + 1 + t}{\alpha + \beta} \right) \neq 0 \quad (0 \leq t < \infty; \Re(\alpha + \beta) > 0).$$

This shows that the function

$$L(z, t) = a_1(t)z + \cdots$$

satisfies the condition $a_1(t) \neq 0$ for all $t \in [0, \infty)$. Furthermore, we have

$$\Re \left(\frac{z(\partial L(z, t)/\partial z)}{\partial L(z, t)/\partial t} \right) = \Re \left(\alpha + \beta + (1+t) \left(1 + \frac{z G''(z)}{G'(z)} \right) \right) > 0 \quad (z \in \mathbb{U}),$$

since G is convex and $\Re(\alpha + \beta) > 0$. Therefore, by virtue of Lemma 6, $L(z, t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z) = G(z) + \frac{1}{\alpha + \beta} z G'(z) = L(z, 0)$$

and

$$L(z, 0) \prec L(z, t) \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

This implies that

$$L(\zeta, t) \notin L(\mathbb{U}, 0) = \phi(\mathbb{U}) \quad (\zeta \in \partial \mathbb{U}; t \in [0, \infty)).$$

Now suppose that the function F is not subordinate to G . Then, by Lemma 3, there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial \mathbb{U}$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F'(z_0) = (1+t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0, t) &= G(\zeta_0) + \frac{1+t}{\alpha+\beta} \zeta_0 G'(\zeta_0) \\ &= F(z_0) + \frac{1}{\alpha+\beta} z_0 F'(z_0) \\ &= \left(\frac{f(z_0)}{z_0} \right)^\alpha \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (7). This contradicts the above observation that $L(\zeta_0, t) \notin \phi(\mathbb{U})$. Therefore, the subordination condition (7) must imply the subordination given by equation (19). Considering $F(z) = G(z)$, we see that the function G is the best dominant. This evidently completes the proof of Theorem 1. ■

Remark 2 We note that δ given by equation (6) in Theorem 1 satisfies the inequality $0 < \delta \leq 1/2$.

We next investigate a dual problem regarding Theorem 1 in the sense that the subordinations are replaced by superordinations.

THEOREM 2 *Let $f, g \in \mathcal{A}_{\alpha, \beta}$. Suppose also that*

$$\Re \left(1 + \frac{z\phi''(z)}{\phi'(z)} \right) > -\delta \quad \left(z \in \mathbb{U}; \phi(z) := \left(\frac{g(z)}{z} \right)^\alpha \right),$$

where δ is given by equation (6). If $(f(z)/z)^\alpha$ is univalent in $\mathbb{U}$ and

$$\left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \in \mathcal{H}[1, 1] \cap \mathcal{Q},$$

then the subordination:

$$\left(\frac{g(z)}{z} \right)^\alpha < \left(\frac{f(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}) \quad (20)$$

implies that

$$\left(\frac{I_{\alpha, \beta}(g)(z)}{z} \right)^\alpha < \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}),$$

where the integral operator $I_{\alpha, \beta}$ is defined by (3). Moreover, the function

$$\left(\frac{I_{\alpha, \beta}(g)(z)}{z} \right)^\alpha$$

is the best subordinant.

Proof The first part of the proof of Theorem 2 is similar to that of Theorem 1 and so we will use the same notation as in the proof of Theorem 1.

Now let us define the functions F and G , as before, by equation (9). We first note that, if the function q is defined by equation (10), by using equation (11) and (12), we obtain

$$\begin{aligned} \phi(z) &= G(z) + \frac{1}{\alpha + \beta} z G'(z) \\ &=: \varphi(G(z), z G'(z)). \end{aligned} \quad (21)$$

After a simple calculation, equation (21) yields the following relationship:

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \alpha + \beta}.$$

Then, by using the same method as in the proof of Theorem 1, we can prove that

$$\Re(q(z)) > 0 \quad (z \in \mathbb{U}),$$

that is, that the function G defined by equation (9) is convex (univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (20) implies that

$$F(z) \prec G(z) \quad (z \in \mathbb{U}) \quad (22)$$

for the functions F and G defined by equation (9). We now consider the function $L(z, t)$ defined by

$$L(z, t) := G(z) + \frac{t}{\alpha + \beta} z G'(z) \quad (z \in \mathbb{U}; 0 \leq t < \infty).$$

Since G is convex and $\Re(\alpha + \beta) > 0$, we can prove easily that $L(z, t)$ is a subordination chain as in the proof of Theorem 1. Therefore, according to Lemma 5, we conclude that the superordination condition (20) must imply the superordination given by equation (22). Furthermore, since the differential equation (21) has the univalent solution G , it is the best subordinant of the given differential superordination. We thus complete the proof of Theorem 2. ■

If we combine Theorem 1 and Theorem 2, we obtain the following differential sandwich-type theorem.

THEOREM 3 *Let $f, g_k \in \mathcal{A}_{\alpha, \beta}$ ($k = 1, 2$). Suppose also that*

$$\Re \left(1 + \frac{z \phi_k''(z)}{\phi_k'(z)} \right) > -\delta \quad \left(z \in \mathbb{U}; \phi_k(z) := \left(\frac{g_k(z)}{z} \right)^\alpha; k = 1, 2 \right), \quad (23)$$

where δ is given by (6). If $(f(z)/z)^\alpha$ is univalent in $\mathbb{U}$ and

$$\left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \in \mathcal{H}[1, 1] \cap \mathcal{Q},$$

then the subordination:

$$\left(\frac{g_1(z)}{z} \right)^\alpha \prec \left(\frac{f(z)}{z} \right)^\alpha \prec \left(\frac{g_2(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}) \quad (24)$$

implies that

$$\left(\frac{I_{\alpha, \beta}(g_1)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, \beta}(g_2)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}), \quad (25)$$

where $I_{\alpha, \beta}$ is the integral operator defined by equation (3). Moreover, the functions

$$\left(\frac{I_{\alpha, \beta}(g_1)(z)}{z} \right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha, \beta}(g_2)(z)}{z} \right)^\alpha$$

are the best subordinant and the best dominant, respectively.

3. Further remarks and consequences

The hypothesis of Theorem 3 that the functions

$$\left(\frac{f(z)}{z} \right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha$$

need to be univalent in $\mathbb{U}$ is not so easy to check. Consequently, we will replace these conditions by another set of conditions in the following result.

COROLLARY 1 Let $f, g_k \in \mathcal{A}_{\alpha, \beta}$ ($k = 1, 2$). Suppose also that the hypothesis (23) of Theorem 3 is satisfied and that

$$\Re \left(1 + \frac{z\psi''(z)}{\psi'(z)} \right) > -\delta \quad \left(z \in \mathbb{U}; \psi(z) := \left(\frac{f(z)}{z} \right)^\alpha \right), \quad (26)$$

where δ is given by (6). Then the subordination:

$$\left(\frac{g_1(z)}{z} \right)^\alpha \prec \left(\frac{f(z)}{z} \right)^\alpha \prec \left(\frac{g_2(z)}{z} \right)^\alpha \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha, \beta}(g_1)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, \beta}(g_2)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}),$$

where $I_{\alpha, \beta}$ is the integral operator defined by (3). Moreover, the functions

$$\left(\frac{I_{\alpha, \beta}(g_1)(z)}{z} \right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha, \beta}(g_2)(z)}{z} \right)^\alpha$$

are the best subdominant and the best dominant, respectively.

Proof In order to prove Corollary 1, we must show that the condition (26) implies the univalence of both $\psi(z)$ and

$$F(z) := \left(\frac{I_{\alpha, \beta}(f)(z)}{z} \right)^\alpha.$$

in $\mathbb{U}$. Since $0 < \delta \leq 1/2$ from Remark 2, the condition (26) means that $\psi(z)$ is a close-to-convex function in $\mathbb{U}$ (see ref. [15]) and hence $\psi(z)$ is univalent in $\mathbb{U}$. Furthermore, by using the same techniques as in the proof of Theorem 1, we can prove the convexity (univalence) of $F(z)$. We omit the details involved and thus deduce Corollary 1 from Theorem 3. ■

Upon setting $\alpha + \beta = 1$ in Theorem 3, we have the following consequence.

COROLLARY 2 Let $f, g_k \in \mathcal{A}_{\alpha, 1-\alpha}$ ($k = 1, 2$). Suppose also that

$$\Re \left(1 + \frac{z\phi_k''(z)}{\phi_k'(z)} \right) > -\frac{1}{2} \quad \left(z \in \mathbb{U}; \phi_k(z) := \left(\frac{g_k(z)}{z} \right)^\alpha; k = 1, 2 \right). \quad (27)$$

If the function $(f(z)/z)^\alpha$ is univalent in $\mathbb{U}$ and

$$(I_{\alpha, 1-\alpha} f(z)/z)^\alpha \in \mathcal{H}[1, 1] \cap \mathcal{Q},$$

then the subordination:

$$\left(\frac{g_1(z)}{z} \right)^\alpha \prec \left(\frac{f(z)}{z} \right)^\alpha \prec \left(\frac{g_2(z)}{z} \right)^\alpha \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha, 1-\alpha}(g_1)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, 1-\alpha}(f)(z)}{z} \right)^\alpha \prec \left(\frac{I_{\alpha, 1-\alpha}(g_2)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}),$$

where the integral operator $I_{\alpha, 1-\alpha}$ is defined by (3) with $\beta = 1 - \alpha$. Moreover, the functions

$$\left(\frac{I_{\alpha, 1-\alpha}(g_1)(z)}{z} \right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha, 1-\alpha}(g_2)(z)}{z} \right)^\alpha$$

are the best subdominant and the best dominant, respectively.

If we put $\alpha + \beta = 1 + i$ in Theorem 3, we are easily led to the following result.

COROLLARY 3 Let $f, g_k \in \mathcal{A}_{\alpha, 1+i-\alpha}$ ($k = 1, 2$). Suppose also that

$$\Re \left(1 + \frac{z\phi_k''(z)}{\phi_k'(z)} \right) > -\frac{3-\sqrt{5}}{4} \quad \left(z \in \mathbb{U}; \phi_k(z) := \left(\frac{g_k(z)}{z} \right)^\alpha; k = 1, 2 \right). \quad (28)$$

If the function $(f(z)/z)^\alpha$ is univalent in $\mathbb{U}$ and

$$\left(\frac{I_{\alpha, 1+i-\alpha} f(z)}{z} \right)^\alpha \in \mathcal{H}[1, 1] \cap \mathcal{Q},$$

then the subordination:

$$\left(\frac{g_1(z)}{z} \right)^\alpha < \left(\frac{f(z)}{z} \right)^\alpha < \left(\frac{g_2(z)}{z} \right)^\alpha \quad (z \in \mathbb{U})$$

implies that

$$\left(\frac{I_{\alpha, 1+i-\alpha}(g_1)(z)}{z} \right)^\alpha < \left(\frac{I_{\alpha, 1+i-\alpha}(f)(z)}{z} \right)^\alpha < \left(\frac{I_{\alpha, 1+i-\alpha}(g_2)(z)}{z} \right)^\alpha \quad (z \in \mathbb{U}),$$

where the integral operator $I_{\alpha, 1+i-\alpha}$ is defined by (3) with $\beta = 1 + i - \alpha$. Moreover, the functions

$$\left(\frac{I_{\alpha, 1+i-\alpha}(g_1)(z)}{z} \right)^\alpha \quad \text{and} \quad \left(\frac{I_{\alpha, 1+i-\alpha}(g_2)(z)}{z} \right)^\alpha$$

are the best subordinant and the best dominant, respectively.

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One non-local problem for the laplace equation in a disc

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This article deals with a non-local problem for the harmonic function in a disc, when the value of the function on one part of its semi-circle and the relation of unknown function values on the semi-circles of concentrated circles on the other part of the circle are given. This problem is reduced to the integral equation with a periodical kernel. To solve it, the analogue of the Wiener–Hopf method for a discrete case is developed. Its use allows one to obtain the problem equivalent to the factorization problem which has a unique solution. An effective solution is constructed.

Keywords: Poisson formula; Laplace equation; Fourier transform

AMS: 42A85; 45E10

1. Introduction

This article investigates a non-local boundary problem for the Laplace equation in a disc. Some boundary value problems for partial differential equations of a composite type were considered by Sjostrand [1]. These were then reduced to new problems for harmonic functions in the two-dimensional case, which are now called non-local problems. Unique existence theorems were obtained by using the theory of Fredholm integral equations. Analogous problems were posed by Bitsadze and Samarski. Unique existence theorems for a harmonic function were obtained in a rectangle [2]. Nowadays, non-local problems are widely investigated for solving various classes of differential equations.

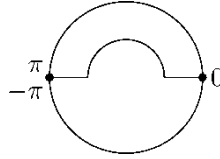
Bitsadze [3] also constructed the harmonic function in a circle, satisfying non-local boundary conditions for $0 \leq \vartheta \leq 2\pi$. Its solution is represented by Fourier series. In this article, a more complicated case is considered. The non-local boundary condition is given for $0 < \vartheta < \pi$, and it is impossible to construct solutions in the same way as in the case of ref. [3]. The analogue of the Wiener–Hopf method was developed for solving this problem. Non-local problems have

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many applications in investigation of baroclinic sea dynamics, in the theory of elasticity and shells [4, 5] and so on.

1.1 Statement of the problem

Let D be a disc with radius R , whose centre coincides with the origin of the coordinates. Consider a finite number of concentric circles, with radii R_1 satisfying the condition $0 < R_1 < R$. Let S be the boundary of D , then $\bar{D} = D \cup S$. Let S_1, S_2 be the parts of S for $0 < \vartheta < \pi, -\pi \leq \vartheta \leq 0$; accordingly, $S = S_1 \cup S_2$.



Consider the non-local problem for the Laplace equation in the disc

$$\Delta u = 0, \quad re^{i\vartheta} \in D, \quad (1)$$

$$\begin{aligned} u(R, \vartheta) - au(R_1, \vartheta) &= f(\vartheta), \quad 0 < \vartheta < \pi, \\ u(R, \vartheta) &= f_1(\vartheta), \quad -\pi \leq \vartheta \leq 0, \end{aligned} \quad (2)$$

where $z = re^{i\vartheta}$ are complex points of the circle, $f \in C([-\pi, \pi])$, $f_1 \in C([-\pi, \pi])$ are the given functions, $0 < a = \text{const.} < 1$. Δ is the Laplace operator written in polar coordinates

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \vartheta^2}.$$

For the investigation of problems (1) and (2), it is necessary to introduce some additional definitions and theorems.

Let L be a class of functions φ integrable in the interval $(-\pi, \pi)$ and periodical with period 2π . Let L^+, L^- be subclasses of the class L , whose functions φ^+, φ^- satisfy the conditions accordingly.

$$\varphi^+(x) = \begin{cases} \varphi(x), & 0 < x < \pi \\ 0, & -\pi \leq x \leq 0 \end{cases}, \quad \varphi^-(x) = \begin{cases} 0, & 0 < x < \pi \\ \varphi(x), & -\pi \leq x \leq 0. \end{cases} \quad (3)$$

DEFINITION 1.1 Let $f, g \in L(-\pi, \pi)$. Integral

$$K(\vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\vartheta - t)g(t)dt = f * g \quad (4)$$

is called a convolution [6–8].

It is obvious that if f or g are periodical functions, then the same can be said of the convolution.

THEOREM 1.2 If $f, g \in L^\pm(-\pi, \pi)$, then $f * g \in L^\pm(-\pi, \pi)$.

Proof Indeed, let $f, g \in L^+(-\pi, \pi)$. According to definition 1.1, $K(\theta) = 0$ for $-\pi \leq \theta \leq 0$, as if $t > 0$

$$f(\vartheta - t) = 0,$$

because $f \in L^+(-\pi, \pi)$, and if $t < 0$

$$g(t) = 0,$$

because $g \in L^+(-\pi, \pi)$. That means $f * g \in L^+(-\pi, \pi)$. Similarly, $f * g \in L^-(-\pi, \pi)$ if $f, g \in L^-(-\pi, \pi)$ can be shown. ■

DEFINITION 1.3 Let $f \in L(-\pi, \pi)$. f_n is called the finite Fourier transform if

$$f_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx. \quad (5)$$

If series $\sum_{n=-\infty}^{\infty} f_n e^{inx}$ is convergent, then it is called a Fourier series of function f and

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}. \quad (6)$$

Let l^-, l^+ and l denote the finite Fourier transform of classes L^-, L^+ and L , accordingly.

THEOREM 1.4 Let $f, g \in L(-\pi, \pi)$, then the finite Fourier transform of function $f * g$ is equal to the multiplication of finite Fourier transforms of functions f and g

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g) \cdot e^{-inx} dx = f_n \cdot g_n \quad (7)$$

Remark 1 If $f, g \in L^\pm(-\pi, \pi)$, then the multiplication $f_n \cdot g_n \in l^\pm$.

Remark 2 If $f_n, g_n \in l^\pm$, then the multiplication $f_n \cdot g_n \in l^\pm$.

To investigate the above problem, let problems (1) and (2) be rewritten as an integral equation. As u is a harmonic function in the disc, let us use the Poisson formula

$$u(r, \vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2)u(R, \theta) d\theta}{R^2 + r^2 - 2Rr \cos(\vartheta - \theta)}, \quad -\pi \leq \vartheta \leq \pi. \quad (8)$$

Taking into account equation (8) from (2), one obtains

$$u(R, \vartheta) - \frac{a}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - R_1^2)u(R, \theta) d\theta}{R^2 + R_1^2 - 2RR_1 \cos(\vartheta - \theta)} = f(\vartheta), \quad 0 < \vartheta < \pi.$$

Let us introduce the following notation:

$$k(\alpha) = \frac{a(R^2 - R_1^2)}{R^2 + R_1^2 - 2RR_1 \cos \alpha}. \quad (9)$$

Taking into account this notation, one gets

$$u(R, \vartheta) - \frac{1}{2\pi} \int_{-\pi}^{\pi} k(\vartheta - \theta)u(R, \theta) d\theta = f(\vartheta), \quad 0 < \vartheta < \pi. \quad (10)$$

This equation represents the analogue of Wiener–Hopf integral equation [9].

Let us introduce the notations:

$$\begin{aligned} \varphi^+(\vartheta) &= \begin{cases} u(R, \vartheta), & 0 < \vartheta < \pi \\ 0, & -\pi \leq \vartheta \leq 0 \end{cases}, & \varphi^-(\vartheta) &= \begin{cases} 0, & 0 < \vartheta < \pi \\ u(R, \vartheta), & -\pi \leq \vartheta \leq 0 \end{cases}, \\ f^+(\vartheta) &= \begin{cases} f(\vartheta), & 0 < \vartheta < \pi \\ 0, & -\pi \leq \vartheta \leq 0 \end{cases}, \\ f^-(\vartheta) &= \begin{cases} 0, & 0 < \vartheta < \pi \\ f_1(\vartheta) - \frac{1}{2\pi} \int_{-\pi}^0 k(\vartheta - \theta) f_1(\theta) d\theta \\ -\frac{1}{2\pi} \int_0^\pi k(\vartheta - \theta) \varphi^+(\theta) d\theta, & -\pi \leq \vartheta \leq 0. \end{cases} \end{aligned} \quad (11)$$

Using these notations, equation (10) will be as follows when $\vartheta \in [-\pi, \pi]$:

$$\begin{aligned} \varphi^+(\vartheta) + \varphi^-(\vartheta) - \frac{1}{2\pi} \int_{-\pi}^\pi k(\vartheta - \theta) \varphi^-(\theta) d\theta - \frac{1}{2\pi} \int_{-\pi}^\pi k(\vartheta - \theta) \varphi^+(\theta) d\theta \\ = f^+(\vartheta) + f^-(\vartheta), \quad -\pi \leq \vartheta \leq \pi. \end{aligned} \quad (12)$$

Let us use the notations

$$\begin{aligned} \varphi_n^\pm &= \frac{1}{2\pi} \int_{-\pi}^\pi \varphi^\pm(\vartheta) e^{-in\vartheta} d\vartheta, \\ f_n^\pm &= \frac{1}{2\pi} \int_{-\pi}^\pi f^\pm(\vartheta) e^{-in\vartheta} d\vartheta, \\ k_n &= \frac{1}{2\pi} \int_{-\pi}^\pi k(\vartheta) e^{-in\vartheta} d\vartheta. \end{aligned} \quad (13)$$

As $k(\vartheta - \theta)e^{-in\vartheta}$ is a periodical function with period 2π , then using the notations of equation (13), equation (12) can be rewritten in the following form:

$$\varphi_n^+ + \varphi_n^- - k_n \varphi_n^+ - k_n \varphi_n^- = f_n^+ + f_n^-, \quad n = 0, \pm 1, \pm 2, \dots,$$

where φ_n^- and f_n^+ are known.

Hence, for unknown φ_n^+ and f_n^- , one obtains

$$\varphi_n^+(1 - k_n) - f_n^- = f_n^+ - \varphi_n^-(1 - k_n), \quad n = 0, \pm 1, \pm 2, \dots \quad (14)$$

Let the following notations be introduced:

$$D_n = 1 - k_n, \quad H_n = f_n^+ - \varphi_n^-(1 - k_n), \quad n = 0, \pm 1, \pm 2, \dots \quad (15)$$

Then, equation (14) will have the form

$$\varphi_n^+ D_n - f_n^- = H_n, \quad n = 0, \pm 1, \pm 2, \dots \quad (16)$$

To show $D_n = 1 - k_n \neq 0$, $n = 0, \pm 1, \pm 2, \dots$, let k_n be calculated. Taking into account equation (9) and using the residue theory, one gets

$$k_n = a \left(\frac{R_1}{R} \right)^{|n|}, \quad n = 0, \pm 1, \pm 2, \dots \quad (17)$$

As $a < 1$, then from equation (17) one gets

$$D_n = 1 - k_n \neq 0, \quad n = 0, \pm 1, \pm 2, \dots \quad (18)$$

THEOREM 1.5 If $1 - k_n \neq 0$, then D_n is represented as follows:

$$D_n = \frac{X_n^+}{X_n^-}, \quad (19)$$

where

$$X_n^\pm = \exp \Gamma_n^\pm,$$

Γ_n^+ and Γ_n^- are represented as

$$\Gamma_n^\pm = \pm \frac{1}{2} \ln D_n - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{\ln D_m (1 - (-1)^{m-n})}{m - n} \quad (20)$$

Proof Indeed, from equation (19) one obtains

$$\ln D_n = \ln(1 - k_n) = \ln X_n^+ - \ln X_n^- = \Gamma_n^+ - \Gamma_n^-. \quad (21)$$

Let the following be denoted:

$$\gamma(\vartheta) = \sum_{m=-\infty}^{\infty} \ln(1 - k_m) e^{im\vartheta} = \sum_{m=-\infty}^{\infty} \ln D_m e^{im\vartheta},$$

then

$$\begin{aligned} \Gamma_n^+ &= \frac{1}{2\pi} \int_0^\pi \gamma(\vartheta) e^{-in\vartheta} d\vartheta = \frac{1}{2\pi} \int_0^\pi \sum_{m=-\infty}^{\infty} \ln D_m e^{i\vartheta(m-n)} d\vartheta \\ &= \frac{1}{2} \ln D_n + \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{\ln D_m (\cos(m-n)\pi - 1)}{m - n} \\ &= \frac{1}{2} \ln D_n - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{\ln D_m (1 - (-1)^{m-n})}{m - n}. \end{aligned} \quad (22)$$

Similarly, one gets

$$\Gamma_n^- = -\frac{1}{2} \ln D_n - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{\ln D_m (1 - (-1)^{m-n})}{m - n}.$$

Now, let the vanishing order of $\Gamma_n^\pm$ at infinity be determined.

As is known, $\ln(1 - k_n) \sim -k_n$, so according to equation (17), the $\ln(1 - k_n)$ vanishes at infinity with a higher order. So, the first sum also vanishes at infinity with a higher order.

Let the vanishing order at infinity of the second sum of equation (20) be determined. To this end, let the following expression be considered:

$$\begin{aligned} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{\ln D_m (1 - (-1)^{m-n})}{m - n} &= -\frac{1}{n} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \ln D_m (1 - (-1)^{m-n}) \\ &\quad + \frac{1}{n} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{m \ln D_m (1 - (-1)^{m-n})}{m - n}. \end{aligned} \quad (23)$$

It is obvious that members of series will be equal to zero for $m - n = 2k$, $k \in \mathbb{Z}$.

Hence, one gets

$$\Gamma_n^\pm = \frac{A_n}{n} + O\left(\frac{1}{n^2}\right),$$

where

$$A_n = \frac{1}{\pi i} \sum_{k=-\infty}^{\infty} \ln D_{n+2k+1}.$$

From equations (16) and (19), one gets

$$\varphi_n^+ X_n^+ - f_n^- X_n^- = H_n X_n^-, \quad n = 0, \pm 1, \pm 2, \dots \quad (24)$$

It can be shown that

$$\varphi_n^+ X_n^+ \in l^+, \quad f_n^- X_n^- \in l^-, \quad H_n X_n^- \in l.$$

Indeed, let the following be denoted:

$$X_n^\pm = e^{\Gamma_n^\pm} = 1 + X_{n0}^\pm, \quad (25)$$

where

$$X_{n0}^\pm = e^{\Gamma_n^\pm} - 1.$$

Now let it be shown that $X_{n0}^\pm = e^{\Gamma_n^\pm} - 1 \in l^\pm$. Let the following be considered:

$$e^{\Gamma_n^+} - 1 - \sum_{m=1}^N \frac{(\Gamma_n^+)^m}{m!} = \sum_{m=N+1}^{\infty} \frac{(\Gamma_n^+)^m}{m!}.$$

Let notation $m - N = k$ be introduced, after which one obtains

$$e^{\Gamma_n^+} - 1 - \sum_{m=1}^N \frac{(\Gamma_n^+)^m}{m!} = \frac{(\Gamma_n^+)^N}{N!} \sum_{k=1}^{\infty} \frac{(\Gamma_n^+)^k N!}{(N+k)!}.$$

Multiplying both parts of the equality by $e^{in\vartheta}$ and summarizing from $-\infty$ to ∞ , one gets

$$\begin{aligned} \left| \sum_{n=-\infty}^{\infty} \left(e^{\Gamma_n^+} - 1 - \sum_{m=1}^N \frac{(\Gamma_n^+)^m}{m!} \right) e^{in\vartheta} \right| &= \left| \sum_{n=-\infty}^{\infty} \frac{(\Gamma_n^+)^N}{N!} e^{in\vartheta} \sum_{k=1}^{\infty} \frac{(\Gamma_n^+)^k N!}{(N+k)!} \right| \\ &\leq \frac{1}{N!} \sum_{n=-\infty}^{\infty} |\Gamma_n^+|^N |e^{\Gamma_n^+} - 1| \longrightarrow 0, \quad \text{when } N \longrightarrow \infty, \end{aligned}$$

as $\sum_{n=-\infty}^{\infty} |\Gamma_n^+|^N$ converges and $|e^{\Gamma_n^+} - 1|$ is limited.

As $\sum_{m=1}^N ((\Gamma_n^+)^m / m!) \in l^+$, therefore $e^{\Gamma_n^+} - 1$ is equal to zero when $-\pi \leq \vartheta \leq 0$, which means $e^{\Gamma_n^+} - 1 \in l^+$. Similarly, it can be shown that $e^{\Gamma_n^-} - 1 \in l^-$.

Let $\varphi_n^+ X_n^+$ be considered:

$$\varphi_n^+ X_n^+ = \varphi_n^+ (1 + X_{n_0}^+) = \varphi_n^+ + \varphi_n^+ X_{n_0}^+.$$

According to Remark 2 of Theorem 1.4

$$\varphi_n^+ X_n^+ \in l^+, \quad f_n^- X_n^- \in l^-, \quad H_n X_n^- \in l.$$

Taking into account equation (20), (21) and (24), one obtains

$$\varphi_n^+ X_n^+ = \frac{1}{2} H_n X_n^- - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{H_m X_m^- (1 - (-1)^{m-n})}{m - n}. \quad (26)$$

Then, from equation (26), one gets

$$\varphi_n^+ = (X_n^+)^{-1} \left(\frac{1}{2} H_n X_n^- - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{H_m X_m^- (1 - (-1)^{m-n})}{m - n} \right). \quad (27)$$

Let the following be denoted:

$$\Psi_n^+ = \frac{1}{2} H_n X_n^- - \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} \frac{H_m X_m^- (1 - (-1)^{m-n})}{m - n}.$$

Let it be shown that $(X_n^+)^{-1} \Psi_n^+ \in l^+$. From equation (25), one gets

$$\frac{1}{X_n^+} = \left(e^{-\Gamma_n^+} - 1 \right) + 1 = X_{n_0}^* + 1,$$

where

$$X_{n_0}^* = e^{-\Gamma_n^+} - 1.$$

As $X_{n_0}^* \in l^+$, therefore

$$(X_n^+)^{-1} \Psi_n^+ = \Psi_n^+ X_{n_0}^* + \Psi_n^+ \in l^+.$$

Similarly, the following can be obtained:

$$(X_n^-)^{-1} \Psi_n^- \in l^-.$$

By virtue of equation (27), one obtains

$$\varphi^+(\vartheta) = \sum_{n=-\infty}^{\infty} (X_n^+)^{-1} \Psi_n^+ e^{in\vartheta}. \quad (28)$$

From equation (28), one can conclude that the degree of vanishing at infinity of the Fourier coefficient of function $\varphi^+(\vartheta)$ is equal to $1/n$, as the vanishing order at infinity of function Ψ_n^+ is $1/n$ and

$$(X_n^+)^{-1} \longrightarrow 1, \quad \text{for } n \longrightarrow \infty.$$

Hence, one can conclude that function $\varphi^+(\vartheta)$ is continuous everywhere, except at the possible points $\vartheta = 0, \pi$, where it may have the first kind of discontinuity or logarithmic singularity.

As the boundary value of the solution of problems (1) and (2) is generally integrable and the functions f , f_1 and $u(R_1, \vartheta)$ are continuous functions in the closed interval $[0, \pi]$, the function $u(R, \vartheta)$ will be continuous in the closed interval $[0, \pi]$. Therefore, for $\vartheta \in [0, \pi]$, when $\vartheta \rightarrow 0$ and $\vartheta \rightarrow \pi$, one obtains

$$\begin{aligned}\lim_{\vartheta \rightarrow 0} u(R, \vartheta) &= \lim_{\vartheta \rightarrow 0} f(\vartheta) + \lim_{\vartheta \rightarrow 0} au(R_1, \vartheta) = f(0) + au(R_1, 0), \\ \lim_{\vartheta \rightarrow \pi} u(R, \vartheta) &= \lim_{\vartheta \rightarrow \pi} f(\vartheta) + \lim_{\vartheta \rightarrow \pi} au(R_1, \vartheta) = f(\pi) + au(R_1, \pi).\end{aligned}$$

Hence, the boundary value of the function $u(R, \vartheta)$ is continuous everywhere on the circumference, except the points $\vartheta = 0, \pi$, where it has the first kind of discontinuity if

$$f(0) + au(R_1, 0) \neq f_1(0), \quad f(\pi) + au(R_1, \pi) \neq f_1(-\pi).$$

Therefore, the solution of problems (1) and (2) is restricted in the closed domain $\bar{D}$ and continuous everywhere on the circumference, except the points $\vartheta = 0, \pi$, where it does not have a limit. ■

The following theorem is true.

THEOREM 1.6 *Let $f \in C([-\pi, \pi])$, $f_1 \in C([-\pi, \pi])$, $0 < a = \text{const.} < 1$. Then there exists the unique solution of problems (1) and (2), which is represented as follows:*

$$u(r, \vartheta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2)u(R, \theta)}{R^2 + r^2 - 2Rr \cos(\vartheta - \theta)} d\theta,$$

where

$$u(R, \theta) = \begin{cases} \varphi^+(\theta), & 0 < \theta < \pi, \\ f_1(\theta), & -\pi \leq \theta \leq 0, \end{cases}$$

where $\varphi^+(\theta)$ is given by formula (28). The solution is restricted in the closed domain $\bar{D}$ and continuous everywhere except at the possible points $\vartheta = 0, \pi$.

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Commutants of the Euler operator and corresponding mean-periodic functions

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The Euler operator $\delta = t(d/dt)$ is considered in the space $C = C(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, and the operators $M: C \rightarrow C$ such that $M\delta = \delta M$ in $C^1(\mathbb{R}_+)$ are characterized. Next, for a non-zero linear functional $\Phi: C(\mathbb{R}_+) \rightarrow \mathbb{C}$ the continuous linear operators M with the invariant hyperplane $\Phi\{f\} = 0$ and commuting with δ in it are also characterized. Further, mean-periodic functions for δ with respect to the functional Φ are introduced and it is proved that they form an ideal in a corresponding convolutional algebra $(C(\mathbb{R}_+), *)$. As an application, unique mean-periodic solutions of Euler differential equations are characterized.

Keywords: Commutant; Riesz–Markov theorem; Invariant hyperplane; Convolutional algebra; Multiplier; Cyclic element; Mean-periodic function

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1. Introduction

Compared with the case of differentiation operator $D = d/dt$ in a space C of continuous functions, the problem of characterizing the continuous linear operators $M: C \rightarrow C$ commuting with the Euler operator $\delta = t(d/dt)$, *i.e.* such that

$$M\delta = \delta M$$

in C^1 , had not been so intensively treated as the corresponding problem for D . Here we can mention only the classical book of Levin [1, Ch. 8 and 9, Theorem 20, pp. 379–380], where δ is considered in spaces of entire functions.

In the operational calculus developed in Elizarraraz and Verde-Star [2] in fact some operators commuting with the Euler operator are found.

Due to the analogy of the considerations for δ and D , a short survey of the results for differentiation operator will be made.

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Bourbaki [3, Chapter 6] seems to be the first to characterize the linear continuous operators $M: C(\mathbb{R}) \rightarrow C(\mathbb{R})$ with $MD = DM$ in $C^1(\mathbb{R})$. These are the operators of the form

$$Mf(t) = \Phi_\tau\{f(t + \tau)\},$$

where Φ is a linear functional on $C(\mathbb{R})$. According to Riesz–Markov theorem ([4, Theorem 4.10.1]) Φ has the form

$$\Phi(f) = \int_\alpha^\beta f(\tau) d\sigma(\tau),$$

where $-\infty < \alpha < \beta < \infty$ and $\sigma(\tau)$ is a Radon measure.

Delsarte [5] introduced the space of the mean-periodic functions determined by the functional Φ as the kernel space of M . For details see also Schwartz [6].

One of the authors (Dimovski [7]) had found the linear continuous operators $M: C(\mathbb{R}) \rightarrow C(\mathbb{R})$, such that the subspace $C_\Phi = \{f \in C(\mathbb{R}), \Phi(f) = 0\}$ is an invariant subspace of M and M commutes with D in C_Φ^1 . It happened that these are the operators of the form

$$Mf = \mu f(t) + m * f,$$

where $\mu = \text{const}$, $m \in C(\mathbb{R})$, and $*$ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma) g(\sigma) d\sigma \right\}.$$

Quite natural is the question about the relationship between the two types of commutants. A partial answer is given by the following theorem (Dimovski and Skórník [8, 9]):

The space of the mean-periodic functions determined by the functional Φ forms an ideal in the convolutional algebra $(C(\mathbb{R}), *)$.

Similar results for the Pommiez operator $\Delta f(z) = [f(z) - f(0)]/z$ are presented by Dimovski and Hristov [10].

An interesting historical survey about commutants of differentiation operator and related operators like the Euler one can be found in the book of Korobeinik [11].

2. General commutant

THEOREM 2.1 *A linear continuous operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ with $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$ commutes with $\delta = t(d/dt)$ in $C^1(\mathbb{R}_+)$ iff it admits a representation of the form*

$$(Mf)(t) = \Phi_\tau\{f(t\tau)\} \quad (1)$$

with a continuous linear functional $\Phi: C(\mathbb{R}_+) \rightarrow \mathbb{C}$.

Proof Consider the one-parameter family T^τ , $0 < \tau < \infty$, of the shift operators defined by

$$(T^\tau f)(t) := f(t\tau), \quad 0 < \tau < \infty. \quad (2)$$

Each of them commutes with $\delta = t(d/dt)$ in $C^1(\mathbb{R}_+)$. Indeed,

$$(\delta T^\tau f)(t) = t f'(t\tau) \tau = t \tau f'(t\tau) = (\delta f)(t\tau) = (T^\tau \delta f)(t).$$

■

LEMMA 2.2 A linear operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ with $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$ commutes with $\delta = t(d/dt)$ in $C^1(\mathbb{R}_+)$ iff M commutes with T^τ for all $\tau, 0 < \tau < \infty$.

Proof First a ‘multiplicative’ version of the Taylor formula is needed. Let f be a polynomial and g be the function defined by

$$g(x) = f(e^x).$$

Then

$$f(t\tau) = g(\ln(t\tau)) = g(\ln t + \ln \tau).$$

Denote $x = \ln t$ and $\xi = \ln \tau$, i.e. $t = e^x$ and $\tau = e^\xi$, and apply the usual Taylor formula for g :

$$f(t\tau) = g(x + \xi) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!} \xi^n. \quad (3)$$

Then

$$g'(x) = \frac{dg(x)}{dx} = \frac{dg(\ln t)}{dt} \cdot \frac{dt}{dx} = \frac{df(t)}{dt} \cdot \frac{de^x}{dx} = f'(t)e^x = tf'(t) = (\delta f)(t). \quad (4)$$

Further,

$$g''(x) = (\delta^2 f)(t), \dots, g^{(n)}(x) = (\delta^n f)(t), \dots \quad (5)$$

Substituting (4) and (5) in (3) gives the desired ‘multiplicative’ Taylor formula:

$$(T^\tau f)(t) = f(t\tau) = \sum_{n=0}^{\infty} (\delta^n f)(t) \frac{(\ln \tau)^n}{n!}. \quad (6)$$

It is true for arbitrary polynomial $f(t)$.

Now suppose that M commutes with the Euler operator δ , i.e. $M\delta = \delta M$. Then, for every $\tau, 0 < \tau < \infty$, (6) implies

$$\begin{aligned} (MT^\tau f)(t) &= M \sum_{n=0}^{\infty} (\delta^n f)(t) \frac{(\ln \tau)^n}{n!} = \sum_{n=0}^{\infty} (M(\delta^n f))(t) \frac{(\ln \tau)^n}{n!} \\ &= \sum_{n=0}^{\infty} (\delta^n (Mf))(t) \frac{(\ln \tau)^n}{n!} = (T^\tau Mf)(t). \end{aligned}$$

In order to prove the opposite implication, suppose $MT^\tau = T^\tau M$ for every $\tau, 0 < \tau < \infty$, and for arbitrary polynomial $f(t)$, and reverse the order in the last chain of equalities as follows:

$$\sum_{n=0}^{\infty} (M(\delta^n f))(t) \frac{(\ln \tau)^n}{n!} = (M(T^\tau f))(t) = (T^\tau (Mf))(t) = \sum_{n=0}^{\infty} (\delta^n (Mf))(t) \frac{(\ln \tau)^n}{n!}.$$

The sums have to coincide for every τ and hence the coefficients of $(\ln \tau)^n$ are equal for arbitrary n . For $n = 1$, it follows that

$$(M(\delta f))(t) = (\delta(Mf))(t). \quad (7)$$

Assuming that (7) is true for polynomials, it follows that it is true for arbitrary $f \in C^1(\mathbb{R}_+)$ since f could be approximated by polynomials. The proof of the lemma is completed. ■

Proof of Theorem 2.1 It is a matter of a direct check to show that the operators of the form (1) commute with δ and here only the proof of the necessity is needed.

If M commutes with δ , then by the lemma

$$MT^\tau f(t) = T^\tau Mf(t), \quad 0 < \tau < \infty. \quad (8)$$

Applying the symmetry property

$$(T^\tau f)(t) = f(t\tau) = f(\tau t) = (T^t f)(\tau) \quad (9)$$

to the right hand side of (8) gives

$$(M(T^\tau f))(t) = (T^t(Mf))(\tau). \quad (10)$$

Define the linear functional Φ as

$$\Phi\{f\} := (Mf)(1).$$

Then, substituting 1 for t in (10) and taking into account that T^1 is the identity operator, one has

$$(M(T^\tau f))(1) = (T^1(Mf))(\tau) = (Mf)(\tau).$$

The left hand side is the value of the functional Φ for the function $g(t) = (T^\tau f)(t)$, and hence

$$(Mf)(\tau) = \Phi_\sigma\{(T^\tau f)(\sigma)\} = \Phi_\sigma\{(T^\sigma f)(\tau)\}.$$

Using (2) and (9), this is in fact the desired representation (1) of the commutant of δ with τ for t , and with the dumb variable σ instead of τ . This completes the proof. ■

The abundance of the operators, commuting with δ in $C(\mathbb{R}_+)$ given by Theorem 2.1, is in sharp contrast to the set of linear operators commuting with δ in $C(\Delta)$, where Δ is a segment $[a, b] \subset \mathbb{R}_+$. Then the only such operators are the trivial ones:

$$Mf(t) = cf(t), \quad c = \text{const.}$$

Such a result for differentiation operator d/dx is shown by Kahane [12]. The corresponding result for the Euler operator δ will be stated in the following theorem.

THEOREM 2.3 *Let $[a, b] \subset \mathbb{R}_+$. Then a continuous linear operator $M: C[a, b] \rightarrow C[a, b]$, such that $M: C^1[a, b] \rightarrow C^1[a, b]$, commutes with the Euler operator δ in $C^1[a, b]$ if and only if it is an operator of the form*

$$Mf(t) = cf(t),$$

with a constant c .

Proof Let $[a, b]$ be an arbitrary segment of $\mathbb{R}_+$ and let $M\delta = \delta M$ in $C^1[a, b]$. Consider the substitution $t = e^x$ as the transformation

$$Sf(t) = f(e^x) =: \tilde{f}(x). \quad (11)$$

Obviously $S: C[a, b] \rightarrow C[\ln a, \ln b]$ and $S: C^1[a, b] \rightarrow C^1[\ln a, \ln b]$. Then, denoting $D := d/dt$, one has as in (4)

$$S\delta f(t) = f'(e^x) = D Sf(t). \quad (12)$$

It is supposed that

$$M\delta f(t) = \delta M f(t).$$

Applying S on the left hand side and using (12) yields

$$SM\delta f(t) = S\delta M f(t) = DSM f(t). \quad (13)$$

Denoting by $\tilde{M}$ the operator

$$\tilde{M} = SMS^{-1}. \quad (14)$$

It is easily seen from (13) and (12) that

$$\tilde{M}D\tilde{f}(x) = D\tilde{M}\tilde{f}(x). \quad (15)$$

This means that the conditions of Kahane's theorem [12] are fulfilled for the operator $\tilde{M}$ in $C[\ln a, \ln b]$ and the result is that

$$\tilde{M}\tilde{f}(x) = c\tilde{f}(x), \quad c = \text{const},$$

which in view of (11) and (14) gives also the desired

$$Mf(t) = cf(t), \quad c = \text{const}.$$

■

3. A general convolution related to the Euler operator

Basic for the theory of differentiation operator d/dt considered in a space $C(\Delta)$ of continuous functions on an interval Δ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma)g(\sigma) d\sigma \right\}, \quad (16)$$

where Φ is a linear functional on $C(\Delta)$. Its properties are studied in detail in [13]. The operation (16) is bilinear, commutative, and associative in $C(\Delta)$. It generalizes the classical Duhamel convolution

$$(f * g)(t) = \int_0^t f(t - \tau)g(\tau) d\tau \quad (17)$$

when the functional Φ in (16) is $\Phi(f) = f(0)$.

In ref. [7] it is shown that any operator of the commutant of d/dt in $C(\Delta)$ with an invariant hyperplane $C_\Phi(\Delta) = \{f \in C(\Delta), \Phi(f) = 0\}$ has the form $Mf(t) = \mu f(t) + (m * f)(t)$ with $\mu = \text{const}$ and $m \in C(\Delta)$.

In order to extend this result to the Euler operator an analogue of the operation (16) is needed. In the literature only the analogue

$$(f * g)(t) = \int_1^t f\left(\frac{t}{\tau}\right) g(\tau) \frac{d\tau}{\tau}$$

of the Duhamel convolution (17) is known (see [2]).

DEFINITION 3.1 *The analytic function*

$$E(\lambda) = \Phi_\tau(\tau^\lambda) \quad (18)$$

is said to be the Euler indicatrix of the functional Φ .

It is also convenient to denote for the rest of this article

$$\varphi_\lambda(t) = \frac{t^\lambda}{E(\lambda)} = \frac{t^\lambda}{\Phi_\tau(\tau^\lambda)}. \quad (19)$$

Here a ‘multiplicative variant’ of (16) is proposed.

THEOREM 3.2 *Let Φ be a continuous non-zero linear functional on $C(\mathbb{R}_+)$. Then the operation*

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\} \quad (20)$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$ such that

$$\Phi(f * g) = 0. \quad (21)$$

Proof According to Riesz–Markov theorem ([4, Theorem 4.10.1])

$$\Phi\{f\} = \int_\alpha^\beta f(\tau) d\sigma(\tau)$$

with $\Delta = [\alpha, \beta] \subset \mathbb{R}_+$ and a Radon measure $\sigma(t)$. Hence, (20) is a separately continuous operation in $C(\Delta)$.

The bilinearity and the commutativity of the operation (20) are almost evident, whereas the associativity of (20) is by no means obvious and needs a proof.

Let $f(t) = t^\mu$ and $g(t) = t^\nu$. Then

$$\begin{aligned} \{t^\mu\} * \{t^\nu\} &= \Phi_\tau \left\{ \int_\tau^t \frac{(t\tau)^\mu}{\sigma^\mu} \sigma^\nu \frac{d\sigma}{\sigma} \right\} = t^\mu \Phi_\tau \left\{ \tau^\mu \int_\tau^t \sigma^{\nu-\mu-1} d\sigma \right\} \\ &= t^\mu \Phi_\tau \left\{ \tau^\mu \frac{t^{\nu-\mu} - \tau^{\nu-\mu}}{\nu - \mu} \right\} = \frac{E(\mu)t^\nu - E(\nu)t^\mu}{\nu - \mu}. \end{aligned}$$

Using this expression, it follows that

$$(\{t^\mu\} * \{t^\nu\}) * \{t^\varkappa\} = \{t^\mu\} * (\{t^\nu\} * \{t^\varkappa\}) \quad (22)$$

because both sides of (22) have one and the same symmetric form

$$t^\mu \frac{E(\nu)E(\varkappa)}{(\mu - \nu)(\mu - \varkappa)} + t^\nu \frac{E(\varkappa)E(\mu)}{(\nu - \varkappa)(\nu - \mu)} + t^\varkappa \frac{E(\mu)E(\nu)}{(\varkappa - \mu)(\varkappa - \nu)}$$

with respect to μ , ν , and $\varkappa$. Then, (22) differentiated m , n , and k times with respect to μ , ν , and $\varkappa$ correspondingly, gives

$$(\{t^\mu (\ln t)^m\} * \{t^\nu (\ln t)^n\}) * \{t^\varkappa (\ln t)^k\} = \{t^\mu (\ln t)^m\} * (\{t^\nu (\ln t)^n\} * \{t^\varkappa (\ln t)^k\}).$$

Next, passing to the limits $\mu \rightarrow +0$, $\nu \rightarrow +0$, and $\varkappa \rightarrow +0$, one gets

$$(\{(\ln t)^m\} * \{(\ln t)^n\}) * \{(\ln t)^k\} = \{(\ln t)^m\} * (\{(\ln t)^n\} * \{(\ln t)^k\}).$$

But the bilinearity of (20) implies for arbitrary polynomials P , Q , and R

$$(\{P(\ln t)\} * \{Q(\ln t)\}) * \{R(\ln t)\} = \{P(\ln t)\} * (\{Q(\ln t)\} * \{R(\ln t)\}).$$

To finish this proof, note that if $t \in \mathbb{R}_+$ then $\ln t$ covers the whole real line $\mathbb{R}$. Then Weierstrass' theorem allows any function in $C(\mathbb{R}_+)$ to be approximated almost uniformly by polynomials of $\ln t$, $t > 0$, *i.e.* by a sequence uniformly convergent to the function on each segment $[a, b] \subset \mathbb{R}_+$. Due to the continuity of the functional Φ , the desired equality holds for every $f, g, h \in C(\mathbb{R}_+)$

$$(f * g) * h = f * (g * h).$$

The second statement (21) of the theorem can be checked as follows: The function

$$h(t, \tau) = \int_{\tau}^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{d\sigma}{\sigma}$$

is antisymmetric with respect to t and τ , *i.e.* $h(t, \tau) = -h(\tau, t)$, and, hence

$$\begin{aligned} \Phi\{f * g\} &= \Phi_t\{(f * g)(t)\} = \Phi_t\Phi_\tau\{h(t, \tau)\} \\ &= \Phi_t\Phi_\tau\{-h(\tau, t)\} = -\Phi_t\Phi_\tau\{h(\tau, t)\} \\ &= -\Phi_\tau\Phi_t\{h(\tau, t)\} = -\Phi_\tau\Phi_\tau\{h(t, \tau)\} = -\Phi\{f * g\}. \end{aligned} \quad (23)$$

Here, the Fubini property of the functional Φ is used, *i.e.* the possibility of interchanging of Φ_t and Φ_τ . At the end, t and τ are also interchanged, since they are 'dumb' variables in the expression. Thus, the last chain of equalities gives $2\Phi\{f * g\} = 0$ and $\Phi\{f * g\} = 0$ holds. ■

4. The commutant of δ in an invariant hyperplane

In this section, another commutant of δ will be described. Here, it is supposed that the operators $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ preserve $C^1(\mathbb{R}_+)$, *i.e.* $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$, and additionally they preserve invariant also a hyperplane

$$C_\Phi := \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}, \quad (24)$$

i.e. $M: C_\Phi \rightarrow C_\Phi$, where $\Phi: C(\mathbb{R}_+) \rightarrow \mathbb{C}$ is an arbitrary non-zero linear functional.

The main result of this section is the explicit representation $Mf = \mu f + m * f$ of any linear continuous operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ with $M: C_\Phi \rightarrow C_\Phi$ and commuting with $\delta = t(d/dt)$ in $C_\Phi^1 := C_\Phi \cap C^1(\mathbb{R}_+)$.

To this end some auxilliary results will be considered.

LEMMA 4.1 A linear operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ with $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$ and $M: C_\Phi(\mathbb{R}_+) \rightarrow C_\Phi(\mathbb{R}_+)$ commutes with the Euler operator δ in $C_\Phi^1(\mathbb{R}_+)$ iff M commutes with L_λ in $C(\mathbb{R}_+)$, where L_λ is the right inverse in $C(\mathbb{R}_+)$ of the perturbed Euler operator $\delta_\lambda = \delta - \lambda$, satisfying the boundary condition $\Phi(L_\lambda f) = 0$.

Proof First an explicit expression for L_λ will be found. Let λ be such that $E(\lambda) \neq 0$. Then

$$L_\lambda f(t) = \int_1^t \left(\frac{t}{\tau}\right)^\lambda f(\tau) \frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)} \Phi_\tau \left\{ \int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma) \frac{d\sigma}{\sigma} \right\}. \quad (25)$$

Indeed, the general solution of the linear differential equation $t(dy/dt) - \lambda y = f(t)$ is $y = t^\lambda \left(c + \int_1^t f(\tau)/\tau^{\lambda+1} d\tau \right)$ with an arbitrary constant c . Then, using the condition $\Phi\{y\} = 0$, one obtains the value

$$c = -\frac{1}{E(\lambda)} \Phi_\tau \left\{ \int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma) \frac{d\sigma}{\sigma} \right\}.$$

Now suppose that $ML_\lambda = L_\lambda M$ in $C(\mathbb{R}_+)$ and $f \in C_\Phi^1(\mathbb{R}_+)$. To prove that

$$h = (M\delta_\lambda - \delta_\lambda M)f = 0,$$

consider

$$L_\lambda h = L_\lambda M\delta_\lambda f - L_\lambda \delta_\lambda Mf = M(L_\lambda \delta_\lambda) f - (L_\lambda \delta_\lambda) Mf = Mf - Mf = 0.$$

But $L_\lambda h = 0$ implies $\delta_\lambda L_\lambda h = 0$, i.e. $h = 0$. Hence $M\delta_\lambda f = \delta_\lambda Mf$.

Conversely, let $M\delta_\lambda f = \delta_\lambda Mf$ for every $f \in C_\Phi^1(\mathbb{R}_+)$. If $g \in C(\mathbb{R}_+)$, then there is a function $f \in C_\Phi^1(\mathbb{R}_+)$, for which $f = L_\lambda g$. After the substitution $f = L_\lambda g$ in $\delta_\lambda Mf = M\delta_\lambda f$, one gets

$$\delta_\lambda (ML_\lambda g) = M\delta_\lambda L_\lambda g = Mg.$$

Since $\Phi\{L_\lambda g\} = 0$, then $\Phi\{ML_\lambda g\} = 0$. But the solution of the equation $\delta_\lambda y = Mg$ with the condition $\Phi\{y\} = 0$ by definition is $y = L_\lambda (Mg)$, which implies

$$ML_\lambda g = L_\lambda Mg$$

in $C(\mathbb{R}_+)$, which completes the proof. ■

LEMMA 4.2 The operator L_λ given by (25) is a convolution operator of the form

$$L_\lambda f = \varphi_\lambda * f = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * f. \quad (26)$$

Proof The equality (26) can be checked directly using (21) and the representation $\int_\tau^t = \int_1^t - \int_1^\tau$. ■

THEOREM 4.3 The commutant of δ in the invariant hyperplane C_Φ coincides with the commutant of any of the operators L_λ in $C(\mathbb{R}_+)$.

Proof Let $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ be a linear operator commuting with L_λ for some $\lambda \in \mathbb{C}$, i.e. $ML_\lambda = L_\lambda M$. First, it will be proved that C_Φ is an invariant hyperplane for M . Indeed, let g be a function from $C(\mathbb{R}_+)$ and f be the solution of the problem

$$\delta f - \lambda f = g, \quad \Phi\{f\} = 0. \quad (27)$$

Then

$$L_\lambda Mg = ML_\lambda g = Mf \quad (28)$$

and hence

$$Mg = (\delta - \lambda)Mf.$$

Using (27) this can be written as

$$M(\delta - \lambda)f = (\delta - \lambda)Mf$$

or, equivalently,

$$(M\delta)f = (\delta M)f.$$

It remains to show that $\Phi\{Mf\} = 0$. This follows using (28) and the representation (26) of L_λ as a convolutional operator, along with the property $\Phi\{p * q\} = 0$ for arbitrary $p, q \in C(\mathbb{R}_+)$ of the convolution (20).

Conversely, let $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ have the hyperplane C_Φ as an invariant subspace and let $M\delta = \delta M$ in C_Φ^1 . One has to prove that $ML_\lambda = L_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$.

Let $f \in C(\mathbb{R}_+)$ be arbitrary and denote $h = (ML_\lambda - L_\lambda M)f$. Then

$$(\delta - \lambda)h = (\delta - \lambda)ML_\lambda f - Mf = M(\delta - \lambda)L_\lambda f - Mf = 0$$

and also

$$\Phi\{h\} = \Phi\{ML_\lambda f\} - \Phi\{L_\lambda Mf\} = 0,$$

according to our assumptions. Since λ is not an eigenvalue, i.e. $E(\lambda) \neq 0$, then $h = 0$, or

$$ML_\lambda f = L_\lambda Mf.$$

The proof is completed. ■

DEFINITION 4.4 A linear operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ is said to be a multiplier of the convolutional algebra $(C(\mathbb{R}_+), *)$ when for arbitrary $f, g \in C(\mathbb{R}_+)$ it holds

$$M(f * g) = (Mf) * g.$$

THEOREM 4.5 A linear operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ with $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$ is a multiplier of the convolution algebra $(C(\mathbb{R}_+), *)$ iff it has a representation of the form

$$Mf(t) = \mu f(t) + (m * f)(t), \quad (29)$$

where $\mu = \text{const}$ and $m \in C(\mathbb{R}_+)$.

Proof The sufficiency is obvious. In order to prove the necessity, the notations from (18) and (19) will be used for convenience.

Let $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ be an arbitrary multiplier of $(C(\mathbb{R}_+), *)$. Applying (26), one has

$$ML_\lambda f = M(\varphi_\lambda * f) = (M\varphi_\lambda) * f = \varphi_\lambda * Mf = L_\lambda Mf, \quad (30)$$

i.e. $ML_\lambda f = L_\lambda Mf$. Also, denoting $e_\lambda = M\varphi_\lambda$, one has $e_\lambda \in C^1(\mathbb{R}_+)$, and (30) gives

$$L_\lambda Mf = e_\lambda * f.$$

It remains to apply the operator $\delta_\lambda = \delta - \lambda$ and the definition of L_λ as the right inverse operator of δ_λ to obtain

$$Mf = \delta_\lambda(e_\lambda * f).$$

The right hand side can be represented in a different way using the identity

$$\delta_\lambda(u * v) = (\delta_\lambda u) * v + \Phi(u)v, \quad (31)$$

which can be checked directly. Then

$$(Mf)(t) = [(\delta_\lambda e_\lambda) * f](t) + \Phi(e_\lambda)f(t),$$

which is the representation (29) with $\mu = \Phi(e_\lambda) = \Phi\{M\varphi_\lambda\}$ and $m(t) = (\delta_\lambda e_\lambda)(t) = [\delta_\lambda M\varphi_\lambda](t)$. Thus, the necessity is proved. ■

THEOREM 4.6 *The function $\varphi_\lambda(t) = t^\lambda / E(\lambda)$ is a cyclic element of the operator L_λ .*

Proof Let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. It is needed to prove that there is a sequence of functions of the form

$$f_n(t) = \sum_{k=0}^n c_{nk} L_\lambda^k \varphi_\lambda(t), \quad n = 1, 2, \dots$$

converging to $f(t)$ uniformly on any segment $[a, b]$ of $\mathbb{R}_+$.

First, it is easy to show by induction that

$$L_\lambda^k \varphi_\lambda(t) = t^\lambda p_k(\ln t), \quad (32)$$

where p_k is a polynomial of degree k , i.e. $p_k(\ln t) = \sum_{s=0}^k a_{ks} (\ln t)^s$.

Indeed, if $k = 1$, then by (26) and (20)

$$\begin{aligned} L_\lambda \varphi_\lambda(t) &= \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * \left\{ \frac{t^\lambda}{E(\lambda)} \right\} = \frac{1}{E^2(\lambda)} \Phi_\tau \left\{ \int_\tau^t \left(\frac{t\tau}{\sigma} \right)^\lambda \sigma^\lambda \frac{d\sigma}{\sigma} \right\} \\ &= \frac{1}{E^2(\lambda)} t^\lambda \Phi_\tau \left\{ \tau^\lambda \int_\tau^t \frac{d\sigma}{\sigma} \right\} = t^\lambda \left[\frac{\Phi_\tau\{\tau^\lambda\}}{E^2(\lambda)} \ln t - \frac{\Phi_\tau\{\tau^\lambda \ln \tau\}}{E^2(\lambda)} \right]. \end{aligned}$$

Next, the inductive step will be made. Suppose that

$$L_\lambda^{k-1} \varphi_\lambda(t) = t^\lambda p_{k-1}(\ln t).$$

Then

$$\begin{aligned} L_\lambda^k \varphi_\lambda(t) &= L_\lambda(L_\lambda^{k-1} \varphi_\lambda(t)) = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * L_\lambda^{k-1} \varphi_\lambda(t) \\ &= \frac{1}{E(\lambda)} \Phi_\tau \left\{ \int_\tau^t \left(\frac{t\tau}{\sigma} \right)^\lambda \sigma^\lambda p_{k-1}(\ln \sigma) \frac{d\sigma}{\sigma} \right\} \\ &= \frac{1}{E(\lambda)} t^\lambda \Phi_\tau \left\{ \tau^\lambda \int_\tau^t p_{k-1}(\ln \sigma) d \ln \sigma \right\}. \end{aligned}$$

The integration of p_{k-1} gives a polynomial q_k of $\ln t$ of degree k and the above chain of equalities can be continued as

$$\begin{aligned} L_\lambda^k \varphi_\lambda(t) &= \frac{1}{E(\lambda)} t^\lambda \Phi_\tau \{ \tau^\lambda [q_k(\ln t) - q_k(\ln \tau)] \} \\ &= t^\lambda \left[\frac{\Phi_\tau \{ \tau^\lambda \}}{E(\lambda)} q_k(\ln t) - \frac{\Phi_\tau \{ \tau^\lambda q_k(\ln \tau) \}}{E(\lambda)} \right], \end{aligned}$$

where the expression in the square brackets is obviously a polynomial p_k of $\ln t$ of degree k , as desired.

Now let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. Consider the function $\tilde{f}(t) = f(t)/t^\lambda$, which is again in $C(\mathbb{R}_+)$. Making the substitution $t = e^x$, $x = \ln t$, the new function $g(x) = \tilde{f}(t)$ is in $C(-\infty, \infty)$. By Weierstrass' theorem, g can be approximated almost uniformly on $(-\infty, \infty)$ by a sequence of polynomials $\{r_n(x)\}_{n=1}^\infty$, $r_n(x) = \sum_{k=0}^n b_{nk} x^k$, i.e. the convergence is uniform on any segment $[a, b] \subset (\mathbb{R}_+)$. Returning to the old variable, $\tilde{f}(t)$ can be approximated by the sequence of polynomials $\{r_n(\ln t) = \sum_{k=0}^n b_{nk} (\ln t)^k\}_{n=1}^\infty$. Finally, multiplying by t^λ and using (32), the desired approximation of $f(t)$ on $(\mathbb{R}_+)$ follows from the representation

$$f_n(t) = t^\lambda r_n(\ln t) = \sum_{k=0}^n b_{nk} t^\lambda (\ln t)^k = \sum_{k=0}^n c_{nk} t^\lambda p_k(\ln t) = \sum_{k=0}^n c_{nk} L_\lambda^k \varphi_\lambda(t).$$

The new coefficients c_{nk} can be calculated from the old ones b_{nk} . Thus, φ_λ is a cyclic element of L_λ in $C(\mathbb{R}_+)$. ■

THEOREM 4.7 A linear operator $M: C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$, such that $M: C^1(\mathbb{R}_+) \rightarrow C^1(\mathbb{R}_+)$, and with an invariant hyperplane $C_\Phi = \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}$ commutes with δ in C_Φ^1 if and only if it has a representation of the form

$$(Mf)(t) = \mu f(t) + (m * f)(t) \quad (33)$$

with a constant $\mu \in \mathbb{C}$ and $m \in C(\mathbb{R}_+)$.

Proof Since $\Phi\{f * g\} = 0$ for $f, g \in C(\mathbb{R}_+)$ (see (10)), then each operator of the form (33) has C_Φ as an invariant subspace. It commutes with δ in C_Φ^1 . Indeed, if $f \in C_\Phi^1$, then (31) gives

$$\delta(m * f) = m * \delta f + \Phi\{f\}m$$

and, using (33),

$$\delta Mf = \mu \delta f + m * (\delta f) + \Phi\{f\}m = \mu \delta f + m * (\delta f) = M \delta f.$$

The sufficiency is proved.

In order to prove the necessity of (33), according to Lemma 4.1, $ML_\lambda = L_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$. As it is shown in [13] (Theorem 1.3.11, p. 33), the commutant of L_λ coincides with the ring of the multipliers of the convolution algebra $(C(\mathbb{R}_+), *)$ since L_λ has a cyclic element. By Theorem 4.6 such a cyclic element is the function $\varphi_\lambda(t) = t^\lambda/E(\lambda)$ for which $L_\lambda f = \varphi_\lambda * f$. The proof is completed. ■

Remark The constant μ and the function $m \in C(\mathbb{R}_+)$ in (29) are uniquely determined. Indeed, assume that $\mu f + m * f = \mu_1 f + m_1 * f$. Take f such that $\Phi(f) \neq 0$. Then, (23) implies $\mu \Phi(f) = \mu_1 \Phi(f)$, and hence $\mu = \mu_1$. From $m * f = m_1 * f$ for arbitrary $f \in C(\mathbb{R}_+)$ it follows that $(m - m_1) * f = 0$, and hence $m = m_1$.

5. Mean-periodic functions for the Euler operator

DEFINITION 5.1 A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to the linear functional Φ if

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically in $\mathbb{R}_+$.

It is clear that the mean-periodic functions with respect to Φ form the kernel space of the operator

$$Mf(t) = \Phi_\tau\{f(t\tau)\}$$

commuting with the Euler operator δ in $C(\mathbb{R}_+)$.

Now a connection between the mean-periodic functions and the convolutional algebra $(C(\mathbb{R}_+), *)$ will be shown.

THEOREM 5.2 The mean-periodic functions for the Euler operator δ with respect to any non-zero functional $\Phi: C(\mathbb{R}_+) \rightarrow \mathbb{C}$ form an ideal in the convolutional algebra $(C(\mathbb{R}_+), *)$.

Proof One need prove only that the convolutional product $(f * g)(t)$ of a mean-periodic function f and an arbitrary function $g \in C(\mathbb{R}_+)$ is a mean-periodic function, too, i.e. it is given that $\Phi_\tau\{f(t\tau)\} = 0$ and then $\Phi_\tau\{(f * g)(t\tau)\} = 0$ is to be shown. By (20)

$$(f * g)(t\tau) = \Phi_\sigma \left\{ \int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\}$$

and

$$\begin{aligned} \Phi_\tau\{(f * g)(t\tau)\} &= \Phi_\tau \Phi_\sigma \left\{ \int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} \\ &= \Phi_\tau \Phi_\sigma \left\{ \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} \\ &\quad + \Phi_\tau \Phi_\sigma \left\{ \int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\}. \end{aligned} \quad (34)$$

Interchanging τ and σ in the first term of (34) and using the Fubini commutational property of the functionals yields

$$\begin{aligned} \Phi_\tau \Phi_\sigma \left\{ \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} &= \Phi_\sigma \Phi_\tau \left\{ \int_\tau^\sigma f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} \\ &= \Phi_\sigma \Phi_\tau \left\{ - \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} \\ &= -\Phi_\tau \Phi_\sigma \left\{ \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\}, \end{aligned}$$

thus obtaining

$$\Phi_\tau \Phi_\sigma \left\{ \int_\sigma^\tau f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} = 0. \quad (35)$$

The second term in (34) also vanishes

$$\Phi_\tau \Phi_\sigma \left\{ \int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{d\eta}{\eta} \right\} = \Phi_\tau \left\{ \int_\tau^{t\tau} \Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right) \right\} g(\eta) \frac{d\eta}{\eta} \right\} = 0 \quad (36)$$

since f is mean-periodic and hence

$$\Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right) \right\} = 0.$$

Finally, equations (34)–(36) give the desired result $\Phi_\tau\{(f * g)(t\tau)\} = 0$. ■

6. Application to the Euler differential equation

Now Theorem 5.2 will be applied to find necessary and sufficient conditions in order the Euler differential equation

$$P(\delta)y(t) = f(t), \quad 0 < t < \infty, \quad (37)$$

to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$. Here, $\delta = t(d/dt)$ is the Euler operator and $P(\mu) = a(\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_k)$ is a polynomial.

THEOREM 6.1 *In order for the Euler differential equation (37) to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$, it is necessary and sufficient no roots of the equation $P(\lambda) = 0$ to be roots of the Euler indicatrix $E(\lambda) = \Phi_\tau(\tau^\lambda)$.*

Proof Consider the Euler differential equation (37). It is clear that in order for y to be a mean-periodic solution, the right hand side, *i.e.* the function $f(t)$, should be mean-periodic, too. Formally, let $Mf(t) = \Phi_\tau\{f(t\tau)\}$. Applying M to (37) and using the commutativity of $\delta = t(d/dt)$ and M yields

$$P(\delta)My(t) = Mf(t).$$

Then from $My = 0$ it follows that $Mf = 0$, *i.e.* the requirement f to be mean-periodic is a necessary condition for existing of a mean-periodic solution y . It can be shown that it is also a sufficient condition, but in general the solution may not be unique. Indeed, if a root μ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda)$, then the function t^μ is a solution of the homogeneous equation $P(\delta)u = 0$, and hence the uniqueness of the solution holds no more.

Now it will be shown that if neither of the roots $\mu_1, \mu_2, \dots, \mu_k$ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$, then there exists a unique mean-periodic solution of the Euler equation $P(\delta)y = f$, provided f is a mean-periodic function with respect to Φ .

Assuming that y is a mean-periodic solution of (37), an explicit expression for y will be obtained. Let P be a polynomial of degree k

$$P(\mu) = a(\mu - \mu_1)(\mu - \mu_2) \cdots (\mu - \mu_k).$$

From the assumption that y is a mean-periodic solution it follows that

$$\Phi\{y\} = \Phi\{\delta y\} = \cdots = \Phi\{\delta^{k-1}y\} = 0. \quad (38)$$

Indeed, the mean-periodicity of y means that

$$\Phi_\tau\{y(t\tau)\} = 0.$$

Applying the operator δ to this identity with respect to t , Theorem 2.1 gives

$$\Phi_\tau\{(\delta^n y)(t\tau)\} = 0, \quad n = 1, 2, \dots, k-1.$$

It remains to put $t = 1$ in order to obtain the boundary conditions (38).

Next, unique solution of (37) is

$$y = \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \cdots L_{\mu_1} f(t). \quad (39)$$

Indeed, the equation (37) can be represented as

$$(\delta - \mu_1)[(\delta - \mu_2) \cdots (\delta - \mu_k)y(t)] = \frac{1}{a} f(t).$$

Denoting the square brackets by $u_1(t)$ yields

$$\delta u_1 - \mu_1 u_1 = \frac{1}{a} f,$$

for u_1 with $\Phi\{u_1\} = 0$, as it follows from (38). This equation has the unique solution $u_1 = (1/a)L_{\mu_1} f$ with L_{μ_1} defined as in Lemma 4.1. Next solve

$$\delta u_2 - \mu_2 u_2 = u_1, \quad \Phi\{u_2\} = 0,$$

for $u_2(t) = (\delta - \mu_3) \cdots (\delta - \mu_k)y(t)$ with the unique solution $u_2 = L_{\mu_2} u_1$. Continuing in the same manner one gets the unique solution (39) of the initial equation (37). Now it is easy to verify that (39) is indeed a mean-periodic solution. It can be written in the form of convolutional product using Lemma 4.2:

$$y = \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \cdots L_{\mu_1} f(t) = \left(\frac{1}{a} \varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \cdots * \varphi_{\mu_1} \right) * f = \varphi * f \quad (40)$$

with $\varphi := (1/a)\varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \cdots * \varphi_{\mu_1}$. It remains to use Theorem 5.2 to assert that the mean-periodicity of f implies the mean-periodicity of y . ■

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Analogue of the Fock space

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We generalize V. Bargmann's theorem for the Fock space.

Keywords: Entire functions; Reproducing kernel; Bergman kernel; Orthosimilar system; Hilbert space; Adjoint operator

1. Introduction

The Fock space F consists of all entire functions g on $\mathbb{C}^n$ such that

$$\|g\|_F^2 \stackrel{\text{def}}{=} \frac{1}{\pi^n} \int_{\mathbb{C}^n} |g(\xi)|^2 e^{-|\xi|^2} dv_n(\xi) < \infty,$$

where v_n is the Lebesgue measure on $\mathbb{C}^n$.

This space is the Hilbert space with inner product

$$(f, g)_F = \frac{1}{\pi^n} \int_{\mathbb{C}^n} f(\xi) \overline{g(\xi)} e^{-|\xi|^2} dv_n(\xi).$$

A system $\{e^{\langle z, \lambda \rangle}, \lambda \in \mathbb{C}^n\}$, $\langle z, \lambda \rangle \stackrel{\text{def}}{=} z_1 \lambda_1 + \cdots + z_n \lambda_n$, is dense in F .

The set

$$\widehat{F} \stackrel{\text{def}}{=} \{\widehat{f} | \widehat{f}(\lambda) \stackrel{\text{def}}{=} (e^{\langle z, \lambda \rangle}, f(z))_F, \forall f \in F, \lambda \in \mathbb{C}^n\}$$

is the Hilbert space of entire function on $\mathbb{C}^n$ with respect to the inner product $(\widehat{f}, \widehat{g})_{\widehat{F}} \stackrel{\text{def}}{=} (g, f)_F$. The following result is well known [1].

THEOREM 1 *The space $\widehat{F}$ coincides with F .*

In this article, we generalize Theorem 1.

The articles [2–5] are also closely related to the theme of our article.

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2. Main results

Denote by H_μ the space of all entire functions g on $\mathbb{C}^n$ such that

$$\|g\|_{H_\mu}^2 \stackrel{\text{def}}{=} \int_{\mathbb{C}^n} |g(\xi)|^2 d\mu(\xi) < \infty,$$

where μ is a Borel measure on $\mathbb{C}^n$.

The space H_μ is the Hilbert space with inner product

$$(f, g)_{H_\mu} = \int_{\mathbb{C}^n} f(\xi) \overline{g(\xi)} d\mu(\xi).$$

We assume that for any point $\xi \in \mathbb{C}^n$, the functional $f \rightarrow f(\xi)$ is a linear continuous functional on H_μ . The Riesz–Fisher theorem implies that for any point $\xi \in \mathbb{C}^n$, there exists an element $K_{H_\mu}(z, \xi) \in H_\mu$ such that

$$f(\xi) = (f(z), K_{H_\mu}(z, \xi)), \quad \forall f \in H_\mu.$$

The function $K_{H_\mu}(z, \xi)$ as function of variables $z, \xi \in \mathbb{C}^n$ is called the Bergman kernel of the space H_μ [6]. The following condition holds [6]:

1. The function $K_{H_\mu}(z, \xi)$ is holomorphic as a function of z and antiholomorphic as a function of ξ .
2. $K_{H_\mu}(z, \xi) = \overline{K_{H_\mu}(\xi, z)}$, $z, \xi \in \mathbb{C}^n$.

Let Ω be a space with measure μ .

DEFINITION 1 [7] *Let H be a Hilbert space. A system of elements $\{y_\omega\}_{\omega \in \Omega} \in H$ is called orthosimilar system in H if any element $x \in H$ can be represented in the form*

$$x = \int_{\Omega} (x, y_\omega)_H y_\omega d\mu.$$

We have

THEOREM 2 *The system $\{K_{H_\mu}(z, \xi)\}_{\xi \in \mathbb{C}^n}$ is orthosimilar system in H_μ , that is, any function $f \in H_\mu$ can be represented in the form*

$$f(\lambda) = \int_{\mathbb{C}^n} (f(z), K_{H_\mu}(z, \xi))_{H_\mu} K_{H_\mu}(\lambda, \xi) d\mu(\xi), \quad \lambda \in \mathbb{C}^n. \quad (1)$$

Proof For $f \in H_\mu$, we have

$$\begin{aligned} f(\lambda) &= (f(\xi), K_{H_\mu}(\xi, \lambda))_{H_\mu} = \int_{\mathbb{C}^n} f(\xi) \overline{K_{H_\mu}(\xi, \lambda)} d\mu(\xi) = \int_{\mathbb{C}^n} f(\xi) K_{H_\mu}(\lambda, \xi) d\mu(\xi) \\ &= \int_{\mathbb{C}^n} (f(z), K_{H_\mu}(z, \xi))_{H_\mu} K_{H_\mu}(\lambda, \xi) d\mu(\xi), \quad \lambda \in \mathbb{C}^n. \end{aligned} \quad (2)$$

Thus, $\{K_{H_\mu}(z, \xi)\}_{\xi \in \mathbb{C}^n}$ is orthosimilar system in H_μ .

Setting $w = \bar{\xi}$ in equation (1), we get

$$f(\lambda) = \int_{\mathbb{C}^n} (f(z), K_{H_\mu}(z, \bar{w}))_{H_\mu} K_{H_\mu}(\lambda, \bar{w}) d\mu(\bar{w}). \quad (3)$$

THEOREM 3 *The set*

$$\tilde{H}_\mu \stackrel{\text{def}}{=} \{\tilde{f} | \tilde{f}(w) = (K_{H_\mu}(z, \bar{w}), f(z))_{H_\mu}, \forall f \in H_\mu, w \in \mathbb{C}^n\}$$

is the Hilbert space of entire functions on $\mathbb{C}^n$ with respect to the inner product $(\tilde{f}, \tilde{g})_{\tilde{H}_\mu} \stackrel{\text{def}}{=} (g, f)_{H_\mu}$; and

$$\|\tilde{f}\|_{\tilde{H}_\mu}^2 = \int_{\mathbb{C}^n} |\tilde{f}(w)|^2 d\mu(\bar{w}), \quad \tilde{f} \in \tilde{H}_\mu. \quad (4)$$

If $g(w)$, $w \in \mathbb{C}^n$ is an entire function with

$$\int_{\mathbb{C}^n} |g(w)|^2 d\mu(\bar{w}) < \infty,$$

then $g \in \tilde{H}_\mu$.

Proof The function $K_{H_\mu}(z, \xi)$ is antiholomorphic as function of ξ , so the function $K_{H_\mu}(z, \bar{w})$ is holomorphic as function of w . It is easy to prove that the function $\tilde{f}(w) = (K_{H_\mu}(z, \bar{w}), f(z))_{H_\mu}$ is an entire function on $\mathbb{C}^n$. It follows from equation (3) that

$$\begin{aligned} \|\tilde{f}\|_{\tilde{H}_\mu}^2 &= \|f\|_{H_\mu}^2 = (f(\lambda), f(\lambda))_{H_\mu} = \left(\int_{\mathbb{C}^n} (f(z), K_{H_\mu}(z, \bar{w}))_{H_\mu} K_{H_\mu}(\lambda, \bar{w}) d\mu(\bar{w}), f(\lambda) \right)_{H_\mu} \\ &= \int_{\mathbb{C}^n} |\tilde{f}(w)|^2 d\mu(\bar{w}), \quad \tilde{f} \in \tilde{H}_\mu. \end{aligned} \quad (5)$$

Let $g(w)$, $w \in \mathbb{C}^n$ be an entire function such that

$$\int_{\mathbb{C}^n} |g(w)|^2 d\mu(\bar{w}) < \infty.$$

It is easy to see that the function $\overline{g(\bar{w})}$ as function of w belongs to the space H_μ . We denote

$$h(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}^n} K_{H_\mu}(z, \bar{w}) \overline{g(\bar{w})} d\mu(\bar{w}), \quad z \in \mathbb{C}^n.$$

Then

$$\begin{aligned} h(z) &= \int_{\mathbb{C}^n} K_{H_\mu}(z, \bar{w}) \overline{g(\bar{w})} d\mu(\bar{w}) = \int_{\mathbb{C}^n} K_{H_\mu}(z, w) \overline{g(\bar{w})} d\mu(w) \\ &= (\overline{g(\bar{w})}, K_{H_\mu}(w, z))_{H_\mu} = \overline{g(\bar{z})}, \quad \forall z \in \mathbb{C}^n. \end{aligned} \quad (6)$$

From equation (6), it follows that

$$g(\bar{z}) = \overline{(\overline{g(\bar{w})}, K_{H_\mu}(w, z))_{H_\mu}} = (K_{H_\mu}(w, z), \overline{g(\bar{w})})_{H_\mu} = (K_{H_\mu}(w, z), h(w))_{H_\mu}, \quad \forall z \in \mathbb{C}^n. \quad (7)$$

Setting $\bar{z} = \xi$, we get

$$g(\xi) = (K_{H_\mu}(w, \bar{\xi}), h(w))_{H_\mu}, \quad \forall \xi \in \mathbb{C}^n.$$

This completes the proof. ■

THEOREM 4 *Suppose that measure μ satisfies the following condition:*

$$d\mu(\bar{w}) = d\mu(w).$$

Then the space $\tilde{H}_\mu$ coincides with H_μ .

The proof is given in Theorem 3.

3. Examples

3.1 Special measure

Let D be a bounded convex domain in $\mathbb{C}^n$ such that $0 \in D$ and the domain $\overline{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C}^n | \bar{z} \in D\}$ coincides with D ; let $H_D(z) \stackrel{\text{def}}{=} \sup_{\xi \in D} \{\operatorname{Re} \langle \xi, \bar{z} \rangle\}$ be the support function of the domain D , and $d\mu(z) = e^{-H_D(z)} dv_n(z)$.

It is easy to prove that $d\mu(\bar{z}) = d\mu(z)$. From Theorem 4, it follows that the space $\tilde{H}_\mu$ coincides with H_μ .

3.2 Fock space

Let

$$d\mu(z) = \frac{1}{\pi^n} e^{-|z|^2} dv_n(z).$$

Then we have $d\mu(\bar{z}) = d\mu(z)$, $H_\mu = F$, $K_{H_\mu}(z, \bar{w}) = e^{\langle z, w \rangle}$, and $\tilde{H}_\mu = \hat{F}$.

Theorem 4 generalizes Theorem 1.

3.3 Radial measure

Let $\varphi(r)$, $r \in [0, +\infty)$, be a convex function such that $\varphi(r)/r \rightarrow +\infty$ as $r \rightarrow +\infty$, and $d\mu(z) = e^{-\varphi(|z|)} dv_n(z)$. It follows from Theorem 4 that the space $\tilde{H}_\mu$ coincides with H_μ .

LEMMA 1 Suppose $d\mu(z) = e^{-\varphi(|z|)} dv_n(z)$; then

$$K_{H_\mu}(z, \xi) = \sum_{|k| \geq 0} \frac{z^k \cdot \bar{\xi}^k}{\|z^k\|_{H_\mu}^2}, \quad z^k = z_1^{k_1} \cdots z_n^{k_n},$$

$$|k| = k_1 + \cdots + k_n, \quad k_i \geq 0, \quad i = 1, \dots, n. \quad (8)$$

Proof The following result is known [6]:

If $K_H(z, \xi)$ is the Bergman kernel of the Hilbert space H and $\{\psi_m(z)\}_{m \in \mathbb{N}}$ is an orthonormal basis in H ; then

$$K_H(z, \xi) = \sum_{m \in \mathbb{N}} \psi_m(z) \cdot \overline{\psi_m(\xi)}.$$

It is known [3] that $\{z^k / \|z^k\|_{H_\mu}\}_{|k| \geq 0}$ is an orthonormal basis in the space H_μ . This completes the proof. $\blacksquare$

Let P_j , $j = 1, \dots, n$, be the following operators on the space H_μ :

$$P_j: f(z) \longrightarrow z_j \cdot f(z); \quad P_j: H_\mu \longrightarrow H_\mu.$$

An adjoint operator P_j^* is defined by the equation

$$(P_j f, g)_{H_\mu} = (f, P_j^* g)_{H_\mu}, \quad f, g \in H_\mu. \quad (9)$$

THEOREM 5 Let $d\mu(z) = e^{-\varphi(|z|)} dv_n(z)$.

If $g(z) = \sum_{|k| \geq 0} g_k \cdot z^k \in H_\mu$, then

$$P_j^* g(z) = \sum_{|k| \geq 0} \frac{\|z_1^{k_1} \cdots z_j^{k_j+1} \cdots z_n^{k_n}\|_{H_\mu}^2}{\|z^k\|_{H_\mu}^2} \cdot g_{(k_1, \dots, k_j+1, \dots, k_n)} \cdot z^k.$$

Proof Let $f, g \in H_\mu$ and

$$f(z) = \sum_{|k| \geq 0} f_k \cdot z^k, \quad g(z) = \sum_{|k| \geq 0} g_k \cdot z^k.$$

The system $\{z^k\}_{|k| \geq 0}$ is an orthogonal basis in H_μ . So, we have

$$(f, g)_{H_\mu} = \sum_{|k| \geq 0} f_k \cdot \overline{g_k} \cdot \|z^k\|_{H_\mu}^2. \quad (10)$$

Then,

$$\begin{aligned} (P_j f, g)_{H_\mu} &= \sum_{|k| \geq 0} f_k \cdot \overline{g_{(k_1, \dots, k_j+1, \dots, k_n)}} \cdot \|z_1^{k_1} \cdots z_j^{k_j+1} \cdots z_n^{k_n}\|_{H_\mu}^2 \\ &= \sum_{|k| \geq 0} f_k \cdot \frac{\|z_1^{k_1} \cdots z_j^{k_j+1} \cdots z_n^{k_n}\|_{H_\mu}^2}{\|z^k\|_{H_\mu}^2} \cdot \overline{g_{(k_1, \dots, k_j+1, \dots, k_n)}} \cdot \|z^k\|_{H_\mu}^2 = (f, P_j^* g)_{H_\mu}. \end{aligned} \quad (11)$$

From equations (10) and (11), it follows that the operator P_j^* has the form: if $g(z) = \sum_{|k| \geq 0} g_k \cdot z^k$, then

$$P_j^* g(z) = \sum_{|k| \geq 0} \frac{\|z_1^{k_1} \cdots z_j^{k_j+1} \cdots z_n^{k_n}\|_{H_\mu}^2}{\|z^k\|_{H_\mu}^2} \cdot g_{(k_1, \dots, k_j+1, \dots, k_n)} \cdot z^k.$$

This completes the proof of Theorem 5. ■

COROLLARY 1 If $\varphi(r) = r^2 + n \cdot \ln \pi$, then $d\mu(z) = (1/\pi^n) e^{-|z|^2} dv_n(z)$, $H_\mu = F$ and

$$\frac{\|z_1^{k_1} \cdots z_j^{k_j+1} \cdots z_n^{k_n}\|_{H_\mu}^2}{\|z^k\|_{H_\mu}^2} = k_j + 1, \quad j = 1, \dots, n.$$

Thus,

$$P_j^* g(z) = \frac{d}{dz_j} g(z), \quad j = 1, \dots, n.$$

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Monotonicity of some modified Bessel function products

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For non-negative real order, the product of modified Bessel functions of first and second kind is shown to be strictly decreasing for positive real arguments. After recalling some established results, only elementary methods are required to complete the proof.

Keywords: Modified Bessel function; Complete monotonicity; Recurrence relation

AMS Subject Classification: 33C10

1. Introduction and motivation

For the modified Bessel functions of first, $I_\nu(u)$, and second, $K_\nu(u)$, kind defined on $(0, \infty)$ and with non-negative real order $\nu \geq 0$, we establish strictly decreasing monotonicity with respect to argument u of the product function $P_\nu(u) = I_\nu(u)K_\nu(u)$ such that $P'_\nu(u) < 0$ on $(0, \infty)$.

Conventional wisdom maintains that the properties and behaviour of many bio-molecules are often controlled by electrostatic (Coulomb) interactions. Consequently, the mathematical models consisting of point charge distributions in piecewise uniform dielectric continua are believed to capture much of the basic physics for important biological phenomena. As a prelude to the development of general but necessarily numerical boundary integral methods, we are interested in heuristic canonical geometries (infinite plane, sphere and infinite right circular cylinder) where a good deal of analytic progress can be made. Standard solutions in the plane geometry are often interpreted with the idea of ‘image’ charges: virtual point singularities in a completely uniform dielectric that generate exactly equivalent potential fields that would arise from actual surface charge distributions induced on discontinuous dielectric boundaries. The optical analogy with a plane mirror is direct and so intuitive that it has given us the appellation ‘image’ charge. Although the corresponding treatments for two parallel planes [1] and for the sphere [2] yield an infinite series of images, the concept remains appealing and physically helpful. We are interested in undertaking similar extensions for the cylindrical geometry, appropriate for simple models of polyelectrolytes and nucleic

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acids [3]. The convergent ‘multiple induction expansion’ deduced by Shaw [4] can be regarded as a formal demonstration that the image charge concept is quite general, but the literature remains silent to our knowledge for the particular cylindrical case. The monotonicity conditions demonstrated here are sufficient for this development.

2. Ancillary results

Without restriction, let $\mathcal{L}_\mu(z)$ denote $I_\mu(z)$, $e^{\mu\pi i}K_\mu(z)$ or any linear combination of these functions with coefficients independent of $z \in \mathbb{C}$ or $\mu \in \mathbb{C}$, then the well-known [5, 9.6.26] recurrence relations

$$2(\mu + 1)\mathcal{L}_{\mu+1}(z) = z(\mathcal{L}_\mu(z) - \mathcal{L}_{\mu+2}(z)), \quad 2\mathcal{L}'_{\mu+1}(z) = \mathcal{L}_\mu(z) + \mathcal{L}_{\mu+2}(z), \quad (1)$$

immediately yield

$$\begin{aligned} 2(v+1)P'_{v+1}(u) = & -\frac{u}{2}((I_v(u) + I_{v+2}(u))(K_v(u) - K_{v+2}(u)) \\ & + (I_v(u) - I_{v+2}(u))(K_v(u) + K_{v+2}(u))) = u(P_{v+2}(u) - P_v(u)). \end{aligned} \quad (2)$$

We begin now by establishing that for sufficiently large real-order v , P_v tends to a stationary value at the origin.

LEMMA 2.1 *For positive real u and real $v > (1/2)$, then $\lim_{u \rightarrow 0} P'_v(u) = 0$.*

Proof Power series expansions about the origin yield the dominant terms [6]:

$$\begin{aligned} I_v(u) &\propto \frac{(1/2u)^v}{\Gamma(v+1)}(1 + O(u^2)) \quad (-v \notin \mathbb{N}^*), \\ K_0(u) &\propto -\ln\left(\frac{1}{2}u\right)(1 + O(u^2)), \\ K_v(u) &\propto \frac{1}{2}\Gamma(|v|)\left(\frac{1}{2}u\right)^{-|v|}(1 + O(u^2)) \quad (v \in \mathbb{Z}^*), \\ K_v(u) &\propto \frac{1}{2}\left(\Gamma(-v)\left(\frac{1}{2}u\right)^v + \Gamma(v)\left(\frac{1}{2}u\right)^{-v}\right)(1 + O(u^2)) \quad (v \notin \mathbb{Z}). \end{aligned}$$

Thus, as $u \rightarrow 0$, we have

$$uP_v(u) \longrightarrow \begin{cases} \frac{u}{2v}, & (v > 0) \\ -u \ln\left(\frac{1}{2}u\right), & (v = 0) \\ \frac{u}{2v} + \frac{\Gamma(-v)}{2\Gamma(v+1)}\left(\frac{u}{2}\right)^{2v+1}, & \left(-\frac{1}{2} < v < 0\right) \end{cases}$$

and the result follows from the recurrence relation (2) where we observe that the additional factor of u is sufficient to suppress the logarithmic divergence. ■

For large arguments, on the other hand, the known asymptotic expansions at fixed order [5, 9.7.1, 9.7.2],

$$I_\mu(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}}(1 + O(z^{-1})) \quad \text{and} \quad K_\mu(z) \sim \frac{\pi \exp(-z)}{\sqrt{2\pi z}}(1 + O(z^{-1})),$$

immediately lead to

$$I_\mu(z)K_\mu(z) \sim \frac{1}{2z} \implies \lim_{z \rightarrow \infty} P'_\mu(z) = 0. \quad (3)$$

In conjunction with Lemma 2.1, the last result (3) essentially substantiates the following observation.

PROPOSITION 2.2 *For positive real u and real $v > 1/2$, the second derivative $P''_v(u)$ admits at least one root in $(0, \infty)$.*

Proof Since, by Lemma 2.1 and equation (3), we have $\lim_{u \rightarrow 0} P'_v(u) = \lim_{u \rightarrow \infty} P'_v(u) = 0$ and $P'_v(u)$ is continuous on $(0, \infty)$, then there exists a compact interval $[a, b]$ with $0 < a < b < \infty$ such that $P'_v(a) = P'_v(b)$. Moreover, $P''_v(u)$ exists on $(0, \infty)$. Hence, by the mean value theorem, there exists $u_* \in (a, b)$ such that $P''_v(u_*) = 0$. ■

For convenience in the subsequent development, we recall the following definition,

DEFINITION 2.3 *A real function $f: (0, \infty) \rightarrow \mathbb{R}$ is said to be completely monotonic (c.m.), if it possesses derivatives $f^{(n)}(x)$ for all $n = 0, 1, 2, 3, \dots$ and if*

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x > 0$.

The following straightforward theorem demonstrated by Miller and Samko [7] will also be useful.

THEOREM 2.4 *If functions $f(x)$ and $g(x)$ are c.m., with λ and σ non-negative real constants, then $\lambda f(x) + \sigma g(x)$ and $f(x)g(x)$ are also c.m.*

In a study of the monotony and convexity properties of the product of linearly independent solutions of certain second-order disconjugate ordinary differential equations, Hartman [8] has also established the following result, which we simply state without proof.

LEMMA 2.5 *The function $uI_v(u)K_v(u)$ is concave on $(0, \infty)$ for $v > 1/2$, thus*

$$(uP_v(u))'' = 2P'_v(u) + uP''_v(u) < 0 \quad (u > 0) \quad \left(v > \frac{1}{2}\right).$$

3. Main result

We are now in a position to establish our main result.

THEOREM 3.1 *For positive real argument u and non-negative real order ν , the product of modified Bessel functions of first and second kind is monotonically decreasing, thus*

$$P_\nu(u) = I_\nu(u)K_\nu(u) \implies P'_\nu(u) < 0 \text{ for all } u \in (0, \infty) \text{ and for all } \nu \geq 0.$$

Proof (i) Case $\nu = 0$. Given the integral representation [5, 9.6.24]

$$K_\nu(u) = \int_0^\infty dv \exp(-u \cosh v) \cosh(\nu v) \quad (u > 0),$$

the strictly increasing behaviour of K_ν with respect to order at fixed u is immediately demonstrated, for

$$\frac{\partial}{\partial \nu} K_\nu(u) = \int_0^\infty dv v \exp(-u \cosh v) \cosh v \sinh(\nu v) > 0 \quad (u > 0). \quad (4)$$

By combining this with the corresponding inequality due to Cochran [9],

$$\frac{\partial}{\partial \nu} I_\nu(u) < 0 \quad (u > 0) \quad (\nu \geq 0), \quad (5)$$

and the well-known special cases of the recurrence (1), we directly obtain

$$P'_0 = I'_0 K_0 + I_0 K'_0 = I_1 K_0 - I_0 K_1 = P_0 \left(\frac{I_1}{I_0} - \frac{K_1}{K_0} \right) < 0$$

as required.

(ii) Case $0 < \nu \leq 1/2$. By verifying equivalence with the Laplace transform of a non-negative function and invoking Bernstein's theorem, Nåsell [10] has established that

$$f_\nu(u) = u^{-\nu} e^{-u} I_\nu(u) \quad (u > 0) \quad \left(\nu > -\frac{1}{2} \right),$$

is c.m. Similarly, Miller and Samko [7] have shown that

$$g_\nu(u) = u^\nu e^u K_\nu(u) \quad (u > 0) \quad \left(0 \leq \nu \leq \frac{1}{2} \right),$$

is also c.m. By applying Theorem 2.4, we immediately deduce that the product $P_\nu(u) = f_\nu(u)g_\nu(u)$ is also c.m. for order $0 \leq \nu \leq 1/2$, which by Definition 2.3, it follows that

$$P'_\nu(u) \leq 0 \quad (u > 0) \quad \left(0 \leq \nu \leq \frac{1}{2} \right). \quad (6)$$

This is consistent with the zero-order case discussed above. To sharpen the inequality (6), we assume, for the sake of contradiction, that there exists some $u_* \in (0, \infty)$ such that $P'_\nu(u_*) = 0$. It follows from the recurrence relation (2) together with the general results (1) that if $\nu \neq 0$, then

$$P_{\nu+1}(u_*) = P_{\nu-1}(u_*) \implies \frac{K_{\nu+1}(u_*)}{K_{\nu-1}(u_*)} = \frac{I_{\nu-1}(u_*)}{I_{\nu+1}(u_*)} \implies \frac{K_\nu(u_*)}{K_{\nu-1}(u_*)} = \frac{I_\nu(u_*)}{I_{\nu+1}(u_*)}. \quad (7)$$

The well-known symmetry property of the K_μ with respect to order [11, 3.71(8)] leads to

$$\frac{K_\nu(u_*)}{K_{\nu-1}(u_*)} = \frac{K_\nu(u_*)}{K_{1-\nu}(u_*)} \leq 1,$$

because $0 < \nu \leq 1/2 \Rightarrow \nu \leq (1 - \nu)$ and the result (4) applies. But the result (5) demands

$$\frac{I_\nu(u_*)}{I_{\nu+1}(u_*)} > 1,$$

and we arrive at a contradiction with equation (7). Hence, we conclude that u_* does not exist and P_ν is strictly decreasing for $0 < \nu \leq 1/2$ as required.

(iii) Case $\nu > 1/2$. Here, we argue by induction over the zeros of P''_ν . First set $u_0 = 0$, then enumerate the set of zeros (non-empty by Proposition 2.2) and list them in ascending order such that

$$P''_\nu(u_n) = 0 \quad u_{n-1} < u_n, \quad (n \in \mathbb{N}^*).$$

Now, define the variable proposition

$$\pi(n): P'_\nu(u) < 0 \quad \text{for } u_{n-1} < u \leq u_n.$$

Our argument proceeds by examining the behaviour of $P'_\nu(u)$ near the origin and then extending to all $u > 0$ by induction on n , although numerical calculations indicate that P''_ν has exactly one root for order $\nu > 1/2$.

Choose ξ such that $u_0 < \xi < u_1$. Suppose, for the sake of contradiction, that $P'_\nu(\xi) \geq 0$, then Lemma 2.5 implies

$$2P'_\nu(\xi) + \xi P''_\nu(\xi) < 0 \implies P'_\nu(\xi) < -\frac{2P'_\nu(\xi)}{\xi} \leq 0.$$

Thus, there exists a compact interval $[\epsilon, \xi]$ with $0 < \epsilon < \xi$ and P''_ν defined on (ϵ, ξ) . From the mean value theorem, there exists $u_* \in (\epsilon, \xi)$ such that $P''_\nu(u_*) = 0$ and $u_0 < u_* < u_1$. But this contradicts the minimality of u_1 such that $P''_\nu(u) \neq 0$ on (u_0, u_1) . Hence, $P'_\nu(\xi) < 0$. Since ξ was chosen arbitrarily on (u_0, u_1) , it follows that $P'_\nu(u) < 0$ for $u \in (u_0, u_1)$.

Thus, $\pi(1)$ is true.

Now, choose ξ' such that $u_0 < \xi' < u_1$ and suppose, for the sake of contradiction, that $P''_\nu(\xi') > 0$. Again the mean value theorem implies the existence of a zero for P''_ν on $u_0 < u_* < u_1$ that contradicts the minimality of u_1 , and we therefore deduce that $P''_\nu(u) \leq 0$ on (u_0, u_1) .

Now, pick $n = k$ and assume, for the sake of induction, that $\pi(k)$ is true. Two possible situations arise.

- (i) if $P''_\nu(u) \leq 0$ for $u_k < u \leq u_{k+1}$, then it follows that $P'_\nu(u) < 0$ on this range, since P_ν is continuous and $P'_\nu(u_k) < 0$ by assumption;
- (ii) otherwise, $P''_\nu(u) > 0$ for $u_k < u \leq u_{k+1}$, and Lemma 2.5 implies

$$P'_\nu(u) < -\frac{u}{2} P''_\nu(u) < 0.$$

Hence, $\pi(k+1)$ is true. Thus, by the principle of mathematical induction, $\pi(n)$ is true for all n and P_ν is strictly decreasing on $(0, \infty)$ for all $\nu > 1/2$. ■

4. Subsidiary results

Appealing again to the general results, equation (1) shows that

$$\begin{aligned}
 2vP_v'' &= -\frac{u}{2}(I'_{v-1}(u) + I'_{v+1}(u))(K_{v-1}(u) - K_{v+1}(u)) \\
 &\quad - v(I_{v-1}(u) + I_{v+1}(u))(K_{v-1}(u) + K_{v+1}(u)) \\
 &\quad - \frac{u}{2}(I_{v-1}(u) - I_{v+1}(u))(K'_{v-1}(u) + K'_{v+1}(u)) \\
 &= 4vP_v - (2v-1)P_{v-1} - (2v+1)P_{v+1}.
 \end{aligned} \tag{8}$$

Differentiating equation (2) directly obtains

$$2v(v^2-1)P_v'' = (v^2-1)(P_{v+1} - P_{v-1}) + u^2 \left(\frac{1}{2}(v-1)P_{v+2} - vP_v + \frac{1}{2}(v+1)P_{v-2} \right)$$

and eliminating P_v'' from equation (8) yields the homogeneous fourth-order linear recurrence relation satisfied by P_v , thus

$$\begin{aligned}
 &\frac{1}{2}u^2(v-1)P_{v+2} + 2(v+1)^2(v-1)P_{v+1} - v(4(v^2-1) + u^2)P_v \\
 &\quad + 2(v+1)(v-1)^2P_{v-1} + \frac{1}{2}u^2(v+1)P_{v-2} = 0.
 \end{aligned}$$

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Hilbert matrices and formulae with binomials

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The inverting of Cauchy/Hilbert matrices leads to sum rules with binomials one can find in Knuth [Knuth, D.E., 1997, *The Art of Computer Programming. Vol.1: Fundamental Algorithms* (Reading MA: Addison–Wesley).]. Generalizations of such formulae are presented here.

Keywords: Binomial coefficients; Combinatorial identities

2000 Mathematics Subject Classifications: 11B65; 05A19

1. Introduction

The Hilbert matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \cdots \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

occurs in best approximation problems in the L^2 sense of square integrable functions, where suitable polynomials are sought in order to replace a given function with a minimal error. This matrix is a special case of Cauchy matrices.

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LEMMA 1.1 Let $n \in \mathbb{N}$, $x \in \mathbb{R} \setminus \{2 - 2n, 3 - 2n, \dots, 2n - 2n\}$, and $H = H(x)$ be an $n \times n$ matrix with entries

$$H_{mq}(x) := \frac{1}{x + m + q - 2}, \quad m, q \in \{1, \dots, n\}.$$

The inverse matrix reads in components

$$G_{pm}(x) = (-1)^{p+m} \frac{pm}{x + p + m - 2} \binom{n}{p} \binom{n}{m} \binom{x + n + p - 2}{n} \binom{x + n + m - 2}{n},$$

and its determinant is

$$\det[G] = \prod_{p=1}^n \frac{p^2}{x + 2p - 2} \binom{x + 2p - 2}{p}^2.$$

Proof Refer Knuth [1], exercise 38, p. 41. ■

From the equality

$$\sum_{m=1}^n G_{pm}(x) H_{mq}(x) = \delta_{p,q},$$

which is Kronecker's symbol equal to one for $p = q$ and zero else, it follows that for $p, q \in \{1, \dots, n\} \subset \mathbb{N}$

$$\begin{aligned} (x + p - 2) \sum_{m=1}^n \frac{(-1)^m}{x + p + m - 2} \binom{x + n + m - 2}{n} \binom{n}{m} \\ = (x + q - 2) \sum_{m=1}^n \frac{(-1)^m}{x + q + m - 2} \binom{x + n + m - 2}{n} \binom{n}{m} \end{aligned} \quad (1)$$

holds and that for $p = q$,

$$\sum_{m=1}^n \frac{(-1)^m m}{(x + p + m - 2)^2} \binom{x + n + m - 2}{n} \binom{n}{m} = \frac{1}{p \binom{x + n + p - 2}{n} \binom{n}{p}}.$$

The evaluation of the sums on the left and right of the first formula does not follow. This is the starting point for the considerations in next section, where more general formulas are derived, which cannot be found in the listed tables of standard compilations.

Remark 1.2 The left, right of equation (1), are independent of p, q respectively, and both are equal to $-\binom{x+n-2}{n}$. Equivalently,

$$\sum_{m=0}^n \frac{(-1)^m}{x + p + m - 2} \binom{n}{m} \binom{x + n + m - 2}{n} = 0$$

holds when $x \neq 2 - p - m$, and $p \in \{1, \dots, n\} \subset \mathbb{N}$. This equality can be shown by Prudnikov *et al.* [2, Vol. 1, No. 4.2.5.47],

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a + kb}{m} = (-1)^n b^n \delta_{n,m} \quad [0 \leq m \leq n],$$

after suitable rearrangements, where due to equation (1) it is sufficient to consider $p = n$.

2. Sums with binomials

In the sequel, we extend formulas given in the tables of standard compilations, in particular, sums involving binomials given by Prudnikov *et al.* [2].

DEFINITION 2.1 *Let $n, m \in \mathbb{N}$. We define the sums*

$$\sum^{n;m} := \sum_{j=0}^n j^m.$$

DEFINITION 2.2 *Let $n \in \mathbb{N}$, $m, p \in \mathbb{Z}$, and $x, y \in \mathbb{R}$. We define alternatively*

$$\begin{aligned}\varphi_m^n(x, y) &:= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+ky}{m}, \\ \varphi_{n;p}^n(x, y) &:= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+ky}{n+p}.\end{aligned}$$

LEMMA 2.3 *The functions given by Definition 2.2 have the property*

$$\begin{aligned}\varphi_{n;p}^n(x, y) &= \frac{x}{n+p} \varphi_{n;p-1}^n(x-1, y) - \frac{yn}{n+p} \varphi_{n-1;p}^{n-1}(x-1+y, y), \\ \varphi_{n;-n}^n(x, y) &= \delta_{0,n}, \quad \varphi_{n;0}^n(x, y) = (-y)^n.\end{aligned}$$

Proof Direct computation. ■

DEFINITION 2.4 *Let $n \in \mathbb{N}$, $m, p \in \mathbb{Z}$, and $a, b, x, y \in \mathbb{R}$. We define alternatively*

$$\begin{aligned}\mathcal{Q}_m^n(a, b, x, y) &:= \sum_{k=0}^n \frac{(-1)^k}{a+bk} \binom{n}{k} \binom{x+ky}{m}, \\ \mathcal{Q}_{n;p}^n(a, b, x, y) &:= \sum_{k=0}^n \frac{(-1)^k}{a+bk} \binom{n}{k} \binom{x+ky}{n+p}.\end{aligned}$$

LEMMA 2.5 *The functions given by Definition 2.4 have the property*

$$\mathcal{Q}_m^n(a, b, x, y) = \frac{y}{mb} \varphi_{m-1}^n(x-1, y) + \frac{1}{m} \left(x - a\frac{y}{b}\right) \mathcal{Q}_{m-1}^n(a, b, x-1, y).$$

Proof This follows directly by the additive splitting

$$\frac{x+ky}{a+bk} = \frac{y}{b} + \frac{x-a(y/b)}{a+bk}.$$
■

THEOREM 2.6 *Let $n \in \mathbb{N}$, and $x, y \in \mathbb{R}$. It holds*

$$\varphi_{n+1}^n(x, y) = (-y)^n \left(x + \frac{n(y-1)}{2}\right).$$

Proof Using that

$$\binom{x+ky}{n+1} = \frac{x+ky}{n+1} \binom{x-1+ky}{n}$$

and Lemma 2.3, one obtains

$$\varphi_{n+1}^n(x, y) = \frac{x(-y)^n}{n+1} + \frac{yn}{n+1} \sum_{k=0}^n (-1)^k \binom{n-1}{k-1} \binom{x-1+ky}{n}.$$

We repeat this procedure $n-1$ times to arrive at

$$\begin{aligned} \varphi_{n+1}^n(x, y) &= \frac{(-y)^n}{n+1} \sum_{j=0}^{n-1} (x + j(y-1)) + \frac{y^n(n-(n-1))}{n+1} \\ &\quad \times \sum_{k=0}^n (-1)^k \binom{n-n}{k-n} \binom{x-n+ky}{n-(n-1)}. \end{aligned}$$

This gives us

$$\varphi_{n+1}^n(x, y) = \frac{(-y)^n}{n+1} \sum_{j=0}^n (x + j(y-1))$$

completing the proof. ■

THEOREM 2.7 *Let $n \in \mathbb{N}$, and $x, y \in \mathbb{R}$. It holds*

$$\begin{aligned} \varphi_{n+2}^n(x, y) &= \frac{(-y)^n}{4} \left(x(2x-2+n(y-1)) + \frac{n(y-1)}{6} (6x-4+3n+1)(y-1) \right) \\ &= (-y)^n \left[\frac{x(x-1)}{2} + \frac{n(3x-1)(y-1)}{6} + \frac{n(3n+1)(y-1)^2}{24} \right]. \end{aligned}$$

Proof By Lemma 2.3 and Theorem 2.6 it follows that

$$\varphi_{n;2}^n(x, y) = \frac{x(-y)^n}{n+2} \left(x-1 + \frac{n(y-1)}{2} \right) - \frac{yn}{n+2} \varphi_{n-1;2}^{n-1}(x-1+y, y).$$

We repeat this procedure n times to obtain that

$$\varphi_{n;2}^n(x, y) = \frac{(-y)^n}{n+2} \sum_{j=0}^n \frac{n+1-j}{n+1} (x+j(y-1)) \left[x-1+j(y-1) + (n-j) \frac{y-1}{2} \right].$$

After some rearrangement in order to use Definition 2.1, one arrives at

$$\begin{aligned} \varphi_{n;2}^n(x, y) &= \frac{(-y)^n}{(n+2)(n+1)} \left[x \left(x-1 + \frac{n(y-1)}{2} \right) \left[(n+1) \sum^{n;0} - \sum^{n;1} \right] \right. \\ &\quad \left. + (y-1) \left(\frac{3x}{2} - 1 + \frac{n(y-1)}{2} \right) \left[(n+1) \sum^{n;1} - \sum^{n;2} \right] \right. \\ &\quad \left. + \frac{(y-1)^2}{2} \left[(n+1) \sum^{n;2} - \sum^{n;3} \right] \right], \end{aligned}$$

and evaluation of the sums completes the proof. ■

Example 2.8 We calculate

$$\varphi_4^2\left(\frac{1}{2}, -\frac{1}{4}\right) = \frac{37}{1024}.$$

DEFINITION 2.9 Let $n, p \in \mathbb{N}$, and $x, y \in \mathbb{R}$. We define

$$A_{n;p}^n := (-y)^{-n} \frac{(n+p)!}{n!} \varphi_{n;p}(x, y).$$

Example 2.10 Let $n, p \in \mathbb{N}$, and $x, y \in \mathbb{R}$. It holds

$$\begin{aligned} A_{n;0}^n(x, y) &= 1, \\ A_{n;1}^n(x, y) &= x \sum_{n;0}^{n;0} + (y-1) \sum_{n;1}^{n;1} = (n+1) \left(x + \frac{n(y-1)}{2} \right). \end{aligned}$$

Proof It is a direct consequence of Theorem 2.6 and Definition 2.1. ■

THEOREM 2.11 Let $n, p \in \mathbb{N}$, and $x, y \in \mathbb{R}$. It holds that

$$A_{n;p}^n(x, y) = \sum_{j=0}^n (x + j(y-1)) A_{n-j;p-1}^{n-j}(x-1 + j(y-1), y).$$

Proof We apply Lemma 2.3 several times and obtain

$$\begin{aligned} \varphi_{n;p}^n(x, y) &= \frac{1}{n+p} \sum_{j=0}^n (-y)^j \left(\prod_{k=0}^{j-1} \frac{n-k}{n-k+p-1} \right) (x + j(y-1)) \\ &\quad \varphi_{n-j;p-1}^{n-j}(x-1 + j(y-1), y), \end{aligned}$$

which can be written as

$$\frac{n!(-y)^n}{(n+p)!} \sum_{j=0}^n (-y)^{j-n} \frac{(n-j+p-1)!}{(n-j)!} (x + j(y-1)) \varphi_{n-j;p-1}^{n-j}(x-1 + j(y-1), y)$$

completing the proof. ■

Remark 2.12 The case $y = 1$ means being a proof of Prudnikov *et al.* [2, Vol. 1, No. 4.2.5.55],

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{a+k}{m} = (-1)^n \binom{a}{m-n}.$$

One can write

$$\varphi_{n;p}^n(x, 1) = \frac{p}{n+p} \sum_{j=0}^n (-1)^j \frac{x}{p} \frac{\binom{n}{j}}{\binom{n+p-1}{j}} (-1)^{n-j} \binom{x-1}{p-1},$$

which is equivalent to [2, No. 4.2.8.1]

$$1 = \frac{p}{n+p} \sum_{j=0}^n \frac{\binom{n}{j}}{\binom{n+p-1}{j}}.$$

Remark 2.13 In Theorem 2.11, alternatively it follows after substitution of $n - j$ by k and translation with respect to the first argument that

$$A_{n;p}^n(x - n(y - 1), y) = \sum_{k=0}^n (x - k(y - 1)) A_{k;p-1}^k(x - 1 - k(y - 1), y).$$

DEFINITION 2.14 Let $n, p \in \mathbb{N}$, and $x, y \in \mathbb{R}$. We define

$$B_{n;p}^n := (x + 1) A_{n;p}^n(x, y).$$

COROLLARY 2.15 Let $n, p \in \mathbb{N}$, and $x, y \in \mathbb{R}$. It holds

$$B_{n;p}^n(x - n(y - 1), y) = (x + 1 - n(y - 1)) \sum_{k=0}^n B_{k;p-1}^k(x - 1 - k(y - 1), y),$$

where

$$B_{k;0}^k(x - k(y - 1), y) = x + 1 - k(y - 1).$$

Remark 2.16 One should keep in mind that

$$\varphi_{n;p}^n(x, y) = (-y)^n \frac{n!}{(n + p)!} \frac{1}{x + 1} B_{n;p}^n(x, y).$$

Before proceeding, we give another version of Theorem 2.11 by the following.

COROLLARY 2.17 Let $n, p \in \mathbb{N}$, $k \in \{0, \dots, p - 1\} \subset \mathbb{N}$, $j_k \in \{0, \dots, n\} \subset \mathbb{N}$, and $x, y \in \mathbb{R}$ to define $x_{j_k}(y) := x - k + (y - 1) \sum_{m=0}^k j_m$.

It holds that

$$A_{n;p}^n(x, y) = \sum_{j_0=0}^n x_{j_0}(y) \sum_{j_1=0}^{n-j_0} x_{j_1}(y) \dots \sum_{j_{p-1}=0}^{n-(j_0+j_1+\dots+j_{p-2})} x_{j_{p-1}}(y).$$

Proof The equality follows by carrying out the recurrence procedure of Theorem 2.11. ■

Example 2.18 The case $p = 2$ in Theorem 2.11 gives us

$$\begin{aligned} A_{n;2}^n &= \sum_{j_0=0}^n (x + (y - 1)j_0) \sum_{j_1=0}^{n-j_0} (x - 1 + (y - 1)(j_0 + j_1)) \\ &= \sum_{j_0=0}^n (x + (y - 1)j_0) \left[(x - 1 + (y - 1)j_0) \sum_{j_1=0}^{n-j_0;0} + (y - 1) \sum_{j_1=0}^{n-j_0;1} \right] \\ &= \sum_{j_0=0}^n (x + (y - 1)j_0)(n + 1 - j_0) \left[x - 1 + j_0(y - 1 + (n - j_0)) \frac{y - 1}{2} \right], \end{aligned}$$

which corresponds to the formulas derived in the proof of Theorem 2.7 for

$$\varphi_{n+2}^n(x, y) = \frac{(-y)^n}{(n + 1)(n + 2)} A_{n;2}^n(x, y).$$

THEOREM 2.19 Let $n \in \mathbb{N}$, $m \in \mathbb{Z}_- \cup \{0, \dots, n+1\}$, and $a, b, x, y \in \mathbb{R}$ have the property that $b \neq 0$, and $a + bk \neq 0$ for $k \in \{0, \dots, n\} \subset \mathbb{N}$. It holds

$$Q_m^n(a, b, x, y) = \frac{y(-y)^n}{bm} \delta_{n, m-1} + \frac{\binom{x - a(y/b)}{m}}{a \binom{n + (a/b)}{n}}.$$

Proof We apply Lemma 2.5, where we see from Prudnikov *et al.* [2, Vol. 1, No. 4.2.5.47] that for $m - 1 \leq n$, there holds

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x - 1 + ky}{m - 1} = (-y)^n \delta_{n, m-1}.$$

The last relation is used to furnish

$$\frac{1}{m} \left(x - \frac{ay}{b} \right) Q_{m-1}^n(a, b, x - 1, y) = \binom{x - \frac{ay}{b}}{m} Q_0^n(a, b, x - m, y).$$

Prudnikov *et al.*'s [2 Vol. 1, No. 4.2.2.45]

$$\sum_{k=0}^n \frac{(-1)^k}{k + a} \binom{n}{k} = \frac{1}{a} \binom{n + a}{n}^{-1}$$

completes the proof. ■

THEOREM 2.20 Let $n \in \mathbb{N}$ and $a, b, x, y \in \mathbb{R}$ have the property $b \neq 0$ and $a + bk \neq 0$ for $k \in \{0, \dots, n\} \subset \mathbb{N}$. It holds

$$Q_{n+2}^n(a, b, x, y) = \frac{y(-y)^n}{b(n+2)} \left(x - 1 + \frac{n(y-1)}{2} + \frac{x - a(y/b)}{n+1} \right) + \frac{\binom{x - a(y/b)}{n+2}}{a \binom{n + (a/b)}{n}}.$$

Proof Lemma 2.5, Theorem 2.6 and Prudnikov *et al.* [2, Vol. 1, No. 4.2.2.45]. ■

LEMMA 2.21 Let $m, n \in \mathbb{N}$, and $a, b, c, d \in \mathbb{R}$. It holds

$$Q_m^n(a, b, x, y) = \frac{y}{bm} \sum_{j=1}^m \left(\prod_{k=1}^{j-1} \frac{x - a(y/b) - (k-1)}{m - k} \right) \varphi_{m-j}^n(x - j, y) + \binom{x - a\frac{y}{b}}{m} \\ \times Q_0^n(a, b, x - m, y),$$

where

$$Q_0^n(a, b, x - m, y) = \frac{1}{a + \binom{n + (a/b)}{n}}, \\ \varphi_{m-j}^n(x - j, 1) = (-1)^n \binom{x - j}{m - n - j}, \\ Q_m^n(a, 0, x, y) = \frac{1}{a} \varphi_m^n(x, y), \\ Q_m^n(0, b, 0, y) = \frac{y}{mb} \varphi_{m-1}^n(-1, y).$$

Proof The use of Lemma 2.5 furnishes the formula for Q_m^n . ■

THEOREM 2.22 *Let $n \in \mathbb{N}$ and $a, b, x, y \in \mathbb{R}$ have the property $b \neq 0$ and $a + bk \neq 0$ for $k \in \{0, \dots, n\} \subset \mathbb{N}$. It holds*

$$\begin{aligned} Q_{n+3}^n(a, b, x, y) &= \frac{y(-y)^n}{4b(n+3)} \left((x-1)(2x-4+n(y-1)) + \frac{n(y-1)}{6}(6x-10+(3n+1)(y-1)) \right) \\ &\quad + \frac{y(-y)^n(x-a(y/b))}{b(n+2)(n+3)} \left(x-2 + \frac{n(y-1)}{2} + \frac{x-1-a(y/b)}{n+1} \right) + \frac{\binom{x-a(y/b)}{n+3}}{a \binom{n+(a/b)}{n}}. \end{aligned}$$

Proof This follows by Theorem 2.7 and Prudnikov *et al.* [2 Vol. 1, No. 4.2.2.45]. ■

DEFINITION 2.23 *Let $n, k \in \mathbb{N}_0$, $j \in \{0, \dots, k\} \subset \mathbb{N}_0$, $\alpha_j \in \mathbb{N}_0$. We define*

$$\sum_{n; \alpha_0, \dots, \alpha_k} := \sum_{j_0=0}^n j_0^{\alpha_0} \sum_{j_1=0}^{j_0} j_1^{\alpha_1} \cdots \sum_{j_k=0}^{j_{k-1}} j_k^{\alpha_k}.$$

Example 2.24 Special values in Definition 2.23 are given by

$$\begin{aligned} \sum_{n; 0,0} &= \sum_{n; 1} + \sum_{n; 0} = \sum_{n+1; 1} = \frac{(n+1)(n+2)}{2}, \\ \sum_{n; 0,1} &= \frac{1}{2} \left\{ \sum_{n; 2} + \sum_{n; 1} \right\} = \frac{n(n+1)(n+2)}{6}, \\ \sum_{n; 1,0} &= \sum_{n; 2} + \sum_{n; 1} = \frac{n(n+1)(n+2)}{3}, \\ \sum_{n; 1,1} &= \frac{1}{2} \left\{ \sum_{n; 3} + \sum_{n; 2} \right\} = \frac{n(n+1)(n+2)(3n+1)}{24}. \end{aligned}$$

LEMMA 2.25 *Let $n, k \in \mathbb{N}_0$ and $\alpha_0, \dots, \alpha_k \in \mathbb{N}_0$. The following hold*

$$\begin{aligned} \sum_{n; 0, \dots, 0, 0} &= \binom{n+k+1}{k+1}, \\ \sum_{n; 0, \dots, 0, 1} &= \binom{n+k+1}{k+2}. \end{aligned}$$

Proof The first and the second formulae are shown by induction and using

$$\sum_{k=m}^n \binom{k}{a} = \binom{n+1}{a+1} - \binom{m}{a+1}$$

by Prudnikov *et al.* [2 Vol. 1, No. 4.2.1.30] completing the proof. ■

THEOREM 2.26 Let $n \in \mathbb{N}$, $k, p \in \mathbb{N}_0$, and $\beta := (\beta_0, \dots, \beta_p) \in \mathbb{N}_0^{p+1}$, $(\gamma_0, \dots, \gamma_k) \in \mathbb{N}_0^{k+1}$ to define with $\alpha := (\beta, 1, \gamma)$ in Definition 2.23 the sum

$$\sum_{n; \beta, 1, \gamma} := \sum_{j_0=0}^n \cdots \sum_{j_{p-1}=0}^{j_{p-1}} \sum_{q=0}^{j_p} q \sum_{m_0=0}^q \sum_{m_1=0}^{m_0} \cdots \sum_{m_k=0}^{m_{k-1}}.$$

It holds that

$$\sum_{n; 0, \dots, 0, 1, 0, \dots, 0} = \frac{(k+2)n}{p+k+4} \binom{n+p+k+3}{p+k+3}.$$

Proof Owing to Lemma 2.25 ($k+1$ zeroes)

$$\sum_{n'=0}^{n''} n' \sum_{n'; 0, \dots, 0} = (k+2) \sum_{n'=0}^{n''} \left[\binom{n'+k+2}{k+2} - \binom{n'+k+1}{k+1} \right],$$

and it remains to apply Prudnikov *et al.* [2 Vol. 1, No. 4.2.1.30] ($p+1$) times completing the proof. ■

Example 2.27 Using the third formula of Example 2.24 it is directly checked that for $k=0=p$, the following holds

$$\sum_{n; 0, 1, 0} = 2 \binom{n+3}{4}.$$

The result in Theorem 2.26 can also be written

$$\sum_{n; 0, \dots, 0, 1, 0, \dots, 0} = (k+2) \binom{n+p+k+3}{p+k+4}.$$

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Generalization of distributional product of Dirac's delta in hypercone

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Let $G = G(m, x)$ be defined by $G(m, x) = \left(\sum_{i=1}^p x_i^2\right)^m - \left(\sum_{i=p+1}^{p+q} x_i^2\right)^m$, $p + q = n$. The hypersurface G is due to Kanathai and Nonlaopon ([Kanthai, A. and Nonlaopon, K., 2003, On the residue of generalized function P^λ . *Thai Journal of Mathematics*, **1**, 49–57]). We observe that putting $m = 1$ we obtain $G = G(1, x) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 = P(x) = P$. The quadratic form P is due to Gelfand and Shilov [Gelfand, I.M. and Shilov, G.E., 1964, *Generalized Function*, Vol. 1 (New York: Academic Press), p. 253]. The hypersurface $P = 0$ is a hypercone with a singular point (the vertex) at the origin. We know that the k th derivative of Dirac's delta in G there exists under conditions depending on n and m , where n is the dimension of the space. In our study, the main purpose is to related distribution product of the Dirac delta with the coefficient corresponding to the double pole of the expansion in the Laurent series of $G^{\lambda+\mu}$, where G^γ is defined by (3). From this we can arrive at a formula in terms of the operator L_m which is defined by (16). Our results are generalizations of formulae that appear in Aguirre [Aguirre, T.M.A., 2000, The distributional product of Dirac's delta in a hypercone. *Journal of Computation and Applied Mathematics*, **115**, 13–21], pp. 20–21.

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1. Introduction

Let $x = (x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q})$ be a point of n -dimensional Euclidean space R^n , where $p + q = n$ dimension of the space, m is a nonnegative integer and $G = G(m, x)$ is defined by

$$G = G(m, x) = \left(\sum_{i=1}^p x_i^2\right)^m - \left(\sum_{i=p+1}^{p+q} x_i^2\right)^m. \quad (1)$$

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The hypersurface G is due to Kanathai and Nonlaopon [1]. We observe that putting $m = 1$ in ref. [1] we obtain

$$G = G(1, x) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^{p+q} x_i^2 = P(x) = P. \quad (2)$$

The quadratic form defined by ref. [2] is due to Gelfand and Shilov [2, p. 253].

The hypersurface $G = 0$ is a generalization of a hypercone $P = 0$ with a singular point (the vertex) at the origin.

We call $\varphi(x)$ the C^∞ function with compact support defined from R^n to R . From ref. [3] the generalized function G_+^λ is defined by

$$(G_+^\lambda, \varphi) = \int_{G>0} (G(m.x))^\lambda \varphi(x) dx \quad (3)$$

and considering [1, p. 54, formula 3.12] we have

$$(G_+^\lambda, \varphi) = \int_0^\infty u^{\lambda+(n/2m)-1} \Phi(\lambda, u) du, \quad (4)$$

where

$$\Phi(\lambda, u) = \frac{1}{4m^2} \int_0^1 (1-t)^\lambda t^{(q/2m)-1} \psi_1(u, ut) dt, \quad (5)$$

$$\psi_1(u, ut) = \int \varphi d\Omega_p d\Omega_q, \quad (6)$$

$d\Omega_p$ and $d\Omega_q$ are elements of surface area on the unit sphere in R^p and R^q , respectively.

Consequently, (G_+^λ, φ) has two sets of singularities, namely

$$\lambda = -1, -2, \dots, -k, \dots \quad (7)$$

and

$$\lambda = -\frac{n}{2m}, -\frac{n}{2m} - 1, \dots, -\frac{n}{2m} - s, \dots \quad (8)$$

where $k = 1, 2, \dots$ and $s = 0, 1, 2, \dots$

Each case has been studied in ref. [1, pp. 49–56] and [3]. From ref. [1], the following formula is valid

$$\lambda = -k \operatorname{Res} G^\lambda = \frac{(-1)^{k-1}}{(k-1)!} \delta_1^{(k-1)}(G) \quad (9)$$

for $p \geq 2m, q \geq 2m, n$ odd and n even if $k < n/2m$ [1, p. 53, formula 3.1], where

$$\delta_1^{(k-1)}(G) = \delta^{(k-1)}(G) \quad (10)$$

if $k-1 < p+q-2m/2$, $\delta_1^{(k-1)}(G)$ is the regularization of $\delta^{(k-1)}(G)$ for $k-1 \geq p+q-2m/2$ and

$$\left\langle \delta_1^{(k-1)}(G), \varphi \right\rangle = \int_0^\infty \left[\left(\frac{1}{2ms^{2m-1}} \cdot \frac{\partial}{\partial s} \right)^{k-1} \left\{ s^{q-2m} \frac{\Psi(r, s)}{2m} \right\} \right]_{s=r} r^{p-1} dr$$

or

$$\left\langle \delta_1^{(k-1)}(G), \varphi \right\rangle = \int_0^\infty \left[\left(\frac{1}{2mr^{2m-1}} \cdot \frac{\partial}{\partial s} \right)^{k-1} \left\{ r^{q-2m} \frac{\Psi(r, s)}{2m} \right\} \right]_{r=s} s^{p-1} ds. \quad (11)$$

For the case when the singular point λ is in both sets (see formulae (7) and (8)), from ref. [3], formula (41) we have the following formula:

$$(G_+^\lambda, \varphi) = \frac{1}{\lambda + (n/2m) + k} \int_0^\infty u^{(\lambda + (n/2m) - 1)} \Phi_0(u) du + \int_0^\infty u^{(\lambda + (n/2m) - 1)} \Phi_1(\lambda, u) du \quad (12)$$

where

$$\Phi_0(u) = \text{Re } s_{\lambda = -n/2m - k, k=0,1,\dots} \Phi(\lambda, u) \quad (13)$$

and $\Phi_1(\lambda, u)$ is a function which is regular at $\lambda = -n/2m - k, k = 0, 1, \dots$

We know from [3] that each of the integrals in (12) may have a simple pole at this value of λ . Therefore (G_+^λ, φ) may have a pole of order two at $\lambda = -n/2m - k, k = 0, 1, \dots$

In the neighborhood of such a point we may expand G_+^λ in the Laurent series

$$G_+^\lambda = \frac{A_{-2}^{(k)}}{\left(\lambda + \frac{n}{2m} + k\right)^2} + \frac{A_{-1}^{(k)}}{\left(\lambda + \frac{n}{2m} + k\right)} + \dots \quad (14)$$

where from [3], $A_{-2}^{(k)}$ is given by

$$\begin{aligned} \langle A_{-2}^{(k)}, \varphi \rangle &= \text{Re } s_{\lambda = -\frac{n}{2m} - k, k=0,1,\dots} \int_0^\infty u^{\lambda + \frac{n}{2m} - 1} \Phi_0(u) du = \frac{1}{k!} \Phi_0^{(k)}(0) \\ &= \frac{1}{(2m)^{2k}} \cdot \frac{1}{\Gamma\left(\frac{n}{2} + k\right)} \cdot \frac{(-1)^k}{\left(\frac{2}{m}(m-1) - 1\right) \dots \left(\frac{2}{m}(m-1) - k\right)} \\ &\quad \cdot \frac{(-1)(-1)^{n+p-m/2m} \pi^{(n/2)-1} \Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{q}{2}\right)} \langle L_m^k \{\delta(x)\}, \varphi \rangle \end{aligned} \quad (15)$$

for $p = 2sm + m, q = 2lm + m, s, l = 0, 1, 2, \dots$, where L_m is the operator defined by

$$L_m = \left(\sum_{i=1}^p x_i^2 \right)^{1-m} \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^{1-m} \left(\sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right). \quad (16)$$

2. The multiplicative product of $\delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G)$

In this section, we give a sense to the multiplicative product

$$\delta^{(k-1)} \left(\left(\sum_{i=1}^p x_i^2 \right)^m - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^m \right) \cdot \delta^{(l-1)} \left(\left(\sum_{i=1}^p x_i^2 \right)^m - \left(\sum_{i=p+1}^{p+q} x_i^2 \right)^m \right) \quad (17)$$

considering essentially formulae (12) and (13). In order to do it, we will consider two cases.

Case 1 n odd and n even if

$$k + l \neq -\frac{n}{2m}, -\frac{n}{2m} - 1, \dots, -\frac{n}{2m} - s, \dots, s = 0, 1, 2, \dots$$

When

$$\lambda = -k \text{ and } \lambda \neq -\frac{n}{2m}, -\frac{n}{2m} - 1, \dots, -\frac{n}{2m} - s, \dots, s = 0, 1, 2, \dots$$

this is always the case when the dimension $n = p + q$ is odd, but is also true if n is even and $k < \frac{n}{2m}$, $m = 1, 2, \dots$ [1, p. 55, formula 3.16], we have

$$(G_+^\lambda, \varphi) = \frac{1}{\lambda + k} \int_0^\infty u^{(\lambda + (n/2m) - 1)} h_0(u) du + \int_0^\infty u^{(\lambda + (n/2m) - 1)} h_1(\lambda, u) du \quad (18)$$

where

$$h_0(u) = \operatorname{Re} s_{\lambda = -k, k=1,2,\dots} \Phi(\lambda, u) \quad (19)$$

and $h_1(\lambda, u)$ is a function which is regular at $\lambda = -k$, $k = 1, 2, \dots$

From (12), we have

$$\begin{aligned} ((\lambda + k)^2 G_+^\lambda, \varphi) &= (\lambda + k) \int_0^\infty u^{(\lambda + (n/2m) - 1)} h_0(u) du \\ &\quad + (\lambda + k)^2 \int_0^\infty u^{(\lambda + (n/2m) - 1)} h_1(\lambda, u) du. \end{aligned} \quad (20)$$

The integrals in (20) are regular functions of λ at $\lambda = -k$, $k = 1, 2, \dots$. Therefore (G_+^λ, φ) has simple poles at such a point and from (18) we obtain

$$\lim_{\lambda \rightarrow -k} (\lambda + k)^2 (G_+^\lambda, \varphi) = 0. \quad (21)$$

Let λ and μ be complex numbers, from (4) and (5) we have

$$(G_+^{\lambda+\mu}, \varphi) = \int_0^\infty u^{(\lambda+\mu + (n/2m) - 1)} \Phi(\lambda, u) du, \quad (22)$$

where

$$\Phi(\lambda + \mu, u) = \frac{1}{4m^2} \int_0^1 (1-t)^{(\lambda+\mu + (q/2m) - 1)} \psi_1(u, ut) dt \quad (23)$$

and $\psi_1(u, ut)$ is defined by (6).

From (22) and considering (7), $(G_+^{\lambda+\mu}, \varphi)$ has two sets of singularities, namely

$$\begin{aligned} \lambda + \mu &= -1, -2, \dots, -l, \dots \\ &\text{and} \end{aligned} \quad (24)$$

$$\lambda + \mu = -\frac{n}{2m}, -\frac{n}{2m} - 1, \dots, -\frac{n}{2m} - j, \dots,$$

where $l = 1, 2, \dots$ and $j = 0, 1, 2, \dots$

From (22) we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -k} (\lambda + k)^2 (G_+^\lambda, \varphi) &= \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -k} (\lambda + k)^2 \int_0^\infty u^{(\lambda + \mu + (n/2m) - 1)} \Phi(\lambda, u) du \\ &= \lim_{\alpha \rightarrow -k-l} \left\{ (\alpha + k + l)^2 \int_0^\infty u^{(\alpha + (n/2m) - 1)} \Phi(\alpha, u) du \right\} \\ &= \lim_{\alpha \rightarrow -s} \left\{ (\alpha + s)^2 \int_0^\infty u^{(\alpha + (n/2m) - 1)} \Phi(\alpha, u) du \right\}, \end{aligned} \quad (25)$$

where $\lambda - l = \alpha$ and $s = k + l$.

From (25), using (21) and considering the condition

$$k + l \neq -\frac{n}{2m}, -\frac{n}{2m} - 1, \dots, -\frac{n}{2m} - s, \dots, s = 0, 1, 2, \dots \quad (26)$$

we obtain

$$\lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -k} (\lambda + k)^2 (G_+^\lambda, \varphi) = 0. \quad (27)$$

Similarly from (22), considering (21) and under condition (26) we have

$$\lim_{\lambda + \mu \rightarrow -(k+l)} (\lambda + \mu + k + l)^2 (G_+^{\lambda + \mu}, \varphi) = \lim_{\gamma \rightarrow -s} (\gamma + s)^2 (G_+^\gamma, \varphi) = 0. \quad (28)$$

On the other hand, from (9) and considering (10) we have

$$\begin{aligned} \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G) &= (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \\ \text{Re } s_{\lambda=-k, k=1,2,\dots} G_+^\lambda \cdot \text{Re } s_{\mu=-l, l=1,2,\dots} G_+^\mu &= (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \\ \lim_{\lambda \rightarrow -k} (\lambda + k) G_+^\lambda \cdot \lim_{\mu \rightarrow -l} (\mu + l) G_+^\mu &= (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \\ \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -k} [(\lambda + k)(\mu + l)] G_+^{\lambda + \mu} &= \end{aligned} \quad (29)$$

Taking into account that

$$[(\lambda + k)(\mu + l)] = \frac{1}{2} \{ (\lambda + \mu + k + l)^2 - [(\lambda + k)^2 + (\mu + l)^2] \}, \quad (30)$$

from (29) we have,

$$\begin{aligned} \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G) &= \frac{1}{2} (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \cdot \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \\ &\quad \times \{ (\lambda + \mu + k + l)^2 - [(\lambda + k)^2 + (\mu + l)^2] \} G_+^{\lambda + \mu}. \end{aligned} \quad (31)$$

Applying (31) to φ , we write

$$\langle \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G), \varphi \rangle = (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \cdot \{L + M + N\}, \quad (32)$$

where

$$L = \frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \langle (\lambda + \mu + k + l)^2 G_+^{\lambda + \mu}, \varphi \rangle, \quad (33)$$

$$M = -\frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \langle (\lambda + k)^2 G_+^{\lambda + \mu}, \varphi \rangle \quad (34)$$

and

$$N = -\frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \left\langle (\mu + l)^2 G_+^{\lambda+\mu}, \varphi \right\rangle. \quad (35)$$

From (31)–(35) and considering (26) and (27), we obtain

$$\langle \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G), \varphi \rangle = 0 \quad (36)$$

for odd n and for even n if $l + k \neq n/2m, n/2m + 1, \dots$

From (36) we arrive at the following theorem.

THEOREM 1 *Let k and l be positive integers and let n be dimension of the space, then*

$$\langle \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G), \varphi \rangle = 0 \quad (37)$$

if n is odd and

$$\langle \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G), \varphi \rangle = 0 \quad (38)$$

if n is even and

$$l + k \neq \frac{n}{2m}, \frac{n}{2m} + 1, \dots, \frac{n}{2m} + h, \dots, \text{ where } h = 0, 1, 2, \dots$$

Case 2 $n = 2mr$, $p = 2sm + m$, $q = 2tm + m$ and $k + l = \frac{n}{2m}, \frac{n}{2m} + 1, \dots, \frac{n}{2m} + h, \dots$, where $s, t, h = 0, 1, 2, \dots$, and $m = 1, 2, \dots$

From (12)–(16), we have

$$\begin{aligned} & \lim_{\lambda+\mu \rightarrow -\frac{n}{2m}-j} \left\langle (\lambda + \mu + \frac{n}{2m} + j)^2 G_+^{\lambda+\mu}, \varphi \right\rangle \\ &= \lim_{\gamma \rightarrow -\frac{n}{2m}-j} \left(\gamma + \frac{n}{2m} + j \right) \int_0^\infty u^{\gamma + \frac{n}{2m}-1} \Phi_0(u) du = \operatorname{Re} s_{\gamma = -\frac{n}{2m}-j} \int_0^\infty u^{\gamma + \frac{n}{2m}-1} \Phi_0(u) du \\ &= \frac{1}{j!} \Phi_0^{(j)}(0) = \frac{1}{(2m)^{2j}} \cdot \frac{1}{\Gamma\left(\frac{n}{2} + j\right)} \cdot \frac{(-1)^k}{\left(\frac{2}{m}(m-1) - 1\right) \cdots \left(\frac{2}{m}(m-1) - j\right)} \\ & \quad \times \left\langle \frac{(-1)(-1)^{(n+p-m/2m)} \pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^j \{\delta(x)\}, \varphi \right\rangle \end{aligned} \quad (39)$$

if $p = 2sm + m$, $q = 2tm + m$ and $s, t = 0, 1, 2, \dots$

From (33), we have

$$\begin{aligned} L &= \frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \left\langle (\lambda + \mu + k + l)^2 G_+^{\lambda+\mu}, \varphi \right\rangle = \frac{1}{2} \lim_{\lambda+\mu \rightarrow -(k+l)} \left\langle (\lambda + \mu + k + l)^2 G_+^{\lambda+\mu}, \varphi \right\rangle \\ &= \frac{1}{2} \lim_{\gamma \rightarrow -(k+l)} \left\langle (\gamma + k + l)^2 G_+^\gamma, \varphi \right\rangle. \end{aligned} \quad (40)$$

Letting $k + l - (n/2m) = j$ in (40) and considering (39), we obtain

$$\begin{aligned} L &= \frac{1}{2} \frac{1}{(2m)^{2j}} \frac{1}{\Gamma\left(\frac{n}{2} + j\right)} \frac{(-1)^j}{\left(\frac{2}{m}(m-1) - 1\right) \dots \left(\frac{2}{m}(m-1) - j\right)} \\ &\quad \times \left\langle \frac{(-1)(-1)^{(n+p-m/2m)} \pi^{\frac{n}{2}-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^j \{\delta(x)\}, \varphi \right\rangle. \end{aligned} \quad (41)$$

From (34) and (35), we have

$$\begin{aligned} M &= -\frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \left\langle (\lambda + k)^2 G_+^{\lambda+\mu}, \varphi \right\rangle \\ &= -\frac{1}{2} \lim_{\alpha \rightarrow -\frac{n}{2m}-j} \left\langle \left(\alpha + \frac{n}{2m} + j\right)^2 G_+^\alpha, \varphi \right\rangle \end{aligned} \quad (42)$$

and

$$\begin{aligned} N &= -\frac{1}{2} \lim_{\lambda \rightarrow -k} \lim_{\mu \rightarrow -l} \left\langle (\mu + l)^2 G_+^{\lambda+\mu}, \varphi \right\rangle \\ &= -\frac{1}{2} \lim_{\beta \rightarrow -\frac{n}{2m}-j} \left\langle \left(\beta + \frac{n}{2m} + j\right)^2 G_+^\beta, \varphi \right\rangle. \end{aligned} \quad (43)$$

From (42) and (43) and considering (13)–(15), we have

$$\begin{aligned} M &= -\frac{1}{2} \lim_{\alpha \rightarrow -\frac{n}{2m}-j} \left(\alpha + \frac{n}{2m} + j\right) \int_0^\infty u^{\alpha+(n/2m)-1} \Phi_0(u) du \\ &= -\frac{1}{2} \operatorname{Re} s_{\alpha=-(n/2m)-j} \int_0^\infty u^{\alpha+(n/2m)-1} \Phi_0(u) du = \frac{1}{j!} \Phi_0^{(j)}(0) \\ &= -\frac{1}{2} \frac{1}{(2m)^{2j}} \cdot \frac{1}{\Gamma\left(\frac{n}{2} + j\right)} \cdot \frac{(-1)^j}{\left(\frac{2}{m}(m-1) - 1\right) \dots \left(\frac{2}{m}(m-1) - j\right)} \\ &\quad \times \left\langle \frac{(-1)(-1)^{(n+p-m/2m)} \pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^j \{\delta(x)\}, \varphi \right\rangle \end{aligned} \quad (44)$$

and

$$\begin{aligned}
 N &= -\frac{1}{2} \lim_{\beta \rightarrow -\frac{n}{2m}-j} \left(\beta + \frac{n}{2m} + j \right) \int_0^\infty u^{\beta+(n/2m)-1} \Phi_0(u) \, du \\
 &= -\frac{1}{2} \operatorname{Re} s_{\beta=-(n/2m)-j} \int_0^\infty u^{\beta+(n/2m)-1} \Phi_0(u) \, du = \frac{1}{j!} \Phi_0^{(j)}(0) \\
 &= -\frac{1}{2} \frac{1}{(2m)^{2j}} \cdot \frac{1}{\Gamma\left(\frac{n}{2} + j\right)} \cdot \frac{(-1)^j}{\left(\frac{2}{m}(m-1)-1\right) \cdots \left(\frac{2}{m}(m-1)-j\right)} \\
 &\quad \times \left\langle \frac{(-1)(-1)^{(n+p-m/2m)} \pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^j \{\delta(x)\}, \varphi \right\rangle \quad (45)
 \end{aligned}$$

if $p = 2sm + m$, $q = 2tm + m$ and $s, t = 0, 1, 2, \dots$

From (32) and considering (41), (44) and (45), we have

$$\begin{aligned}
 &\langle \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G), \varphi \rangle \\
 &= (-1)^{k-1} (k-1)! \cdot (-1)^{l-1} (l-1)! \cdot \left[-\frac{1}{2} \frac{1}{(2m)^{2j}} \cdot \frac{1}{\Gamma\left(\frac{n}{2} + j\right)} \right] \\
 &\quad \times \frac{(-1)^j}{\left(\frac{2}{m}(m-1)-1\right) \cdots \left(\frac{2}{m}(m-1)-j\right)} \\
 &\quad \times \left\langle \frac{(-1)(-1)^{(n+p-m/2m)} \pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^j \{\delta(x)\}, \varphi \right\rangle \quad (46)
 \end{aligned}$$

for n even, $p = 2sm + m$, $q = 2tm + m$ and $k + l = \frac{n}{2m}, \frac{n}{2m} + 1, \dots, \frac{n}{2m} + h, \dots$, where s, t and $h = 0, 1, 2, \dots$

From (46) we arrive at the following formula:

$$\begin{aligned}
 \delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G) &= -\frac{1}{2} (-1)^{k+l-2} (-1)^{k+l-(n/2m)} (-1)^{(n+p-m/2m)} (k-1)! \cdot (l-1)! \\
 &\quad \cdot \frac{1}{(2m)^{2j}} \cdot \frac{1}{\Gamma\left(\frac{n}{2m} + k + l - \frac{n}{2m}\right)} \\
 &\quad \cdot \frac{(-1)^j}{\left(\frac{2}{m}(m-1)-1\right) \cdots \left(\frac{2}{m}(m-1) - \left(k + l - \frac{n}{2m}\right)\right)} \\
 &\quad \times \frac{(-1) \pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)} L_m^{k+l-(n/2m)} \{\delta(x)\} \quad (47)
 \end{aligned}$$

for n even, $p = 2sm + m$, $q = 2tm + m$ and $k + l = \frac{n}{2m}, \frac{n}{2m} + 1, \dots, \frac{n}{2m} + h, \dots$, where s, t and $h = 0, 1, 2, \dots$

We know that formulae (9) and (10) are valid for $k < n/2m$ when n is even and apply it to each factor of the product of the left hand side of (47). Thus, we arrive at the following theorem.

THEOREM 2 *Let k and l be positive integers and let n be dimension of the space such that $n = 2mr$, $m = 1, 2, \dots$ and $r = 1, 2, \dots$. Let L_m be the operator defined by (16), then the following formula*

$$\delta^{(k-1)}(G) \cdot \delta^{(l-1)}(G) = B_{k,l,n,p,q} L_m^{k+l-\frac{n}{2m}} \{\delta(x)\} \quad (48)$$

is valid under the following conditions

$$0 \leq k + l - \frac{n}{2m} < \frac{n}{2m} \quad (49)$$

and

$$p = 2ms + m, \quad q = 2mt + m, \quad (50)$$

where $s, t = 0, 1, 2, \dots$, and

$$\begin{aligned} & B_{k,l,n,p,q} \\ &= -\frac{1}{2} (-1)^{k+l-2} (-1)^{k+l-(n/2m)} (-1)^{(n+p-m/2m)} (k-1)! \cdot (l-1)! \\ & \cdot \frac{1}{(2m)^{2(k+l-(n/2m))}} \frac{1}{\Gamma(k+l)} \frac{1}{\left(\frac{2}{m}(m-1)-1\right) \cdots \left(\frac{2}{m}(m-1)-\left(k+l-\frac{n}{2m}\right)\right)} \\ & \times \frac{(-1)\pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)}. \end{aligned} \quad (51)$$

Letting $k = l$ in (48) and considering (51) we have

$$\delta^{(k-1)}(G) \cdot \delta^{(k-1)}(G) = B_{k,k,n,p,q} L_m^{2k-\frac{n}{2m}} \{\delta(x)\}, \quad (52)$$

where

$$\begin{aligned} & B_{k,k,n,p,q} = -\frac{1}{2} (-1)^{(2k-(n/2m))} (-1)^{(n+p-m/2m)} ((k-1)!)^2 \frac{1}{(2m)^{2(2k-(n/2m))}} \cdot \frac{1}{\Gamma(2k)} \\ & \cdot \frac{1}{\left(\frac{2}{m}(m-1)-1\right) \cdots \left(\frac{2}{m}(m-1)-\left(2k-\frac{n}{2m}\right)\right)} \\ & \times \frac{(-1)\pi^{(n/2)-1}}{\Gamma\left(\frac{p}{2}\right)} \frac{\Gamma\left(\frac{p}{2m}\right) \Gamma\left(\frac{q}{2m}\right)}{\Gamma\left(\frac{q}{2}\right)}. \end{aligned} \quad (53)$$

The formulae (52) and (53) are generalizations of the product $\delta^{(k-1)}(G) \cdot \delta^{(k-1)}(G)$ given by Aguirre in [4, pp. 20–21], formulae (54) and (55). In fact letting $m = 1$ in (48)–(51) and

considering (2), we obtain

$$\begin{aligned}
 \delta^{(k-1)}(P) \cdot \delta^{(l-1)}(P) &= \delta^{(k-1)}(G(1, x)) \cdot \delta^{(l-1)}(G(1, x)) \\
 &= B_{k,l,n,p,q} L_m^{(k+l-(n/2m))} \{\delta(x)\}_{m=1} \\
 &= -\frac{1}{2} (-1)^{k+l-2} (-1)^{(q-1/2)} (k-1)! \cdot (l-1)! \cdot \frac{1}{(2m)^{2(k+l-(n/2))}} \\
 &\quad \cdot \frac{1}{\Gamma(k+l)} \cdot \frac{(-1)\pi^{(n/2)-1}}{\left(k+l-\frac{n}{2}\right)!} L_1^{k+l-(n/2)} \{\delta(x)\}
 \end{aligned} \tag{54}$$

and

$$\begin{aligned}
 \delta^{(k-1)}(P) \cdot \delta^{(k-1)}(P) &= \delta^{(k-1)}(G(1, x)) \cdot \delta^{(k-1)}(G(1, x)) = B_{k,l,n,p,q} L_m^{k+k-(n/2m)} \{\delta(x)\}_{m=1} \Big| \\
 &= -\frac{1}{2} (-1)^{2k-2} (-1)^{(q-1/2)} ((k-1)!)^2 \cdot \frac{1}{(2m)^{2(2k-(n/2))}} \\
 &\quad \cdot \frac{1}{\Gamma(2k)} \cdot \frac{(-1)\pi^{(n/2)-1}}{\left(2k-\frac{n}{2}\right)!} L_1^{2k-(n/2)} \{\delta(x)\},
 \end{aligned} \tag{55}$$

where

$$L_1^j = L^j = \left\{ \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) - \left(\sum_{i=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \right) \right\}^j \tag{56}$$

is the ultrahyperbolic operator iterated j -times.

The formulae (54) and (55) appear in ref. [4, pages 21 and 22, formulae (54) and (55), respectively].

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Planar harmonic convolution operators generated by hypergeometric functions

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Let $S_{\hat{H}}$ be the family of all planar harmonic, univalent and sense-preserving mappings $f = h + \bar{g}$ where h and g are analytic functions in the open unit disk. The purpose of this article is to investigate connections between the theory of harmonic mappings in the plane and hypergeometric functions by applying certain planar harmonic convolution operators on various subclasses of $S_{\hat{H}}$.

Keywords: Planar harmonic mappings; Hypergeometric functions; Harmonic convolution operators; Harmonic starlike; Harmonic convex; Harmonic close-to-convex; Harmonic typically real

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1. Introduction

Harmonic mappings in the plane are complex-valued harmonic functions $w = f(z)$ which map the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ onto a domain $D \subset \mathbb{C}$. The major difference of such mappings from analytic functions is the canonical representation $f = h + \bar{g}$ where h and g are analytic functions in Δ . Let $\hat{H}$ be the collection of all harmonic mappings of the form $f = h + \bar{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad |B_1| < 1 \quad (1)$$

are analytic functions in Δ . A necessary and sufficient condition for $f = h + \bar{g}$ to be locally univalent and sense-preserving in Δ is that $|h'(z)| > |g'(z)|$ in Δ . See Clunie and Sheil-Small [1]. Denote by $S_{\hat{H}}$ the subclass of $\hat{H}$ that are univalent and sense-preserving in Δ . Note that $(f - \overline{B_1 f})/(1 - |B_1|^2) \in S_{\hat{H}}$ whenever $f \in S_{\hat{H}}$. Thus we may restrict our

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attention to the subclass $S_{\hat{H}}^0$ of $S_{\hat{H}}$ defined by

$$S_{\hat{H}}^0 := \{f = h + \bar{g} \in S_{\hat{H}} : g'(0) = B_1 = 0\}.$$

The classes $S_{\hat{H}}^0$ and $S_{\hat{H}}$ were first studied in [1].

Analogous to the well-known subclasses of the family S of all analytic, univalent and normalized functions in Δ , there are several subclasses of $S_{\hat{H}}^0$ and $S_{\hat{H}}$. Note that $S \subset S_{\hat{H}}^0 \subset S_{\hat{H}}$. A function $f \in S_{\hat{H}}$ ($f \in S_{\hat{H}}^0$) is said to be in $S_{\hat{H}}^*$ ($S_{\hat{H}}^{*0}$ respectively), the class of all harmonic starlike functions in Δ , if the range $f(\Delta)$ is starlike with respect to the origin. Likewise $f \in S_{\hat{H}}$ ($f \in S_{\hat{H}}^0$) is said to be in $K_{\hat{H}}$ ($K_{\hat{H}}^0$ respectively), the class of all harmonic convex functions in Δ , if $f(\Delta)$ is a convex domain. Analytically, for $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq 1$ we have

$$f \in S_{\hat{H}}^* \iff \frac{\partial}{\partial \theta} (\arg (f(re^{i\theta}))) > 0,$$

$$f \in K_{\hat{H}} \iff \frac{\partial}{\partial \theta} \left(\arg \left(\frac{\partial}{\partial \theta} f(re^{i\theta}) \right) \right) > 0.$$

The order α ($0 \leq \alpha < 1$) in $S_{\hat{H}}^*$ and $K_{\hat{H}}$ is defined by replacing '0' on the right side of the above inequalities by α . Denote the corresponding subclasses of the functions which are harmonic starlike of order α and harmonic convex of order α , respectively, by $S_{\hat{H}}^*(\alpha)$ and $K_{\hat{H}}(\alpha)$. Note that $S_{\hat{H}}^*(0) = S_{\hat{H}}$ and $K_{\hat{H}}(0) = K_{\hat{H}}$. Similar to the class C of all close-to-convex analytic functions in Δ , let $C_{\hat{H}}$ and $C_{\hat{H}}^0$ denote the subclasses, respectively, of $S_{\hat{H}}$ and $S_{\hat{H}}^0$ such that for any $f \in C_{\hat{H}}$ or $f \in C_{\hat{H}}^0$, $f(\Delta)$ is close-to-convex in Δ . Recall that a domain D is close-to-convex if the complement of D can be written as a union of non-increasing half-lines. For more information, see refs. [2–4].

We shall consider the functions ϕ_1 and ϕ_2 defined by

$$\phi_1(z) := zF(a_1, b_1; c_1; z), \quad \phi_2(z) := zF(a_2, b_2; c_2; z) \quad (z \in \Delta) \quad (2)$$

where $F(a, b, c; z)$ is a well-known Gaussian hypergeometric function; see, for example, [5] and [6]. A hypergeometric function $F(a, b; c; z)$ is analytic in Δ and plays an important role in the theory of univalent functions. See, for example, the works by de Branges [7], Dziok and Srivastava [8], Miller and Mocanu [5], Owa and Srivastava [9], Ruscheweyh and Singh [10], and Srivastava and Manocha [6].

Corresponding to $f = h + \bar{g} \in \hat{H}$ and ϕ_1, ϕ_2 given by (2), we define convolution operator $\Omega : \hat{H} \rightarrow \hat{H}$ by

$$\Omega(f) := \Omega(a_1, b_1, c_1, a_2, b_2, c_2; f) = f \star (\phi_1 + \overline{\phi_2}) = h \star \phi_1 + \overline{g \star \phi_2}, \quad (3)$$

where $\star$ denotes the usual Hadamard or convolution product of power series. For any function $f = h + \bar{g} \in \hat{H}$ given by (1), we have $\Omega(f) = H + \bar{G} \in \hat{H}$ where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n z^n, \quad G(z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n z^n, \quad (4)$$

and where $|B_1| < 1$. Note that we may represent $f = h + \bar{g} \in \hat{H}$ as the Hadamard product according to

$$f(z) = f(z) \star \left[\phi_1(1, 1; 1; z) + \overline{\phi_2(1, 1; 1; z)} \right] = f(z) \star \left[\frac{z}{1-z} + \overline{\frac{z}{1-z}} \right].$$

We also define the convolution operator $L: \hat{H} \rightarrow \hat{H}$ by

$$L(f) := L(b_1, c_1, b_2, c_2; f) = f \star (\varphi_1 + \overline{\varphi_2}) = h \star \varphi_1 + \overline{g \star \varphi_2}, \quad (5)$$

where φ_1 and φ_2 are incomplete beta functions defined by

$$\varphi_1(z) := \varphi(b_1, c_1; z) = zF(1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(b_1)_{n-1}}{(c_1)_{n-1}} z^n, \quad (6)$$

$$\varphi_2(z) := \varphi(b_2, c_2; z) = zF(1, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(b_2)_{n-1}}{(c_2)_{n-1}} z^n, \quad |B_1| < 1, \quad (7)$$

and where $c_1, c_2 \neq 0, -1, -2, \dots$. An incomplete beta function $\varphi(b, c; z)$ has an analytic continuation to the z -plane cut along the positive real line from 1 to ∞ . Note that $\varphi(b, 1; z) = z/(1-z)^b$ and $\varphi(2, 1; z)$ is the Koebe function. We also observe that if $b, c \neq 0, -1, -2, \dots$, then $L(b, b, b, b; f)$ is the identity and $L(b, c, b, c; f)$ has a continuous inverse of the operator $L(c, b, c, b; f)$. In particular, convolution of an analytic function with an incomplete beta function was first studied by Carlson and Shaffer [11].

The connections of hypergeometric functions with the theory of univalent functions are well-known in the literature, but the corresponding connections with harmonic mappings have not yet received much attention. The purpose of this article is to investigate the following:

1.1 Main problem

Under what restrictions on the parameters a_1, b_1, c_1, a_2, b_2 , and c_2 are the convolution operators $\Omega: \hat{H} \rightarrow \hat{H}$ and $L: \hat{H} \rightarrow \hat{H}$ map various subclasses of $S_{\hat{H}}^0$ ($S_{\hat{H}}^0$) into various subclasses of $S_{\hat{H}}$ (respectively $S_{\hat{H}}^0$)? More precisely, this article investigates connections between various subclasses of $S_{\hat{H}}$ and hypergeometric functions by applying convolution operators Ω and L on certain subclasses of $S_{\hat{H}}$ and $S_{\hat{H}}^0$.

Throughout this article, we will frequently use the notations

$$\Omega(f) := \Omega(a_1, b_1, c_1, a_2, b_2, c_2; f), \quad L(f) := L(b_1, c_1, b_2, c_2; f)$$

and a well-known formula

$$F(a, b, c; 1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \operatorname{Re}(c-a-b) > 0, \quad c \neq 0, -1, -2, \dots \quad (8)$$

2. Preliminaries and key lemmas

First four Lemmas are due to Clunie and Sheil-Small [1].

LEMMA 1 *A harmonic function $f = h + \bar{g} \in \hat{H}$ is locally univalent and sense-preserving in Δ if only if the Jacobian*

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0, \quad z \in \Delta$$

or equivalently $|g'(z)| < |h'(z)|$ ($z \in \Delta$).

LEMMA 2 Suppose H and G are analytic functions in Δ with $|G'(0)| < |H'(0)|$ and that $H + \varepsilon G \in C$ for each $\varepsilon (|\varepsilon| = 1)$. Then $H + \bar{G} \in C_{\hat{H}}$.

LEMMA 3 A function $f = h + \bar{g} \in K_{\hat{H}}$ with h and g given by (1) if and only if the analytic functions $h(z) - e^{it}g(z)$ ($0 \leq t \leq 2\pi$) are convex in the direction $t/2$ and f is suitably normalized.

LEMMA 4 If $f = h + \bar{g} \in K_{\hat{H}}^0$ is given by (1), then $|A_n| \leq (n+1)/2$, $|B_n| \leq (n-1)/2$, $n = 2, 3, \dots$

The proof of next lemma is available in [12].

LEMMA 5 If $q(z) = z + \sum_{n=2}^{\infty} t_n z^n$ is analytic in Δ , then

- (i) q maps onto a starlike domain if $\sum_{n=2}^{\infty} n|t_n| \leq 1$,
- (i) q maps onto a convex domain if $\sum_{n=2}^{\infty} n^2|t_n| \leq 1$.

A complex-valued function f harmonic in Δ is said to be typically real provided that $f(z)$ is real if and only if z is real. Suppose the class $T_{\hat{H}}$ consists of all sense-preserving typically real harmonic functions $f = h + \bar{g} \in \hat{H}$. The subclass of $T_{\hat{H}}$ with $g'(0) = 0$ is denoted by $T_{\hat{H}}^0$. The results in the next two lemmas may be found in [1] and [13], respectively.

LEMMA 6 If $f = h + \bar{g} \in C_{\hat{H}}^0$ ($S_{\hat{H}}^{*0}$ or $T_{\hat{H}}^0$) with h and g as given by (1), then

$$|A_n| \leq \frac{(2n+1)(n+1)}{6} \quad (n = 2, 3, \dots),$$

$$|B_n| \leq \frac{(2n-1)(n-1)}{6} \quad (n = 2, 3, \dots).$$

LEMMA 7 If $f = h + \bar{g} \in C_{\hat{H}}$ is given by (1), then

$$|A_n| \leq \frac{(2n+1)(n+1)}{6} + \frac{(2n-1)(n-1)}{6} |B_1| \quad (n = 2, 3, \dots),$$

$$|B_n| \leq \frac{(2n-1)(n-1)}{6} + \frac{(2n+1)(n+1)}{6} |B_1| \quad (n = 1, 2, \dots).$$

The results in the next lemma were found in [14] and [15].

LEMMA 8 If $f = h + \bar{g}$ with h and g of the form (1), then

(i)

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |B_n| \leq 1 \implies f \in S_{\hat{H}}^*(\alpha),$$

(ii)

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |B_n| \leq 1 \implies f \in K_{\hat{H}}(\alpha).$$

For $\alpha = 0$, the corresponding results in Lemma 8 were found in [16] and [17]. Let $S_{RH}^*(\alpha)$ and $K_{RH}(\alpha)$ denote, respectively, the subclasses of $S_H^*(\alpha)$ and $K_H(\alpha)$ consisting of functions $f = h + \bar{g}$ where

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n, \quad |B_1| < 1. \quad (9)$$

Let $S_{RH}^* := S_{RH}^*(0)$, $K_{RH} := K_{RH}(0)$. The results in the next Lemma, found in [14] and [15], provide the coefficient characterizations of the functions in $S_{RH}^*(\alpha)$ and $K_{RH}(\alpha)$.

LEMMA 9 If $f = h + \bar{g}$ with h and g of the form (9), then

(i)

$$f \in S_{RH}^*(\alpha) \iff \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} |B_n| \leq 1,$$

(ii)

$$f \in K_{RH}(\alpha) \iff \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} |A_n| + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} |B_n| \leq 1.$$

LEMMA 10 If $a, b, c > 0$, then

(i)

$$F(a+k, b+k; c+k; 1) = \frac{(c)_k}{(c-a-b-k)_k} F(a, b; c; 1),$$

for $k = 0, 1, 2, \dots$, if $c > a + b + k$

(ii)

$$\sum_{n=2}^{\infty} (n-1) \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \frac{ab}{c-a-b-1} F(a, b; c; 1) \quad \text{if } c > a + b + 1$$

(iii)

$$\sum_{n=2}^{\infty} (n-1)^2 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \left[\frac{(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1} \right] F(a, b; c; 1) \quad \text{if } c > a + b + 2$$

(iv)

$$\sum_{n=2}^{\infty} (n-1)^3 \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} = \left[\frac{(a)_3 (b)_3}{(c-a-b-3)_3} + \frac{3(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1} \right] F(a, b; c; 1) \quad \text{if } c > a + b + 3.$$

Proof (i) The proof follows by using (8) and a well-known property $\Gamma(d+k) = (d)_k \Gamma(d)$.

- (ii) In view of (8) and by an application of Part (i) of this lemma, Part (ii) immediately follows.
- (iii) Again applying (8), we have

$$\begin{aligned}
 \sum_{n=2}^{\infty} (n-1)^2 \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} &= \frac{ab}{c} \sum_{n=3}^{\infty} (n-2) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} \\
 &\quad + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} \\
 &= \frac{(a)_2(b)_2}{(c)_2} F(a+2, b+2; c+2; 1) \\
 &\quad + \frac{ab}{c} F(a+1, b+1; c+1; 1).
 \end{aligned}$$

The result follows by using Part (i) of this lemma.

- (iv) The proof is similar to the proof of Part (iii) and so it is omitted. ■

3. Main results

We first determine the conditions which ensures that the operator Ω maps $\hat{H}$ into $S_{\hat{H}}$.

THEOREM 1 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1), $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty} |A_n| + \sum_{n=1}^{\infty} |B_n| \leq 1, \quad |B_1| < 1,$$

(ii)

$$\sum_{j=1}^2 \left[\frac{|a_j b_j|}{c_j - |a_j| - |b_j| - 1} + 1 \right] F(|a_j|, |b_j|; c_j; 1) \leq 2,$$

then $\Omega(f)$ is sense-preserving, harmonic, and univalent in Δ ; and so $\Omega(\hat{H}) \subset S_{\hat{H}}$.

Proof Note that $|A_n| \leq 1$ and $|B_n| \leq 1$ for all $n \geq 2$ by the given condition (i). Therefore, in view of Lemma 1 we observe that $\Omega(f)$ defined by (3) is locally univalent and sense-preserving in Δ provided that the Jacobian

$$J_{\Omega(f)}(z) = (|h \star \phi'_1(z)| + |g \star \phi'_2(z)|)(|h \star \phi'_1(z)| - |g \star \phi'_2(z)|) \geq 0.$$

However, it suffices to prove that $|h \star \phi'_1(z)| - |g \star \phi'_2(z)| \geq 0$. In view of Lemma 10(i), it follows that

$$|h \star \phi'_1(z)| - |g \star \phi'_2(z)| \geq 1 - \sum_{n=2}^{\infty} n \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=1}^{\infty} n \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$$

$$\begin{aligned}
&= 2 - \left[\frac{|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} + 1 \right] F(|a_1|, |b_1|; c_1; 1) \\
&\quad - \left[\frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1} + 1 \right] F(|a_2|, |b_2|; c_2; 1)
\end{aligned}$$

is non-negative by the given condition (ii). To show that $\Omega(f) = H + \bar{G}$ given by (3) is univalent in Δ , we assume that $z_1, z_2 \in \Delta$, $z_1 \neq z_2$. Then by using (4) we have

$$\begin{aligned}
\left| \frac{\Omega(f)(z_1) - \Omega(f)(z_2)}{H(z_1) - H(z_2)} \right| &\geq 1 - \left| \frac{G(z_1) - G(z_2)}{H(z_1) - H(z_2)} \right| \\
&\geq 1 - \frac{\sum_{n=1}^{\infty} n(|a_2|)_{n-1}(|b_2|)_{n-1}/((c_2)_{n-1}(1)_{n-1})}{1 - \sum_{n=2}^{\infty} n(|a_1|)_{n-1}(|b_1|)_{n-1}/((c_1)_{n-1}(1)_{n-1})} \\
&\geq 1 - \frac{|G'(z)|}{|H'(z)|}.
\end{aligned}$$

Since by the first part of the theorem, $\Omega(f) = H + \bar{G}$ is locally univalent and sense-preserving in Δ , an application of Lemma 1 shows that the last inequality is strictly positive and, therefore, $\Omega(f)$ is univalent in Δ . ■

We next investigate the conditions which guarantee that a harmonic convex function maps the unit disk Δ into a harmonic close-to-convex domain.

THEOREM 2 *If $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 2$ for $j = 1, 2$ are such that*

$$\begin{aligned}
&\left[\frac{(|a_1|)_2(|b_1|)_2}{2(c_1 - |a_1| - |b_1| - 2)_2} + \frac{2|a_1 b_1|}{c_1 - |a_1| - |b_1| - 1} + 1 \right] F(|a_1|, |b_1|; c_1; 1) \\
&\quad + \left[\frac{(|a_2|)_2(|b_2|)_2}{2(c_2 - |a_2| - |b_2| - 2)_2} + \frac{|a_2 b_2|}{c_2 - |a_2| - |b_2| - 1} \right] F(|a_2|, |b_2|; c_2; 1) \leq 2,
\end{aligned} \tag{10}$$

then $\Omega(K_{\bar{H}}^0) \subset C_{\bar{H}}^0$.

Proof Let $f = h + \bar{g} \in K_{\bar{H}}^0$ with h and g of the form (1) with $B_1 = 0$. We need to show that $\Omega(f) = H + \bar{G} \in C_{\bar{H}}^0$, where H and G given by (4) with $B_1 = 0$ are analytic functions in Δ . Note that $|H'(0)| = 1 > |B_1| = |G'(0)|$. Therefore, in view of Lemma 2, it suffices to prove that $H + \varepsilon G = z + \sum_{n=2}^{\infty} t_n z^n \in C$ for every ε ($|\varepsilon| = 1$), where

$$t_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} A_n + \varepsilon \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} B_n. \tag{11}$$

As an application of Lemma 5(i), we only need to prove that $\sum_{n=2}^{\infty} n|t_n| \leq 1$. Using Lemma 4 we have

$$\sum_{n=2}^{\infty} n|t_n| \leq \frac{1}{2} \sum_{n=2}^{\infty} \left[n(n+1) \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + n(n-1) \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \right].$$

Writing

$$n(n+1) = (n-1)^2 + 3(n-1) + 2, \quad n(n-1) = (n-1)^2 + (n-1),$$

and using Lemma 10 (i–iii) and (10), it is a routine matter to show that $\sum_{n=2}^{\infty} n|t_n| \leq 1$. ■

THEOREM 3 If $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 3$ for $j = 1, 2$ are such that

$$\begin{aligned} & \left[\frac{(|a_1|)_3(|b_1|)_3}{3(c_1 - |a_1| - |b_1| - 3)_3} + \frac{5(|a_1|)_2(|b_1|)_2}{2(c_1 - |a_1| - |b_1| - 2)_2} + \frac{4|a_1b_1|}{c_1 - |a_1| - |b_1| - 1} + 1 \right] \\ & \times F(|a_1|, |b_1|; c_1; 1) \\ & + \left[\frac{(|a_2|)_3(|b_2|)_3}{3(c_2 - |a_2| - |b_2| - 3)_3} + \frac{3(|a_2|)_2(|b_2|)_2}{2(c_2 - |a_2| - |b_2| - 2)_2} + \frac{|a_2b_2|}{c_2 - |a_2| - |b_2| - 1} \right] \\ & \times F(|a_2|, |b_2|; c_2; 1) \leq 2, \end{aligned}$$

then $\Omega(C_{\hat{H}}^0) \subset C_{\hat{H}}^0$, $\Omega(S_{\hat{H}}^{*0}) \subset C_{\hat{H}}^0$ and $\Omega(T_{\hat{H}}^0) \subset C_{\hat{H}}^0$.

Proof Let $f = h + \bar{g} \in C_{\hat{H}}^0$ ($S_{\hat{H}}^{*0}$ or $T_{\hat{H}}^0$) with h and g of the form (1). We need to show that $\Omega(f) = H + \bar{G} \in C_{\hat{H}}^0$, where H and G given by (4) are analytic functions in Δ . Using Lemma 2, Lemma 5(i) and proceeding similar to the proof of Theorem 2, it suffices to prove that $\sum_{n=2}^{\infty} n|t_n| \leq 1$, where t_n is given by (11). Using Lemma 6 and identities

$$\begin{aligned} n(2n+1)(n+1) &= 2(n-1)^3 + 9(n-1)^2 + 13(n-1) + 6, \\ n(2n-1)(n-1) &= 2(n-1)^3 + 3(n-1)^2 + (n-1), \end{aligned}$$

it follows by applications of Lemma 10(i-iv) that $\sum_{n=2}^{\infty} n|t_n| \leq 1$, by the given condition. ■

Letting $a_1 = a_2 = 1$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$ in Theorem 3, we obtain the following result for the harmonic convolution operator L defined by (5).

COROLLARY 1 Let $b \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$, and $c > |b| + 4$. If

$$\left[\frac{4(|b|)_3}{(c - |b| - 4)_3} + \frac{8(|b|)_2}{(c - |b| - 3)_2} + \frac{5|b|}{c - |b| - 2} + 1 \right] \varphi(b; c; 1) \leq 2,$$

then $L(C_{\hat{H}}^0) \subset C_{\hat{H}}^0$, $L(S_{\hat{H}}^{*0}) \subset C_{\hat{H}}^0$, $L(T_{\hat{H}}^0) \subset C_{\hat{H}}^0$.

Using Lemma 7 we can extend the above theorem for the full class $C_{\hat{H}}$. However, the corresponding statement and proof for $C_{\hat{H}}$ becomes very lengthy and messy. We, therefore, look at the special case when $a_1 = a_2 = a$, $b_1 = b_2 = b$, and $c_1 = c_2 = c$ so that $\phi_1 = \phi_2 = \phi$ and $\Omega(f) = f \star (\phi + \bar{\phi}) = h \star \phi + g \star \bar{\phi}$ for any $f = h + \bar{g} \in C_{\hat{H}}$. For the next result, we use the notation $\Omega(f) = \Omega(a, b, c; f)$.

THEOREM 4 If $a, b \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{R}$, $c > |a| + |b| + 3$, then a sufficient condition for $\Omega(a, b, c; C_{\hat{H}}) \subset C_{\hat{H}}$ is such that

$$\begin{aligned} & \left[\frac{2(|a|)_3(|b|)_3}{(c - |a| - |b| - 3)_3} + \frac{12(|a|)_2(|b|)_2}{(c - |a| - |b| - 2)_2} + \frac{15|ab|}{c - |a| - |b| - 1} + 3 \right] \\ & \times F(|a|, |b|; c; 1) \leq \frac{6}{1 - |B_1|} \end{aligned}$$

for all $f \in C_{\hat{H}}$, where $|B_1| = |f_{\bar{z}}(0)| < 1$.

Proof Let $f = h + \bar{g} \in C_{\hat{H}}$ with h and g of the form (1). Then

$$\begin{aligned}\Omega(a, b, c; f)(z) &= H(z) + \overline{G(z)} \\ &= z + \overline{B_1 z} + \sum_{n=2}^{\infty} \left[\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} A_n z^n + \overline{\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} B_n z^n} \right].\end{aligned}$$

We need to show that $H + \bar{G} \in C_{\hat{H}}$, where H and G are analytic functions in Δ and $|H'(0)| = 1 > |B_1| = |G'(0)|$. In view of Lemma 2, it suffices to prove that

$$\frac{H + \varepsilon G}{1 + \varepsilon B_1} = z + \sum_{n=2}^{\infty} \ell_n z^n \in C$$

for every $\varepsilon(|\varepsilon| = 1)$, where

$$\ell_n := \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \left[\frac{A_n + \varepsilon B_n}{1 + \varepsilon B_1} \right] \quad (12)$$

As an application of Lemma 5(i), we only need to show that $\sum_{n=2}^{\infty} n|\ell_n| \leq 1$. Using Lemma 7, the identity

$$n(2n^2 + 1) = 2(n-1)^3 + 6(n-1)^2 + 7(n-1) + 3,$$

we have

$$\begin{aligned}\sum_{n=2}^{\infty} n|\ell_n| &\leq \frac{2(1 + |B_1|)}{6(1 - |B_1|)} \sum_{n=2}^{\infty} n(2n^2 + 1) \frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\ &= \frac{(1 + |B_1|)}{3(1 - |B_1|)} \left[\frac{2(|a|)_3(|b|)_3}{(c - |a| - |b| - 1)_3} + \frac{12(|a|)_2(|b|)_2}{(c - |a| - |b| - 1)_2} \right. \\ &\quad \left. + \frac{15|ab|}{c - |a| - |b| - 1} + 3 \right] F(|a|, |b|; c; 1) - \frac{(1 + |B_1|)}{(1 - |B_1|)}.\end{aligned}$$

The last inequality is bounded above by one because of the given condition. ■

We next determine sufficient conditions for a mapping f that maps $\hat{H}$ into a harmonic close-to-convex domain.

THEOREM 5 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1), $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j|$ for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| \leq 1, |B_1| < 1,$$

(ii)

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1| < 3, \quad \text{then } \Omega(f) \in C_{\hat{H}}.$$

Proof Note that $\Omega(f) = H + \bar{G}$, where H and G are given by (4). Then

$$\omega(z) := \frac{H(z) + \varepsilon G(z)}{1 + \varepsilon B_1(z)} = z + \sum_{n=2}^{\infty} s_n$$

where

$$s_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{A_n}{1 + \varepsilon B_1} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{\varepsilon B_n}{1 + \varepsilon B_1}.$$

From the given condition (i), it follows that $|A_n| \leq 1/n$ and $|B_n| \leq 1/n \forall n \geq 2$. We, therefore, have

$$\begin{aligned} \sum_{n=2}^{\infty} n|s_n| &\leq \sum_{n=2}^{\infty} \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{1}{1 - |B_1|} + \sum_{n=2}^{\infty} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{1}{1 - |B_1|} \\ &= [F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) - 2] \frac{1}{1 - |B_1|} \leq 1, \end{aligned}$$

by the given condition (ii). As an application of Lemma 5(i), it follows that ω is starlike in Δ and therefore $\omega \in C$. Hence by Lemma 2, $\Omega(f) = H + \bar{G}$ is in $C_{\hat{H}}$. ■

Lemma 9(i) together with the above theorem yield the following

COROLLARY 2 *If $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j|$ for $j = 1, 2$ are such that*

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) < 3,$$

then $\phi_1 + \varepsilon \bar{\phi}_2 \in C_{\hat{H}}$ for each $\varepsilon (|\varepsilon| < 1)$.

Proof Note that for each $\varepsilon (|\varepsilon| < 1)$, we have

$$f(z) := \frac{z}{1-z} + \varepsilon \overline{\left(\frac{z}{1-z} \right)} = z + \sum_{n=2}^{\infty} z^n + \sum_{n=1}^{\infty} \bar{\varepsilon} \bar{z}^n \in S_{\hat{H}},$$

and $\Omega(f) = f \tilde{*} (\phi_1 + \bar{\phi}_2) = h * \phi_1 + \overline{g * \phi_2} = \phi_1 + \varepsilon \bar{\phi}_2$. Also

$$\sum_{n=2}^{\infty} n|A_n| + \sum_{n=1}^{\infty} n|B_n| = |\varepsilon| + (1 + |\varepsilon|) \sum_{n=2}^{\infty} n \leq 1,$$

because $|\varepsilon| \leq (1 - \sum_{n=2}^{\infty} n) / (1 + \sum_{n=2}^{\infty} n) < 1$. The result follows from the above theorem. ■

Analogous to Theorem 5, we now obtain sufficient conditions for a mapping f in $\hat{H}$ into a harmonic convex domain.

THEOREM 6 *If $f = h + \bar{g} \in \hat{H}$ with h and g of the form (1), $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ are such that*

(i)

$$\sum_{n=2}^{\infty} n^2 |A_n| + \sum_{n=1}^{\infty} n^2 |B_n| \leq 1, |B_1| < 1,$$

(ii)

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1| < 3,$$

then $\Omega(f) \in K_{\hat{H}}$.

Proof If $\Omega(f) = H + \bar{G}$, where H and G are given by (4), then

$$\varsigma(z) := \frac{H(z) - e^{it}G(z)}{1 - B_1 e^{it}} = z + \sum_{n=2}^{\infty} \rho_n z^n,$$

where

$$\rho_n := \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \frac{A_n}{1 - e^{it}B_1} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \frac{e^{it}B_n}{1 - e^{it}B_1}.$$

On the other hand, condition (i) yields $|A_n| \leq 1/n^2$ and $|B_n| \leq 1/n^2$, $\forall n \geq 2$ and so using the arguments similar to the proof of Theorem 5, it is easy to prove that $\sum_{n=2}^{\infty} n^2 |\rho_n| \leq 1$, by the condition (ii). It therefore follows from Lemma 5(ii) that ς is convex in Δ and therefore by Lemma 3, $\Omega(f) \in K_{\hat{H}}$. ■

Using Lemma 9(ii), the above theorem gives the following

COROLLARY 3 If $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$ and $|B_1| < 1$ are such that

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 3 - |B_1|,$$

then $\Omega(K_{R\hat{H}}) \subset K_{\hat{H}}$.

The next result is more general than the above corollary and the method of proof is also different.

THEOREM 7 If $a_j, b_j \in \mathbb{C} \setminus \{0\}$, $c_j \in \mathbb{R}$, $c_j > |a_j| + |b_j| + 1$ for $j = 1, 2$, and $0 \leq \alpha < 1$ are such that

$$F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) \leq 2,$$

then $\Omega(K_{R\hat{H}}(\alpha)) \subset K_{\hat{H}}(\alpha)$.

Proof Let $f = h + \bar{g} \in K_{R\hat{H}}(\alpha)$ with h and g are of the form (9). It follows from Lemma 9(ii) that

$$|A_n| \leq \frac{1 - \alpha}{n(n - \alpha)} \quad \forall n \geq 2, \quad |B_n| \leq \frac{1 - \alpha}{n(n + \alpha)} \quad \forall n \geq 1, \quad |B_1| < 1.$$

Note that

$$(\Omega(f))(z) = z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \overline{\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| z^n}. \quad (13)$$

In view of Lemma 8(ii), $\Omega(f) \in K_{\hat{H}}(\alpha)$ because

$$\begin{aligned} P_1 &:= \sum_{n=2}^{\infty} \frac{n(n - \alpha)}{1 - \alpha} \left| \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| \right| + \sum_{n=1}^{\infty} \frac{n(n + \alpha)}{1 - \alpha} \left| \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| \right| \\ &\leq \sum_{n=2}^{\infty} \frac{(|a_1|)_{n-1}(|b_1|)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{(|a_2|)_{n-1}(|b_2|)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \\ &= F(|a_1|, |b_1|; c_1; 1) + F(|a_2|, |b_2|; c_2; 1) - 1 \leq 1, \end{aligned}$$

by the given condition. ■

We now dilute the restrictions on complex coefficients and obtain a different sufficient condition for harmonic convolution operator Ω applied on $K_{R\hat{H}}(\alpha)$.

THEOREM 8 *If $a_1, b_1 > -1, a_1 b_1 < 0, a_2, b_2 \in \mathbb{C} \setminus \{0\}, c_1 > a_1 + b_1, c_2 > |a_2| + |b_2|$, then a sufficient condition for $\Omega(K_{R\hat{H}}(\alpha)) \subset K_{\hat{H}}(\alpha)$ is that*

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0.$$

Proof If $f = h + \bar{g} \in K_{R\hat{H}}(\alpha)$ with h and g in (9), then $\Omega(f)$ in (13) can be rewritten as

$$z + \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{(a_1 + 1)_{n-2} (b_1 + 1)_{n-2}}{(c_1 + 1)_{n-2} (1)_{n-1}} |A_n| z^n + \sum_{n=1}^{\infty} \frac{\overline{(a_2)_{n-1} (b_2)_{n-1}}}{(c_2)_{n-1} (1)_{n-1}} |B_n| z^n.$$

In view of Lemma 8(ii), $\Omega(f) \in K_{\hat{H}}(\alpha)$ if $P_2 \leq 1$ where

$$P_2 := \frac{|a_1 b_1|}{c_1} \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha} \left| \frac{(a_1 + 1)_{n-2} (b_1 + 1)_{n-2}}{(c_1 + 1)_{n-2} (1)_{n-1}} |A_n| \right| \\ + \sum_{n=1}^{\infty} \frac{n(n+\alpha)}{1-\alpha} \left| \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} |B_n| \right|.$$

Following the steps of the preceeding theorem and using Lemma 10(i) and the given condition, it can easily be proved that $P_2 \leq 1$. ■

Our next result will parallel the above theorem for the case of harmonic starlike functions of order α and so its proof is omitted. However, note that the proof of the next theorem uses Lemma 9(i), Lemma 8(i) and Lemma 10(i).

THEOREM 9 *If $a_1, b_1 > -1, a_1 b_1 < 0, a_2, b_2 \in \mathbb{C} \setminus \{0\}, c_1 > a_1 + b_1, c_2 > |a_2| + |b_2|$, then a sufficient condition for $\Omega(S_{R\hat{H}}^*(\alpha)) \subset S_{\hat{H}}^*(\alpha)$ is that*

$$F(a_1, b_1; c_1; 1) - F(|a_2|, |b_2|; c_2; 1) \geq 0.$$

Finally, by imposing stronger conditions on a_1, a_2, b_1, b_2, c_1 , and c_2 we obtain a characterization for the operator Ω which maps $S_{RH}^*(\alpha)$ into itself.

THEOREM 10 *If $a_j, b_j > 0, c_j > a_j + b_j$ for $j = 1, 2$, then a necessary and sufficient condition for $\Omega(S_{R\hat{H}}^*(\alpha)) \subset S_{R\hat{H}}^*(\alpha)$ is that*

$$F(a_1, b_1; c_1; 1) + F(a_2, b_2; c_2; 1) \leq 2. \quad (14)$$

Proof If $f = h + \bar{g} \in S_{R\hat{H}}^*(\alpha)$ with h and g given in (9), then $\Omega(f)$ has the form (13). As an application of Lemma 9(i) it follows that $\Omega(f) \in S_{R\hat{H}}^*(\alpha)$ if and only if $P_3 \leq 1$ where

$$P_3 := \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha} \left| \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} |A_n| \right| + \sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} \left| \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} |B_n| \right|.$$

Again using Lemma 9(i) and following the procedure as in Theorem 8, it is straightforward to show that $P_3 \leq 1$ if and only if the expression (14) holds. ■

Using similar arguments and Lemma 9(ii) in place of Lemma 9(i) in the previous theorem, we obtain the following characterization for the operator Ω which maps $K_{RH}(\alpha)$ onto itself.

THEOREM 11 *If $a_j, b_j > 0, c_j > a_j + b_j$ for $j = 1, 2$, then $\Omega(K_{RH}(\alpha)) \subset K_{RH}(\alpha)$ if and only if (14) is satisfied.*

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Fourier and other series containing associated Legendre functions for incomplete Epstein–Hubbell integrals and functions related to elliptic integrals

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Fourier series containing associated Legendre functions are given for functions related to the Legendre elliptic integrals and for incomplete Epstein–Hubbell integrals. A number of definite and indefinite integrals related to these series are derived. Additional series of Legendre functions and integrals are derived from a binomial expansion of the reciprocal of the delta amplitude. A power series expansion for the elliptic integral of the third kind in terms of the parameter α is obtained. Series of associated Legendre functions are given for general monomials.

Keywords: Epstein–Hubbell integrals; Legendre functions; Fourier series; Elliptic integrals

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1. Introduction

Recently [1], Fourier expansions in terms of the amplitude ϕ were given for some Legendre incomplete elliptic integrals, the Fourier coefficients containing Legendre functions of the second kind and half-integral degree $Q_{m-1/2}(\omega)$, $m \in \mathbb{N}_0$. These Fourier series were obtained from an integral representation for $Q_{m-1/2}(\omega)$ which gives directly a Fourier expansion for the reciprocal of the delta amplitude $\Delta^{-1}(\phi, k)$, where $\Delta(\phi, k) \equiv \sqrt{1 - k^2 \sin^2 \phi}$. In this article the more general integral representation for the associated Legendre function $Q_{m-1/2}^\mu(\omega)$ $m \in \mathbb{N}_0$, $\mu \in \mathbb{C}$ is exploited to obtain Fourier series for more general algebraic expressions. On term-by-term integration these expansions yield various series for functions related to the Legendre elliptic integrals. In particular, Fourier expansions are obtained for incomplete Epstein–Hubbell elliptic-type integrals, and a number of integrals and various other Legendre function expansions can be obtained. Additional series and integrals resulting from binomial expansion of $\Delta^{-1}(\phi, k)$ will be derived here, including a power series expansion in the parameter α for the complete elliptic integral of the third kind, $\Pi(\alpha, k)$.

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The associated Legendre function $Q_{m-1/2}^\mu(\omega)$ of the second kind and half-integral degree, where $\mu, \omega \in \mathbb{C}$ and $m \in \mathbb{N}_0$, has the well-known integral representation [2]:

$$Q_{m-1/2}^\mu(\omega) = \frac{\exp(\mu\pi i)\Gamma(\mu+1/2)}{\sqrt{2\pi}}(\omega^2-1)^{\mu/2} \int_0^\pi \frac{\cos m\phi \, d\phi}{(\omega - \cos \phi)^{\mu+1/2}} \quad (1)$$

$$\left[\operatorname{Re}(\mu) > -\frac{1}{2}, m \in \mathbb{N}_0 \right].$$

For evaluation of (1), a single cut is taken along the real axis from $-\infty$ to $+1$, hence $Q_{m-1/2}^\mu(\omega)$ as defined here is continuous on the real axis for $\omega > 1$. An alternative is to cut the real axis from $-\infty$ to -1 and from 1 to ∞ , with the functions $Q_{m-1/2}^\mu(\omega)$ then being continuous on the real axis for $-1 < \omega < 1$. For $\mu = j$ where $j \in \mathbb{N}_0$, equation (1) reduces to

$$Q_{m-1/2}^j(\omega) = \frac{(-1)^j(2j)!}{2^{2j+1/2}j!}(\omega^2-1)^{j/2} \int_0^\pi \frac{\cos m\phi}{(\omega - \cos \phi)^{j+1/2}} d\phi \quad [j, m \in \mathbb{N}_0]. \quad (2)$$

Equation (1) yields immediately the Fourier expansion for $(\omega - \cos \phi)^{-\mu-1/2}$:

$$\frac{1}{(\omega - \cos \phi)^{\mu+1/2}} = \frac{2^{1/2} \exp(-\mu\pi i)}{\pi^{1/2}(\omega^2-1)^{\mu/2}\Gamma(\mu+1/2)} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2}^\mu(\omega) \cos m\phi \quad (3)$$

where in equation (3) ε_m is the Neumann factor, with $\varepsilon_m = 1$ for $m = 0$ and $\varepsilon_m = 2$ otherwise.

Relatively simple transformations of the above equations yield important results. For $\omega = 1/k^2$ where $(k \in \mathbb{R}, 0 < k^2 < 1)$ in equation (3) yields

$$\frac{1}{(1 - k^2 \cos \phi)^{\mu+1/2}} = \frac{2^{1/2} \exp(-\mu\pi i)}{\pi^{1/2}k(1 - k^4)^{\mu/2}\Gamma(\mu+1/2)} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2}^\mu\left(\frac{1}{k^2}\right) \cos m\phi \quad (4)$$

The factor $(1 - k^2 \cos \phi)^{-j-1/2}$ where $j \in \mathbb{N}_0$ is central to the definition of the Epstein–Hubbell integrals $\Omega_j(k)$ [3].

Making the substitutions $\theta = 2\phi$ and $\bar{k}^2 \equiv 2k^2/(1 + k^2)$ in equation (3) for $\mu = j \in \mathbb{N}_0$, then renaming back the variables with $\theta \rightarrow \phi$ and $\bar{k} \rightarrow k$ gives:

$$\frac{1}{(1 - k^2 \cos^2 \phi)^{j+1/2}} = \frac{(-1)^j 2^{2j+1} j!}{\pi k (2j)! (1 - k^2)^{j/2}} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2}^j\left(\frac{2 - k^2}{k^2}\right) \cos 2m\phi \quad (5)$$

with $k \in \mathbb{R}$ and $k^2 < 1$. The substitution $\psi = \pi/2 - \phi$ in (5) and then $\psi \rightarrow \phi$ yields

$$\frac{1}{(1 - k^2 \sin^2 \phi)^{j+1/2}} = \frac{(-1)^j 2^{2j+1} j!}{\pi k (2j)! (1 - k^2)^{j/2}} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m Q_{m-1/2}^j\left(\frac{2 - k^2}{k^2}\right) \cos 2m\phi. \quad (6)$$

1.1 Fourier series involving the delta amplitude

For the special case where $j = 0$ equation (6) gives

$$\frac{1}{\Delta(\phi, k)} = \frac{2}{\pi k} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m Q_{m-1/2}\left(\frac{2 - k^2}{k^2}\right) \cos 2m\phi \quad [0 < k < 1] \quad (7)$$

where $\Delta(\phi, k) \equiv (1 - k^2 \sin^2 \phi)^{1/2}$ is the delta amplitude of the theory of elliptic integrals and functions. From (7) it is straightforward to obtain [1]:

$$\begin{aligned} \Delta(\phi, k) = & \frac{k}{\pi} \left[\left(\frac{2-k^2}{k^2} \right) Q_{-1/2} \left(\frac{2-k^2}{k^2} \right) - Q_{1/2} \left(\frac{2-k^2}{k^2} \right) \right] \\ & + \frac{k}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m} \left[Q_{m+1/2} \left(\frac{2-k^2}{k^2} \right) - Q_{m-3/2} \left(\frac{2-k^2}{k^2} \right) \right] \cos 2m\phi. \end{aligned} \quad (8)$$

Equation (8) can be rearranged using the recurrence relations for the associated Legendre functions to give the more compact alternative forms:

$$\Delta(\phi, k) = \frac{k'}{\pi k} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m Q_{m-1/2}^{-1} \left(\frac{2-k^2}{k^2} \right) \cos 2m\phi \quad (9)$$

$$\Delta(\phi, k) = \frac{k'}{\pi k} \sum_{m=0}^{\infty} \frac{(-1)^m \varepsilon_m}{m^2 - 1/4} Q_{m-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \cos 2m\phi \quad (10)$$

where k' is the complimentary modulus to k . The substitution $\phi \rightarrow \phi \pm \pi/2$ in (10) gives immediately:

$$\hat{\Delta}(\phi, k) = \frac{k'}{\pi k} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{m^2 - 1/4} Q_{m-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \cos 2m\phi \quad (11)$$

where $\hat{\Delta}(\phi, k) \equiv \sqrt{1 - k^2 \cos^2 \phi}$.

Fourier expansions [1] for the Legendre elliptic integrals $F(\phi, k)$ and $E(\phi, k)$ are obtained on integration of (7) and (8):

$$F(\phi, k) = \frac{2\phi}{\pi k} Q_{-1/2} \left(\frac{2-k^2}{k^2} \right) + \frac{2}{\pi k} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} Q_{m-1/2} \left(\frac{2-k^2}{k^2} \right) \sin 2m\phi \quad (12)$$

$$\begin{aligned} E(\phi, k) = & \frac{k\phi}{\pi} \left[\left(\frac{2-k^2}{k^2} \right) Q_{-1/2} \left(\frac{2-k^2}{k^2} \right) - Q_{1/2} \left(\frac{2-k^2}{k^2} \right) \right] \\ & + \frac{k}{4\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \left[Q_{m+1/2} \left(\frac{2-k^2}{k^2} \right) - Q_{m-3/2} \left(\frac{2-k^2}{k^2} \right) \right] \sin 2m\phi. \end{aligned} \quad (13)$$

Integration of (10) gives an alternative more compact Fourier expansion for $E(\phi, k)$:

$$E(\phi, k) = -\frac{4k'\phi}{\pi k} Q_{-1/2}^1 \left(\frac{2-k^2}{k^2} \right) + \frac{k'}{\pi k} \sum_{m=1}^{\infty} \frac{(-1)^m}{m(m^2 - 1/4)} Q_{m-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \sin 2m\phi. \quad (14)$$

2. Incomplete Epstein–Hubbell integrals

The usual definition for the Epstein–Hubbell generalized elliptic-type integrals is [3]

$$\Omega_j(k) = \int_0^\pi (1 - k^2 \cos \phi)^{-j-1/2} d\phi, \quad 0 \leq k < 1, \quad j \in \mathbb{N}_0. \quad (15)$$

A number of generalizations of these integrals have been examined, including [4]

$$\Omega_j^m(k) = \int_0^\pi \cos m\phi (1 - k^2 \cos \phi)^{-j-1/2} d\phi, \quad 0 \leq k < 1, \quad j, m \in \mathbb{N}_0, \quad (16)$$

and this which can be related to other generalizations [5–8] given in the literature. The function $\Omega_j^m(k)$ can also be expressed in terms of associated Legendre functions of the second kind [4]

$$\Omega_j^m(k) = \frac{(-1)^j 2^{2j+1/2} j!}{(2j)! k (1 - k^4)^{j/2}} \mathcal{Q}_{m-1/2}^j \left(\frac{1}{k^2} \right) \quad (17)$$

or in terms of the Gauss hypergeometric function [4]

$$\Omega_j^m(k) = \pi \left(\frac{k^2}{8} \right)^m \frac{(2j+2m)! j!}{(2j)!(j+m)!m!} F \left(\frac{j+m}{2} + \frac{3}{4}, \frac{j+m}{2} + \frac{1}{4}; m+1; k^4 \right). \quad (18)$$

By analogy with incomplete Legendre elliptic integrals, (16) can be further generalized to define incomplete Epstein–Hubbell integrals such that

$$\Omega_j^m(\phi, k) = \int_0^\phi \cos m\psi (1 - k^2 \cos \psi)^{-j-1/2} d\psi, \quad 0 \leq k < 1, \quad j, m \in \mathbb{N}_0, \quad \phi \in \mathbb{R}. \quad (19)$$

From equations (4) and (19) it follows immediately that the factor $(1 - k^2 \cos \psi)^{-j-1/2}$ can be expanded as a cosine Fourier series in ψ with the coefficients being the complete generalized Epstein–Hubbell integrals:

$$(1 - k^2 \cos \psi)^{-j-1/2} = \frac{\Omega_j(k)}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \Omega_j^n(k) \cos n\psi. \quad (20)$$

Substituting (20) in (19) and integrating term by term [9] gives the Fourier expansion for the incomplete Epstein–Hubbell integrals:

$$\begin{aligned} \Omega_j^m(\phi, k) &= \frac{\sin m\phi \Omega_j(k)}{m\pi} + \frac{\phi \Omega_j^m(k)}{\pi} + \sum_{n=1}^{\infty} \frac{\Omega_j^n(k)}{\pi(n+m)} \sin(n+m)\phi \\ &+ \sum_{n=1}^{m-1} \frac{\Omega_j^n(k)}{\pi(m-n)} \sin(m-n)\phi + \sum_{n=m+1}^{\infty} \frac{\Omega_j^n(k)}{\pi(n-m)} \sin(n-m)\phi. \end{aligned} \quad (21)$$

2.1 Relation to the incomplete Legendre elliptic integrals

Epstein and Hubbell [3] derived well-known relations between the first two Epstein–Hubbell integrals and the complete Legendre elliptic integrals of the first and second kinds:

$$\Omega_0(k) = \frac{2}{\sqrt{1+k^2}} K \left(\sqrt{\frac{2k^2}{1+k^2}} \right) \quad (22)$$

$$\Omega_1(k) = \frac{2}{(1-k^2)\sqrt{1+k^2}} E \left(\sqrt{\frac{2k^2}{1+k^2}} \right). \quad (23)$$

The defining formula (19) for the incomplete Epstein–Hubbell integrals gives the corresponding formulas:

$$\Omega_0(\phi, k) = \frac{1}{\sqrt{1+k^2}} \int_0^\phi \frac{d\psi}{(1 - (2k^2/(1+k^2)) \cos^2(\psi/2))^{1/2}} \quad (24)$$

$$\Omega_1(\phi, k) = \frac{1}{(1+k^2)^{3/2}} \int_0^\phi \frac{d\psi}{(1 - (2k^2/(1+k^2)) \cos^2(\psi/2))^{3/2}} \quad (25)$$

and after some reduction these yield:

$$\Omega_0(\phi, k) = \Omega_0(k) - \frac{2}{\sqrt{1+k^2}} F\left(\frac{\pi - \phi}{2}, \sqrt{\frac{2k^2}{1+k^2}}\right) \quad (26)$$

$$\begin{aligned} \Omega_1(\phi, k) = \Omega_1(k) - \frac{2E((\pi - \phi)/2, \sqrt{2k^2/(1+k^2)})}{(1+k^2)^{1/2}(1-k^2)} \\ + \frac{2k^2 \sin \phi}{(1+k^2)^{3/2}(1-k^2)(1 - (2k^2/(1+k^2)) \cos^2(\phi/2))^{1/2}} \end{aligned} \quad (27)$$

3. Indefinite integrals

Many indefinite integrals involving the delta amplitude $\Delta(\phi, k)$ can be evaluated in closed form [2, 10, 11]. Often these integrals can be alternatively expressed as Fourier series using the results derived above, yielding Fourier expansions for the various closed form expressions. An example is

$$\begin{aligned} \int_0^\phi \frac{d\psi}{\Delta^{2n+1}(\psi, k)} &= \frac{(-1)^n 2^{2n+1} n! \phi}{\pi k (2n)! (1 - k^2)^{n/2}} Q_{-1/2}^n \left(\frac{2 - k^2}{k^2} \right) \\ &+ \frac{(-1)^n 2^{2n+1} n!}{\pi k (2n)! (1 - k^2)^{n/2}} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} Q_{m-1/2}^n \\ &\times \left(\frac{2 - k^2}{k^2} \right) \sin 2m\phi \quad [n \in \mathbb{N}_0]. \end{aligned} \quad (28)$$

The standard closed form expressions [2, 10, 11] corresponding to (28) become progressively more intricate as the value of n increases.

Integrals of the general form

$$\int \left\{ \begin{array}{c} \sin^{2n} \phi \\ \sin^{2n-1} \phi \\ \cos^{2n} \phi \\ \cos^{2n-1} \phi \end{array} \right\} \frac{d\phi}{\Delta(\phi, k)} \quad [n \in \mathbb{N}] \quad (29)$$

can also be evaluated in closed form [2, 10, 11]. The corresponding Fourier series can always be obtained by expanding the numerator in (29) as a finite sum of cosines [10] and then

employing equation (6). Some special cases are:

$$\int \frac{\sin \phi}{\Delta(\phi, k)} d\phi \equiv \frac{1}{k} \ln(\Delta(\phi, k) - k \cos \phi) \quad (30)$$

and

$$\int \frac{\cos \phi}{\Delta(\phi, k)} d\phi \equiv \frac{1}{k} \sin^{-1}(k \sin \phi) \quad (31)$$

which yield the respective Fourier expansions

$$\begin{aligned} \frac{1}{k} \ln \left(\frac{\Delta(\phi, k) - k \cos \phi}{1 - k} \right) &= \frac{1}{\pi k} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m Q_{m-1/2} \left(\frac{2 - k^2}{k^2} \right) \\ &\times \left(\frac{1 - \cos(2m+1)\phi}{2m+1} - \frac{1 - \cos(2m-1)\phi}{2m-1} \right) \end{aligned} \quad (32)$$

and

$$\begin{aligned} \frac{1}{k} \sin^{-1}(k \sin \phi) &= \frac{1}{\pi k} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m Q_{m-1/2} \left(\frac{2 - k^2}{k^2} \right) \\ &\times \left(\frac{\sin(2m-1)\phi}{2m-1} + \frac{\sin(2m+1)\phi}{2m+1} \right). \end{aligned} \quad (33)$$

Integrals of the form

$$\int \Delta(\phi, k) \times \begin{Bmatrix} \sin^{2n} \phi \\ \sin^{2n-1} \phi \\ \cos^{2n} \phi \\ \cos^{2n-1} \phi \end{Bmatrix} d\phi \quad (n \in \mathbb{N}) \quad (34)$$

can also be evaluated in closed form [2, 10, 11] and these integrals can be reduced to pairs of cases covered by equation (29) by employing the identities

$$\Delta(\phi, k) \equiv \frac{1 - k^2 \sin^2 \phi}{\Delta(\phi, k)} \quad (35)$$

$$\Delta(\phi, k) \equiv \frac{k'^2 + k^2 \cos^2 \phi}{\Delta(\phi, k)}. \quad (36)$$

All of these Fourier series can be repeatedly integrated term by term any finite number of times [9] to give additional Fourier series. In particular, integrating (12) and (14) with respect to ϕ yields

$$\begin{aligned} \int_0^\phi F(\psi, k) d\psi &= \frac{\phi^2}{\pi k} Q_{-1/2} \left(\frac{2 - k^2}{k^2} \right) \\ &+ \frac{1}{\pi k} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} Q_{m-1/2} \left(\frac{2 - k^2}{k^2} \right) (1 - \cos 2m\phi) \end{aligned} \quad (37)$$

and

$$\begin{aligned} \int_0^\phi E(\psi, k) d\psi &= -\frac{2k'\phi^2}{\pi k} Q_{-1/2}^1 \left(\frac{2 - k^2}{k^2} \right) \\ &+ \frac{k'}{\pi k} \sum_{m=1}^{\infty} \frac{(-1)^m}{2m^2(m^2 - 1/4)} Q_{m-1/2}^1 \left(\frac{2 - k^2}{k^2} \right) (1 - \cos 2m\phi). \end{aligned} \quad (38)$$

4. Definite integrals

4.1 Integrals from the Fourier expansions

The various Fourier series derived above yield definite integrals in closed form when the range of integration is chosen such that only a few terms survive on term by term integration of the series multiplied by an angular term. Some examples are

$$\int_0^{\pi/2} \sin 2n\phi F(\phi, k) d\phi = \frac{(-1)^n}{2nk} \left(Q_{n-1/2} \left(\frac{2-k^2}{k^2} \right) - Q_{-1/2} \left(\frac{2-k^2}{k^2} \right) \right) \quad (39)$$

$$\int_0^{\pi/2} \sin 2n\phi E(\phi, k) d\phi = \frac{(-1)^n k'}{nk} \left(\frac{1}{4n^2 - 1} Q_{n-1/2}^1 \left(\frac{2-k^2}{k^2} \right) + Q_{-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \right) \quad (40)$$

$$\int_0^{\pi/2} \frac{\cos 2n\phi}{\Delta(\phi, k)} d\phi = \frac{(-1)^n}{k} Q_{n-1/2} \left(\frac{2-k^2}{k^2} \right) \quad (41)$$

$$\int_0^{\pi/2} \Delta(\phi, k) \cos 2n\phi d\phi = \frac{2k'(-1)^n}{k(4n^2 - 1)} Q_{n-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \quad (42)$$

$$\int_0^{\pi/2} \frac{\cos 2n\phi}{\hat{\Delta}(\phi, k)} d\phi = \frac{1}{k} Q_{n-1/2} \left(\frac{2-k^2}{k^2} \right) \quad (43)$$

$$\int_0^{\pi/2} \hat{\Delta}(\phi, k) \cos 2n\phi d\phi = \frac{2k'}{k(4n^2 - 1)} Q_{n-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \quad (44)$$

4.2 Definite integrals from binomial expansion of $\Delta^{-1}(\phi, k)$

Integrals of the form given in (29) and (34), with the range of integration being $0 \rightarrow \pi/2$, can be evaluated as special cases of the corresponding indefinite integrals, and are sometimes tabulated as such [11]. Such integrals are also readily evaluated as series by binomially expanding $\Delta^{-1}(\phi, k)$ or $\Delta(\phi, k)$ and integrating term by term, and this seems to be the usual form in which they are given in standard tables of integrals. The integral from $0 \rightarrow \pi/2$ of a general product of sines and cosines is [2]

$$\int_0^{\pi/2} \sin^\alpha \phi \cos^\beta \phi d\phi = \frac{1}{2} B \left(\frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) \quad [\operatorname{Re}(\alpha) > -1, \operatorname{Re}(\beta) > -1] \quad (45)$$

and binomially expanding $\Delta^{-1}(\phi, k)$ and employing (45) gives the quite general formula [2]:

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin^\alpha \phi \cos^\beta \phi}{\Delta(\phi, k)} d\phi &\equiv \int_0^K \operatorname{sn}^\alpha u \operatorname{cn}^\beta u du \\ &= \frac{1}{2} B \left(\frac{\alpha+1}{2}, \frac{\beta+1}{2} \right) F \left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2 \right) \\ &\quad \times [\operatorname{Re}(\alpha) > -1, \operatorname{Re}(\beta) > -1, |k| < 1]. \end{aligned} \quad (46)$$

For the cases $\beta = 0$, $\alpha = 0$ and $\alpha = 1$ in (46), then from the tabulations of Prudnikov *et al.* [10] the hypergeometric functions can be reduced to Legendre functions. This gives the three

integrals below, which appear to have been neglected

$$\int_0^{\pi/2} \frac{\sin^\alpha \phi}{\Delta(\phi, k)} d\phi = \frac{1}{2k^{(\alpha+1)/2}} Q_{(\alpha-1)/2} \left(\frac{1+k^2}{2k} \right) \quad [\operatorname{Re}(\alpha) > -1, 0 < k < 1] \quad (47)$$

$$\begin{aligned} \int_0^{\pi/2} \frac{\cos^\beta \phi}{\Delta(\phi, k)} d\phi &= \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{\beta+1}{2}\right) \left(\frac{k'}{k}\right)^{\beta/2} P_{-1/2}^{-\beta/2}(1-2k^2) \\ &\times [\operatorname{Re}(\beta) > -1, 0 < k < 1] \end{aligned} \quad (48)$$

$$\begin{aligned} \int_0^{\pi/2} \frac{\sin \phi \cos^\beta \phi}{\Delta(\phi, k)} d\phi &= \frac{2^{\beta/2} \Gamma((\beta+1)/2) \exp(\beta\pi i/2) (1-k^2)^{\beta/4}}{\sqrt{\pi} k^{1+\beta/2}} Q_{\beta/2}^{-\beta/2} \left(\frac{1}{k} \right) \\ &\times [\operatorname{Re}(\beta) > -1, 0 < k < 1]. \end{aligned} \quad (49)$$

Equations (47) to (49), together with the identity $\sin^2 \phi + \cos^2 \phi = 1$, allow any integral of the form

$$I_1(j, l, k) = \int_0^{\pi/2} \frac{\sin^j \phi \cos^l \phi}{\Delta(\phi, k)} d\phi \quad [j, l \in \mathbb{N}_0, 0 < k < 1] \quad (50)$$

to be evaluated as a finite sum of Legendre functions.

The functions $\sin(m+1)\phi$ and $\cos n\phi$ can be expanded using the Chebyshev polynomials such that

$$\cos n\phi = T_n(\cos \phi) \quad (n \in \mathbb{N}_0) \quad (51)$$

$$\sin(m+1)\phi = \sin \phi U_m(\cos \phi) \quad (m \in \mathbb{N}_0) \quad (52)$$

where $T_n(x)$ and $U_m(x)$ are the Chebyshev polynomials of the first and second kinds, respectively. Hence any integral of the more general form

$$I_2(j, l, m, n, k) = \int_0^{\pi/2} \frac{\sin^j(m+1)\phi \cos^l n\phi}{\Delta(\phi, k)} d\phi \quad [j, l, m, n \in \mathbb{N}_0, 0 < k < 1] \quad (53)$$

can be expressed as a sum of terms of the form (50) and hence expressed as a finite sum of Legendre functions.

4.3 Some examples involving power series

The integral

$$\int_0^{\pi/2} \frac{\sin^{2n} \phi}{\Delta(\phi, k)} d\phi = \frac{1}{2k^{n+1/2}} Q_{n-1/2} \left(\frac{1+k^2}{2k} \right) \quad (54)$$

is tabulated by Gradshteyn and Ryzhik [2] (for twice the above range of integration) in the equivalent series form:

$$\int_0^\pi \frac{\sin^{2n} \phi}{\Delta(\phi, k)} d\phi = \frac{\pi}{2^n} \sum_{j=0}^{\infty} \frac{(2j-1)!!(2n+2j-1)!!}{2^{2j} j!(n+j)!} k^{2j} \quad [k^2 < 1]. \quad (55)$$

This expansion can be obtained directly from the binomial expansion formula (46) and it does not appear that this series has hitherto been recognized as a Legendre function.

The integral (41) follows immediately from the Fourier series expansion of $\Delta^{-1}(\phi, k)$. Substituting the hypergeometric representation of the Legendre function [12] in (41) gives

$$\int_0^{\pi/2} \frac{\cos 2n\phi}{\Delta(\phi, k)} d\phi = \frac{(-1)^n \pi^{1/2} \Gamma(n+1/2) k^{2n}}{2^{n+1/2} (2-k^2)^{n+1/2} n!} F\left(\frac{n}{2} + \frac{1}{4}, \frac{n}{2} + \frac{3}{4}; n+1; \frac{k^4}{(2-k^2)^2}\right) \quad (56)$$

The hypergeometric function in (56) can be transformed using the quadratic transformation [10, 12]:

$$F\left(a, a + \frac{1}{2}; c; z\right) = (1 \pm \sqrt{z})^{-2a} F\left(2a, c - \frac{1}{2}; 2c - 1; \frac{\pm 2\sqrt{z}}{1 \pm \sqrt{z}}\right) \quad (57)$$

and this yields

$$\int_0^{\pi/2} \frac{\cos 2n\phi}{\Delta(\phi, k)} d\phi = \frac{(-1)^n \pi^{1/2} \Gamma(n+1/2) k^{2n}}{2^{2n+1} n!} F\left(\frac{n+1}{2}, \frac{n+1}{2}; 2n+1; k^2\right). \quad (58)$$

Expanding the hypergeometric function in (58) as a power series gives:

$$\int_0^{\pi/2} \frac{\cos 2n\phi}{\Delta(\phi, k)} d\phi = \frac{(-1)^n \pi}{2} \sum_{j=0}^{\infty} \frac{[(2n+2j-1)!!]^2}{(2n+j)! j!} \left(\frac{k}{2}\right)^{2n+2j}. \quad (59)$$

The integral (41) is given in this series form (59) by Prudnikov *et al.* [10] and is also given as an equivalent series by Byrd and Friedman [11].

The integral (42) follows immediately from the Fourier expansion of $\Delta(\phi, k)$. Substituting the hypergeometric representation of the associated Legendre function in (42) gives

$$\begin{aligned} \int_0^{\pi/2} \Delta(\phi, k) \cos 2n\phi d\phi &= \frac{(-1)^{n+1} \pi^{1/2} (1-k^2) \Gamma(n+3/2) k^{2n}}{(4n^2-1) 2^{n-3/2} n! (2-k^2)^{n+3/2}} \\ &\times F\left(\frac{n}{2} + \frac{3}{4}, \frac{n}{2} + \frac{5}{4}; n+1; \frac{k^4}{(2-k^2)^2}\right). \end{aligned} \quad (60)$$

Employing the transformation (57) in (60) gives

$$\begin{aligned} \int_0^{\pi/2} \Delta(\phi, k) \cos 2n\phi d\phi &= \frac{(-1)^{n+1} \pi^{1/2} (1-k^2) \Gamma(n+3/2) k^{2n}}{(4n^2-1) 2^{2n} n!} \\ &\times F\left(n + \frac{1}{2}, n + \frac{3}{2}; 2n+1; k^2\right) \end{aligned} \quad (61)$$

and expanding the hypergeometric function in (61) gives the series solution

$$\begin{aligned} \int_0^{\pi/2} \Delta(\phi, k) \cos 2n\phi d\phi &= \frac{(-1)^{n+1} \pi (1-k^2)}{2(4n^2-1)} \\ &\times \sum_{j=0}^{\infty} \frac{(2n+2j-1)!! (2n+2j+1)!!}{(2n+j)! j!} \left(\frac{k}{2}\right)^{2j+2n}. \end{aligned} \quad (62)$$

This series has been numerically checked against both the Legendre function formula (42) and direct numerical evaluation of the integral. An alternative form is obtained by employing

$\Delta(\phi, k)$ in the form given in (9) and proceeding as above. This yields the series

$$\int_0^{\pi/2} \Delta(\phi, k) \cos 2n\phi \, d\phi = \frac{(-1)^{n+1}\pi}{2} \sum_{j=0}^{\infty} \frac{(2n+2j-3)!!(2n+2j-1)!!}{(2n+j)!j!} \left(\frac{k}{2}\right)^{2j+2n}. \quad (63)$$

Equation (63) is given by Prudnikov *et al.* [10] and in a rearranged form by Byrd and Friedman [11]. Both (62) and (63) have been checked by numerical evaluation.

5. Fourier expansions resulting in series with products of Legendre functions

If two functions $f(\phi, p)$ and $g(\phi, q)$ can be expressed as two cosine or two sine series of the form

$$f(\phi, p) = \sum_{m=0}^{\infty} a_m(p) \varepsilon_m \begin{Bmatrix} \cos 2m\phi \\ \sin 2m\phi \end{Bmatrix} \quad (64)$$

$$g(\phi, q) = \sum_{n=0}^{\infty} b_n(q) \varepsilon_n \begin{Bmatrix} \cos 2n\phi \\ \sin 2n\phi \end{Bmatrix} \quad (65)$$

then from the elementary properties of Fourier series

$$\sum_{m=0}^{\infty} \varepsilon_m a_m(p) b_m(q) = \frac{2}{\pi} \int_0^{\pi/2} f(\phi, p) g(\phi, q) \, d\phi. \quad (66)$$

Substituting various Fourier expansions derived above into (66) results in expansions in products of Legendre functions. One of the simplest examples results from $f(\phi, k) = \Delta^{-1}(\phi, k)$ and $g(\phi, k) = \Delta(\phi, k)$, which from (7) and (10) yields

$$\frac{8k'}{\pi^2 k^2} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{4m^2 - 1} Q_{m-1/2} \left(\frac{2-k^2}{k^2} \right) Q_{m-1/2}^1 \left(\frac{2-k^2}{k^2} \right) = 1. \quad (67)$$

Other examples are

$$\frac{2}{\pi} \int_0^{\pi/2} \left(\frac{k'}{\Delta(\phi, k)} \times \frac{1}{\Delta(\phi, k)} \right) d\phi = 1 \quad (68)$$

$$\frac{2}{\pi} \int_0^{\pi/2} \left(\frac{2\Delta(\phi, k)}{(2-k^2)} \times \Delta(\phi, k) \right) d\phi = 1 \quad (69)$$

$$\frac{2}{\pi} \int_0^{\pi/2} \left(\frac{\pi}{2\Delta(\phi, p)} \times \frac{1}{\Delta(\phi, q)} \right) d\phi = \frac{1}{\sqrt{1-p^2}} K \left(\sqrt{\frac{q^2-p^2}{1-p^2}} \right) \quad (70)$$

$$[0 < p^2 < q^2 < 1]$$

and therefore, respectively

$$\frac{4k'}{\pi^2 k^2} \sum_{m=0}^{\infty} \varepsilon_m \left(Q_{m-1/2} \left(\frac{2-k^2}{k^2} \right) \right)^2 = 1 \quad (71)$$

$$= \frac{32(1-k^2)}{\pi^2 k^2 (2-k^2)} \sum_{m=0}^{\infty} \frac{\varepsilon_m}{(4m^2-1)^2} \left(Q_{m-1/2}^1 \left(\frac{2-k^2}{k^2} \right) \right)^2 = 1 \quad (72)$$

$$\frac{2}{\pi pq} \sum_{m=0}^{\infty} \varepsilon_m Q_{m-1/2} \left(\frac{2-p^2}{p^2} \right) Q_{m-1/2} \left(\frac{2-q^2}{q^2} \right) = \frac{1}{\sqrt{1-p^2}} K \left(\sqrt{\frac{q^2-p^2}{1-p^2}} \right) \quad (73)$$

$$[0 < p^2 < q^2 < 1]$$

6. Binomial expansions giving Legendre series

The various integrals in section above can be alternatively evaluated by expanding one of the products binomially and then employing equations (47)–(49). For example

$$\frac{1}{\Delta(\phi, q)} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} q^{2n} \sin^{2n} \phi \quad (74)$$

and hence

$$\int_0^{\pi/2} \left(\frac{1}{\Delta(\phi, p)} \times \frac{1}{\Delta(\phi, q)} \right) d\phi = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} q^{2n} \int_0^{\pi/2} \frac{\sin^{2n} \phi}{\Delta(\phi, p)} d\phi \quad (75)$$

Hence from (47) and (73)

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n+1}(n!)^2} \frac{q^{2n}}{p^{n+1/2}} Q_{n-1/2} \left(\frac{1+p^2}{2p} \right) = \frac{1}{\sqrt{1-p^2}} K \left(\sqrt{\frac{q^2-p^2}{1-p^2}} \right) \quad (76)$$

$$[0 < p < q < 1].$$

Expanding $\Delta(\phi, p)$ binomially gives the alternative series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n+1}(n!)^2} \frac{p^{2n}}{q^{n+1/2}} Q_{n-1/2} \left(\frac{1+q^2}{2q} \right) = \frac{1}{\sqrt{1-p^2}} K \left(\sqrt{\frac{q^2-p^2}{1-p^2}} \right) \quad (77)$$

$$[0 < p < q < 1].$$

The integral

$$\int_0^{\pi/2} \left(\frac{1}{\Delta(\phi, k)} \times \frac{1}{\hat{\Delta}(\phi, k)} \right) d\phi = \frac{1}{1-k^2/2} K \left(\frac{k^2}{2-k^2} \right) \quad (78)$$

can be expressed as a Legendre series by expanding $\hat{\Delta}^{-1}(\phi, k)$ binomially. This gives initially

$$\sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} k^{2n} \int_0^{\pi/2} \frac{\cos^{2n} \phi}{\Delta(\phi, k)} d\phi = \frac{1}{1-k^2/2} K \left(\frac{k^2}{2-k^2} \right) \quad (79)$$

and from (48) this yields after some reduction a series expansion in terms of Legendre functions of the first kind:

$$\frac{1}{1-k^2/2} K \left(\frac{k^2}{2-k^2} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \Gamma^2 \left(\frac{n+1}{2} \right) \frac{(kk')^n}{n!} P_{-1/2}^{-n}(1-2k^2) \quad (80)$$

7. A new expansion for the complete elliptic integral of the third kind

The complete elliptic integral of the third kind is given by

$$\Pi(\alpha, k) = \int_0^{\pi/2} \frac{d\phi}{(1 - \alpha^2 \sin^2 \phi) \Delta(\phi, k)} \quad (81)$$

For $\alpha^2 < 1$ the factor $(1 - \alpha^2 \sin^2 \phi)^{-1}$ containing the parameter α can be expanded as a geometric series to give

$$\Pi(\alpha, k) = \sum_{j=0}^{\infty} \alpha^{2j} \int_0^{\pi/2} \frac{\sin^{2j} \phi d\phi}{\Delta(\phi, k)} \quad (82)$$

and from (54) this gives

$$\Pi(\alpha, k) = \frac{1}{2} \sum_{j=0}^{\infty} \frac{\alpha^{2j}}{k^{j+1/2}} \mathcal{Q}_{j-1/2} \left(\frac{1+k^2}{2k} \right). \quad (83)$$

This expansion allows both repeated integration of $\Pi(\alpha, k)$ with respect to the parameter α and a very large number of integrals of the form

$$\int f(\alpha) \Pi(\alpha, k) d\alpha \quad (84)$$

to be evaluated as Legendre series. An obvious example is

$$\int \alpha^\lambda \Pi(\alpha, k) d\alpha = \frac{1}{2} \sum_{j=0}^{\infty} \frac{\alpha^{2j+\lambda+1}}{(2j+\lambda+1)k^{j+1/2}} \mathcal{Q}_{j-1/2} \left(\frac{1+k^2}{2k} \right) \quad (85)$$

No integrals with respect to the parameter are listed in the standard tables [2, 10, 11].

8. Expansions for monomials

Substituting the values $\phi = 0$ and $\phi = \pi$ in the Fourier series (3), followed by the respective substitutions $z = (\omega + 1)/(\omega - 1)$ and $z = (\omega - 1)/(\omega + 1)$ yields for $\text{Re}(\mu) > -1/2$:

$$z^{\mu/2} = \frac{2}{\pi^{1/2}} \frac{\exp(-\mu\pi i)}{\Gamma(\mu + 1/2)} \frac{1}{(z - 1)^{1/2}} \sum_{m=0}^{\infty} \varepsilon_m \mathcal{Q}_{m-1/2}^\mu \left(\frac{z+1}{z-1} \right) \quad (86)$$

$$z^{\mu/2} = \frac{2}{\pi^{1/2}} \frac{\exp(-\mu\pi i)}{\Gamma(\mu + 1/2)} \frac{1}{(1 - z)^{1/2}} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m \mathcal{Q}_{m-1/2}^\mu \left(\frac{1+z}{1-z} \right). \quad (87)$$

The substitution $z \rightarrow 1/z$ in (86) and (87) gives, respectively

$$z^{-(\mu+1)/2} = \frac{2}{\pi^{1/2}} \frac{\exp(-\mu\pi i)}{\Gamma(\mu + 1/2)} \frac{1}{(1 - z)^{1/2}} \sum_{m=0}^{\infty} \varepsilon_m \mathcal{Q}_{m-1/2}^\mu \left(\frac{1+z}{1-z} \right) \quad (88)$$

$$z^{-(\mu+1)/2} = \frac{2}{\pi^{1/2}} \frac{\exp(-\mu\pi i)}{\Gamma(\mu + 1/2)} \frac{1}{(z - 1)^{1/2}} \sum_{m=0}^{\infty} (-1)^m \varepsilon_m \mathcal{Q}_{m-1/2}^\mu \left(\frac{z+1}{z-1} \right). \quad (89)$$

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Laplace transforms and the Riemann zeta function

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In this article we examine certain formulae applying to the Riemann zeta function in the critical strip $0 < \operatorname{Re}(s) < 1$, and by means of the Laplace transform we demonstrate new relations between existing formulae. Müntz has proposed a formula for the Riemann zeta function in the critical strip, which involves an arbitrary function satisfying certain conditions. From this formula it is clear that a resolution of the Riemann hypothesis may hinge on a successful method for dealing with the sum-integral difference. The Euler–MacLaurin summation formula is one such device. Here, we generate new expressions for certain Laplace transforms involving the sum-integral difference. Subsequently, the formulae thus far established are generalized using an arbitrary positive number m , which makes apparent the dependence of classical analysis on formulae and results associated with the exponential function which is characterized by the particular case $m = 1$.

Keywords: Riemann zeta function; Representations in the critical strip; Laplace transform; Sum-integral difference

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1. Introduction

For $\operatorname{Re}(s) > 1$, the Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1)$$

and the following equation due to Euler

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad (2)$$

where p ranges over all primes ($p = 2, 3, 5, 7, 11, \dots$) indicates the existence of a deep-lying connection between the theory of the zeta function and prime number theory. The Riemann

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hypothesis asserts that, ‘all the complex zeros of the function which is the analytical continuation of equation (1) in the critical strip $0 < \operatorname{Re}(s) < 1$ lie on the line $\operatorname{Re}(s) = 1/2$ ’. In his 1974 book Edwards [1, p. 164] states: ‘Even today, more than a hundred years later, one cannot really give any solid reasons for saying that the truth of the Riemann hypothesis is probable’. The reader is referred to Conrey [2] for a highly readable and comprehensive account of many recent developments. The remarkable general formula due to Müntz [3, p. 28] and which applies for any function $f(x)$ which is such that both the function and its first derivative are bounded in any finite interval and both are $O(x^{-\alpha})$ for certain constants $\alpha > 1$ as x tends to infinity. Under these assumptions, Müntz shows that the analytical continuation of equation (1) in the critical strip $0 < \operatorname{Re}(s) < 1$ is given by

$$\zeta(s) \int_0^\infty x^{s-1} f(x) dx = \int_0^\infty x^{s-1} \left\{ \sum_{n=1}^\infty f(nx) - \int_0^\infty f(ux) du \right\} dx, \quad (3)$$

and this is a valid formula for $\zeta(s)$ applying in the critical strip $0 < \operatorname{Re}(s) < 1$ for any function $f(x)$ satisfying the stated conditions. The sum-integral difference appearing in the right-hand side of equation (3) is a fundamental quantity of mathematical analysis, which is normally calculated using the Euler–MacLaurin summation formula [4, p. 14] and involves both Bernoulli numbers and polynomials, which in turn are intimately bound up with the exponential function. Using Laplace transforms, this article first explores a number of formal connections relating various integral expressions for the Riemann zeta function and then derives new expressions for the Laplace transforms of certain functions involving the sum-integral difference.

In the following section we state a number of well-known results for the Riemann zeta function. In the section thereafter, by means of the Laplace transform, we establish connections between the various standard formulae for the zeta function. In section 4 we obtain new expressions for the Laplace transforms of functions involving the sum-integral difference. These Laplace transform expressions are new. Remaining sections of the paper provide a possible generalization of the results thus far given, involving an arbitrary positive constant m . In particular in section 5, we give a general formula for the zeta function and in section 6 we generalize the previously given Laplace transform expressions involving the sum-integral difference. These latter results emphasize the extent to which existing mathematical analysis, such as formulae involving Bernoulli numbers, the Euler–MacLaurin summation formulae and even the Laplace transform itself, are all associated with the particular case $m = 1$ corresponding to the exponential function. We conclude with certain formulae which utilize standard Laplace transforms involving Bessel functions, and provide particular instances of results which are consistent with those given in preceding sections.

We comment that in order to derive the formulae given here a number of formal manipulations are necessary. The justification for these formal manipulations, such as interchanges of integrations over infinite intervals both with themselves and with summations of infinite series are discussed at length by both Bromwich [5, pp. 499–506] and Carslaw [6, pp. 188–214]. It is not easy to provide general conditions under which such interchanges are legitimate, and a case-by-case analysis is necessary. However, typically it is usually necessary to establish uniform convergence of both summations and integrals, and sometimes the absolute convergence of an integral is required. Throughout the article we make extensive formal interchanges of the operations of integration and summation, and we assume throughout that these processes are legitimate without going into the full details. However, the final results are cross-checked from several perspectives.

2. Preliminaries and basic equations

In this section, we state some standard formulae for the Riemann zeta function which can all be found in chapter 2 of Titchmarsh [3]. The reader may also consult Ivic [7]. Throughout the article our interest is primarily those formulae applying in the critical strip $0 < \text{Re}(s) < 1$. In particular, we have the standard formula

$$\Gamma(s)\zeta(s) = \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx = \int_0^\infty x^{s-1} \phi_1(x) dx, \quad (4)$$

which is valid in the critical strip $0 < \text{Re}(s) < 1$ and the function $\phi_1(x)$ is defined by

$$\phi_1(x) = \sum_{n=1}^\infty e^{-nx} - \int_0^\infty e^{-ux} du = \frac{1}{e^x - 1} - \frac{1}{x}. \quad (5)$$

The function designated here as $\phi_1(x)$ is a fundamental function in mathematical analysis. It may be viewed as the generating function for the Bernoulli numbers, thus

$$\frac{1}{e^x - 1} - \frac{1}{x} = -\frac{1}{2} + \sum_{k=1}^\infty \frac{B_{2k}}{(2k)!} x^{2k-1}, \quad (6)$$

where $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ etc, and $B_{2k+1} = 0$ for $k \geq 1$. These are fundamental to the Euler–MacLaurin summation formula for expressions involving the sum-integral difference. It also admits an expansion in partial fractions and the Legendre expansion, thus [8, p. 280]

$$\frac{1}{e^x - 1} - \frac{1}{x} = -\frac{1}{2} + 2 \sum_{k=1}^\infty \frac{x}{x^2 + 4\pi^2 k^2}, \quad (7)$$

$$\frac{1}{e^x - 1} - \frac{1}{x} = -\frac{1}{2} + 2 \int_0^\infty \frac{\sin(xt)}{e^{2\pi t} - 1} dt. \quad (8)$$

Because of the last formula, the function $\phi_1(x\sqrt{2\pi})$ is self-reciprocal with respect to the sine transform, which is used in Titchmarsh [3, p. 23] to deduce the functional equation.

Another standard expression for the zeta function in the critical strip is

$$\frac{\Gamma(s/2)\zeta(s)}{2} = \int_0^\infty x^{s-1} \phi_2(x) dx, \quad (9)$$

which on making the substitution $y = x^2/\pi$ assumes the more familiar form

$$\frac{\Gamma(s/2)\zeta(s)}{\pi^{s/2}} = \int_0^\infty y^{s/2-1} \phi(y) dy. \quad (10)$$

Here we designate the function appearing in equation (9) as $\phi_2(x)$, thus

$$\phi_2(x) = \sum_{n=1}^\infty e^{-(nx)^2} - \int_0^\infty e^{-(ux)^2} du = \sum_{n=1}^\infty e^{-(nx)^2} - \frac{\sqrt{\pi}}{2x}, \quad (11)$$

while the function appearing in equation (10) denoted by $\phi(x)$ is given by

$$\phi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x} - \frac{1}{2\sqrt{x}}, \quad (12)$$

and we have the relation $\phi(x) = \phi_2(\sqrt{\pi x})$. The function $\phi(x)$ is related to the theta functions and is known to satisfy the functional relation [9, p. 124]

$$\phi(x) = \frac{1}{\sqrt{x}} \phi\left(\frac{1}{x}\right). \quad (13)$$

In section 5 we generalize equations (4) and (9) in an obvious manner. The above results are special cases of the previously stated general formula due to Müntz and equations (4) and (9) arise from equation (3) on taking $f(x) = e^{-x}$ and $f(x) = e^{-x^2}$, respectively.

In the following section we show how the formulae (4), (9) and (10) are all connected by formally applying the Laplace transform, in the sense that application of the Laplace transform gives rise to the well-known functional equation [9, chapter 2]

$$2^{1-s} \Gamma(s) \zeta(s) \cos\left(\frac{\pi s}{2}\right) = \pi^s \zeta(1-s). \quad (14)$$

In making these connections, the following standard gamma function formulae are useful,

$$2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \pi^{1/2} \Gamma(2z), \quad (15)$$

$$\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad (16)$$

which may found in Whittaker and Watson [9, p. 239, 240]. Finally in this section, we state two straightforward Laplace transforms [10]

$$\int_0^\infty e^{-pt} t^\nu dt = \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \quad \operatorname{Re}(\nu) > -1, \quad (17)$$

$$\int_0^\infty e^{-pt} \phi(t) dt = \sum_{n=1}^\infty \frac{1}{p + \pi n^2} - \frac{1}{2} \left(\frac{\pi}{p}\right)^{1/2} = \left(\frac{\pi}{p}\right)^{1/2} \phi_1(2(\pi p)^{1/2}), \quad (18)$$

where $\phi(x)$ and $\phi_1(x)$ are defined by equations (12) and (5), respectively, and in deriving equation (18) we have used the partial fraction expansion equation (7) for $\phi_1(x)$.

3. Formal application of the Laplace transform

In this section, we show that the functional equation (14) emerges from a formal application of the Laplace transform to the identities listed in the previous section. First from equation (10) we have on making the substitution $x = ty$ ($t > 0$)

$$\frac{\Gamma(s/2) \zeta(s)}{(\pi t)^{s/2}} = \int_0^\infty y^{s/2-1} \phi(ty) dy,$$

and on taking the Laplace transform of this equation with respect to t , assuming the order of integration may be interchanged, we may readily deduce, after using equations (17) and (18) and making a simple change of integration variable,

$$\frac{\Gamma(s/2) \Gamma(1-s/2)}{(2\pi)^s} \zeta(s) = \int_0^\infty x^{(1-s)-1} \phi_1(x) dx,$$

which using equation (4) can be shown to be equivalent to the functional equation (14). If we apply the same process to the representation (4) then ultimately we may deduce the functional

equation (14) as follows. Using the substitution $x = ty$, equation (4) gives

$$\frac{\Gamma(s)\zeta(s)}{t^s} = \int_0^\infty y^{s-1} \phi_1(ty) dy, \quad (19)$$

while from the Legendre expansion (8) we have

$$\phi_1(t) = 2 \int_0^\infty \sin(xt) \phi_1(2\pi x) dx, \quad (20)$$

where in deducing equation (20) we have utilized the well-known integral

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

Thus, from equation (20) we find that the Laplace transform becomes

$$\int_0^\infty e^{-pt} \phi_1(t) dt = 2 \int_0^\infty \frac{x \phi_1(2\pi x)}{(p^2 + x^2)} dx. \quad (21)$$

This unusual looking formula, which is valid only for $\text{Re}(p) > 0$, may be readily confirmed numerically using any of the standard packages such as MAPLE. On taking the Laplace transform of equation (20) and using equation (21) we have

$$\Gamma(s)\Gamma(1-s)\zeta(s) = 2^s \pi^{s-1} \int_0^{\pi/2} (\tan \theta)^s d\theta \int_0^\infty u^{-s} \phi_1(u) du, \quad (22)$$

which on using the standard integral [11, p. 369]

$$\int_0^{\pi/2} (\tan \theta)^s d\theta = \frac{\pi}{2 \cos(s\pi/2)},$$

and equation (4) itself, can be shown to reduce to the functional equation (14).

We may also connect equation (4) with the general Müntz formula (3), other than in the obvious manner already cited, in the following way. Suppose that the function $f(x)$ in equation (3) is given as the Laplace transform of a certain function $g(x)$, thus

$$f(x) = \int_0^\infty e^{-xt} g(t) dt, \quad (23)$$

then the Mellin transform of $f(x)$ becomes

$$\int_0^\infty x^{s-1} \left(\int_0^\infty e^{-xt} g(t) dt \right) dx = \int_0^\infty g(t) \left(\int_0^\infty e^{-xt} x^{s-1} dx \right) dt = \Gamma(s) \int_0^\infty \frac{g(t)}{t^s} dt, \quad (24)$$

again assuming of course the orders of integration may be reversed. Now for the right-hand side of equation (3) we have

$$\sum_{n=1}^\infty f(nx) - \frac{1}{x} \int_0^\infty f(\eta) d\eta = \int_0^\infty g(t) \left\{ \frac{1}{e^{xt} - 1} - \frac{1}{xt} \right\} dt = \int_0^\infty g(t) \phi_1(xt) dt, \quad (25)$$

where again we have reversed the various orders of summation and integration. Thus from equations (3), (24) and (25) we may deduce equation (4) as follows,

$$\Gamma(s)\zeta(s) \int_0^\infty \frac{g(t)}{t^s} dt = \int_0^\infty x^{s-1} \left(\int_0^\infty g(t) \phi_1(xt) dt \right) dx = \int_0^\infty \xi^{s-1} \phi_1(\xi) d\xi \int_0^\infty \frac{g(t)}{t^s} dt,$$

on reversing the orders of integration and making the substitution $\xi = xt$.

4. Laplace transforms involving the sum-integral difference

The Müntz formula equation (3) indicates that a resolution of the Riemann hypothesis may well hinge on knowing the behaviour of the function $F(x)$ which is defined by

$$F(x) = \sum_{n=1}^{\infty} f(nx) - \int_0^{\infty} f(ux) du, \quad (26)$$

and which we loosely refer to as the sum-integral difference. In this section we derive expressions for the Laplace transforms of the two functions $F(1/t)$ and $F(1/t^{1/2})$, which are new results. Again our method is formal, but we may confirm the veracity of the results obtained in three ways; by checking special cases, using the Euler–MacLaurin summation formula and finally by formally inverting and re-tracing our steps. We again note that the sum-integral difference is usually analyzed by means of the Euler–MacLaurin summation formula [4, p. 14] involving the Bernoulli polynomials [11, p. 1076].

From equation (26) on taking the Laplace transform of the function $F(1/x)$, we have

$$\begin{aligned} \int_0^{\infty} e^{-pt} F\left(\frac{1}{t}\right) dt &= \int_0^{\infty} e^{-pt} \left\{ \sum_{n=1}^{\infty} f\left(\frac{n}{t}\right) - \int_0^{\infty} f\left(\frac{u}{t}\right) du \right\} dt \\ &= \sum_{n=1}^{\infty} \int_0^{\infty} e^{-pn/\xi} f(\xi) \frac{n}{\xi^2} d\xi - \frac{1}{p^2} \int_0^{\infty} f(\xi) d\xi \\ &= \frac{1}{p} \int_0^{\infty} f(\xi) \frac{d}{d\xi} \left\{ \sum_{n=1}^{\infty} e^{-pn/\xi} - \frac{\xi}{p} \right\} d\xi \\ &= \frac{1}{p} \int_0^{\infty} f(\xi) \frac{d}{d\xi} \phi_1\left(\frac{p}{\xi}\right) d\xi \\ &= \frac{1}{p} \left[f(\xi) \phi_1\left(\frac{p}{\xi}\right) \right]_0^{\infty} - \frac{1}{p} \int_0^{\infty} f'(\xi) \phi_1\left(\frac{p}{\xi}\right) d\xi, \end{aligned}$$

where in the first line we have made the substitutions $\xi = n/t$ in the summation and $\xi = u/t$ in the infinite integral, and in the last line the prime denotes differentiation with respect to ξ . Now on assuming that the conditions on the function $f(x)$ which apply to the Müntz formula also apply here, and noting that $\phi_1(\infty)$ is zero we may deduce

$$\int_0^{\infty} e^{-pt} F\left(\frac{1}{t}\right) dt = -\frac{1}{p} \int_0^{\infty} f'(\xi) \phi_1\left(\frac{p}{\xi}\right) d\xi, \quad (27)$$

for $F(x)$ defined by equation (26) and $\phi_1(x)$ defined by equation (5).

First, we observe that equation (27) is trivially correct in the case when $f(x) = e^{-x}$. In this case $F(x) = \phi_1(x)$ and the equality of equation (27) follows trivially from the substitution $t = \xi/p$. Secondly, from equations (7) and (27) we may readily deduce

$$\int_0^{\infty} e^{-pt} F\left(\frac{1}{t}\right) dt = \frac{1}{p} \int_0^{\infty} f'(\xi) \left\{ \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2p\xi}{p^2 + (2\pi n\xi)^2} \right\} d\xi,$$

and formal term by term inversion yields

$$F\left(\frac{1}{t}\right) = \frac{1}{2} \int_0^{\infty} f'(\xi) d\xi - 2 \int_0^{\infty} f'(\xi) \left(\sum_{n=1}^{\infty} \frac{\sin 2\pi n\xi t}{2\pi n} \right) d\xi,$$

which on using the following formula from Rademacher [4, p. 16]

$$B_1(t - [t]) = -2 \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{2\pi n},$$

yields

$$F\left(\frac{1}{t}\right) = -\frac{f(0)}{2} + \int_0^{\infty} f'(\xi) B_1(\xi t - [\xi t]) d\xi.$$

This latter formula coincides precisely with the Euler-MacLaurin summation formula [4, p. 14] in the case $a = 0$, $b = \infty$, $k = q = 1$ and noting that the function $h(u)$ in ref. [4] is our function $f(u/t)$. Finally, we may confirm the veracity of equation (27) by term by term Laplace inversion; thus on writing equation (27) as

$$\int_0^{\infty} e^{-pt} F\left(\frac{1}{t}\right) dt = - \int_0^{\infty} f'(\xi) \left\{ \sum_{n=1}^{\infty} \frac{e^{-np/\xi}}{p} - \frac{\xi}{p^2} \right\} d\xi,$$

we have

$$F\left(\frac{1}{t}\right) = \int_0^{\infty} f'(\xi) \left\{ \sum_{n=1}^{\infty} H\left(\frac{t-n}{\xi}\right) - \xi t \right\} d\xi,$$

where $H(x)$ denotes the usual Heaviside function. On further simplification and performing an integration by parts for the final term, the required expression follows.

$$F\left(\frac{1}{t}\right) = - \sum_{n=1}^{\infty} \int_{n/t}^{\infty} f'(\xi) d\xi - t \int_0^{\infty} f(\xi) d\xi = \sum_{n=1}^{\infty} f\left(\frac{n}{t}\right) - t \int_0^{\infty} f(\xi) d\xi.$$

Now proceeding in a completely analogous manner to the above, we may also formally establish

$$\int_0^{\infty} e^{-pt} F\left(\frac{1}{t^{1/2}}\right) dt = -\frac{1}{p} \int_0^{\infty} f'(\xi) \phi_2\left(\frac{p^{1/2}}{\xi}\right) d\xi = -\frac{1}{p} \int_0^{\infty} f'(\xi) \phi\left(\frac{p}{\pi \xi^2}\right) d\xi, \quad (28)$$

where $\phi_2(x)$ and $\phi(x)$ are defined by equations (11) and (12), respectively, and we may proceed to verify equation (28) in much the same manner as we have attempted to confirm the validity of equation (27). Since these results are expressed in a more general setting in the final two sections of the article, we do not present the details.

Of course the formulae presented here do give rise to a number of curious relations between infinite integrals. However, it is of some considerable interest to confirm that the result arising from equation (27) with $f(x) = e^{-x^2}$ ($F(x) = \phi_2(x)$) coincides with that obtained from equation (28) with $f(x) = e^{-x}$ ($F(x) = \phi_1(x)$). This may be confirmed as follows. Under these circumstances we have from equation (27)

$$\int_0^{\infty} e^{-pt} \phi_2\left(\frac{1}{t}\right) dt = \frac{2}{p} \int_0^{\infty} \xi e^{-\xi^2} \phi_1\left(\frac{p}{\xi}\right) d\xi, \quad (29)$$

whereas from equation (28) we have

$$\int_0^{\infty} e^{-qt} \phi_1\left(\frac{1}{t^{1/2}}\right) dt = \frac{1}{q} \int_0^{\infty} e^{-\xi} \phi_2\left(\frac{q^{1/2}}{\xi}\right) d\xi, \quad (30)$$

where in the latter equation, in order to avoid confusion we have used q as the Laplace transform variable. Now in the integral on the right-hand side of equation (30) we make the

substitution $\xi = tq^{1/2}$ from which it is clear that $q = p^2$ so that equation (30) becomes

$$\int_0^\infty e^{-pt} \phi_2\left(\frac{1}{t}\right) dt = p \int_0^\infty e^{-p^2 t} \phi_1\left(\frac{1}{t^{1/2}}\right) dt, \quad (31)$$

and the right-hand side of equation (29) arises from the right-hand side of equation (31) with the substitution $t = (\xi/p)^2$, thus confirming the equivalence of equation (29) and equation (30).

As a more interesting example, we consider

$$f(x) = \frac{1}{\pi} \frac{\epsilon}{(x^2 + \epsilon^2)}, \quad (32)$$

where $\epsilon > 0$ and this happens to be a well-known delta convergent sequence. In this case the sum-integral difference function defined by equation (26) can be shown to become

$$F(x) = \frac{1}{x} \left(\frac{1}{e^{2\pi\epsilon/x} - 1} - \frac{x}{2\pi\epsilon} \right) = \frac{\phi_1(2\pi\epsilon/x)}{x}, \quad (33)$$

and in deriving equation (33) we have utilized the series (7). We note that in this case, on using the representation (4), the Müntz formula (3) can be shown to yield the functional equation (14), as noted in ref. [3, p. 29]. From equations (27), (32) and (33) we obtain

$$\int_0^\infty e^{-pt} t \phi_1(2\pi\epsilon t) dt = \frac{2\epsilon}{\pi p} \int_0^\infty \frac{\xi}{(\xi^2 + \epsilon^2)^2} \phi_1\left(\frac{p}{\xi}\right) d\xi,$$

which on making the respective substitutions $x = 2\pi\epsilon t$ and $y = p/2\pi\xi$ in the two integrals, we may readily deduce

$$\int_0^\infty e^{-qx} x \phi_1(x) dx = 4q \int_0^\infty \frac{y \phi_1(2\pi y)}{(q^2 + y^2)^2} dy, \quad (34)$$

where $q = p/2\pi\epsilon$ and which we observe is entirely consistent with the result obtained from (21) by formal differentiation with respect to p .

We may also relate equation (5) to the Müntz formula (3) in the following manner. From equation (3) and the definition (4) we have on making the substitution $x = y/t$ in both integrals we have (assuming $t > 0$)

$$\zeta(s) \int_0^\infty y^{s-1} f\left(\frac{y}{t}\right) dy = \int_0^\infty y^{s-1} F\left(\frac{y}{t}\right) dy. \quad (35)$$

On taking the Laplace transform with respect to t , the left-hand side involves

$$\int_0^\infty e^{-pt} \left\{ \int_0^\infty y^{s-1} f\left(\frac{y}{t}\right) dy \right\} dt = \frac{\Gamma(s+1)}{p^{s+1}} \int_0^\infty \xi^{s-1} f(\xi) d\xi, \quad (36)$$

where we have made the progressive substitutions $\xi = y/t$ and $z = py/\xi$. For the right-hand side of equation (35) we have

$$\int_0^\infty e^{-pt} \left\{ \int_0^\infty y^{s-1} F\left(\frac{y}{t}\right) dy \right\} dt = -\frac{1}{p^{s+1}} \int_0^\infty \xi^s f'(\xi) \left\{ \int_0^\infty z^{s-1} \phi_1(z) dz \right\} d\xi.$$

Now assuming that $f(x)$ satisfies the conditions of the Müntz formula, we have by integration by parts

$$-\int_0^\infty \xi^s f'(\xi) d\xi = s \int_0^\infty \xi^{s-1} f(\xi) d\xi, \quad (37)$$

and therefore from the above equations we may deduce equation (4). Thus assuming all the numerous changes of the order of integration may be justified, we may conclude that the formula (27) is consistent with the well-established result (4).

5. General formula for the Riemann zeta function

In this section, we generalize equations (4) and (9) in the following obvious manner. For $m > 0$ we may readily deduce from the general Müntz formula (3) with $f(x) = e^{-x^m}$

$$\frac{\Gamma(s/m)}{m} \zeta(s) = \int_0^\infty x^{s-1} \left\{ \sum_{n=1}^\infty e^{-(nx)^m} - \int_0^\infty e^{-(ux)^m} du \right\} dx, \quad (38)$$

as an expression for the Riemann zeta function valid in the critical strip $0 < \operatorname{Re}(s) < 1$. Thus, we might introduce functions $\phi_m(x)$ defined by

$$\phi_m(x) = \sum_{n=1}^\infty e^{-(nx)^m} - \int_0^\infty e^{-(ux)^m} du, \quad (39)$$

which are consistent with equations (5) and (11). We note that the infinite integral in equation (39) is

$$\int_0^\infty e^{-(ux)^m} du = \frac{1}{mx} \int_0^\infty e^{-\xi} \xi^{1/m-1} d\xi = \frac{\Gamma(1/m+1)}{x}, \quad (40)$$

where we have made the substitution $\xi = (ux)^m$. From equations (39) and (40) we obtain

$$\frac{\Gamma(s/m)}{m} \zeta(s) = \int_0^\infty x^{s-1} \phi_m(x) dx, \quad (41)$$

as an expression for the zeta function valid in the critical strip. We emphasize that while (41) is an obvious consequence of the Müntz formula (3), it nevertheless does tend to highlight the fact that much of the existing formulae for the Riemann zeta function arise from the special cases $m = 1$ and $m = 2$. The case $m = 1$ is important because of the obvious fact that the function $\phi_1(x)$ admits a simple finite expression, whereas the case $m = 2$, although not summable in finite form, is related to the theta functions.

As a simple illustration of the above results we consider the case $m = 1/2$ and exploit the standard Laplace transform

$$e^{-p^{1/2}} = \frac{1}{2\pi^{1/2}} \int_0^\infty e^{-pt} \frac{e^{-1/4t}}{t^{3/2}} dt, \quad (42)$$

so that from equation (39) we have

$$\phi_{1/2}(x) = \frac{1}{2\pi^{1/2}} \int_0^\infty \frac{e^{-1/4t}}{t^{3/2}} \left\{ \frac{1}{e^{xt} - 1} - \frac{1}{xt} \right\} dt, \quad (43)$$

assuming as usual that the order of the various processes may be interchanged. Thus from equations (43) and (41) with $m = 1/2$, we obtain on making the substitution $\xi = xt$

$$2\Gamma(2s) \zeta(s) = \frac{1}{2\pi^{1/2}} \int_0^\infty \frac{e^{-1/4t}}{t^{s+3/2}} dt \int_0^\infty \xi^{s-1} \phi_1(\xi) d\xi, \quad (44)$$

from which on using equation (4) we may deduce

$$\Gamma(2s) = \frac{\Gamma(s)}{4\pi^{1/2}} \int_0^\infty \frac{e^{-1/4t}}{t^{s+3/2}} dt. \quad (45)$$

On making the substitution $v = 1/4t$ we obtain

$$\Gamma(2s) = \frac{2^{2s-1}}{\pi^{1/2}} \Gamma(s) \int_0^\infty e^{-v} v^{s+1/2-1} dv = \frac{2^{2s-1}}{\pi^{1/2}} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right), \quad (46)$$

which of course is the duplication formula (15) for the gamma function.

Similarly, from the Laplace transform [10, p. 261]

$$e^{-p^{1/3}} = \frac{3^{1/2}}{\pi} \int_0^\infty e^{-pt/27} \frac{K_{1/3}(2/t^{1/2})}{t^{3/2}} dt, \quad (47)$$

we may deduce from equation (39)

$$\phi_{1/3}(x) = \frac{3^{1/2}}{\pi} \int_0^\infty \frac{K_{1/3}(2/t^{1/2})}{t^{3/2}} \phi_1\left(\frac{xt}{27}\right) dt, \quad (48)$$

where as usual $K_\nu(x)$ denotes the modified Bessel function of the second kind. On first making the substitution $\xi = xt/27$ and then $y = 2/t^{1/2}$ we find that equation (38) simplifies to

$$\Gamma(3s) = \frac{\Gamma(s)3^{3s-1/2}}{\pi 2^{2s}} \int_0^\infty y^{2s} K_{1/3}(y) dy, \quad (49)$$

which on using the standard Mellin transform

$$\int_0^\infty x^{s-1} K_\nu(x) dx = 2^{s-2} \Gamma\left(\frac{s}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{s}{2} - \frac{\nu}{2}\right), \quad (50)$$

which is valid for $\text{Re}(s) > \nu > 0$ [11, p. 684] and becomes

$$\Gamma(3s) = \frac{3^{3s-1/2}}{2\pi} \Gamma(s) \Gamma\left(s + \left(\frac{1}{3}\right)\right) \Gamma\left(s + \left(\frac{2}{3}\right)\right). \quad (51)$$

This is the multiplication-theorem of Gauss and Legendre for $n = 3$ [9, p. 240].

It may also be worthwhile noting that equation (41) arises from the general Müntz formula (3), other than in the obvious manner, on assuming that the function $f(x)$ has the form

$$f(x) = \int_0^\infty e^{-(xt)^m} g(t) dt, \quad (52)$$

for some function $g(x)$. In a sense equation (52) is a generalized Laplace transform. Now from the left-hand side of equation (3) we have

$$\int_0^\infty x^{s-1} f(x) dx = \frac{\Gamma(s/m)}{m} \int_0^\infty \frac{g(t)}{t^s} dt, \quad (53)$$

where we have interchanged the order of integration and we have made the substitution $\xi = xt$. Similarly for the right-hand side of equation (3) we have

$$\sum_{n=1}^\infty f(nx) - \int_0^\infty f(ux) du = \int_0^\infty g(t) \phi_m(xt) dt,$$

and therefore again on making the substitution $\xi = xt$ we have

$$\int_0^\infty x^{s-1} \left\{ \sum_{n=1}^\infty f(nx) - \int_0^\infty f(ux) du \right\} dx = \int_0^\infty \frac{g(t)}{t^s} dt \int_0^\infty \xi^{s-1} \phi_m(\xi) d\xi, \quad (54)$$

and equation (41) follows on using equation (53) and equation (54) in conjunction with equation (3). In the following section we generalize the Laplace transform formulae (27) and (28).

6. Generalization of the Laplace transform involving the sum-integral difference

We now use the formalism of the previous section and with $F(x)$ defined by equation (26) we may show precisely as in section 4 that the Laplace transform simplifies to become

$$\int_0^\infty e^{-pt} F\left(\frac{1}{t^{1/m}}\right) dt = \frac{1}{p} \int_0^\infty f(\xi) \frac{d}{d\xi} \left\{ \sum_{n=1}^\infty e^{-p(n/\xi)^m} - \int_0^\infty e^{-p(u/\xi)^m} du \right\} d\xi, \quad (55)$$

so that with $\phi_m(x)$ defined by equation (39) we have from equation (55)

$$\begin{aligned} \int_0^\infty e^{-pt} F\left(\frac{1}{t^{1/m}}\right) dt &= \frac{1}{p} \int_0^\infty f(\xi) \frac{d}{d\xi} \left[\phi_m\left(\frac{p^{1/m}}{\xi}\right) \right] d\xi \\ &= \frac{1}{p} \left[f(\xi) \phi_m\left(\frac{p^{1/m}}{\xi}\right) \right]_0^\infty - \frac{1}{p} \int_0^\infty f'(\xi) \phi_m\left(\frac{p^{1/m}}{\xi}\right) d\xi, \end{aligned} \quad (56)$$

and again on assuming that $f(x)$ satisfies the conditions of the Müntz formula, we may conclude from equation (56) that

$$\int_0^\infty e^{-pt} F\left(\frac{1}{t^{1/m}}\right) dt = -\frac{1}{p} \int_0^\infty f'(\xi) \phi_m\left(\frac{p^{1/m}}{\xi}\right) d\xi, \quad (57)$$

giving the appropriate generalization of equations (27) and (28) for arbitrary $m > 0$.

Again equation (57) is trivially satisfied in the event $f(x) = e^{-x^m}$ ($F(x) = \phi_m(x)$) and also produces a number of curious results, such as that arising from $f(x) = e^{-x^k}$ and $F(x) = \phi_k(x)$, thus from equation (57) we obtain

$$q^m \int_0^\infty e^{-tq^m} \phi_k\left(\frac{1}{t^{1/m}}\right) dt = q^k \int_0^\infty e^{-tq^k} \phi_m\left(\frac{1}{t^{1/k}}\right) dt, \quad (58)$$

where we have written $p = q^m$ to bring out the inherent symmetry in the relation. Alternatively, equation (58) may be written as

$$\int_0^\infty e^{-t} \phi_k\left(\frac{q}{t^{1/m}}\right) dt = \int_0^\infty e^{-t} \phi_m\left(\frac{q}{t^{1/k}}\right) dt, \quad (59)$$

which applies for positive k , m and q .

7. Some examples involving Bessel functions

In this section we show that some of the results of preceeding sections are consistent with some standard Laplace transform formulae involving Bessel functions. First we examine the Laplace transform [10, p. 138].

$$\int_0^\infty e^{-pt} t^{\nu/2} J_\nu(at^{1/2}) dt = \left(\frac{a}{2}\right)^\nu \frac{1}{p^{\nu+1}} e^{-a^2/4p}, \quad (60)$$

valid for $\text{Re}(\nu) > -1$. In particular when $\nu = 0$ we have

$$\int_0^\infty e^{-pt} J_0(at^{1/2}) dt = \frac{1}{p} e^{-a^2/4p}, \quad (61)$$

and therefore with $f(x) = J_0(2x^{1/2})$ we have

$$\int_0^\infty e^{-pt} F(t) dt = \frac{1}{p} \phi_1\left(\frac{1}{p}\right), \quad (62)$$

where in the summation we have used equation (61) with $a = 2n^{1/2}$, and in the integral we have used equation (11) with $a = 2u^{1/2}$. We may now use equation (62) to confirm the validity of equation (27) in the case when $f(x) = J_0(2x^{1/2})$ as follows. From the right-hand side of equation (27) we have

$$-\frac{1}{p} \int_0^\infty f'(\xi) \phi_1\left(\frac{p}{\xi}\right) d\xi = \frac{1}{p^2} \int_0^\infty F(t) \left(\int_0^\infty e^{-\xi t/p} J_1(2\xi^{1/2}) \xi^{1/2} d\xi \right) dt,$$

where we have assumed the validity of interchanging the orders of integration and we have used $J'_0 = -J_1$. Now on using equation (60) with $\nu = 1$, $a = 2$ and replacing p by t/p , we may conclude

$$-\frac{1}{p} \int_0^\infty f'(\xi) \phi_1\left(\frac{p}{\xi}\right) d\xi = \int_0^\infty F(t) \frac{e^{-p/t}}{t^2} dt = \int_0^\infty e^{-p\xi} F\left(\frac{1}{\xi}\right) d\xi,$$

confirming the validity of equation (27) when $f(x) = J_0(2x^{1/2})$.

Secondly, from the standard Laplace transform [10, p. 41])

$$\int_0^\infty e^{-pt} t^{\nu-1} e^{-a/t} dt = 2 \left(\frac{a}{p}\right)^{\nu/2} K_\nu[2(ap)^{1/2}], \quad (63)$$

which is valid for $\text{Re}(a) > 0$, we may deduce

$$(ax)^{\nu/2} K_\nu[2(ax)^{1/2}] = \frac{x^\nu}{2} \int_0^\infty e^{-xt} t^{\nu-1} e^{-a/t} dt, \quad (64)$$

and therefore with $f(x) = x^{\nu/2} K_\nu(2x^{1/2})$ we have

$$F(x) = \frac{x^\nu}{2} \int_0^\infty e^{-xt} t^{\nu-1} \phi_1\left(\frac{1}{t}\right) dt = \frac{1}{2} \int_0^\infty e^{-\xi} \xi^{\nu-1} \phi_1\left(\frac{x}{\xi}\right) d\xi, \quad (65)$$

where the last result follows on making the substitution $\xi = xt$. Thus from the Müntz formula (3) we see from equation (65) that the right-hand side becomes

$$\int_0^\infty x^{s-1} F(x) dx = \frac{1}{2} \int_0^\infty e^{-\xi} \xi^{s+\nu-1} \left(\int_0^\infty y^{s-1} \phi_1(y) dy \right) d\xi.$$

Thus from the definition of the gamma function and equation (5) we may deduce

$$\int_0^\infty x^{s-1} F(x) dx = \frac{1}{2} \Gamma(s) \Gamma(s+\nu) \zeta(s). \quad (66)$$

Now for the left-hand side of equation (3) we have

$$\int_0^\infty x^{s-1} f(x) dx = \int_0^\infty x^{s-1+\nu/2} K_\nu(2x^{1/2}) dx = \int_0^\infty \left(\frac{y}{2}\right)^{2s+\nu-1} K_\nu(y) dy,$$

where we have made the substitution $y = 2x^{1/2}$. On using the Mellin transform (50) with s replaced by $2s + \nu$, we may conclude

$$\int_0^\infty x^{s-1} f(x) dx = \frac{1}{2} \Gamma(s) \Gamma(s+\nu), \quad (67)$$

which is entirely consistent with equation (66) and the Müntz formula.

8. Conclusions

We have focused our attention on expressions for the Riemann zeta function which apply in the critical strip $0 < \operatorname{Re}(s) < 1$. We have shown that two standard formulae, namely (4) and (10), can by a simple Laplace transformation, be transformed into the functional equation for the zeta function, namely (14). We have also shown that (4) also emerges from the general Müntz formula (3), other than in the obvious manner, if the function $f(x)$ is assumed to be the Laplace transform of another function. The Müntz formula (3) highlights the fundamental importance of being able to deal with expressions involving the sum-integral difference, and in the latter parts of the article we have established formulae for the Laplace transforms of certain functions involving the sum-integral difference. In the final sections of the article the standard formulae such as (4) and (10) and the corresponding results for the Laplace transforms of the sum-integral differences (27) and (28), are couched in a general formulation involving the functions $\phi_m(x)$ defined by equation (39). This formulation highlights the extent to which classical analysis relies on the special case arising from the exponential function and given by $m = 1$, since it is only this case for which the function $\phi_m(x)$ has a known simple finite form. Nevertheless, the wider structure provides a good deal of insight into existing mathematical analysis, and even the Laplace transform itself can be viewed as a special case of the more general transform (52). The general formula for the Laplace transform of a function involving the sum-integral difference is (57), which embodies many curious relations between infinite integrals such as (58) or (59). The formula (57) and the special cases (27) and (28) are believed to be new, and may be relevant to the Riemann hypothesis, in the sense that the majority of formulae for the Riemann zeta function which apply in the critical strip involve the sum-integral difference. These formulae provide a possible alternative approach to the Euler–MacLaurin summation formula, which along with Bernoulli numbers and polynomials are all consequences arising from the particular analysis associated with the exponential function for the case $m = 1$.

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An integral operator associated with the Hurwitz–Lerch Zeta function and differential subordination

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The main object of this article is to introduce and investigate an integral operator $J_{s,b}(f)$ defined, by using the Hurwitz–Lerch Zeta function, on the various subclasses of the class of normalized analytic functions f in the open unit disk $\mathbb{U}$. Using the technique of differential subordination, an interesting property of the general integral operator $J_{s,b}(f)$ is obtained. Some applications of the results presented here are also considered.

Keywords: Analytic functions; Univalent functions; Convex functions; Starlike functions; Differential subordination; Hurwitz–Lerch Zeta function; Riemann and Hurwitz (or generalized) Zeta functions; Polylogarithmic function; Jung–Kim–Srivastava integral operator; Hadamard product (or convolution); Multiplier transformations; Koebe function of order α

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1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function f in the class $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order α* , if it satisfies the following inequality:

$$\Re \left(\frac{z f'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1). \quad (2)$$

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For convenience, we write

$$\mathcal{S}^*(0) =: \mathcal{S}^*,$$

where $\mathcal{S}^*$ denotes the class of starlike functions in $\mathbb{U}$ (see for details [1]).

Jung *et al.* [2] introduced the following one-parameter family of integral operators:

$$I^\sigma(f)(z) := \frac{2^\sigma}{z\Gamma(\sigma)} \int_0^z \left(\log\left(\frac{z}{t}\right)\right)^{\sigma-1} f(t) dt \quad (\sigma > 0; f \in \mathcal{A}), \quad (3)$$

for which they showed that

$$I^\sigma(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^\sigma a_k z^k. \quad (4)$$

The Jung–Kim–Srivastava integral operator $I^\sigma(f)$ is closely related to some multiplier transformations studied earlier by Flett [3] (see also [1] and [4]).

We begin our present investigation by recalling that a general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ defined by (*cf.*, *e.g.*, [5, p. 121 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad (5)$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s \in \mathbb{C} \text{ when } |z| < 1; \quad \Re(s) > 1 \text{ when } |z| = 1)$$

contains, as its *special* cases, not only the Riemann and Hurwitz (or generalized) Zeta functions:

$$\zeta(s) = \Phi(1, s, 1) \quad \text{and} \quad \zeta(s, a) = \Phi(1, s, a) \quad (6)$$

and the Lerch Zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i \xi}}{n^s} = e^{2\pi i \xi} \Phi(e^{2\pi i \xi}, s, 1) \quad (7)$$

$$(\xi \in \mathbb{R}; \quad \Re(s) > 1),$$

but also such other important functions of analytic number theory as the Polylogarithmic function:

$$\text{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z \Phi(z, s, 1) \quad (8)$$

$$(s \in \mathbb{C} \text{ when } |z| < 1; \quad \Re(s) > 1 \text{ when } |z| = 1)$$

and the Lipschitz–Lerch Zeta function (*cf.* 5, p. 122, Eq. 30 (11)):

$$\phi(\xi, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i \xi}}{(n+a)^s} = \Phi(e^{2\pi i \xi}, s, a) =: L(\xi, s, a) \quad (9)$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad \Re(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z}; \quad \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}),$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet’s famous theorem on primes in arithmetic progressions. Several interesting properties and characteristics of the Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ can

be found in the recent investigations by (for example) Choi and Srivastava [6], Ferreira and Lopez [7], Lin and Srivastava [8], Luo and Srivastava [9], and others.

Now, for $f \in \mathcal{A}$, $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $s \in \mathbb{C}$, we define the function $G_{s,b}(z)$ by

$$G_{s,b}(z) := (1+b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}). \quad (10)$$

We also denote by

$$J_{s,b}(f) : \mathcal{A} \longrightarrow \mathcal{A}$$

the linear operator defined by

$$\begin{aligned} J_{s,b}(f)(z) &:= G_{s,b}(z) * f(z) \\ (z \in \mathbb{U}; f \in \mathcal{A}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}) \end{aligned} \quad (11)$$

in terms of the Hadamard product (or convolution).

We note that

$$J_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^s a_k z^k. \quad (12)$$

Remark 1 It follows from (11) and (12) that one can define the operator $J_{s,b}(f)$ for $b \in \mathbb{C} \setminus \mathbb{Z}^-$. Therefore, we may use the following limit relationship:

$$J_{s,0}(f)(z) := \lim_{b \rightarrow 0} \{J_{s,b}(f)(z)\}. \quad (13)$$

Remark 2 For $f \in \mathcal{A}$ and $z \in \mathbb{U}$, we find from the definitions (11) and (12) that

$$J_{0,b}(f)(z) = f(z), \quad (14)$$

$$J_{1,0}(f)(z) = \int_0^z \frac{f(t)}{t} dt =: A(f)(z), \quad (15)$$

$$J_{1,1}(f)(z) = \frac{2}{z} \int_0^z f(t) dt =: L(f)(z), \quad (16)$$

$$J_{1,\gamma}(f)(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt =: L_\gamma(f)(z) \quad (\gamma > -1), \quad (17)$$

and

$$J_{\sigma,1}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^\sigma a_k z^k =: I^\sigma(f)(z) \quad (\sigma > 0), \quad (18)$$

where $A(f)$ and $L(f)$ are the integral operators introduced earlier by Alexander [10] and Libera [11], respectively, $L_\gamma(f)$ is a familiar generalization of the integral operator

$$L_\gamma(f) \quad (\gamma \in \mathbb{N} := \{1, 2, 3, \dots\})$$

introduced by Bernardi [12], and $I^\sigma(f)$ denotes the Jung–Kim–Srivastava integral operator defined already by (3) and (4).

Remark 3 For $f \in \mathcal{A}$, $z, t_j \in \mathbb{U}$ ($j = 1, \dots, n$), $n \in \mathbb{N}$ and $b \in \mathbb{C} \setminus \mathbb{Z}^-$, we have

$$J_{2,0}(f)(z) = \int_0^z \frac{1}{t_1} \int_0^{t_1} \frac{f(t_2)}{t_2} dt_2 dt_1, \quad (19)$$

$$J_{n,0}(f)(z) = \int_0^z \frac{1}{t_1} \int_0^{t_1} \frac{1}{t_2} \int_0^{t_2} \cdots \frac{1}{t_{n-1}} \int_0^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n dt_{n-1} \cdots dt_1, \quad (20)$$

$$J_{2,b}(f)(z) = \frac{(1+b)^2}{z^b} \int_0^z \frac{1}{t_1} \int_0^{t_1} t_2^{b-1} f(t_2) dt_2 dt_1, \quad (21)$$

and

$$J_{n,b}(f)(z) = \frac{(1+b)^n}{z^b} \int_0^z \frac{1}{t_1} \int_0^{t_1} \frac{1}{t_2} \int_0^{t_2} \cdots \frac{1}{t_{n-1}} \int_0^{t_{n-1}} t_n^{b-1} f(t_n) dt_n dt_{n-1} \cdots dt_1. \quad (22)$$

By applying the definition (12), we can show that

$$J_{s+1,b}(f)(z) = \frac{1+b}{z^b} \int_0^z t^{b-1} J_{s,b}(f)(t) dt, \quad (23)$$

which, upon differentiating both sides with respect to z , yields the following useful identity for the operator $J_{s,b}(f)$:

$$z J'_{s+1,b}(f)(z) = (1+b) J_{s,b}(f)(z) - b J_{s+1,b}(f)(z). \quad (24)$$

We shall need each of the following definitions in our investigation of the two-parameter family of linear operators $J_{s,b}(f)$.

DEFINITION 1 Let the functions $f(z)$ and $F(z)$ be analytic in $\mathbb{U}$. The function $f(z)$ is said to be subordinate to the function $F(z)$, written symbolically as

$$f(z) \prec F(z) \quad (z \in \mathbb{U}),$$

if there exists a function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| \leq 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)).$$

Furthermore, if the function $F(z)$ is univalent in $\mathbb{U}$, then

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \iff f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

DEFINITION 2 Let $\Psi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C}$ be analytic in a domain $\mathbb{D}$ of the complex z -plane. Also let the function $h(z)$ be univalent in $\mathbb{U}$. If the function $p(z)$ is analytic in $\mathbb{U}$ with

$$(p(z), z p'(z)) \in \mathbb{D} \quad (z \in \mathbb{U}),$$

then we say that $p(z)$ satisfies a first-order differential subordination when

$$\Psi(p(z), zp'(z); z) \prec h(z) \quad (z \in \mathbb{U}). \quad (25)$$

DEFINITION 3 A univalent function $q(z)$ is called the dominant of the differential subordination (25), if

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all functions $p(z)$ satisfying the subordination (25). Moreover, if

$$\tilde{q}(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all dominants of (25), then we say that $\tilde{q}(z)$ is the best dominant of (25).

2. Differential subordination associated with the operator $J_{s,b}(f)$

We first recall the following lemma due to Miller and Mocanu [13] (see also [14, p. 132]), which will be required in our investigation.

LEMMA Let the function $q(z)$ be univalent in $\mathbb{U}$ and let the functions θ and ϕ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set

$$Q(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + Q(z)$$

and suppose that either

(i) $h(z)$ is convex in $\mathbb{U}$

or

(ii) $Q(z)$ is starlike in $\mathbb{U}$.

Assume also that

(iii)

$$\Re \left(\frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

If the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = q(0)$, $p(\mathbb{U}) \subset \mathbb{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \quad (26)$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant of (26).

We now prove the following result.

THEOREM Let $b \geq 0$ and $\alpha \in [0, 1)$. Suppose also that

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} \prec h(z) \quad (z \in \mathbb{U}) \quad (27)$$

for all $f \in \mathcal{A}$ satisfying the following condition:

$$\frac{J_{s+j,b}(f)(z)}{z} \neq 0 \quad (j = 0, 1),$$

$h(z)$ being given by

$$h(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)[1 + b + (1 - 2\alpha - b)z]}. \quad (28)$$

Then

$$J_{s+1,b}(f)(z) \in \mathcal{S}^*(\alpha)$$

and the constant α is the best possible.

Proof If we choose

$$p(z) = \frac{z J'_{s+1,b}(f)(z)}{J_{s+1,b}(f)(z)},$$

then (24) becomes

$$[p(z) + b]J_{s+1,b}(f)(z) = (b + 1)J_{s,b}(f)(z). \quad (29)$$

Thus, from the identities (24) and (29), we have

$$\frac{z J'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = \left(p(z) + \frac{z p'(z)}{p(z) + b} \right), \quad (30)$$

which, in light of the hypothesis (27) of the Theorem, yields the following subordination relation:

$$p(z) + \frac{z p'(z)}{p(z) + b} < h(z) \quad (z \in \mathbb{U}). \quad (31)$$

Let us now choose the functions

$$q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad \theta(w) = w \quad \text{and} \quad \phi(w) = \frac{1}{w + b}.$$

Then $\theta(w)$ and $\phi(w)$ are both analytic inside the domain

$$\mathbb{D}^* = \mathbb{C} \setminus \{-b\},$$

which contains $q(\mathbb{U})$, $q(0) = 1$, and

$$\phi(w) \neq 0 \quad \text{when} \quad w \in q(\mathbb{U}).$$

Next we define the function $Q(z)$ by

$$Q(z) = zq'(z)\phi(q(z)) = \frac{2(1 - \alpha)z}{(1 - z)[1 + b + (1 - 2\alpha - b)z]}. \quad (32)$$

Then, since [cf. equation (28)]

$$\theta(q(z)) + Q(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} + \frac{2(1 - \alpha)z}{(1 - z)[1 + b + (1 - 2\alpha - b)z]} =: h(z), \quad (33)$$

we readily observe that

$$\frac{zQ'(z)}{Q(z)} = \frac{1}{1-z} - \frac{(1-2\alpha-b)z}{1+b+(1-2\alpha-b)z}. \quad (34)$$

Furthermore, we have

$$\phi(q(z)) = \frac{1-z}{1+b+(1-2\alpha-b)z}. \quad (35)$$

It follows from (33) and (34) that $Q(z)$ is starlike in $\mathbb{U}$ and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{1}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) \quad (z \in \mathbb{U}). \quad (36)$$

Since

$$\Re(\phi(q(z))) > 0 \quad (z \in \mathbb{U}),$$

we have

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

The hypothesis

$$\frac{J_{s+1,b}(f)(z)}{z} \neq 0$$

shows that the function $p(z)$ is analytic in $\mathbb{U}$, with

$$p(0) = q(0) = 1,$$

and the hypothesis

$$\frac{J_{s,b}(f)(z)}{z} \neq 0$$

shows that

$$-b \notin p(\mathbb{U}).$$

Therefore,

$$p(\mathbb{U}) \subset \mathbb{D}^*.$$

By the above Lemma, we deduce that

$$p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

that is, that

$$J_{s+1,b}(f)(z) \in \mathcal{S}^*(\alpha),$$

and $q(z)$ is the best dominant of (31), which implies that the constant α is the best possible. This evidently completes our proof of the Theorem. ■

3. Applications of the Theorem

Putting $s = 0$ in the Theorem, we get the following result for the integral operator $L_b(f)$ given by (17).

COROLLARY 1 *Let $b \geq 0$ and $\alpha \in [0, 1)$. Suppose also that*

$$\frac{zf'(z)}{f(z)} \prec h(z) \quad (z \in \mathbb{U}; f \in \mathcal{A}) \quad (37)$$

for all $f \in \mathcal{A}$ satisfying the following conditions:

$$\frac{f(z)}{z} \neq 0 \quad \text{and} \quad \frac{L_b(f)(z)}{z} \neq 0 \quad (z \in \mathbb{U}),$$

$h(z)$ being given by (28). Then

$$L_b(f)(z) \in \mathcal{S}^*(\alpha)$$

and the constant α is the best possible.

Remark 4 By setting $\alpha = 0$ in Corollary 1, we have the result due to Oros [15].

Putting $b = 1$ in the Theorem, we obtain the following property for the Jung–Kim–Srivastava integral operator $I^s(f)$ ($s \geq 0$) given by (3), (4) and (18).

COROLLARY 2 *Let $s \geq 0$ and $\alpha \in [0, 1)$. Suppose also that*

$$\frac{z(I^s(f)(z))'}{I^s(f)(z)} \prec h(z) \quad (z \in \mathbb{U}; f \in \mathcal{A}) \quad (38)$$

for all $f \in \mathcal{A}$ satisfying the following conditions:

$$\frac{I^{s+j}(f)(z)}{z} \neq 0 \quad (j = 0, 1),$$

$h(z)$ being given by (28). Then

$$I^{s+1}(f)(z) \in \mathcal{S}^*(\alpha)$$

and the constant α is the best possible.

By taking

$$f(z) = f_0(z) = \frac{z}{1-z}$$

in the Theorem, we readily obtain the following property for the Hurwitz–Lerch Zeta function $\Phi(z, s, b)$ defined by (5).

COROLLARY 3 Let $b > 0$ and $\alpha \in [0, 1)$. Suppose also that

$$K(z, s, b) := \frac{z \Phi'(z, s, b)}{\Phi(z, s, b) - b^{-s}} \quad (z \in \mathbb{U}), \quad (39)$$

where, for convenience,

$$\Phi'(z, s, b) = \frac{\partial}{\partial z} \{\Phi(z, s, b)\}.$$

If

$$\frac{\Phi(z, s + j, b) - b^{-s-j}}{z} \neq 0 \quad (j = 0, 1)$$

and

$$K(z, s, b) < h(z) \quad (z \in \mathbb{U}), \quad (40)$$

where $h(z)$ is given by (28), then

$$\Re(K(z, s + 1, b)) > \alpha \quad (z \in \mathbb{U})$$

and the constant α is the best possible.

The following example would provide an interesting illustration and clarification for Corollary 1.

Example For $b \geq 0$ and $\alpha \in [0, 1)$, we can show that the function $f_1(z)$ defined by

$$f_1(z) = z \left[1 + \left(\frac{1 - 2\alpha - b}{1 + b} \right) z \right] (1 - z)^{2\alpha-3} \quad (z \in \mathbb{U}) \quad (41)$$

satisfies the hypotheses of Corollary 1. We thus find from Corollary 1 that

$$\begin{aligned} L_b(f_1)(z) &= \frac{1+b}{z^b} \int_0^z f_1(t) t^{b-1} dt \\ &= \frac{1}{z^b} \int_0^z [1 + b + (1 - 2\alpha - b)t] t^b (1 - t)^{2\alpha-3} dt \\ &= \frac{1}{z^b} \int_0^z [(1+b)(1-t) + 2(1-\alpha)t] t^b (1 - t)^{2\alpha-3} dt \\ &= \frac{z}{(1-z)^{2(1-\alpha)}} =: K_\alpha(z), \end{aligned} \quad (42)$$

where $K_\alpha(z)$ denotes the *Koebe function of order α* . Thus we have the following condition:

$$\frac{L_b(f_1)(z)}{z} = \frac{K_\alpha(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Of course, it is well known that

$$K_\alpha(z) = L_b(f_1)(z) \in \mathcal{S}^*(\alpha).$$

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A simple approach to Jacobi polynomials

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Starting from the Rodrigues formula polynomials are introduced by carrying out the innermost differentiations. The key is a generating function for them that is simpler than that of Jacobi polynomials leading to short and transparent derivations of recursion relations and addition theorems for Jacobi polynomials.

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1. Introduction and results

Jacobi polynomials are the most general of the classical polynomials. They play a role in rotation matrices [1] and in exact solutions of the Eckart potential in supersymmetric quantum mechanics [2] and, with complex argument and parameters, in the trigonometric Rosen–Morse potential [2], cases of the few exact solutions in quantum mechanics. Here our goal is to introduce polynomials by reworking the Rodrigues formula [3, 4]

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} ((1-x)^{n+\alpha} (1+x)^{n+\beta}) \quad (1)$$

in a natural way by carrying out the innermost differentiations in equation (1). This at once leads to the generalized Rodrigues formula [3]

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^{n-\mu}}{dx^{n-\mu}} \{ (1-x)^{n+\alpha-\mu} (1+x)^{n+\beta-\mu} (-2)^\mu \mu! \cdot P_\mu^{(\alpha-\mu+n, \beta-\mu+n)}(x) \}, \quad \mu = 0, 1, \dots, n; \quad n = 0, 1, \dots, \quad (2)$$

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where the polynomials in question are the Jacobi polynomials on the rhs of equation (2). The generating function of these polynomials [4],

$$\sum_{v=0}^{\infty} (-2y)^v P_v^{(\alpha+n-v, \beta+n-v)}(x) = (1+y(1-x))^{n+\beta} (1-y(1+x))^{n+\alpha}, \quad |y| < \frac{1}{2} \quad (3)$$

is much simpler than the usual one for Jacobi polynomials, especially if we set $n = 0$, allowing for shorter and more transparent derivations of many known properties of Jacobi polynomials and new ones. Among the recursions relations that we obtain are

$$4(v+1)P_{v+1}^{(\alpha, \beta)}(x) = -2[(\beta+v+1)(1-x) - (\alpha+v+1)(1+x)]P_v^{(\alpha, \beta)}(x) - (2v+2+\alpha+\beta)(1-x^2)P_{v-1}^{(\alpha+1, \beta+1)}(x); \quad (4)$$

$$4(v+1)P_{v+1}^{(\alpha-1, \beta-1)}(x) + [4vx + 2(\beta+v)(1-x) - 2(\alpha+v)(1+x)]P_v^{(\alpha, \beta)}(x) - (v+1+\alpha+\beta)(1-x^2)P_{v-1}^{(\alpha+1, \beta+1)}(x) = 0; \quad (5)$$

$$4 \frac{dP_v^{(\alpha, \beta)}(x)}{dx} = 2(2v+\alpha+\beta)P_{v-1}^{(\alpha, \beta)}(x) + [(\alpha+v)(x-1) + (\beta+v)(x+1)]P_{v-2}^{(\alpha+1, \beta+1)}(x); \quad (6)$$

$$4(v+\alpha+\beta) \frac{dP_v^{(\alpha, \beta)}(x)}{dx} = 2(v+\alpha+\beta)(2v+\alpha+\beta)P_{v-1}^{(\alpha, \beta)}(x) + 2[(\alpha+v)(x-1) + (\beta+v)(x+1)] \frac{dP_{v-1}^{(\alpha, \beta)}(x)}{dx}. \quad (7)$$

A product formula is given by ($0 \leq v_2 \leq v_1$)

$$P_{v_1}^{(\alpha, \beta)}(x) P_{v_2}^{(\alpha+v_1-v_2, \beta+v_1-v_2)}(x) = \sum_{0 \leq v \leq (v_1+v_2)/2} 2^{-2v} \binom{v_1+v_2-2v}{v_1-v} \sum_{k=0}^v \binom{\alpha+v_1}{k} \binom{\beta+v_1}{v-k} \cdot (1+x)^{2k} (1-x)^{2(v-k)} P_{v_1+v_2-2v}^{(\alpha+2v-v_2-k, \beta+v-v_2+k)}(x). \quad (8)$$

An elegant parameter addition formula is

$$P_v^{(\alpha_1+\alpha_2, \beta_1+\beta_2)}(x) = \sum_{k=0}^v P_k^{(\alpha_1-k, \beta_1-k)}(x) P_{v-k}^{(\alpha_2+k, \beta_2+k)}(x). \quad (9)$$

The generating function also leads to expansion formulas for Jacobi polynomials, such as

$$P_v^{(\alpha, \beta)}(x) = \left(\frac{-1}{2}\right)^v \sum_{0 \leq j+2k \leq v} (-1)^j \binom{\beta-\alpha}{j} \binom{\beta+v}{k} x^{2k} (1+x)^j G_{v-j-2k}^{(\beta+v-k)}(x), \quad (10)$$

$$P_v^{(\alpha, \beta)}(x) = \left(\frac{-1}{2}\right)^v \sum_{0 \leq j+2k \leq v} \binom{\beta-\alpha}{j} \binom{\alpha+v}{k} x^{2k} (1-x)^j G_{v-j-2k}^{(\alpha+v-k)}(x). \quad (11)$$

Here the real polynomials $G_v^{(\beta)}(x)$ are directly related to the ultraspherical or Gegenbauer polynomials [3, 4]:

$$i^v G_v^{(-\beta)}(-ix) = C_v^{\beta}(x). \quad (12)$$

In the next section where, for brevity's sake, proofs are merely outlined, we introduce the polynomials in question and link them to Jacobi polynomials. Then we sketch a derivation of their generating function, the key result from which recursion relations and addition theorems follow.

2. Outline of proofs

We carry out the innermost differentiation of the Rodrigues formula, equation (1),

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^{n-1}}{dx^{n-1}} \cdot ((1-x)^{n+\alpha-1} (1+x)^{n+\beta-1} (-2) P_1^{(\alpha+n-1, \beta+n-1)}(x)), \quad (13)$$

the first step of the inductive proof of equation (2). For the step $\mu \rightarrow \mu + 1$, we carry out the innermost differentiation in equation (2) obtaining

$$\begin{aligned} P_n^{(\alpha, \beta)}(x) &= \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^{n-\mu-1}}{dx^{n-\mu-1}} \left\{ (1-x)^{n+\alpha-\mu-1} (1+x)^{n+\beta-\mu-1} (-2)^\mu \mu! \right. \\ &\quad \cdot \left[(1-x^2) \frac{d P_\mu^{(\alpha+n-\mu, \beta+n-\mu)}(x)}{dx} - (\alpha+n+\mu)(1+x) P_\mu^{(\alpha+n-\mu, \beta+n-\mu)}(x) \right. \\ &\quad \left. \left. + (\beta+n+\mu)(1-x) P_\mu^{(\alpha+n-\mu, \beta+n-\mu)}(x) \right] \right\} \\ &= \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^{n-\mu-1}}{dx^{n-\mu-1}} \left\{ (1-x)^{n+\alpha-\mu-1} (1+x)^{n+\beta-\mu-1} \right. \\ &\quad \cdot (-2)^{\mu+1} (\mu+1)! P_{\mu+1}^{(\alpha+n-\mu-1, \beta+n-\mu-1)}(x) \left. \right\}, \quad (14) \end{aligned}$$

upon using the recursive ODE

$$\begin{aligned} (1-x^2) \frac{d}{dx} P_\mu^{(\alpha+n-\mu, \beta+n-\mu)}(x) &+ [(\beta+n+\mu)(1-x) - (\alpha+n+\mu)(1+x)] \\ &\times P_\mu^{(\alpha+n-\mu, \beta+n-\mu)}(x) \\ &= -2(\mu+1) P_{\mu+1}^{(\alpha+n-\mu-1, \beta+n-\mu-1)}(x). \quad (15) \end{aligned}$$

Substituting the Rodrigues formula (1) for the polynomials $P_v^{(\alpha+n-v, \beta+n-v)}(x)$ in equation (3) and denoting the generating function by $H(x, y; \alpha, \beta)$,

$$\begin{aligned} H(x, y; \alpha, \beta) &= (1+x)^{-n-\beta} (1-x)^{-n-\alpha} \sum_{v=0}^{\infty} \frac{[(1-x^2)y]^v}{v!} \frac{d^v}{dx^v} ((1+x)^{\beta+n} (1-x)^{\alpha+n}) \\ &= (1+x)^{-n-\beta} (1-x)^{-n-\alpha} [1+x+y(1-x^2)]^{n+\beta} [1-x-y(1-x^2)]^{n+\alpha} \\ &= (1+y(1-x))^{n+\beta} (1-y(1+x))^{n+\alpha} = \sum_{v=0}^{\infty} (-2y)^v P_v^{(\alpha+n-v, \beta+n-v)}(x), \quad (16) \end{aligned}$$

we sum the series recognizing the Taylor expansion of the function $(1+x)^{n+\beta} (1-x)^{n+\alpha}$ at x with increment $(1-x^2)y$. Equation (3) is a special case of more general generating functions in a closed form (due to Srivastava and Singhal, 1973) in ref. [4, p. 82].

Differentiating the generating function for $n = 0$ with respect to the variable y leads to the recursive partial differential equation (PDE)

$$\begin{aligned} \frac{\partial}{\partial y} H(x, y; \alpha, \beta) &= \{(1-x)\beta[1-y(1+x)] - (1+x)\alpha[1+y(1-x)]\} \\ &\cdot H(x, y; \alpha-1, \beta-1), \end{aligned} \quad (17)$$

then expanding in a power series according to equation (16), comparing like powers of the variable y and finally shifting $\alpha \rightarrow \alpha + v + 1$, $\beta \rightarrow \beta + v + 1$ gives the recursion relation (4).

If we rewrite the recursive PDE (17) in the alternative form of a PDE

$$\begin{aligned} [1+y(1-x)][1-y(1+x)] \frac{\partial}{\partial y} H(x, y; \alpha, \beta) \\ = \{(1-x)\beta[1-y(1+x)] - (1+x)\alpha[1+y(1-x)]\} H(x, y; \alpha, \beta) \end{aligned} \quad (18)$$

and translate this relation similarly into Jacobi polynomials we find the recursion relation (5).

If we differentiate the generating function for $n = 0$ with respect to the variable x we obtain another recursive PDE

$$\frac{\partial}{\partial x} H(x, y; \alpha, \beta) = \{-y\beta[1-y(1+x)] - y\alpha[1+y(1-x)]\} H(x, y; \alpha-1, \beta-1) \quad (19)$$

from which the recursion relation (6) follows similarly.

Using the recursive ODE [3]

$$\frac{dP_v^{(\alpha, \beta)}(x)}{dx} = \frac{1}{2}(v + \alpha + \beta + 1)P_{v-1}^{(\alpha+1, \beta+1)}(x) \quad (20)$$

in the last term of equation (6) we obtain the recursion (7) involving the same parameters.

The product formula (8) follows from an addition theorem in the variable y for the generating function (for $n = 0$)

$$\begin{aligned} H(x, y_1; \alpha, \beta) H(x, y_2; \alpha, \beta) \\ &= [(1+y_1(1-x)(1+y_2(1-x))]^\beta [(1-y_1(1+x)(1-2(1+x))]^\alpha \\ &= [1+(y_1+y_2)(1-x)]^\beta \left\{ 1 + \frac{y_1 y_2 (1-x)^2}{1+(y_1+y_2)(1-x)} \right\}^\beta \\ &\quad \cdot [1-(y_1+y_2)(1+x)]^\alpha \left\{ 1 + \frac{y_1 y_2 (1+x)^2}{1-(y_1+y_2)(1+x)} \right\}^\alpha \\ &= H(x, y_1+y_2; \alpha, \beta) \sum_{j,k=0}^{\infty} \binom{\alpha}{k} \binom{\beta}{j} \left(\frac{y_1 y_2 (1-x)^2}{1+(y_1+y_2)(1-x)} \right)^j \left(\frac{y_1 y_2 (1+x)^2}{1-(y_1+y_2)(1+x)} \right)^k \\ &= \sum_{j,k=0}^{\infty} \binom{\alpha}{k} \binom{\beta}{j} H(x, y_1+y_2; \alpha-k, \beta-j) y_1^{j+k} y_2^{j+k} (1-x)^{2j} (1+x)^{2k}. \end{aligned} \quad (21)$$

Expanding into Jacobi polynomials according to equation (16), comparing like powers of $y_1 y_2$ and shifting $\alpha \rightarrow \alpha + v_1$, $\beta \rightarrow \beta + v_1$ yields the product formula (8). An alternative proof

starts directly from binomial expansions of the product of generating functions

$$H(x, y_1; \alpha, \beta) H(x, y_2; \alpha, \beta) = \sum_{j,k=0}^{\infty} \binom{\alpha}{j} \binom{\beta}{k} H(x, y_1 + y_2; j, k) y_1^{\alpha+\beta-j-k} y_2^{\alpha+\beta-j-k} \cdot (1-x)^{2(\beta-k)} (1+x)^{2(\alpha-j)}. \quad (22)$$

The multiplicative identity for the generating function,

$$H(x, y; \alpha_1 + \alpha_2, \beta_1 + \beta_2) = H(x, y; \alpha_1, \beta_1) H(x, y; \alpha_2, \beta_2), \quad (23)$$

translates into the parameter addition theorem (9).

If we organize the generating function for $n = 0$ somewhat differently and make a binomial expansion we obtain

$$\begin{aligned} H(x, y; \alpha, \beta) &= (1 - 2xy - y^2 + x^2y^2)^{\beta} (1 - y(1+x))^{\alpha-\beta} \\ &= (1 - 2xy - y^2)^{\beta} \left(1 + \frac{x^2y^2}{1 - 2xy - y^2}\right)^{\beta} (1 - y(1+x))^{\alpha-\beta} \\ &= (1 - y(1+x))^{\alpha-\beta} \sum_{k=0}^{\infty} \binom{\beta}{k} (xy)^{2k} G(x, y; \beta - k). \end{aligned} \quad (24)$$

Here we have introduced the generating function $G(x, y; \beta)$ for the polynomials $G_v^{(\beta)}(x)$ by

$$G(x, y; \beta) \equiv (1 - 2xy - y^2)^{\beta} = \sum_{v=0}^{\infty} y^v G_v^{(\beta)}(x), \quad |y| < 1. \quad (25)$$

Now we expand the monomial and the generating function in power series, compare like powers of the variable y and obtain the expansion (10). Formula (10) generalizes the parameters in the expansion of Jacobi polynomials into Gegenbauer polynomials [3]. An alternative way of writing equation (3) for $n = 0$ as

$$H(x, y; \alpha, \beta) = (1 - 2xy - y^2 + x^2y^2)^{\alpha} (1 + y(1-x))^{\beta-\alpha}$$

and expanding according to equation (16) leads to the expansion (11).

3. Conclusion

Our method of introducing polynomials by carrying out the innermost differentiations of the Rodrigues formula for Jacobi polynomials leads to a simple generating function for Jacobi polynomials that we exploit to derive numerous recursion relations and an addition theorem in a simple and transparent way. This method complements, extends and often simplifies considerably standard methods of deriving properties of Jacobi polynomials.

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Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces

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In this article we study the boundedness and the compactness for an operator related to composition operators, differentiation operators and multiplication operators from Bergman type spaces to Bers type spaces.

Keywords: Composition operator; Differentiation operator; Multiplication operator; Bergman type space; Bers type space

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$ and $f \in H(\mathbb{D})$, we say that f belongs to the Bers type space [1], denoted by H_α^∞ , if

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

H_α^∞ is sometimes called growth space [2]. The little Bers type space, denoted by $H_{\alpha,0}^\infty$, is the set of $f \in H(\mathbb{D})$ with $\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f(z)| = 0$. For more information on the Bers-type space, see [1–3].

A positive continuous function ϕ on $[0, 1)$ is called a normal function, if there exist s and t , $0 < s < t$, such that $(\phi(r)/(1 - r)^s) \downarrow 0$, $(\phi(r)/(1 - r)^t) \uparrow \infty$ as $r \rightarrow 1^-$ (see, for example, [4, 5]).

For $0 < p < \infty$ and a normal function ϕ , let $A^p(\phi)$, called the Bergman type space, denote the space of all $f \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^p \frac{\phi^p(|z|)}{1 - |z|} dA(z) < \infty,$$

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where dA denotes the normalized Lebesgue area measure on the unit disk $\mathbb{D}$ such that $A(\mathbb{D}) = 1$. Note that if $\phi(r) = (1 - r)^{1/p}$, $A^p(\phi)$ is the classical Bergman space A^p .

Throughout this article, φ denotes a nonconstant analytic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. It is well known that the composition operator is bounded on the Hardy space, the Bergman space and the Bloch space. The composition operator was studied extensively by many people, see, for example, [6–8] and references therein.

Let u be a fixed analytic function on $\mathbb{D}$. A linear operator uC_φ , called a weighted composition operator, is defined by $uC_\varphi f = u(f \circ \varphi)$, where $f \in H(\mathbb{D})$. If $\varphi(z) = z$, we get the multiplication operator M_u , i.e. $M_u f = uf$. Recently, there are many people who have studied the weighted composition operator, see, for example, [9–13] and references therein.

Let D be the differentiation operator and n be a nonnegative integer, i.e. we have $Df = f'$, $D^n f = f^{(n)}$, $f \in H(\mathbb{D})$. The differentiation operator is typically unbounded on many analytic function spaces. Now, we define a linear operator $D_{\varphi,u}^n$ as following

$$D_{\varphi,u}^n f = u \cdot f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

This operator is called the generalized weighted composition operator, since it includes many known operators. If $n = 0$, then we get the weighted composition operator uC_φ . If $n = 0$ and $u(z) \equiv 1$, then we get the composition operator C_φ . If $n = 1$, $u(z) = \varphi'(z)$, then $D_{\varphi,u}^n = DC_\varphi$, which was studied in refs. [14, 15]. When $n = 1$ and $u(z) = 1$, then $D_{\varphi,u}^n = C_\varphi D$ which was studied in refs. [14, 16]. If we put $n = 1$ and $\varphi(z) = z$, then $D_{\varphi,u}^n = M_u D$, i.e. the product of differentiation operator and multiplication operator.

In this article, we study the operator $D_{\varphi,u}^n$ from the Bergman type space to the Bers type space. Sufficient and necessary conditions for the boundedness and the compactness of $D_{\varphi,u}^n$ are given. The boundedness and the compactness of the operator DM_u from the Bergman type space to the Bers type space are also characterized. The following theorems are our main results.

THEOREM 1 *Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded if and only*

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} < \infty. \quad (1)$$

THEOREM 2 *Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is compact if and only if $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded and*

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{(1 - |z|^2)^\beta |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} = 0. \quad (2)$$

THEOREM 3 *Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^\infty$ is bounded if and only if $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded and $u \in H_{\beta,0}^\infty$.*

THEOREM 4 *Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^\infty$ is compact if and only if*

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\beta |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} = 0. \quad (3)$$

THEOREM 5 Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$ and ϕ is normal on $[0, 1)$. Then $DM_u : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta - (1/p)} |u'(z)|}{\phi(|z|)} < \infty \quad \text{and} \quad \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^{\beta - (1/p) - 1} |u(z)|}{\phi(|z|)} < \infty. \quad (4)$$

THEOREM 6 Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$ and ϕ is normal on $[0, 1)$. Then $DM_u : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded if and only if $DM_u : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{\beta} |u'(z)| = 0$.

THEOREM 7 Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$ and ϕ is normal on $[0, 1)$. Then the following statements are equivalent.

- (1) $DM_u : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact;
- (2) $DM_u : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is compact;
- (3)

$$\lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\beta - (1/p)} |u'(z)|}{\phi(|z|)} = 0 \quad \text{and} \quad \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^{\beta - (1/p) - 1} |u(z)|}{\phi(|z|)} = 0. \quad (5)$$

As corollaries, we obtain characterizations of the boundedness and the compactness of the operators DC_{φ} , $C_{\varphi}D$, $M_u D$ and uC_{φ} from the Bergman type space to the Bers type space, i.e. we obtain the following corollaries.

COROLLARY 1 Suppose that $0 < p < \infty$, $0 < \beta < \infty$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then

- (1) $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded if and only if $\sup_{z \in \mathbb{D}} ((1 - |z|^2)^{\beta} |\varphi'(z)|) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) < \infty$;
- (2) $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact if and only if $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $\lim_{|\varphi(z)| \rightarrow 1^-} ((1 - |z|^2)^{\beta} |\varphi'(z)|) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) = 0$;
- (3) $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded if and only if $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^{\beta} |\varphi'(z)| = 0$;
- (4) $DC_{\varphi} : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is compact if and only if $\lim_{|z| \rightarrow 1^-} ((1 - |z|^2)^{\beta} |\varphi'(z)|) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) = 0$.

COROLLARY 2 Suppose that $0 < p < \infty$, $0 < \beta < \infty$, ϕ is normal on $[0, 1)$ and φ is an analytic self-map of $\mathbb{D}$. Then

- (1) $C_{\varphi}D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded if and only if $C_{\varphi}D : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded if and only if $\sup_{z \in \mathbb{D}} ((1 - |z|^2)^{\beta}) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) < \infty$;
- (2) $C_{\varphi}D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact if and only if $C_{\varphi}D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $\lim_{|\varphi(z)| \rightarrow 1^-} ((1 - |z|^2)^{\beta}) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) = 0$;
- (3) $C_{\varphi}D : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is compact if and only if $\lim_{|z| \rightarrow 1^-} ((1 - |z|^2)^{\beta}) / (\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+1}) = 0$.

COROLLARY 3 Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$ and ϕ is normal on $[0, 1)$. Then

- (1) $M_u D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded if and only if $\sup_{z \in \mathbb{D}} \phi^{-1}(|z|)(1 - |z|^2)^{\beta - (1/p) - 1} |u(z)| < \infty$;

- (2) $M_u D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact if and only if $M_u D : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is compact if and only if $\lim_{|z| \rightarrow 1^-} \phi^{-1}(|z|)(1 - |z|^2)^{\beta-(1/p)-1}|u(z)| = 0$;
- (3) $M_u D : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded if and only if $M_u D : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $u \in H_{\beta,0}^{\infty}$.

COROLLARY 4 Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and ϕ is an analytic self-map of $\mathbb{D}$. Then

- (1) $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded if and only if $\sup_{z \in \mathbb{D}} ((1 - |z|^2)^{\beta}|u(z)|)/(\phi(|\phi(z)|)(1 - |\phi(z)|^2)^{(1/p)}) < \infty$.
- (2) $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact if and only if $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $\lim_{|\phi(z)| \rightarrow 1^-} ((1 - |z|^2)^{\beta}|u(z)|)/(\phi(|\phi(z)|)(1 - |\phi(z)|^2)^{(1/p)}) = 0$.
- (3) $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded if and only if $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $u \in H_{\beta,0}^{\infty}$.
- (4) $uC_{\phi} : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is compact if and only if $\lim_{|z| \rightarrow 1^-} ((1 - |z|^2)^{\beta}|u(z)|)/(\phi(|\phi(z)|)(1 - |\phi(z)|^2)^{(1/p)}) = 0$.

Especially, let $\phi(r) = (1 - r)^{1/p}$, we obtain the corresponding results for these operators from Bergman spaces to Bers type spaces.

Throughout the article, C denotes a positive constant which may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Proof of main result

In this section we give the proofs of the former theorems. For this purpose, we give some auxiliary results which will be used in proving the main results of the paper. They are incorporated in the lemmas which follow.

LEMMA 1 Let n be a nonnegative integer. Suppose that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. If $f \in A^p(\phi)$ and $z \in \mathbb{D}$, then

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{A^p(\phi)}}{\phi(|z|)(1 - |z|^2)^{(1/p)+n}}. \quad (6)$$

Proof For $a \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(a, r) = \{z \in \mathbb{D} : \beta(a, z) < r\}$ be the Bergman metric disk centered at a with radius r . Since $\phi(r)$ is normal, from [8] it is not difficult to check that for a fixed $r \in (0, 1)$ we have $\phi(|z|) \asymp \phi(|a|)$, $z \in \mathbb{D}(a, r)$. From [4], we see that

$$\|f\|_{A^p(\phi)} \asymp |f(0)| + \cdots + |f^{(n-1)}(0)| + \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p (1 - |z|^2)^{pn} \frac{\phi^p(|z|)}{1 - |z|} dA(z) \right)^{(1/p)}. \quad (7)$$

For $0 < r < 1$ and $z \in \mathbb{D}$, by using (7) and Lemmas 4.3.3 and 4.3.4 of [8], the subharmonicity of $|f^{(n)}(z)|^p$ yields

$$|f^{(n)}(z)|^p \leq \frac{C}{(1 - |z|^2)^2} \int_{\mathbb{D}(z, r)} |f^{(n)}(w)|^p dA(w)$$

$$\begin{aligned}
&\leq \frac{C}{(1 - |z|^2)^{pn+1} \phi^p(|z|)} \int_{\mathbb{D}} \frac{\phi^p(|w|)}{1 - |w|} (1 - |w|^2)^{pn} |f^{(n)}(w)|^p dA(w) \\
&\leq \frac{C \|f\|_{A^p(\phi)}^p}{(1 - |z|^2)^{pn+1} \phi^p(|z|)}.
\end{aligned}$$

From which we get (6). ■

The following criterion for the compactness follows by standard arguments similar to those outlined in Proposition 3.11 of [6].

LEMMA 2 *Let n be a nonnegative integer. Suppose that $0 < p < \infty$, $0 < \beta < \infty$, $u \in H(\mathbb{D})$, ϕ is normal on $[0, 1)$ and that φ is an analytic self-map of $\mathbb{D}$. The operator $D_{\varphi, u}^n(DM_u) : A^p(\phi) \rightarrow H_\beta^\infty$ is compact if and only if $D_{\varphi, u}^n(DM_u) : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $A^p(\phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D_{\varphi, u}^n f_k\|_{H_\beta^\infty} \rightarrow 0$ ($\|DM_u f_k\|_{H_\beta^\infty} \rightarrow 0$) as $k \rightarrow \infty$.*

Proof of Theorem 1 Assume (1). Then, for any $z \in \mathbb{D}$ and $f \in A^p(\phi)$, (6) implies

$$\begin{aligned}
(1 - |z|^2)^\beta |(D_{\varphi, u}^n f)(z)| &= (1 - |z|^2)^\beta |u(z)| |f^{(n)}(\varphi(z))| \\
&\leq C \frac{|u(z)| (1 - |z|^2)^\beta}{\phi(|\varphi(z)|) (1 - |\varphi(z)|^2)^{(1/p)+n}} \|f\|_{A^p(\phi)}.
\end{aligned} \tag{8}$$

Taking the supremum in (8) for z over $\mathbb{D}$, then employing condition (1), we see that $D_{\varphi, u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded.

Conversely, suppose that $D_{\varphi, u}^n : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded. Then it is clear that $u \in H_\beta^\infty$. For a fixed $w \in \mathbb{D}$, let

$$f_w(z) = \frac{(1 - |w|^2)^{t+1}}{\phi(|w|)(1 - \bar{w}z)^{(1/p)+t+1}}. \tag{9}$$

By [17, Lemma 1.4.10], $M_p(f_w, r) \leq C((1 - |w|^2)^{t+1})/(\phi(|w|)(1 - r|w|)^{t+1})$. Since ϕ is normal, from Lemma 6 of [5], we obtain that $f_w \in A^p(\phi)$ and $\sup_{w \in \mathbb{D}} \|f_w\|_{A^p(\phi)} < \infty$ (or see [4, 12]). For $\lambda \in \mathbb{D}$, we have

$$C \|D_{\varphi, u}^n\|_{A^p(\phi) \rightarrow H_\beta^\infty} \geq \|D_{\varphi, u}^n f_{\varphi(\lambda)}\|_{H_\beta^\infty} \geq \frac{C_{p,t,n} (1 - |\lambda|^2)^\beta |u(\lambda)| |\varphi(\lambda)|^n}{\phi(|\varphi(\lambda)|) (1 - |\varphi(\lambda)|^2)^{(1/p)+n}}, \tag{10}$$

where $C_{p,t,n} = ((1/p) + t + 1) \cdots ((1/p) + t + n)$, $C_{p,t,0} = 1$. From (10), we have

$$\begin{aligned}
\sup_{|\varphi(\lambda)| > (1/2)} \frac{(1 - |\lambda|^2)^\beta |u(\lambda)|}{\phi(|\varphi(\lambda)|) (1 - |\varphi(\lambda)|^2)^{(1/p)+n}} &\leq \sup_{|\varphi(\lambda)| > (1/2)} 2^n \frac{(1 - |\lambda|^2)^\beta |u(\lambda)| |\varphi(\lambda)|^n}{\phi(|\varphi(\lambda)|) (1 - |\varphi(\lambda)|^2)^{(1/p)+n}} \\
&\leq \sup_{|\varphi(\lambda)| > (1/2)} C \|D_{\varphi, u}^n\|_{A^p(\phi) \rightarrow H_\beta^\infty} < \infty.
\end{aligned} \tag{11}$$

Since ϕ is normal, $u \in H_\beta^\infty$ gives

$$\begin{aligned}
&\sup_{|\varphi(\lambda)| \leq (1/2)} \frac{(1 - |\lambda|^2)^\beta |u(\lambda)|}{\phi(|\varphi(\lambda)|) (1 - |\varphi(\lambda)|^2)^{(1/p)+n}} \\
&\leq \frac{4^{(1/p)+n}}{\phi((1/2)) 3^{(1/p)+n}} \sup_{|\varphi(\lambda)| \leq (1/2)} (1 - |\lambda|^2)^\beta |u(\lambda)| < \infty.
\end{aligned} \tag{12}$$

Combining (11), (12) implies (1). The proof is completed. ■

Proof of Theorem 2 Suppose that $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded. From the proof of Theorem 1, we see that $u \in H_{\beta}^{\infty}$. Assume that the condition (2) holds. For any $\varepsilon > 0$, there exist $\delta \in (0, 1)$, such that

$$\frac{(1 - |z|^2)^{\beta} |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} < \varepsilon$$

whenever $\delta < |\varphi(z)| < 1$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $A^p(\phi)$ such that $\sup_{k \in \mathbb{N}} \|f_k\|_{A^p(\phi)} < \infty$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. Let $K = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$. Then we have

$$\begin{aligned} \|D_{\varphi,u}^n f_k\|_{H_{\beta}^{\infty}} &= \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} |(D_{\varphi,u}^n f_k)(z)| \\ &\leq \sup_{z \in K} (1 - |z|^2)^{\beta} |u(z)| |f_k^{(n)}(\varphi(z))| + C \sup_{z \in \mathbb{D} \setminus K} \frac{(1 - |z|^2)^{\beta} |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{-(1/p)-n}} \|f_k\|_{A^p(\phi)} \\ &\leq C \sup_{z \in K} |f_k^{(n)}(\varphi(z))| + C\varepsilon \|f_k\|_{A^p(\phi)}. \end{aligned} \quad (13)$$

Assumption $f_k \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of $\mathbb{D}$ along with Cauchy's estimate gives that $f_k^{(n)} \rightarrow 0$ as $k \rightarrow \infty$ on compact subsets of $\mathbb{D}$. Letting $k \rightarrow \infty$ in (13), and using the fact that ε is an arbitrary positive number, we obtain $\lim_{k \rightarrow \infty} \|D_{\varphi,u}^n f_k\|_{H_{\beta}^{\infty}} = 0$. Applying Lemma 2 yields that $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact.

Now, suppose that $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact. So it is bounded. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \rightarrow 1^-$ as $k \rightarrow \infty$. Take

$$f_k(z) = \frac{(1 - |\varphi(z_k)|^2)^{t+1}}{\phi(|\varphi(z_k)|)(1 - \overline{\varphi(z_k)}z)^{(1/p)+t+1}}. \quad (14)$$

From the proof of Theorem 1 we see that $\sup_{k \in \mathbb{N}} \|f_k\|_{A^p(\phi)} < \infty$. Moreover, f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \rightarrow \infty$. Since $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is compact, by Lemma 2 we have $\lim_{k \rightarrow \infty} \|D_{\varphi,u}^n f_k\|_{H_{\beta}^{\infty}} = 0$. Similar to the proof of Theorem 1, we obtain

$$\|D_{\varphi,u}^n f_k\|_{H_{\beta}^{\infty}} \geq \frac{C_{p,t,n}(1 - |z_k|^2)^{\beta} |u(z_k)| |\varphi(z_k)|^n}{\phi(|\varphi(z_k)|)(1 - |\varphi(z_k)|^2)^{(1/p)+n}},$$

where $C_{p,t,n} = ((1/p) + t + 1) \cdots ((1/p) + t + n)$, $C_{p,t,0} = 1$. From the above inequality, the result follows. $\blacksquare$

Proof of Theorem 3 The necessity is obvious. Now we assume that $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$ is bounded and $u \in H_{\beta,0}^{\infty}$ holds. For each polynomial $p(z)$, we get

$$(1 - |z|^2)^{\beta} |(D_{\varphi,u}^n p)(z)| = (1 - |z|^2)^{\beta} |u(z)| |p^{(n)}(\varphi(z))|. \quad (15)$$

Since $\sup_{w \in \mathbb{D}} |p^{(n)}(w)| < \infty$, from (15) it follows that $D_{\varphi,u}^n p \in H_{\beta,0}^{\infty}$. From the set of all polynomials is dense in $A^p(\phi)$, we have that for every $f \in A^p(\phi)$ there is a sequence of polynomials $(p_k)_{k \in \mathbb{N}}$ such that $\|f - p_k\|_{A^p(\phi)} \rightarrow 0$ as $k \rightarrow \infty$. Hence

$$\|D_{\varphi,u}^n f - D_{\varphi,u}^n p_k\|_{H_{\beta}^{\infty}} \leq \|D_{\varphi,u}^n\|_{A^p(\phi) \rightarrow H_{\beta}^{\infty}} \|f - p_k\|_{A^p(\phi)} \longrightarrow 0$$

as $k \rightarrow \infty$, by using the boundedness of the operator $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta}^{\infty}$. Since $H_{\beta,0}^{\infty}$ is a closed subset of H_{β}^{∞} , we obtain $D_{\varphi,u}^n(A^p(\phi)) \subset H_{\beta,0}^{\infty}$. Therefore $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^{\infty}$ is bounded. The sufficiency is proved. $\blacksquare$

Proof of Theorem 4 From [3], we know that a closed set K in $H_{\beta,0}^\infty$ is compact if and only if it is bounded and satisfies $\lim_{|z| \rightarrow 1^-} \sup_{f \in K} (1 - |z|^2)^\beta |f(z)| = 0$. Assume (3). Let $f \in A^p(\phi)$. Since

$$(1 - |z|^2)^\beta |D_{\varphi,u}^n f(z)| \leq C \frac{(1 - |z|^2)^\beta |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} \|f\|_{A^p(\phi)}.$$

Taking the supremum in the above inequality over all $f \in A^p(\phi)$ such that $\|f\|_{A^p(\phi)} \leq 1$, then letting $|z| \rightarrow 1^-$, we obtain $\lim_{|z| \rightarrow 1^-} \sup_{\|f\|_{A^p(\phi)} \leq 1} (1 - |z|^2)^\beta |D_{\varphi,u}^n f(z)| = 0$. Hence the operator $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^\infty$ is compact.

Necessity. Assume that $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^\infty$ is compact. By using the boundedness of $D_{\varphi,u}^n : A^p(\phi) \rightarrow H_{\beta,0}^\infty$, we get $u \in H_{\beta,0}^\infty$. If $\|\varphi\|_\infty < 1$,

$$\begin{aligned} & \lim_{|z| \rightarrow 1^-} \frac{(1 - |z|^2)^\beta |u(z)|}{\phi(|\varphi(z)|)(1 - |\varphi(z)|^2)^{(1/p)+n}} \\ & \leq \frac{C}{\phi(\|\varphi\|_\infty)(1 - \|\varphi\|_\infty^2)^{(1/p)+n}} \lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\beta |u(z)| = 0, \end{aligned}$$

from which the result follows in this case.

If $\|\varphi\|_\infty = 1$, let $(\varphi(z_k))_{k \in \mathbb{N}}$ be a sequence such that $\lim_{k \rightarrow \infty} |\varphi(z_k)| = 1^-$. We can take a subsequence of $(\varphi(z_k))_{k \in \mathbb{N}}$ if necessary (we use the same notation $(\varphi(z_k))_{k \in \mathbb{N}}$). Taking the functions f_k defined in (14), by the proof of Theorem 2 we get (2). Equation (2) together with $u \in H_{\beta,0}^\infty$ imply (3). The proof is completed. ■

Proof of Theorem 5 Suppose that (4) holds. For any $z \in \mathbb{D}$ and $f \in A^p(\phi)$, by Lemma 1 we have

$$\begin{aligned} (1 - |z|^2)^\beta |(DM_u f)(z)| & \leq C \left(\frac{(1 - |z|^2)^{\beta-(1/p)} |u'(z)|}{\phi(|z|)} \right. \\ & \quad \left. + \frac{(1 - |z|^2)^{\beta-(1/p)-1} |u(z)|}{\phi(|z|)} \right) \|f\|_{A^p(\phi)}. \end{aligned} \quad (16)$$

Taking the supremum in (16) for z over $\mathbb{D}$, then using (4) we obtain that $DM_u : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded.

Conversely, suppose that $DM_u : A^p(\phi) \rightarrow H_\beta^\infty$ is bounded. Then, taking $f(z) = z$ and $f(z) = 1$ gives $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u(z) + u'(z)z| < \infty$ and $\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |u'(z)| < \infty$. Thus $u \in H_\beta^\infty$. For $w \in \mathbb{D}$, taking the functions f_w defined in (9), then we have

$$\begin{aligned} C \|DM_u\| & \geq \|DM_u f_w\|_{H_\beta^\infty} \geq \frac{(1 - |w|^2)^{\beta-(1/p)} |u'(w)|}{\phi(|w|)} \\ & \quad - \frac{((1/p) + t + 1)(1 - |w|^2)^{\beta-(1/p)-1} |u(w)\overline{w}|}{\phi(|w|)}. \end{aligned}$$

From which it follows that

$$\frac{(1 - |w|^2)^{\beta-(1/p)} |u'(w)|}{\phi(|w|)} \leq C \|DM_u\| + \frac{((1/p) + t + 1)(1 - |w|^2)^{\beta-(1/p)-1} |u(w)\overline{w}|}{\phi(|w|)}. \quad (17)$$

Further, for $w \in \mathbb{D}$, let

$$g_w(z) = \frac{(1 - |w|^2)^{t+2}}{\phi(|w|)(1 - \overline{w}z)^{(1/p)+t+2}} - \frac{(1 - |w|^2)^{t+1}}{\phi(|w|)(1 - \overline{w}z)^{(1/p)+t+1}}.$$

Then $\sup_{w \in \mathbb{D}} \|g_w\|_{A^p(\phi)} < \infty$, $g_w(w) = 0$ and $g'_w(w) = (\overline{w})/(\phi(|w|)(1 - |w|^2)^{(1/p)+1})$. Thus,

$$C \|DM_u\| \geq \|DM_u g_w\|_{H_\beta^\infty} \geq \frac{(1 - |w|^2)^{\beta-(1/p)-1} |u(w)\overline{w}|}{\phi(|w|)},$$

i.e. we have $\sup_{w \in \mathbb{D}} ((1 - |w|^2)^{\beta-(1/p)-1} |u(w)\overline{w}|)/(\phi(|w|)) < \infty$. Thus for a fixed δ , $0 < \delta < 1$,

$$\sup_{|w| > \delta} \frac{(1 - |w|^2)^{\beta-(1/p)-1} |u(w)|}{\phi(|w|)} < \infty. \quad (18)$$

For $w \in \mathbb{D}$ such that $|w| \leq \delta$, since ϕ is normal, from $u \in H_\beta^\infty$ we have

$$\sup_{|w| \leq \delta} \frac{(1 - |w|^2)^{\beta-(1/p)-1} |u(w)|}{\phi(|w|)} \leq \frac{C}{\phi(\delta)(1 - \delta^2)^{(1/p)+1}} \sup_{|w| \leq \delta} (1 - |w|^2)^\beta |u(w)| < \infty. \quad (19)$$

The first inequality in (4) follows from (17), the second one follows from (18) and (19). This completes the proof of the theorem. $\blacksquare$

Proof of Theorem 6 Note that $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\beta |u'(z)| = 0$ implies $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\beta |u(z)| = 0$, we get Theorem 6 from Theorem 5. Since the proof is similar to Theorem 3, we omit the details. $\blacksquare$

Proof of Theorem 7 (1) $\Rightarrow$ (3). Suppose that $DM_u : A^p(\phi) \rightarrow H_\beta^\infty$ is compact. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|z_n| \rightarrow 1^-$ as $n \rightarrow \infty$. Let $f_n(z) = ((1 - |z_n|^2)^{t+1})/(\phi(|z_n|)(1 - \overline{z_n}z)^{(1/p)+t+1})$, $n \in \mathbb{N}$. Then $\sup_{n \in \mathbb{N}} \|f_n\|_{A^p(\phi)} < \infty$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Since DM_u is compact, by Lemma 2 we have $\|DM_u f_n\|_{H_\beta^\infty} \rightarrow 0$ as $n \rightarrow \infty$. Thus

$$\|DM_u f_n\|_{H_\beta^\infty} \geq \frac{((1/p) + t + 1)(1 - |z_n|^2)^{\beta-(1/p)-1} |u(z_n)\overline{z_n}|}{\phi(|z_n|)} - \frac{(1 - |z_n|^2)^{\beta-(1/p)} |u'(z_n)|}{\phi(|z_n|)}.$$

We obtain

$$\lim_{|z_n| \rightarrow 1^-} \frac{((1/p) + t + 1)(1 - |z_n|^2)^{\beta-(1/p)-1} |u(z_n)\overline{z_n}|}{\phi(|z_n|)} = \lim_{|z_n| \rightarrow 1^-} \frac{(1 - |z_n|^2)^{\beta-(1/p)} |u'(z_n)|}{\phi(|z_n|)} \quad (20)$$

if one of these two limits exists.

Next, let $g_n(z) = ((1 - |z_n|^2)^{t+2})/(\phi(|z_n|)(1 - \overline{z_n}z)^{(1/p)+t+2}) - ((1 - |z_n|^2)^{t+1})/(\phi(|z_n|)(1 - \overline{z_n}z)^{(1/p)+t+1})$, $n \in \mathbb{N}$. Then $(g_n)_{n \in \mathbb{N}}$ is a bounded sequence in $A^p(\phi)$, $g_n \rightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \rightarrow \infty$. Hence by Lemma 2,

$$\frac{(1 - |z_n|^2)^{\beta-(1/p)-1} |u(z_n)\overline{z_n}|}{\phi(|z_n|)} \leq \|DM_u g_n\|_{H_\beta^\infty} \longrightarrow 0 \quad (21)$$

as $n \rightarrow \infty$. From (20) and (21), we get the desired result.

(3) $\Rightarrow$ (2). From Lemma 1 we have

$$\begin{aligned} (1 - |z|^2)^\beta |(DM_u f)(z)| &\leq C \left(\frac{(1 - |z|^2)^{\beta-(1/p)} |u'(z)|}{\phi(|z|)} \right. \\ &\quad \left. + \frac{(1 - |z|^2)^{\beta-(1/p)-1} |u(z)|}{\phi(|z|)} \right) \|f\|_{A^p(\phi)}. \end{aligned}$$

Similar to the proof of Theorem 4, the condition (5) implies that $DM_u : A^p(\phi) \rightarrow H_{\beta,0}^\infty$ is compact.

(2) $\Rightarrow$ (1) is obvious. This completes the proof of Theorem 7. ■

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Hermite expansions and products of tempered distributions

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The space of tempered distributions $\mathcal{S}'$ can be realized as a sequence spaces by means of the Hermite representation theorem. In this article, we introduce and study two new products of tempered distributions based on this Hermite representation theorem. In particular, we obtain the products $[H]\delta = \delta/2$, $[\delta]vp(1/x) = -\delta'$ and $[\delta^{(r)}]vp(1/x) = -(\delta^{(r+1)})/(r+1)$ for even r .

Keywords: Tempered distributions; Product of distributions; Hermite functions; Hermite expansion of tempered distributions

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1. Introduction

Owing to the large employment of Schwartz distributions in the natural science and other mathematical fields, where products of distributions often appear, the problem of extending the ordinary multiplication of functions to distributions has been the objective of many studies (see [1] and references therein). There is, however, no canonical way to define such products. A possible approach to define a product of a pair of distributions is to approximate one of them by smooth functions, multiply this approximation by the other distribution and pass to a limit. In the case of the sequential approach [2, p. 242], the approximation is done by convolution with δ -sequences. In this work, we propose to take the approximation given by the Fourier–Hermite expansion of the distribution.

More precisely, by multiplying tempered distributions our methods are based on the Hermite representation theorem for $\mathcal{S}'$ [2, p. 182; 3, 4, example 7, p. 260; 5, p. 143], which establishes

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that every $S \in \mathcal{S}'$ can be represented by a Hermite series

$$S = \sum_{n=0}^{\infty} b_n h_n, \quad (1)$$

where $\{h_n\}$ are the Hermite functions in $\mathbb{R}$, $b_n = \langle S, h_n \rangle$ and the equality is in the weak sense.

In this context, we say that there exists the product $[S]T$ of the tempered distributions S and T if $\sum_{k=0}^{\infty} c_k h_k \in \mathcal{S}'$, where the coefficients c_k are given by

$$c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m b_n \langle T, h_n h_k \rangle. \quad (2)$$

The product $[S]T$ of S and T is, by definition, $\sum_{k=0}^{\infty} c_k h_k$. Symmetrically, we define the product $S[T]$. These products are not commutative, but it is clear that $S[T] = [T]S$. Moreover, these products are extensions of the well-known product of the spaces $\mathcal{O}_M$ of multipliers and $\mathcal{S}'$. We show several examples of pairs of distributions for which the defined product exists and prove the following formulae: $[H]\delta = \delta/2$, $[\delta]vp(1/x) = -\delta'$ and $[\delta^{(r)}]vp(1/x) = -(\delta^{(r+1)})/(r+1)$ for even r .

The plan of exposition is as follows: In section 2, we have compiled some basic facts on Hermite functions and Hermite expansions of tempered distributions. For the convenience of the reader, we enunciate without proof the relevant material, thus making our exposition self-contained. We also calculate the Hermite expansions of some important distributions. In section 3, we introduce the definitions of the products of tempered distributions and study its properties. In section 4, we calculate some products to illustrate how to use this method and its advantages. Finally, in the appendix, we prove two technical formulas.

2. Preliminaries

Throughout this article, we shall use freely concepts and notations of Antosik *et al.* [2], Schwartz [3, 4] and Zemanian [6, 7].

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the space of infinitely differentiable functions such that together with all its derivatives are of rapid decrease.

For each $m \in \mathbb{N} \cup \{0\}$, we consider $\|\cdot\|_m$ the norm of $\mathcal{S}$ given by

$$\|\varphi\|_m = \left(\int_{-\infty}^{\infty} \frac{1}{2^m} \left| \left(-\frac{d^2}{dx^2} + x^2 + 1 \right)^m \varphi(x) \right|^2 dx \right)^{1/2}.$$

We observe that $\mathcal{S}$ endowed with this family of norms is a sequentially complete locally convex space, and its dual space $\mathcal{S}'$ is the space of tempered distributions.

The *Hermite polynomials* $H_n(x)$ are defined by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \quad (3)$$

for $n \in \mathbb{N} \cup \{0\}$ or equivalently

$$H_n(x) = 2^{-n/2} \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (\sqrt{2}x)^{n-2k}}{k!(n-2k)!}. \quad (4)$$

The *Hermite functions* $h_n(x)$ are defined by

$$h_n(x) = (\sqrt{2\pi}n!)^{-1/2} e^{-(1/4)x^2} H_n(x) \quad (5)$$

for $n \in \mathbb{N} \cup \{0\}$. We enumerate some properties of the Hermite functions that will be mentioned later:

- (i) $h_n \in \mathcal{S}$ for all $n \in \mathbb{N} \cup \{0\}$;
- (ii) h_n is an even (odd) function if n is even (odd);
- (iii) $\sqrt{n+1}h_{n+1}(x) + 2h'_n(x) = \sqrt{n}h_{n-1}(x)$ for all $n \in \mathbb{N} \cup \{0\}$;
- (iv) $\sqrt{n+1}h_{n+1}(x) = xh_n(x) - \sqrt{n}h_{n-1}(x)$ for all $n \in \mathbb{N} \cup \{0\}$;
- (v) $\{h_n : n \in \mathbb{N} \cup \{0\}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

The Hermite representation theorem for $\mathcal{S}$ establishes a topological isomorphism from $\mathcal{S}$ onto the space of sequences $\mathbf{s}$, given by

$$\mathbf{s} = \left\{ (a_n) \in \ell^2 : \sum_{n=0}^{\infty} (n+1)^{2m} |a_n|^2 < \infty, \text{ for all } m \in \mathbb{N} \cup \{0\} \right\}.$$

Then $\mathbf{s}$ is a locally convex space with the following sequences of norms

$$|(a_n)|_m = \left(\sum_{n=0}^{\infty} (n+1)^{2m} |a_n|^2 \right)^{1/2}.$$

THEOREM 2.1 *Let $\mathbf{h} : \mathcal{S} \rightarrow \mathbf{s}$ be the mapping $\mathbf{h}(\varphi) = (\langle \varphi, h_n \rangle)$, where $\langle \varphi, h_n \rangle = \int_{-\infty}^{\infty} \varphi(x)h_n(x) dx$ for $n \in \mathbb{N} \cup \{0\}$. Then $\mathbf{h}$ is a topological isomorphism.*

Let us denote by $\mathbf{s}'$ the set

$$\{(b_n) : \text{for some } (C, m) \in \mathbb{R} \times \mathbb{N}, |b_n| \leq C(n+1)^m \text{ for all } n\}.$$

It is clear that $\mathbf{s}'$ is the (topological) dual space of $\mathbf{s}$.

THEOREM 2.2 *Let $\mathbf{H} : \mathcal{S}' \rightarrow \mathbf{s}'$ be the mapping $\mathbf{H}(T) = (\langle T, h_n \rangle)$. Then $\mathbf{H}$ is a topological isomorphism. Moreover, if $T \in \mathcal{S}'$, we have that*

$$T = \sum_{n=0}^{\infty} \langle T, h_n \rangle h_n$$

in the weak sense, and for all $\varphi \in \mathcal{S}$,

$$\langle T, \varphi \rangle = \langle \mathbf{H}(T), \mathbf{h}(\varphi) \rangle = \sum_{n=0}^{\infty} \langle T, h_n \rangle \langle \varphi, h_n \rangle.$$

We say that $\mathbf{H}(T)$ is the sequence of the *Hermite coefficients* of the tempered distribution T .

Now, we calculate the Hermite coefficients for the product $\varphi \cdot T$ with respect to the Hermite coefficients of $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}'$.

PROPOSITION 2.3 *Let $\varphi \in \mathcal{S}$ and $T \in \mathcal{S}'$ be such that $\mathbf{h}(\varphi) = (a_n)$ and $\mathbf{H}(T) = (b_m)$. Then $\mathbf{H}(\varphi \cdot T) = (c_k)$, where c_k are given by*

$$c_k = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x) dx. \quad (6)$$

Proof Let us first recall that $\langle \varphi \cdot T, \phi \rangle = \langle T, \varphi \phi \rangle$ for $\phi \in \mathcal{S}$. Let $\varphi = \sum_{n=0}^{\infty} a_n h_n$ and $T = \sum_{n=0}^{\infty} b_n h_n$ be the Hermite expansions of φ in $\mathcal{S}$ and T in $\mathcal{S}'$, respectively. Then,

$$\langle \varphi T, h_k \rangle = \langle T, \varphi h_k \rangle = \left\langle T, \sum_{n=0}^{\infty} a_n h_n h_k \right\rangle = \sum_{n=0}^{\infty} a_n \langle T, h_n h_k \rangle.$$

Since $h_n h_k = \sum_{m=0}^{\infty} \left(\int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x) dx \right) h_m$, we have

$$\langle \varphi T, h_k \rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \int_{-\infty}^{\infty} h_m(x) h_n(x) h_k(x) dx. \quad \blacksquare$$

Now, we calculate the Hermite coefficients of some important tempered distributions [2, 8–10].

(1) *The delta distribution* [2, p. 191]

$$\langle \delta, h_n \rangle = h_n(0) = \begin{cases} \frac{(-1)^{n/2}}{\sqrt[4]{2\pi}} \sqrt{\frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n}} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (7)$$

(2) *The constant distribution* **1** [2, p. 190]

$$\langle 1, h_n \rangle = \int_{-\infty}^{\infty} h_n(x) dx = \begin{cases} \sqrt[4]{8\pi} \sqrt{\frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n}} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \quad (8)$$

(3) *The x_+^p distribution* [10, p. 162]. We recall that $\langle x_+^p, \phi \rangle = \int_0^{\infty} x^p \phi(x) dx$.

$$\langle x_+^p, h_n \rangle = \begin{cases} (\sqrt{2\pi} n!)^{-1/2} 2^p \Gamma\left(\frac{p+1}{2}\right) W_n(2p+1) & \text{for } n \text{ even,} \\ (\sqrt{2\pi} n!)^{-1/2} 2^{p+1} \Gamma\left(\frac{p+2}{2}\right) W_n(2p+1) & \text{for } n \text{ odd,} \end{cases} \quad (9)$$

where $W_n(x)$ are polynomials such that $W_0(x) = W_1(x) = 1$ and

$$W_{n+2}(x) = x W_n(x) + n(n-1) W_{n-2}(x).$$

Note that if $p = 0$, then x_+^p is the Heaviside distribution H .

(4) *The distribution principal value of $1/x$* [2, p. 193]. A straightforward computation from the recursive expression for the Hermite coefficients of $1/x$ [2, p. 193] yields

$$\left\langle \nu p \left(\frac{1}{x} \right), h_n \right\rangle = \begin{cases} 0 & \text{for } n \text{ even,} \\ 2 \frac{\sqrt{\pi}}{\sqrt{n} h_{n-1}(0)} \left(\sum_{i \text{ even}}^{n-1} (-1)^{i/2} h_i^2(0) \right) & \text{for } n \text{ odd.} \end{cases} \quad (10)$$

(5) *The δ' distribution*

$$\langle \delta', h_n \rangle = -h'_n(0) = \sqrt{n}h_{n-1}(0). \quad (11)$$

(6) *The δ^+ distribution.* We recall that $\delta^+ = vp(1/x) - i\pi\delta$. Applying equations (7) and (10), we have

$$\langle \delta^+, h_n \rangle = \begin{cases} (-i\pi) \frac{(-1)^{n/2}}{\sqrt[4]{2\pi}} \sqrt{\frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n}} & \text{for } n \text{ even,} \\ 2 \frac{\sqrt{\pi}}{\sqrt{n}h_{n-1}(0)} \left(\sum_{i \text{ even}}^{n-1} (-1)^{i/2} h_i^2(0) \right) & \text{for } n \text{ odd.} \end{cases} \quad (12)$$

(7) *The sgn distribution* [2, p. 194]. Some easy algebraic manipulations provide

$$\langle \text{sgn}, h_n \rangle = \begin{cases} 0 & \text{for } n \text{ even,} \\ \frac{4(-1)^{(n-1)/2}}{\sqrt{n}\sqrt{\pi}h_{n-1}(0)} \left(\sum_{i \text{ even}}^{n-1} (-1)^{i/2} h_i^2(0) \right) & \text{for } n \text{ odd.} \end{cases} \quad (13)$$

3. Hermite products for tempered distributions

Let S and T be in $\mathcal{S}'$ with $\mathbf{H}(S) = (b_n)$ and $\mathbf{H}(T) = (e_m)$. The Hermite representation theorem for $\mathcal{S}'$ ensures that the following definitions are well posed.

Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$c_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m b_n \langle T, h_n h_k \rangle$$

and that $(c_k) \in \mathcal{S}'$. We define the *left Hermite product* $[S] \cdot T \in \mathcal{S}'$ by

$$[S] \cdot T = \sum_{k=0}^{\infty} c_k h_k. \quad (14)$$

Suppose that for all $k \in \mathbb{N} \cup \{0\}$ there exists

$$d_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m e_n \langle S, h_n h_k \rangle$$

and that $(d_k) \in \mathcal{S}'$. We define the *right Hermite product* $S \cdot [T] \in \mathcal{S}'$ by

$$S \cdot [T] = \sum_{k=0}^{\infty} d_k h_k. \quad (15)$$

We observe that $[S] \cdot T = T \cdot [S]$.

Remark We have

$$c_k = \lim_{m \rightarrow \infty} \sum_{i=0}^m \sum_{n=0}^{\infty} b_i e_n C(n, i, k)$$

and

$$d_k = \lim_{m \rightarrow \infty} \sum_{n=0}^m \sum_{i=0}^{\infty} e_n b_i C(n, i, k),$$

where $C(n, i, k) = \int_{-\infty}^{\infty} h_n(x) h_i(x) h_k(x) dx$.

Here, and subsequently, S_m and T_m denote the partial sums $\sum_{n=0}^m b_n h_n$ and $\sum_{i=0}^m e_n h_n$, respectively.

PROPOSITION 3.1 *The products $[S]T$ and $S[T]$ are not commutative and are not partially associative.*

Proof Let us show that $[\delta']\delta \neq \delta'[\delta]$. In fact, from $\langle \delta, h_n h_k \rangle = 0$ for odd n and $\langle \delta', h_n \rangle = 0$ for even n , we have

$$\langle [\delta']\delta, h_k \rangle = \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta', h_n \rangle \langle \delta, h_n h_k \rangle = 0.$$

On the other hand,

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta, h_n \rangle \langle \delta', h_n \phi_k \rangle &= \lim_{m \rightarrow \infty} \sum_{\text{even } n}^m -h_n(0) \phi_n(0) h'_k(0) \\ &= -h'_k(0) \lim_{m \rightarrow \infty} \sum_{\text{even } n}^m \pi^{-1/2} \frac{1}{2} \cdots \frac{n-1}{n} \\ &= \pm \infty. \end{aligned}$$

We conclude that $[\delta']\delta = 0$ and $[\delta]\delta'$ does not exist; consequently, the Hermite products are not commutative.

We claim that $[\varphi]([\delta']\delta) \neq [\varphi\delta']\delta$ for $\varphi \in \mathcal{S}$ such that $\varphi'(0) \neq 0$. In fact, we have that $[\varphi\delta']\delta$ does not exist, because for even k

$$\begin{aligned} \sum_{n=0}^{\infty} \langle \varphi\delta', h_n \rangle \langle \delta, h_n h_k \rangle &= \sum_{\text{even } n}^{\infty} \langle \delta', \varphi h_n \rangle \langle \delta, h_n h_k \rangle \\ &= -\varphi'(0) h_k(0) \sum_{\text{even } n}^m \frac{1}{\sqrt{\pi}} \frac{1}{2} \cdots \frac{n-1}{n} \\ &= \pm \infty. \end{aligned}$$

On other hand, $\varphi([\delta']\delta) = \varphi 0 = 0$. This proves that the Hermite products are not partially associative. ■

The Hermite products satisfy the Leibnitz rule.

PROPOSITION 3.2 *Let S and T be in $\mathcal{S}'$. Then,*

$$([S]T)' = [S']T + [S]T'$$

and

$$(S[T])' = S'[T] + S[T'].$$

Proof Let $k \in \mathbb{N} \cup \{0\}$. We have

$$\begin{aligned} \langle ([S]T)', h_k \rangle &= -\langle [S] \cdot T, h'_k \rangle \\ &= \lim_{m \rightarrow \infty} \langle S_m T, -h'_k \rangle \\ &= \lim_{m \rightarrow \infty} \langle (S_m T)', h_k \rangle \\ &= \lim_{m \rightarrow \infty} \langle S'_m T + S_m T', h_k \rangle \\ &= \langle [S']T + [S]T', h_k \rangle. \end{aligned}$$

Let $\mathcal{O}_M$ denote the set of multipliers of the space $\mathcal{S}'$. The Hermite products $[S]T$ and $S[T]$ extend the product of elements of $\mathcal{O}_M$ and $\mathcal{S}'$.

PROPOSITION 3.3 *Let $T \in \mathcal{S}'$ and $f \in \mathcal{O}_M$. Then, $[T]f = [f] \cdot T = fT$.*

Proof Let $k \in \mathbb{N} \cup \{0\}$. We have

$$\begin{aligned} \langle f[T], h_k \rangle &= \lim_{m \rightarrow \infty} \langle fT_m, h_k \rangle \\ &= \lim_{m \rightarrow \infty} \langle T_m, fh_k \rangle \\ &= \langle T, fh_k \rangle \\ &= \langle fT, h_k \rangle, \end{aligned}$$

and Theorem 2.2 shows that $f[T] = fT$.

By the regularity theorem for $\mathcal{S}'$, there exists an $r \in \mathbb{N} \cup \{0\}$ and $g \in C(\mathbb{R})$ polynomially bounded such that $T = g^{(r)}$. Then,

$$\begin{aligned} \langle [f]T, h_k \rangle &= \lim_{m \rightarrow \infty} \langle f_m T, h_k \rangle \\ &= \lim_{m \rightarrow \infty} (-1)^r \int_{-\infty}^{\infty} g(x) (f_m h_k)^{(r)}(x) dx \\ &= \lim_{m \rightarrow \infty} (-1)^r \sum_{i=0}^r \binom{r}{i} \int_{-\infty}^{\infty} g(x) f_m^{(i)}(x) h_k^{(r-i)}(x) dx, \end{aligned}$$

where $f_m = \sum_{k=0}^m \langle f, h_k \rangle h_k$.

Let us observe that $\mathcal{S}$ is dense in $\mathcal{M} = \{h \in C(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^\alpha h(x)| < \infty \text{ for all } \alpha \in \mathbb{N} \cup \{0\}\}$ with the topology induced by the semi-norms $\|h\|_\alpha = \sup_{x \in \mathbb{R}} |x^\alpha h(x)|$. By this density, we have that for all $\varphi \in \mathcal{M}$,

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} f_m^{(i)}(x) \varphi(x) dx = \int_{-\infty}^{\infty} f^{(i)}(x) \varphi(x) dx.$$

Since $g\phi_k^{(r-i)} \in \mathcal{M}$, it follows that

$$\lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} g(x) f_m^{(i)}(x) h_k^{(r-i)}(x) dx = \int_{-\infty}^{\infty} g(x) f^{(i)}(x) h_k^{(r-i)}(x) dx.$$

Combining the above equalities, we obtain that

$$\begin{aligned} \langle [f]T, h_k \rangle &= (-1)^r \sum_{i=0}^r \binom{r}{i} \int_{-\infty}^{\infty} g(x) f^{(i)}(x) h_k^{(r-i)}(x) dx \\ &= \langle T, fh_k \rangle \\ &= \langle fT, h_k \rangle, \end{aligned}$$

and the result follows from Theorem 2.2. ■

4. Some examples of Hermite products

- (1) $[H] \cdot \delta = \delta \cdot [H] = (1/2)\delta$.

From the parity of the Hermite functions and Proposition 3.3, we have

$$\begin{aligned} \langle [H] \cdot \delta, \phi_k \rangle &= \lim_{m \rightarrow \infty} \sum_{n \text{ even}}^m \langle H, h_n \rangle \langle \delta, h_n h_k \rangle \\ &= \sum_{n \text{ even}}^m \frac{1}{2} \langle 1, h_n \rangle \langle \delta, h_n h_k \rangle \\ &= \frac{1}{2} \langle [1]\delta, h_k \rangle \\ &= \frac{1}{2} \langle \delta, h_k \rangle. \end{aligned}$$

- (2) The products $[\delta] \cdot \delta$ and $\delta \cdot [\delta]$ do not exist. For k even, from formula (7), we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta, h_n \rangle \langle \delta, h_n h_k \rangle &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta, h_n \rangle^2 \langle \delta, h_k \rangle \\ &= h_k(0) \lim_{m \rightarrow \infty} \sum_{n \text{ even}}^m \frac{1}{\sqrt{2\pi}} \frac{1}{2} \frac{3}{4} \cdots \frac{n-1}{n} \\ &= \pm\infty. \end{aligned}$$

This proves that $[\delta] \cdot \delta$ does not exist.

- (3) $[\delta] \cdot vp(1/x) = vp(1/x) \cdot [\delta] = -\delta' [11]$. It suffices to show that $\langle [\delta] \cdot vp(1/x), h_k \rangle = \langle \delta', h_k \rangle$ for k odd. In fact, if k is even, we have that $\langle -\delta', h_k \rangle = 0$ and

$$\left\langle [\delta] \cdot vp\left(\frac{1}{x}\right), h_k \right\rangle = \lim_{m \rightarrow \infty} \sum_{n \text{ even}}^m \langle \delta, h_n \rangle \left\langle vp\left(\frac{1}{x}\right), h_n h_k \right\rangle = 0$$

because $h_n h_k$ is even.

Let us observe the following formula (proof in the appendix) for k odd,

$$\left\langle vp\left(\frac{1}{x}\right), h_n h_k \right\rangle = \begin{cases} \frac{h_n(0)}{\sqrt{k} h_{k-1}(0)} & \text{for } n > k, \\ 0 & \text{for } n \leq k. \end{cases} \quad (16)$$

Else, from formula (16) and $\sum_{n \text{ even}}^{k-1} h_n^2(0) = k h_{k-1}^2(0)$, it follows that

$$\begin{aligned} \left\langle [\delta] \cdot vp\left(\frac{1}{x}\right), h_k \right\rangle &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta, h_n \rangle \left\langle vp\left(\frac{1}{x}\right), h_n h_k \right\rangle \\ &= \frac{1}{\sqrt{k} h_{k-1}(0)} \sum_{n \text{ even}}^{k-1} h_n^2(0) \\ &= \frac{1}{\sqrt{k} h_{k-1}(0)} k h_{k-1}^2(0) \\ &= \langle -\delta', h_k \rangle. \end{aligned}$$

- (4) Let $T \in \mathcal{S}'$ such that for even n , $\langle T, h_n \rangle = 0$. Then, $[T] \cdot \delta = \delta \cdot [T] = 0$.

In fact,

$$\langle [T] \cdot \delta, h_k \rangle = \lim_{m \rightarrow \infty} \sum_{\text{odd } n}^m \langle T, h_n \rangle \langle \delta, h_n h_k \rangle = 0$$

because $\langle \delta, h_n h_k \rangle = h_n(0) h_k(0) = 0$ for odd n .

- (5) $[\delta^{(r)}] \cdot vp(1/x) = -(\delta^{(r+1)})/(r+1)$, for r even.

It suffices to show that $\langle [\delta^{(r)}] \cdot vp(1/x), h_k \rangle = \langle -(\delta^{(r+1)})/(r+1), h_k \rangle$ for odd k . In fact, if k is even, we have $\langle -(\delta^{(r+1)})/(r+1), h_k \rangle = 0$ and

$$\left\langle [\delta^{(r)}] \cdot vp\left(\frac{1}{x}\right), \phi_k \right\rangle = \lim_{m \rightarrow \infty} \sum_{n \text{ even}}^m h_n^{(r)}(0) \left\langle vp\left(\frac{1}{x}\right), h_n h_k \right\rangle = 0$$

because $h_n h_k$ is even.

For k odd, from formula (16) it follows that

$$\begin{aligned} \left\langle [\delta^{(r)}] \cdot vp\left(\frac{1}{x}\right), \phi_k \right\rangle &= \lim_{m \rightarrow \infty} \sum_{n=0}^m \langle \delta^{(r)}, h_n \rangle \left\langle vp\left(\frac{1}{x}\right), h_n h_k \right\rangle \\ &= \sum_{n=0}^{k-1} h_n^{(r)}(0) \frac{h_n(0)}{\sqrt{k} h_{k-1}(0)}. \end{aligned}$$

Applying the formula (proof in the appendix)

$$\sum_{n=0}^{k-1} h_n^{(r)}(0) h_n(0) = \frac{1}{r+1} \sqrt{k} h_{k-1}(0) h_k^{(r+1)}(0), \quad (17)$$

we obtain

$$\begin{aligned} \left\langle [\delta^{(r)}] \cdot vp\left(\frac{1}{x}\right), \phi_k \right\rangle &= \frac{1}{r+1} h_k^{(r+1)}(0) \\ &= \left\langle -\frac{\delta^{(r+1)}}{r+1}, h_k \right\rangle. \end{aligned}$$

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Appendix A

A.1 Proof of formula (16)

From the recurrence relations for Hermite functions, we have

$$h_k(x) = \frac{x h_{k-1}(x)}{\sqrt{k}} - \frac{\sqrt{k-1}}{\sqrt{k}} h_{k-2}(x).$$

By the above formula and the orthogonality of the Hermite functions, we have

$$\begin{aligned} \left\langle vp \left(\frac{1}{x} \right), h_n h_k \right\rangle &= \int_{-\infty}^{\infty} \frac{1}{x} h_n(x) h_{k-1}(x) dx \\ &= \frac{1}{\sqrt{k}} \int_{-\infty}^{\infty} h_n(x) h_{k-1}(x) dx - \frac{\sqrt{k-1}}{\sqrt{k}} \int_{-\infty}^{\infty} \frac{1}{x} h_n(x) h_{k-2}(x) dx \\ &= \frac{1}{\sqrt{k}} \delta_{n,k-1} - \frac{\sqrt{k-1}}{\sqrt{k}} \left\langle vp \left(\frac{1}{x} \right), h_n h_{k-2} \right\rangle. \end{aligned}$$

Combining the above recurrence relation and formula (7), it is easy to check that for k odd

$$\left\langle vp \left(\frac{1}{x} \right), h_n h_k \right\rangle = \begin{cases} \frac{h_n(0)}{\sqrt{k} h_{k-1}(0)} & \text{for } n > k, \\ 0 & \text{for } n \leq k. \end{cases}$$

Appendix B

B.1 Proof of formula (17)

From the recurrence relations for Hermite functions, we have

$$\begin{aligned} \frac{\sqrt{k}}{x-y} (h_k(x)h_{k-1}(y) - h_k(y)h_{k-1}(x)) &= h_{k-1}(x)h_{k-1}(y) \\ &+ \sqrt{k-1} (h_{k-1}(x)h_{k-2}(y) - h_{k-1}(y)h_{k-2}(x)). \end{aligned}$$

Applying this formula k times yields

$$\sum_{n=0}^{k-1} h_n(x)h_n(y) = \frac{\sqrt{k}}{x-y} (h_k(x)h_{k-1}(y) - h_k(y)h_{k-1}(x)).$$

Taking $y = 0$ in the above formula, we obtain

$$\sum_{n=0}^{k-1} h_n(x)h_n(0) = \frac{\sqrt{k}}{x} h_k(x)h_{k-1}(0). \quad (\text{B1})$$

Our next claim is that

$$\lim_{x \rightarrow 0} \frac{d^r}{dx^r} \frac{h_k(x)}{x} = \frac{h_k^{(r+1)}(0)}{r+1}. \quad (\text{B2})$$

In fact,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{d^r}{dx^r} \frac{h_k}{x}(x) &= \lim_{x \rightarrow 0} \sum_{i=0}^r \binom{r}{i} h_k^{(r-i)}(x) \left(\frac{1}{x}\right)^{(i)}(x) \\ &= \sum_{i=0}^r \binom{r}{i} \lim_{x \rightarrow 0} h_k^{(r-i)}(x) \frac{(-1)^i i!}{x^{i+1}}. \end{aligned}$$

Applying $i + 1$ times the L'Hospital rule to each term of the right side, we obtain (B2).

Finally, differentiating r times identity (B1) and making use of (B2), we get

$$\sum_{n=0}^{k-1} h_n^{(r)}(0)h_n(0) = \frac{1}{r+1} \sqrt{k} h_{k-1}(0) h_k^{(r+1)}(0).$$

A note on Bernoulli identities associated with the Weierstrass $\wp$ -function

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Various types of Bernoulli identities have been discussed by Euler, Ramanujan, Rademacher, Eie, and others. In particular, by using certain identities involving zeta functions, Eie constructed several Bernoulli identities systematically. There are many expressions for the Bernoulli number B_{2k} , which express B_{2k} as a sum of the products of lower-order Bernoulli numbers in two terms or in three terms. In this article, we show that there is no unique method to calculate the Bernoulli number B_{2k} as a sum of the products of lower-order Bernoulli numbers in three terms. By employing a technique which is substantially different from the method of Eie's proof, we derive several Bernoulli identities through the properties of the Weierstrass $\wp$ -function. We also discuss the relationship between these and other known Bernoulli identities.

Keywords: Bernoulli numbers; Weierstrass $\wp$ -function; Euler's identity; Zeta function; Eisenstein series; Bernoulli identities

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1. Introduction

For the familiar Bernoulli numbers

$$B_k \ (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \quad \mathbb{N} := \{1, 2, 3, \dots\})$$

defined by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \quad (|z| < 2\pi), \quad (1)$$

it is well known that

$$B_0 = 1, B_1 = -\frac{1}{2}, \quad \text{and} \quad B_{2k+1} = 0 \quad (k \in \mathbb{N}). \quad (2)$$

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Furthermore, the first eight Bernoulli numbers of even orders are given by

$$\begin{aligned} B_2 &= \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, \\ B_{12} &= -\frac{691}{2730}, B_{14} = \frac{6}{7}, \quad \text{and} \quad B_{16} = -\frac{3617}{510}. \end{aligned} \quad (3)$$

Various identities involving the Bernoulli numbers B_k as well as the Bernoulli polynomials $B_k(x)$ defined by

$$\left(\frac{z}{e^z - 1} \right) \cdot e^{xz} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!} \quad (|z| < 2\pi), \quad (4)$$

so that, obviously,

$$B_k(0) =: B_k \quad (k \in \mathbb{N}_0), \quad (5)$$

have been discussed by (among others) Euler and Ramanujan (see, for example, Berndt [1, 2]), Rademacher [3], Eie *et al.* ([4] and [5]) (see also the recent works [6] and [7]). In this article we focus on the following family of Bernoulli identities.

Euler's identity:

$$B_{2k} = -\frac{1}{2k+1} \sum_{m=1}^{k-1} \frac{(2k)!}{(2m)!(2k-2m)!} B_{2m} B_{2k-2m} \quad (k \in \mathbb{N} \setminus \{1\}). \quad (6)$$

Rademacher's identity:

$$\begin{aligned} \frac{B_{2k}}{k} &= -3 \frac{(k-1)(2k-3)}{(2k+1)(k-3)} \\ &\cdot \sum_{m=2}^{k-2} \frac{(2k-4)!}{(2m-2)!(2k-2m-2)!} \frac{B_{2m}}{m} \frac{B_{2k-2m}}{k-m} \quad (k \in \mathbb{N} \setminus \{1, 2, 3\}). \end{aligned} \quad (7)$$

In equations (6) and (7), we say that the Bernoulli numbers B_{2k} are expressed by means of sums of the products of lower-order Bernoulli numbers in two terms. Rademacher's identity (7) is based upon the properties of the Weierstrass $\wp$ -function. More recently, by the use of certain identities among zeta functions, Eie *et al.* ([4], [5]) showed how to construct the Bernoulli identities systematically. They also extended the expression of B_{2k} to a sum of the products of lower-order Bernoulli numbers in three terms (see, for details, [4] and [5]). Thus the expressions for the Bernoulli numbers B_{2k} are getting more involved. In our present investigation, we make use of a technique based upon the various properties of the Weierstrass $\wp$ -function in order to prove the following expressions for the Bernoulli numbers B_{2k} in three terms:

$$\begin{aligned} &\frac{B_{2k}}{2k} \left(\frac{1}{5!} - \frac{2k^2 - 5k}{(2k-2)(2k-3)(2k-4)(2k-5)} \right) - \frac{1}{80} \left(\frac{B_{2k-4}}{2k-4} \right) \\ &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-6)!}{(2m-2)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \quad (k \in \mathbb{N} \setminus \{1, \dots, 5\}) \end{aligned} \quad (8)$$

and

$$\begin{aligned} &\frac{B_{2k}}{2k} \left(\frac{1}{5!} - \frac{2k^2 - 5k}{(2k-2)(2k-3)(2k-4)(2k-5)} \right) - \frac{1}{80} \left(\frac{B_{2k-4}}{2k-4} \right) \\ &= 3 \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \quad (k \in \mathbb{N} \setminus \{1, \dots, 5\}). \end{aligned} \quad (9)$$

Our technique for proving the Bernoulli identities (8) and (9) is remarkably different from Eie's method based upon certain zeta-function identities.

2. Preliminary results involving the Weierstrass $\wp$ -function and the Eisenstein series

For the purpose of proving our main Bernoulli identities (8) and (9), we first derive some properties of the Weierstrass $\wp$ -function and the Eisenstein series (see, for details, [3]).

Let

$$\Omega := \left\{ m_1 z_1 + m_2 z_2 : z_1, z_2 \in \mathbb{C} \setminus \{0\}, \frac{z_1}{z_2} \notin R, \text{ and } m_1, m_2 \in \mathbb{Z} \right\} \quad (10)$$

be a lattice in $\mathbb{C}$. Suppose also that

$$G_r(z_1, z_2) = \sum_{\Omega^*} \frac{1}{(m_1 z_1 + m_2 z_2)^r} \quad (r \in \mathbb{N} \setminus \{1, 2\}; \Omega^* := \Omega \setminus \{0\}). \quad (11)$$

Such sums as $G_r(z_1, z_2)$ given by (11) are called Eisenstein series. In fact, for these Eisenstein series, we have

$$G_r(z_1, z_2) = 0 \quad (r = 3, 5, 7, \dots) \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} G_{2k}(2\pi, i\lambda) = (-1)^{k+1} \frac{B_{2k}}{(2k)!}. \quad (12)$$

The Weierstrass $\wp$ -function defined by

$$\wp(u) = \frac{1}{u^2} + \sum_{\Omega^*} \left(\frac{1}{[u - (m_1 z_1 + m_2 z_2)]^2} - \frac{1}{(m_1 z_1 + m_2 z_2)^2} \right) \quad (13)$$

is a meromorphic function in the complex u -plane and has poles of second order at the points of Ω^* . For

$$|u| < \min_{\Omega^*} |m_1 z_1 + m_2 z_2|,$$

we have the following expansion of $\wp(u)$ in a Laurent series:

$$\wp(u) = \frac{1}{u^2} + \sum_{k=2}^{\infty} (2k-1) G_{2k}(z_1, z_2) u^{2k-2}, \quad (14)$$

which, upon differentiation with respect to u , yields

$$\wp'(u) = -\frac{2}{u^3} + 2 \cdot 3 G_4(z_1, z_2) u + 4 \cdot 5 G_6(z_1, z_2) u^3 + \dots \quad (15)$$

By the doubly-periodic property of $\wp(u)$ and Liouville's theorem, we obtain the Weierstrass differential equation as follows:

$$[\wp'(u)]^2 = 4[\wp(u)]^3 - 60 G_4(z_1, z_2) \wp(u) - 140 G_6(z_1, z_2). \quad (16)$$

3. Expansion of the Bernoulli numbers B_{2k} in three terms

As usual, we use the notation $\wp^{(n)}(u)$ to represent the n th derivative of the Weierstrass function $\wp(u)$. We also conveniently abbreviate $G_r(z_1, z_2)$ by G_r .

LEMMA 1 *The following relations hold true for the higher-order derivatives of the Weierstrass $\wp$ -function:*

$$\wp^{(4)}(u) = 5! \cdot ([\wp(u)]^3 - 9 G_4 \wp(u) - 14 G_6) \quad (17)$$

and

$$\wp^{(5)}(u) = 5! \cdot (3[\wp(u)]^2 \cdot \wp'(u) - 9 G_4 \wp'(u)). \quad (18)$$

Proof Upon differentiating (16) with respect to u and then dividing both sides by $\wp'(u)$, we obtain

$$\wp^{(2)}(u) = 6[\wp(u)]^2 - 30 G_4. \quad (19)$$

Now we differentiate (19) once again. We thus get

$$\wp^{(3)}(u) = 12\wp(u) \cdot \wp'(u). \quad (20)$$

Further differentiation of (20) and substitutions from (16) and (19) would finally yield the assertion (17) of Lemma 1.

The assertion (18) of Lemma 1 follows similarly from (17). ■

LEMMA 2 *Each of the following expressions holds true for the Bernoulli numbers B_{2k} :*

$$\sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-4)!}{(2m-2)!(2n-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} = -\frac{1}{6} \left(\frac{(k-3)(2k+1)}{(k-1)(2k-3)} \frac{B_{2k}}{2k} \right) \quad (21)$$

and

$$\sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-2)!(2n-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} = -\frac{1}{12} \left(\frac{(k-3)(2k+1)}{(k-1)(2k-3)} \frac{B_{2k}}{2k} \right). \quad (22)$$

Proof Equation (21) is just a slightly modified version of Rademacher's identity (7). To prove the assertion (22) of Lemma 2, we need the following summation identity (which exhibits the symmetry property of the summation indices m and n):

$$\sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-2)!(2n-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} = \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-3)!(2n-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \quad (23)$$

as well as the following combinatorial identity:

$$\binom{2k-4}{2m-2} = \binom{2k-5}{2m-2} + \binom{2k-5}{2m-3}. \quad (24)$$

Thus, for $k \in \mathbb{N} \setminus \{1, 2, 3\}$, we find that

$$\begin{aligned}
 & 2 \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-2)!(2n-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \\
 &= \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-2)!(2n-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \\
 &\quad + \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-3)!(2n-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \\
 &= \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \left(\frac{(2k-5)!}{(2m-2)!(2n-3)!} + \frac{(2k-5)!}{(2m-3)!(2n-2)!} \right) \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \\
 &= \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-4)!}{(2m-2)!(2n-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n}. \tag{25}
 \end{aligned}$$

By substituting the relation (25) into the left-hand side of (21), we readily arrive at the assertion (22) of Lemma 2. ■

By applying Lemma 1 and Lemma 2, we now prove our main theorem stated below.

THEOREM *For the Bernoulli numbers B_{2k} , the three-term expressions (8) and (9) hold true.*

Proof First of all, we substitute the Laurent series (14) into (17) and multiply both sides of the resulting equation by $u^6/5!$. We thus obtain

$$\begin{aligned}
 & 1 + \sum_{k=3}^{\infty} \frac{(2k-1)!}{5!(2k-6)!} G_{2k} u^{2k} \\
 &= \left(1 + \sum_{k=2}^{\infty} (2k-1) G_{2k} u^{2k} \right)^3 - 9G_4 \left(u^4 + \sum_{k=2}^{\infty} (2k-1) G_{2k} u^{2k+4} \right) - 14G_6 u^6 \\
 &= 1 + 3 \sum_{k=2}^{\infty} (2k-1) G_{2k} u^{2k} + 3 \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} (2m-1)(2n-1) G_{2m} G_{2n} u^{2m+2n} \\
 &\quad + \sum_{m=2}^{\infty} \sum_{n=2}^{\infty} \sum_{\ell=2}^{\infty} (2m-1)(2n-1)(2\ell-1) G_{2m} G_{2n} G_{2\ell} u^{2m+2n+2\ell} \\
 &\quad - 9G_4 \sum_{k=4}^{\infty} (2k-5) G_{2k-4} u^{2k} - 9G_4 u^4 - 14G_6 u^6. \tag{26}
 \end{aligned}$$

By comparing the coefficients of u^{2k} on the two sides of (26), we get the following identity involving Eisenstein series:

$$\begin{aligned}
 & \left(\frac{(2k-1)!}{5!(2k-6)!} - 3(2k-1) \right) G_{2k} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1)(2\ell-1) G_{2m} G_{2n} G_{2\ell} \\
 &+ 3 \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1) G_{2m} G_{2n} - 9(2k-5) G_4 G_{2k-4} \quad (k \in \mathbb{N} \setminus \{1, \dots, 5\}).
 \end{aligned} \tag{27}$$

Now, in view of the limit relationship (12), by taking

$$\lim_{\lambda \rightarrow \infty} G_{2k}(2\pi, i\lambda)$$

on both sides of (27), we get

$$\begin{aligned}
 & \left(\frac{(2k-1)!}{5!(2k-6)!} - 3(2k-1) \right) (-1)^{k+1} \frac{B_{2k}}{(2k)!} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1)(2\ell-1) (-1)^{k+3} \frac{B_{2m}}{(2m)!} \frac{B_{2n}}{(2n)!} \frac{B_{2\ell}}{(2\ell)!} \\
 &+ 3 \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1) (-1)^{k+2} \frac{B_{2m}}{(2m)!} \frac{B_{2n}}{(2n)!} \\
 &+ 9(2k-5) (-1)^{k+2} \frac{1}{30} \left(\frac{B_{2k-4}}{(2k-4)!} \right).
 \end{aligned} \tag{28}$$

The Bernoulli identity (8) follows easily by multiplying both sides of (28) by $(2k-6)!$ and appealing to the assertion (21) of Lemma 2.

We next consider the assertion (18) of Lemma 1, substitute the Laurent series (14) into it, and multiply both sides of the resulting equation by $u^7/6!$. We thus find that

$$\begin{aligned}
 & -1 + \sum_{k=4}^{\infty} \frac{(2k-1)!}{6!(2k-7)!} G_{2k} u^{2k} \\
 &= \frac{1}{2} \left(-2 + \sum_{k=2}^{\infty} (2k-1)(2k-2) G_{2k} u^{2k} \right) \left(1 + \sum_{k=2}^{\infty} (2k-1) G_{2k} u^{2k} \right)^2 \\
 &\quad - \frac{3}{2} G_4 \left(-2u^4 + \sum_{k=2}^{\infty} (2k-1)(2k-2) G_{2k} u^{2k+4} \right) \\
 &= \frac{1}{2} \left(-2 + \sum_{k=2}^{\infty} (2k-1)(2k-2) G_{2k} u^{2k} \right) \left(1 + \sum_{k=2}^{\infty} (2k-1) G_{2k} u^{2k} \right)^2 \\
 &\quad - \frac{3}{2} G_4 \left(-2u^4 + \sum_{k=4}^{\infty} (2k-5)(2k-6) G_{2k-4} u^{2k} \right).
 \end{aligned} \tag{29}$$

By comparing the coefficients of u^{2k} in (29), we have the following identity involving Eisenstein series:

$$\begin{aligned}
 & \left(\frac{(2k-1)!}{6!(2k-7)!} - \frac{1}{2}(2k-1)(2k-2) + 2(2k-1) \right) G_{2k} + \frac{3}{2} G_4(2k-5)(2k-6) G_{2k-4} \\
 &= \frac{1}{2} \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1)(2\ell-1)(2\ell-2) G_{2m} G_{2n} G_{2\ell} \\
 & \quad - \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1) G_{2m} G_{2n} \\
 & \quad + \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} (2m-1)(2n-1)(2n-2) G_{2m} G_{2n} \quad (k \in \mathbb{N} \setminus \{1, \dots, 5\}). \tag{30}
 \end{aligned}$$

Once again, in view of the limit relationship (12), taking the appropriate limits on both sides of (30) just as in our derivation of (28) from (27), and then multiplying the resulting equation by $(2k-7)!$, we obtain

$$\begin{aligned}
 & \left(\frac{1}{6!} - \frac{1}{2} \frac{1}{(2k-3) \cdots (2k-6)} + \frac{2}{(2k-2) \cdots (2k-6)} \right) \frac{B_{2k}}{2k} + \frac{1}{480} \left(\frac{B_{2k-4}}{2k-4} \right) \\
 &= \frac{1}{2} \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 & \quad + \frac{1}{(2k-4)(2k-5)(2k-6)} \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-4)!}{(2m-2)!(2n-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \\
 & \quad - \frac{1}{(2k-5)(2k-6)} \sum_{\substack{m+n=k \\ m,n \in \mathbb{N} \setminus \{1\}}} \frac{(2k-5)!}{(2m-2)!(2n-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n}. \tag{31}
 \end{aligned}$$

By applying Lemma 2, we find from (31) that

$$\begin{aligned}
 & \left(\frac{1}{6!} - \frac{1}{12(2k-5)(2k-6)} + \frac{1}{6(2k-4)(2k-5)(2k-6)} \right. \\
 & \quad \left. + \frac{1}{2(2k-3) \cdots (2k-6)} - \frac{2}{(2k-2) \cdots (2k-6)} \right) \frac{B_{2k}}{2k} + \frac{1}{480} \left(\frac{B_{2k-4}}{2k-4} \right) \\
 &= \frac{1}{2} \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell}. \tag{32}
 \end{aligned}$$

Finally, by applying the elementary identity:

$$\begin{aligned}
 & \frac{1}{6!} - \frac{1}{12(2k-5)(2k-6)} + \frac{1}{6(2k-4)(2k-5)(2k-6)} \\
 & \quad + \frac{1}{2(2k-3) \cdots (2k-6)} - \frac{2}{(2k-2) \cdots (2k-6)} \\
 &= \frac{1}{6} \left(\frac{1}{5!} - \frac{2k^2 - 5k}{(2k-2)(2k-3)(2k-4)(2k-5)} \right) \tag{33}
 \end{aligned}$$

to the left-hand side of (32), we complete our proof of the Bernoulli identity (9) as well. ■

4. Concluding remarks and observations

It is natural to ask whether or not the seemingly different Bernoulli identities (8) and (9) are equivalent. An answer to this question is provided by the following assertion.

PROPOSITION *The Bernoulli identities (8) and (9) are equivalent.*

Our demonstration of the equivalence of the Bernoulli identities (8) and (9) follows at once from the summation identity (34) asserted by Lemma 3 below.

LEMMA 3 *The following summation identity holds true:*

$$\begin{aligned}
 & 3 \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-6)!}{(2m-2)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell}. \tag{34}
 \end{aligned}$$

Proof By trivially interchanging the summation indices m , n , and ℓ , we get

$$\begin{aligned}
 & \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-3)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-3)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell}, \tag{35}
 \end{aligned}$$

so that

$$\begin{aligned}
 & 3 \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &+ \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-2)!(2n-3)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &+ \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-7)!}{(2m-3)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \left(\frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-3)!} + \frac{(2k-7)!}{(2m-2)!(2n-3)!(2\ell-2)!} \right. \\
 &\quad \left. + \frac{(2k-7)!}{(2m-3)!(2n-2)!(2\ell-2)!} \right) \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} [(2\ell-2) + (2n-2) + (2m-2)] \\
 &\quad \cdot \frac{(2k-7)!}{(2m-2)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell} \\
 &= \sum_{\substack{m+n+\ell=k \\ m,n,\ell \in \mathbb{N} \setminus \{1\}}} \frac{(2k-6)!}{(2m-2)!(2n-2)!(2\ell-2)!} \frac{B_{2m}}{2m} \frac{B_{2n}}{2n} \frac{B_{2\ell}}{2\ell}. \tag{36}
 \end{aligned}$$

Our proof of Lemma 3 (and hence also of the above Proposition) is thus completed by means of (36). ■

We conclude this investigation by remarking that, by appealing appropriately to the various useful properties and characteristics of the Weierstrass $\wp$ -function and its such higher-order derivatives as $\wp^{(4)}$ and $\wp^{(5)}$, we have succeeded in expressing the Bernoulli numbers B_{2k} as a sum of the products of lower-order Bernoulli numbers in three terms. Our methodology and techniques are capable of producing *further* analogous results for the Bernoulli numbers B_{2k} . The details involved are being left as an exercise for the interested reader.

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Covariant symbolic calculus for Toeplitz operators on the sphere

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We consider Toeplitz operators of order zero on the unit sphere of $\mathbb{C}^n$ and describe its covariant symbolic calculus defined by the Berezin quantization. These Toeplitz operators are written as Bessel integral operators. We derive a new explicit formula in terms of Bessel functions for the composition of covariant symbols and a star product.

Keywords: Toeplitz operators; Berezin quantization; Bessel integral operators; Star product

AMS 2000 Mathematics Subject Classification codes: 33C10; 58J40; 53D55

1. Introduction

We first recall the subjects we are dealing with: Toeplitz operators on the sphere and Berezin's theory of covariant symbols.

1.1 Toeplitz operators on the sphere

Let X be the unit sphere of $\mathbb{C}^n$ and dS the element of the surface on X , invariant by rotation. We endow the Hilbert space $L^2(X, dS)$ with the inner product:

$$(f, g) = \int_X f(x) \overline{g(x)} dS(x).$$

Let $\mathbf{O} := L^2(X, dS) \cap \text{Ker } \bar{\partial}_b$ be the Hardy space. The Szegő projection is the orthogonal projection $\mathbf{S} : L^2(X, dS) \rightarrow \mathbf{O}$. We consider Toeplitz operators of order zero, *i.e.* operators of the form $A = \mathbf{S}MS$, where M is an operator of multiplication by an essentially bounded function defined on X . It is known that the space of Toeplitz operators of order zero is an algebra that we will denote by $T^0(X)$. For further details, see ref. [11] and [12].

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1.2 Berezin's theory of covariant symbols

Throughout this article $\hbar$ is a positive constant.

(a) Let $H_{\hbar}$ be a Hilbert space endowed with the inner product $(\cdot, \cdot)_{\hbar}$ and $(M; d\mu_{\hbar})$ be a measured space. If there exists a family $\{e_{\alpha}^{(\hbar)} \in H_{\hbar} \setminus \{0\}; \text{ for every } \alpha \in M\}$ such that the following two properties are fulfilled:

1. Let $\hat{f}(\alpha) := (f, e_{\alpha}^{(\hbar)})$, the map $f \rightarrow \hat{f}$ is an embedding from $H_{\hbar}$ into $L^2(M; d\mu_{\hbar})$.
2. $(f, g)_{\hbar} = \int_M (f, e_{\alpha}^{(\hbar)})(e_{\alpha}^{(\hbar)}, g) d\mu_{\hbar}(\alpha)$ for every $f, g \in H_{\hbar}$ (Parseval identity).

Then we have a covariant symbolic calculus on $\mathbf{B}(H_{\hbar})$, the space of bounded linear operators on $H_{\hbar}$. More precisely, we define the covariant symbol of an operator $A \in \mathbf{B}(H_{\hbar})$ by

$$\sigma_A^{(\hbar)}(\alpha) := \frac{(Ae_{\alpha}^{(\hbar)}, e_{\alpha}^{(\hbar)})_{\hbar}}{(e_{\alpha}^{(\hbar)}, e_{\alpha}^{(\hbar)})_{\hbar}} \quad \text{for every } \alpha \in M.$$

If we further suppose that the function $\alpha \rightarrow |e_{\alpha}^{(\hbar)}|_{\hbar}$ is continuous and as $(e_{\alpha}^{(\hbar)}, e_{\alpha}^{(\hbar)})_{\hbar} > 0$, then this symbol may be extended in a neighborhood of the diagonal of $M \times M$ to the function

$$\sigma_A^{(\hbar)}(\alpha, \beta) := \frac{(Ae_{\alpha}^{(\hbar)}, e_{\beta}^{(\hbar)})_{\hbar}}{(e_{\alpha}^{(\hbar)}, e_{\beta}^{(\hbar)})_{\hbar}}.$$

The function $\theta_{\hbar}(\alpha, \beta) := (e_{\alpha}^{(\hbar)}, e_{\beta}^{(\hbar)})_{\hbar}$ will be called the ‘indicator of orthogonality in $H'_{\hbar}$ ’.

For every $f \in H_{\hbar}$, $A, B \in \mathbf{B}(H_{\hbar})$, $c \in \mathbb{C}$ and $\alpha, \beta \in M$, we have the following formulas:

$$\sigma_{A+cB}^{(\hbar)} = \sigma_A^{(\hbar)} + c\sigma_B^{(\hbar)}, \quad \sigma_{Id}^{(\hbar)} \equiv 1 \quad \text{and} \quad \sigma_{A^*}^{(\hbar)}(\alpha, \beta) = \overline{\sigma_A^{(\hbar)}(\beta, \alpha)}.$$

$$\widehat{Af}(\alpha) = \int_M \hat{f}(\beta) \sigma_A^{(\hbar)}(\beta, \alpha) \theta_{\hbar}(\beta, \alpha) d\mu_{\hbar}(\beta).$$

$$\sigma_A^{(\hbar)} \# \sigma_B^{(\hbar)}(\alpha, \beta) = \int_M \sigma_B^{(\hbar)}(\alpha, \gamma) \sigma_A^{(\hbar)}(\gamma, \beta) \frac{\theta_{\hbar}(\alpha, \gamma) \theta_{\hbar}(\gamma, \beta)}{\theta_{\hbar}(\alpha, \beta)} d\mu_{\hbar}(\gamma).$$

Notice that the product $\theta_{\hbar} \sigma_A^{(\hbar)}$ is defined in the whole $M \times M$.

The first useful application of this theory is a symbolic calculus (hence functional calculus) of operators. A second application is that this symbolic calculus allows us to build a star product [1].

(b) Berezin applied this method to the so-called ‘quantization of Kähler manifolds’ [2, 3, 4]:

- M is a Kähler manifold with the metric $ds = \sum g_{j\bar{k}} dz_j \wedge d\bar{z}_k$.
- $d\mu := \det [g_{j\bar{k}}] \times \prod \frac{dz_j \wedge d\bar{z}_j}{2\pi i}$.
- H is the Hilbert subspace of functions of $L^2(M, d\mu)$ which are analytic.
- $e_{\alpha}^{(\hbar)}$ ’s are obtained by freezing one variable in the Bergman kernel: $B(\cdot, \alpha)$.

Indeed, in this situation, H is a closed subspace of $L^2(M, d\mu)$ and the orthogonal projection from $L^2(M, d\mu)$ onto H is called the Bergman projection whose Schwartz kernel is denoted by B .

- $\theta(\alpha, \beta)$ is the Bergman kernel $B(\beta, \alpha)$.

A parameter $\hbar > 0$ may be added in order to get a semi-classical approach of the deformation quantization. A particular case of this construction is the Wick quantization:

- $M := \mathbb{C}^n$.
- $d\mu_{\hbar} := \pi^{-n} e^{-|z|^2/\hbar} dz$, where dz is the Lebesgue measure.
- $H_{\hbar}$ is the Fock space, i.e. $L^2(\mathbb{C}^n, d\mu_{\hbar}) \cap \text{Ker } \partial$.
- $e_{\alpha}^{(\hbar)}$ s are defined by $e_z^{(\hbar)}(u) := e^{\bar{z} \cdot u/\hbar}$.
- $\theta_{\hbar}(\alpha, \beta) = e^{\bar{z} \cdot u/\hbar}$.

In this article, we will use a similar quantization procedure:

- $M := \mathbb{C}^n$.
- $d\mu_{\hbar} := \varpi_{\hbar} dV$, where $\varpi_{\hbar}$ is a weight function defined by a Bessel function and dV is the element of volume of $\mathbb{C}^n$.
- $H_{\hbar} := \mathbf{O}$ is the Hardy space.
- $e_{\alpha}^{(\hbar)}$ s are coherent states defined by an exponential function.
- $\theta_{\hbar}(\alpha, \beta)$ is a Bessel function.

In the quantization of Kähler manifolds, the space H is defined to be the intersection of $L^2(M, d\mu)$ with the space of analytic functions on M so that the embedding $H \rightarrow L^2(M, d\mu)$ is simply the inclusion; besides, the supercomplete family is obtained by freezing one variable in the Bergman kernel.

In our construction where $H = \mathbf{O}$ and $L^2(M, d\mu) = L^2(\mathbb{C}^n; \varpi_{\hbar} dV)$, the latter space is called ‘a model’ for H : there is no inclusion of H into $L^2(M, d\mu)$. The embedding is defined by a so-called ‘wave packets transformation’. Moreover, the space $\mathbb{C}^n$ may be regarded as the phase space of X so that this quantization deals with microlocal calculus. This situation is thus slightly different from Berezin’s one.

Let us notice that the use of Bessel functions will make it easier to compute explicitly covariant symbols of usual Toeplitz operators and to give an explicit formula of star product.

2. Quantization of the Szegő kernel

We will first recall some notations about Bessel functions we will use. For usual definitions and properties of Bessel functions, see refs. [5, 6]. $I_n(z)$ is the modified Bessel function of the first kind and of order n and $K_0(z)$ is the modified Bessel function of the third kind of order zero, also called ‘Macdonald function’.

For every positive integer n , we set

$$I_n^-(z) := z^{-n} I_n(z) = 2^{-n} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!(k+n)!}, \quad (1)$$

which is an entire function, and we have

$$\frac{d}{dz} I_n^-(z) = z I_{n+1}^-(z). \quad (2)$$

We are going to define an embedding from $\mathbf{O}$ into another Hilbert space. For that, we need the following.

DEFINITION 2.1 *Let $\varpi_{\hbar}$ be the weight function defined on $\mathbb{C}^n$ by*

$$\varpi_{\hbar}(z) := (\pi \hbar)^{-2n} K_0(2|z|/\hbar). \quad (3)$$

From general properties of Bessel functions, we deduce that this weight function is a continuous, positive-valued, radial function defined on $\mathbb{C}^n \setminus \{0\}$, and its asymptotic is given by

$$\varpi_{\hbar}(z) \stackrel{z \rightarrow \infty}{\sim} \frac{1}{2\pi} (\pi \hbar)^{1/2-2n} |z|^{-1/2} e^{-2|z|/\hbar} \left(\sum_{k=0}^N \frac{(-\hbar)^k}{k!} \frac{\Gamma(1/2+k)}{(4|z|)^k} + O(|z|^{-N-1}) \right)$$

for every positive integer N .

$L^2(\mathbb{C}^n; \varpi_{\hbar} dV)$ is the Hilbert space endowed with the inner product

$$\langle f, g \rangle_{\hbar} := \int_{\mathbb{C}^n} f(z) \overline{g(z)} \varpi_{\hbar}(z) dV(z).$$

We proceed with the Toeplitz–Berezin quantization by defining coherent states depending on the parameters z varying in $\mathbb{C}^n$:

$$e_z^{(\hbar)}(x) := \exp\left(\frac{x \cdot \bar{z}}{\hbar}\right) \quad \text{for every } z \in \mathbb{C}^n.$$

They are functions in $\mathbf{O}$.

PROPOSITION 2.2 The map $f \rightarrow \hat{f}$ defined by

$$\hat{f}(z) := (f, e_z^{(\hbar)}) = \int_X e^{\bar{x} \cdot z / \hbar} f(x) dS(x) \quad \text{for every } f \in \mathbf{O}$$

is an embedding from $\mathbf{O}$ into $L^2(\mathbb{C}^n; \varpi_{\hbar} dV)$.

Proof This is a consequence of Theorem 1 in ref. [7] which states that this transform is one-to-one from $\mathbf{O}$ onto $L^2(\mathbb{C}^n; \varpi_{\hbar} dV) \cap \text{Ker } \bar{\partial}$. ■

We also define the dual coherent states by

$$\varepsilon_x^{(\hbar)}(z) := \overline{e_z^{(\hbar)}(x)} = \exp\left(\frac{z \cdot \bar{x}}{\hbar}\right) \quad \text{for every } x \in X.$$

For every multi-indices $\alpha, \beta \in \mathbb{N}^n$, we will write $\alpha \leq \beta$ iff $\alpha_j \leq \beta_j$ for $j = 1, \dots, n$.

LEMMA 2.3 Let $\alpha, \beta \in \mathbb{C}^n$, $p, q \in \mathbb{C}$ and $z, u \in \mathbb{C}^n \setminus \{0\}$. Then

$$J := \int_X x^\alpha \bar{x}^\beta (x \cdot \bar{z})^p (\bar{x} \cdot u)^q dS(x) = 0 \quad \text{if } |\alpha| + p \neq |\beta| + q.$$

If $|\alpha| + p = |\beta| + q$, then we have the following two equalities:

If $\alpha \geq \beta$ then

$$J = \frac{u^\alpha}{\bar{z}^\beta} \frac{2\pi^n (p + |\alpha| - |\beta|)!}{(p + |\alpha| + n - 1)!} \sum_{|\gamma|=p} \frac{p!}{\gamma!} \frac{(\alpha + \gamma)!}{(\alpha - \beta + \gamma)!} \bar{z}^\gamma u^\gamma. \quad (4)$$

If $\beta \geq \alpha$ then

$$J = \frac{\bar{z}^\beta}{\bar{z}^\alpha} \frac{2\pi^n (q + |\beta| - |\alpha|)!}{(q + |\beta| + n - 1)!} \sum_{|\delta|=q} \frac{q!}{\delta!} \frac{(\beta + \delta)!}{(\beta - \alpha + \delta)!} \bar{z}^\delta u^\delta. \quad (5)$$

In particular, we have the following:

If $\beta = 0$ then

$$J = 2\pi^n u^\alpha (\bar{z} \cdot u)^p \frac{(p + |\alpha|)!}{(p + |\alpha| + n - 1)!}. \quad (6)$$

If $\alpha = 0$ then

$$J = 2\pi^n \bar{z}^\beta (\bar{z} \cdot u)^q \frac{(q + |\beta|)!}{(q + |\beta| + n - 1)!} \quad (7)$$

Proof We have $J = \int_X f(x)g(x) dS(x)$, where $f(x) := x^\alpha (x\bar{z})^p$ and $g(x) := \bar{x}^\beta (\bar{x} \cdot u^q)$; both functions are harmonic and homogeneous, respectively, of order $|\alpha| + p$ and $|\beta| + q$. Then, $J = 0$ if $|\alpha| + p \neq |\beta| + q$. We suppose now that $|\alpha| + p = |\beta| + q$:

$$\begin{aligned} J &= \int_X x^\alpha \left(\sum_{|\gamma|=p} \frac{p!}{\gamma!} x^\gamma \bar{z}^\gamma \right) \bar{x}^\beta \left(\sum_{|\delta|=q} \frac{q!}{\delta!} \bar{x}^\delta u^\delta \right) dS(x) \\ &= \sum_{|\gamma|=p} \sum_{|\delta|=q} \frac{p!}{\gamma!} \frac{q!}{\delta!} \bar{z}^\gamma u^\delta \int_X x^{\alpha+\gamma} \bar{x}^{\beta+\delta} dS(x). \end{aligned}$$

The last integral does not vanish if and only if $\alpha + \gamma = \beta + \delta$.

If $\alpha \geq \beta$ and for fixed γ , we keep only $\delta = \alpha - \beta + \gamma$ so that

$$\begin{aligned} J &= \sum_{|\gamma|=p} \frac{p!}{\gamma!} \frac{(|\alpha| - |\beta| + p)!}{(\alpha - \beta + \gamma)!} \bar{z}^\gamma u^{\alpha-\beta+\gamma} \frac{2\pi^n (\alpha + \gamma)!}{(n - 1 + |\alpha| + p)!} \\ &= u^{\alpha-\beta} \frac{2\pi^n (|\alpha| - |\beta| + p)!}{(n - 1 + |\alpha| + p)!} \sum_{|\gamma|=p} \frac{p!}{\gamma!} \frac{(\alpha + \gamma)!}{(\alpha - \beta + \gamma)!} \bar{z}^\gamma u^\gamma. \end{aligned}$$

Similarly, if $\beta \geq \alpha$ and for fixed δ , we keep only $\gamma = \beta + \delta - \alpha$ so that we get the other formula. ■

LEMMA 2.4 Let $\alpha, \beta \in \mathbb{N}^n$, $p, q \in \mathbb{N}$, and $z, u \in \mathbb{C}^n \setminus \{0\}$, then

$$J := \int_{\mathbb{C}^n} w^\alpha \bar{w}^\beta (w \cdot \bar{z})^p (\bar{w} \cdot u)^q K_0(2|w|/\hbar) dV(w) = 0 \quad \text{if } |\alpha| + p \neq |\beta| + q.$$

If $|\alpha| + p = |\beta| + q$ and if $\alpha \geq \beta$ or $\beta \geq \alpha$, then

$$J = \frac{1}{4} \hbar^{2n+2|\alpha|+2p} (n - 1 + |\alpha| + p)!^2 \times \int_X x^\alpha \bar{x}^\beta (x \cdot \bar{z})^p (\bar{x} \cdot u)^q dS(x). \quad (8)$$

In particular, we have

If $\alpha = 0$ then

$$J = \frac{\pi^n}{2} \hbar^{2n+2p} \bar{z}^\beta (\bar{z} \cdot u)^q (n - 1 + |\beta| + q)!. \quad (9)$$

If $\beta = 0$ then

$$J = \frac{\pi^n}{2} \hbar^{2n+2q} u^\alpha (\bar{z} \cdot u)^p (n - 1 + |\alpha| + p)!. \quad (10)$$

Proof We express J in polar coordinates:

$$J = \int_0^\infty r^{2n-1+|\alpha|+|\beta|+p+q} K_0(2r/\hbar) dr \int_X x^\alpha x^\beta (x \cdot \bar{z})^p (\bar{x} \cdot u)^q dS(x).$$

We use the Heaviside formula [6, XIII.21(8)] and Lemma 2.3, so that we get the claimed formula. $\blacksquare$

Let us denote the following oscillatory integral by $S_\hbar(x, \bar{y})$:

$$S_\hbar(x, \bar{y}) = \left\langle \varepsilon_x^{(\hbar)}, \varepsilon_y^{(\hbar)} \right\rangle_\hbar = (\pi \hbar)^{-2n} \int_{\mathbb{C}^n} e^{(x \cdot \bar{z} + \bar{y} \cdot z)/\hbar} K_0(2|z|/\hbar) dV(z) \quad (11)$$

THEOREM 2.5 $S_\hbar(x, \bar{y})$ is the same as the Szegő kernel.

Remark Actually, this kernel does not depend on $\hbar$.

Proof Let I be the integral in equation (11). We expand the exponential function into a power series, and we will use the binomial and the multinomial formulas. We may commute integrations and summations because the integrals are oscillatory and the series are absolutely convergent, so that we get

$$\begin{aligned} I &= \sum_{m=0}^\infty \frac{\hbar^{-m}}{m!} \int_{\mathbb{C}^n} (x \cdot \bar{z} + \bar{y} \cdot z)^m K_0\left(\frac{2|z|}{\hbar}\right) dV(z) \\ &= \sum_{m=0}^\infty \frac{\hbar^{-m}}{m!} \int_{\mathbb{C}^n} \left(\sum_{k=0}^m \binom{m}{k} (x \cdot \bar{z})^k (\bar{y} \cdot z)^{m-k} \right) K_0\left(\frac{2|z|}{\hbar}\right) dV(z) \\ &= \sum_{m=0}^\infty \frac{\hbar^{-m}}{m!} \int_{\mathbb{C}^n} \left(\sum_{k=0}^m \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha \bar{z}^\alpha \right) \left(\sum_{|\beta|=m-k} \frac{(m-k)!}{\beta!} \bar{y}^\beta z^\beta \right) \right) K_0\left(\frac{2|z|}{\hbar}\right) dV(z) \\ &= \sum_{m=0}^\infty \hbar^{-m} \sum_{k=0}^m \sum_{|\alpha|=k} \sum_{|\beta|=m-k} \frac{x^\alpha \bar{y}^\beta}{\alpha! \beta!} \int_{\mathbb{C}^n} \bar{z}^\alpha z^\beta K_0\left(\frac{2|z|}{\hbar}\right) dV(z). \end{aligned}$$

Following Lemma 2.4, the last integral does not vanish if and only if $\alpha = \beta$, so that, for fixed α , we only keep the term where $\beta = \alpha$, which implies $|\alpha| = k = m - k$, i.e. $m = 2k$. As k is a nonnegative integer, we need only to keep in the first sum those m which are even, $m = 2m'$ with $m' \in \mathbb{N}$, say, and we deduce

$$\begin{aligned} I &= \sum_{m'=0}^\infty \hbar^{-2m'} \sum_{|\alpha|=m'} \frac{x^\alpha \bar{y}^\alpha}{\alpha! \alpha!} \int_{\mathbb{C}^n} \bar{z}^\alpha z^\alpha K_0\left(\frac{2|z|}{\hbar}\right) dV(z) \\ &= \sum_{m'=0}^\infty \hbar^{-2m'} \sum_{|\alpha|=m'} \frac{x^\alpha \bar{y}^\alpha}{\alpha! \alpha!} \frac{1}{4} \hbar^{2n+2|\alpha|} (n-1+|\alpha|)!^2 \frac{2\pi^n \alpha!}{(n-1+|\alpha|)!} \\ &= \sum_{m'=0}^\infty \hbar^{2n} \frac{\pi^n}{2} \frac{(n-1+m')!}{m'!} \sum_{|\alpha|=m'} \frac{m'!}{\alpha!} x^\alpha \bar{y}^\alpha, \end{aligned}$$

so that

$$S_\hbar(x, \bar{y}) = (\pi \hbar)^{-2n} I = \frac{1}{2\pi^n} \sum_{m'=0}^\infty \frac{(n-1+m')!}{m'!} (x \cdot \bar{y})^{m'}.$$

Now, we test this kernel on the monomial y^γ , $\gamma \in \mathbb{N}^n$. We again use the multinomial formula

$$\int_X S_{\bar{h}}(x, \bar{y}) y^\gamma dS(y) = \frac{1}{2\pi^n} \sum_{m'=0}^{\infty} \frac{(n-1+m')!}{m'!} \sum_{|\alpha|=m'} \frac{m'!}{\alpha!} x^\alpha \int_X \bar{y}^\alpha y^\gamma dS(y).$$

The last integral vanishes except for $\alpha = \gamma$ and only when $m' = |\gamma|$; therefore,

$$\int_X S_{\bar{h}}(x, \bar{y}) y^\gamma dS(y) = \frac{1}{2\pi^n} (n-1+|\gamma|)! \frac{1}{\gamma!} x^\gamma \frac{2\pi^n \gamma!}{(n-1+|\gamma|)!} = x^\gamma.$$

As monomials are dense in $\mathbf{O}$, the proof is complete. ■

There is a very short proof of this theorem using the K_0 -transform of the exponential function if we assume the known formula for the Szegő kernel [7]; but here we wanted to stress the calculus with Bessel functions.

COROLLARY 2.6 *The family $\{e_z^{(\bar{h})}; \text{ for every } z \in \mathbb{C}^n\}$ is a supercomplete family for $\mathbf{O}$ in the sense of Berezin.*

Proof For every $f, g \in \mathbf{O}$, we have the Parseval formula

$$\begin{aligned} & \int_{\mathbb{C}^n} (f; e_z^{(\bar{h})})(e_z^{(\bar{h})}; g) \varpi_{\bar{h}}(z) dV(z) \\ &= \int_{\mathbb{C}^n} \left(\int_X e^{\bar{x} \cdot z / \bar{h}} f(x) dS(x) \right) \left(\int_X e^{y \cdot \bar{z} / \bar{h}} \overline{g(y)} dS(y) \right) \varpi_{\bar{h}}(z) dV(z) \\ &= \int_X \left(\int_X \left(\int_{\mathbb{C}^n} e^{\bar{x} \cdot z / \bar{h}} e^{y \cdot \bar{z} / \bar{h}} \varpi_{\bar{h}}(z) dV(z) \right) f(x) dS(x) \right) \overline{g(y)} dS(y) \\ &= \int_X \left(\int_X S_{\bar{h}}(y, \bar{x}) f(x) dS(x) \right) \overline{g(y)} dS(y) \\ &= \int_X f(y) \overline{g(y)} dS(y) \\ &= (f; g). \end{aligned}$$

We may have used a Fubini-like theorem because the integral with respect to z is oscillatory. ■

3. Covariant symbolic calculus

Relative to the supercomplete family $\{e_z^{(\bar{h})}; z \in \mathbb{C}^n\}$, we may consider the following.

DEFINITION 3.1 *The covariant symbol of a bounded linear operator $A : \mathbf{O} \rightarrow \mathbf{O}$ is defined, for every $z \in \mathbb{C}^n$, by*

$$\sigma_A^{(\bar{h})}(\bar{z}; z) = \frac{(Ae_z^{(\bar{h})}, e_z^{(\bar{h})})}{(e_z^{(\bar{h})}, e_z^{(\bar{h})})}. \quad (12)$$

The following proposition is of crucial importance to our calculus.

PROPOSITION 3.2

$$\theta_{\hbar}(\bar{z}, u) := (e_z^{(\hbar)}, e_u^{(\hbar)}) = (2\pi)^n I_{n-1}^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right). \quad (13)$$

Proof By definition, $\theta_{\hbar}(\bar{z}, u) = \int_X e^{(x \cdot \bar{z} + \bar{x} \cdot u)/\hbar} dS(x)$. We expand the exponential function into a power series and use commutations of sums which are justified by the facts that either the series are absolutely convergent or the integrals are performed on the sphere. Then, as in the proof of Theorem 1.5, we use the binomial and the multinomial formulas so that

$$\begin{aligned} \theta_{\hbar}(\bar{z}, u) &= \int_X e^{(x \cdot \bar{z} + \bar{x} \cdot u)/\hbar} dS(x) \\ &= \int_X \sum_{m=0}^{\infty} \frac{\hbar^{-m} (x \cdot \bar{z} + \bar{x} \cdot u)^m}{m!} dS(x) \\ &= \int_X \sum_{m=0}^{\infty} \frac{\hbar^{-m}}{m!} \left(\sum_{k=0}^m \binom{m}{k} (x \cdot \bar{z})^k (\bar{x} \cdot u)^{m-k} \right) dS(x) \\ &= \int_X \sum_{m=0}^{\infty} \frac{\hbar^{-m}}{m!} \left(\sum_{k=0}^m \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} x^{\alpha} \bar{z}^{\alpha} \right) \left(\sum_{|\beta|=m-k} \frac{(m-k)!}{\beta!} \bar{x}^{\beta} u^{\beta} \right) \right) dS(x) \\ &= \sum_{m=0}^{\infty} \hbar^{-m} \sum_{k=0}^m \sum_{|\alpha|=k} \sum_{|\beta|=m-k} \frac{\bar{z}^{\alpha}}{\alpha!} \frac{u^{\beta}}{\beta!} \int_X x^{\alpha} \bar{x}^{\beta} dS(x). \end{aligned}$$

For fixed α , we only keep $\beta = \alpha$ and $k = m' = m/2$ with m even.

$$\begin{aligned} \theta_{\hbar}(\bar{z}, u) &= \sum_{m'=0}^{\infty} \hbar^{-2m'} \sum_{|\alpha|=m'} \frac{\bar{z}^{\alpha}}{\alpha!} \frac{u^{\alpha}}{\alpha!} \frac{2\pi^n \alpha!}{(n-1+m')!} \\ &= 2\pi^n \sum_{m'=0}^{\infty} \frac{\hbar^{-2m'} (\bar{z} \cdot u)^{m'}}{m'!(n-1+m')!} = 2\pi^n 2^{n-1} I_{n-1}^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right), \end{aligned}$$

following formula (1). ■

As $|e_z^{(\hbar)}|^2 = (2\pi)^n I_{n-1}^-(2|z|/\hbar)$, this shows that the coherent states are continuous with respect to the parameter z , *i.e.* the map $z \rightarrow |e_z^{(\hbar)}|$ is continuous. Therefore, the covariant symbol may be extended to a neighborhood of the diagonal $z = u$ of $\mathbb{C}^n \times \mathbb{C}^n$ by

$$\sigma_A^{(\hbar)}(\bar{z}, u) := \frac{(Ae_z^{(\hbar)}, e_u^{(\hbar)})}{(e_z^{(\hbar)}, e_u^{(\hbar)})}. \quad (14)$$

We describe the location of the singularities of the covariant symbols.

Besides, the zeros of the Bessel function $I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)$ are well known and they become the singularities of the extended covariant symbols. Let $0 < \lambda_1 < \lambda_2 < \dots$ be the sequence of positive numbers such that $\{i\lambda_k; k = 1, 2, \dots\}$ is the set of zeros of $I_{n-1}^-(\zeta)$. Then the extended covariant symbol $\sigma_A^{(\hbar)}(\bar{z}, u)$ is defined on $\mathbb{C}^n \times \mathbb{C}^n \setminus Z$ with

$$Z := \left\{ (\bar{z}; u) \in \mathbb{C}^n \times \mathbb{C}^n \text{ such that } \sqrt{\bar{z} \cdot u} = \frac{i\hbar}{2} \lambda_k \text{ for some } k \right\}.$$

We now give the expected rules for symbolic calculus.

PROPOSITION 3.3 For every $z, u \in \mathbb{C}^n$ such that $\arg(u \cdot \bar{z}) < \pi$, $A, B \in T_h^0(X)$ and $f, g \in \mathbf{O}$, we have the following formulas:

$$\widehat{Af}(z) = \left(\frac{\pi}{2} \hbar^2\right)^{-n} \int_{\mathbb{C}^n} \sigma_A^{(\hbar)}(\bar{w}, z) \hat{f}(w) I_{n-1}^- \left(\frac{2\sqrt{\bar{w} \cdot z}}{\hbar} \right) K_0 \left(\frac{2|w|}{\hbar} \right) dV(w) \quad (15)$$

$$\sigma_{Id}^{(\hbar)} \equiv 1 \quad \text{and} \quad \sigma_{A*}^{(\hbar)}(\bar{z}, u) = \overline{\sigma_A^{(\hbar)}(\bar{u}, z)} \quad (16)$$

$$\begin{aligned} \sigma_A^{(\hbar)} \# \sigma_B^{(\hbar)}(\bar{z}, u) &= \sigma_{AB}^{(\hbar)}(\bar{z}, u) = \frac{(\pi \hbar/2)^{-n}}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \int_{\mathbb{C}^n} \sigma_B^{(\hbar)}(\bar{z}, w) \sigma_A^{(\hbar)}(\bar{w}, u) \times \dots \\ &\times I_{n-1}^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right) I_{n-1}^- \left(\frac{2\sqrt{\bar{w} \cdot u}}{\hbar} \right) K_0 \left(\frac{2|w|}{\hbar} \right) dV(w). \end{aligned} \quad (17)$$

Proof This follows from formulas in ref. [2] and from Proposition 2.2. ■

COROLLARY 3.4 Let α be a multi-index, $\sigma_\alpha(\bar{z}, u) := u^\alpha$ and $\sigma_{\bar{\alpha}}(\bar{z}, u) := \bar{z}^\alpha$. Let $\sigma(\bar{z}, u)$ be the covariant symbol of some Toeplitz operator. Then, $\sigma_\alpha \# \sigma = \sigma_\alpha \times \sigma$ and $\sigma \# \sigma_{\bar{\alpha}} = \sigma \times \sigma_{\bar{\alpha}}$.

This is an illustration of a result in [13].

LEMMA 3.5 Let M and N be two nonnegative integers. Then, we have

$$I_M^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right) \times I_N^- \left(\frac{2\sqrt{\bar{w} \cdot u}}{\hbar} \right) = 2^{-M-N} \sum_{m=0}^{\infty} \hbar^{-2m} (\bar{w} \cdot u)^m \sum_{r=0}^m c_{m,r} \omega^r \quad (18)$$

with

$$\omega := \frac{\bar{z} \cdot w}{\bar{w} \cdot u} \quad \text{and} \quad c_{m,r} := \frac{1}{r!} \frac{1}{(r+M)!} \frac{1}{(m-r)!} \frac{1}{(m-r+N)!}.$$

Proof Direct computations with formula (1). ■

PROPOSITION 3.6 Let α and β be two multi-indices, M and N two non-negative integers, and d and ε two integers such that

$$D := |\alpha| - |\beta| = 2d + \varepsilon \quad \text{with} \quad \begin{cases} \varepsilon = 0 & \text{if } D \text{ is even} \\ \varepsilon = 1 & \text{if } D \text{ is odd} \end{cases}.$$

Let $J := \int_{\mathbb{C}^n} w^\alpha \bar{w}^\beta I_M^-(2\sqrt{\bar{z} \cdot w}/\hbar) I_N^-(2\sqrt{\bar{w} \cdot u}/\hbar) K_0(2|w|/\hbar) dV(w)$. Then we have the following:

if $\alpha \geq \beta$ then

$$J = \frac{u^\alpha}{u^\beta} \frac{\pi^n}{2} \hbar^{2n+|\beta|} 2^{-M-N} \sum_{m=0}^{\infty} \hbar^{-2(m-d)} C_m S_{m-d}(\bar{z}, u) \quad (19)$$

with

$$C_m = \frac{(m-d+|\alpha|+n-1)!}{(m-d)!(m-d+M)!(m-d+\varepsilon+N)!}, \quad S_p(\bar{z}, u) = \sum_{|\gamma|=p} \frac{(\alpha+\gamma)!}{(\alpha+\gamma-\beta)!} \frac{p!}{\gamma!} \bar{z}^\gamma u^\gamma$$

if $\beta \geq \alpha$ then

$$J = \frac{\bar{z}^\beta}{\bar{z}^\alpha} \frac{\pi^n}{2} \hbar^{2n+|\alpha|} 2^{-M-N} \sum_{m=0}^{\infty} \hbar^{-2(m-d)} \tilde{C}_m \tilde{S}_{m-d}(\bar{z}, u) \quad (20)$$

with

$$\tilde{C}_m = \frac{(m-d+|\beta|+\varepsilon+n-1)!}{(m-d+M)!(m+d+\varepsilon)!(m+d+\varepsilon+N)!}, \quad \tilde{S}_q(\bar{z}, u) = \sum_{|\delta|=q} \frac{(\beta+\delta)!}{(\beta+\delta-\alpha)!} \frac{q!}{\delta!} \bar{z}^\delta u^\delta$$

In particular, let 1_j be the multi-index with 1 at the j th place and zero elsewhere. Then,

$$\text{if } \alpha = 0, \quad \beta = 0 \quad \text{and} \quad N = n-1 \quad \text{then } J = \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_M^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (21)$$

$$\text{if } \alpha = 0, \quad \beta = 0 \quad \text{and} \quad M = n-1 \quad \text{then } J = \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_N^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (22)$$

$$\text{if } \alpha = 1_j, \quad \beta = 0 \quad \text{and} \quad N = n-1 \quad \text{then } J = u_j \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_M^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (23)$$

$$\text{if } \alpha = 0, \quad \beta = 1_j \quad \text{and} \quad M = n-1 \quad \text{then } J = \bar{z}_j \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_N^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (24)$$

$$\text{if } \alpha = 1_j, \quad \beta = 0 \quad \text{and} \quad N = n \quad \text{then } J = u_j \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_{M+1}^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (25)$$

$$\text{if } \alpha = 0, \quad \beta = 1_j \quad \text{and} \quad M = n \quad \text{then } J = \bar{z}_j \left(\frac{\pi}{2}\right)^n \hbar^{2n} I_{N+1}^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar} \right) \quad (26)$$

Proof Prior to Lemma 3.5 we get

$$J = 2^{-M-N} \sum_{m'=0}^{\infty} \hbar^{-2m'} c_{m',r} J_{m',r} \quad \text{with}$$

$$J_{m',r} := \int_{\mathbb{C}^n} w^\alpha \bar{w}^\beta (\bar{z} \cdot w)^r (\bar{w} \cdot u)^{m'-r} K_0 \left(\frac{2|w|}{\hbar} \right) dV(w).$$

By Lemmas 2.2 and 2.3, if $|\alpha| + r \neq |\beta| + m' - r$ then $J_{m',r} = 0$, otherwise we have

$$J_{m',r} = \frac{1}{4} \hbar^{2n+|\alpha|+|\beta|+m'} \Gamma \left(n + \frac{|\alpha| + |\beta| + m'}{2} \right)^2 \times \frac{u^\alpha}{u^\beta} \frac{2\pi^n (D+m')!}{(|\alpha| + m' + n - 1)!} S_{m'}(\bar{z}, u)$$

with

$$S_{m'}(\bar{z}, u) := \sum_{|\gamma|=m'} \frac{(\alpha + \gamma)!}{(\alpha + \gamma - \beta)!} \frac{m'!}{\gamma!} \bar{z}^\gamma u^\gamma.$$

Suppose we have $\alpha \geq \beta$. This condition implies that $2r = m' - D$. As $r \in \mathbb{C}$, then m' and D must be of the same parity, $m' = 2m + \varepsilon$ and $D = 2d + \varepsilon$, say, with $m', d \in \mathbb{C}$ and $\varepsilon = 0$ or 1. Therefore, we need only to keep those m' such that $m' = 2m + \varepsilon$, so that $r = m - d$.

We get

$$J_{m',r} = \frac{1}{4} \hbar^{2n+2|\alpha|+2m-2d} (n-1+|\alpha|+m-d)!^2 \times \frac{u^\alpha}{u^\beta} \frac{2\pi^n (m+d+\varepsilon)!}{(n-1+|\alpha|+m-d)!} S_{m-d}(\bar{z}, u)$$

and

$$c_{m',r} = \frac{1}{(m-d)!} \frac{1}{(m-d+M)!} \frac{1}{(m+d+\varepsilon)!} \frac{1}{(m+d+\varepsilon+N)!},$$

so that

$$c_{m',r} J_{m',r} = \frac{\pi^n}{2} \frac{\hbar^{2n+2|\alpha|+2m-2d} (n-1+|\alpha|+m-d)!}{(m-d)!(m-d+M)!(m+d+\varepsilon+N)!} \frac{u^\alpha}{u^\beta} S_{m-d}(\bar{z}, u).$$

Then putting all together, we deduce the claimed formula.

The case $\beta \geq \alpha$ can be proved in the same way. ■

THEOREM 3.7 *Let α be a multi-index, $B \in T^0(X)$ and $(\bar{z}, u) \in \mathbb{C}^n \times \mathbb{C}^n \setminus Z$.*

1. *If $A = \text{'multiplication by } x^\alpha\text{'}$, then*

$$\sigma_A^{(\hbar)}(\bar{z}, u) = \left(\frac{2}{\hbar}\right)^{|\alpha|} u^\alpha \frac{I_{n-1+|\alpha|}^-(2\sqrt{\bar{z} \cdot u}/\hbar)}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)}. \quad (27)$$

2. *If $A = B \circ x^\alpha$, then*

$$\sigma_A^{(\hbar)}(\bar{z}, u) = \left(\frac{2}{\hbar}\right)^{|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left(\frac{\hbar}{2}\right)^{|\beta|} u^{\alpha-\beta} \frac{I_{n-1+|\alpha|-|\beta|}^-(2\sqrt{\bar{z} \cdot u}/\hbar)}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \times \left(\frac{\partial}{\partial \bar{z}}\right)^\beta \sigma_B(\bar{z}, u). \quad (28)$$

Proof Both assertions are proved by induction: formula (27) is obvious for $\alpha = 0$. We suppose the formula true for some α and we will show it for $\alpha + 1_j$. Let $A = x^\alpha \circ x_j$. From formula (17) and the inductive hypothesis applied to α and 1_j , we get

$$\begin{aligned} \sigma_A^{(\hbar)}(\bar{z}, u) &= \left(\frac{2}{\hbar}\right)^{1+|\alpha|} u^\alpha \frac{(\pi \hbar/2)^{-n}}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \int_{\mathbb{C}^n} w_j I_n^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right) I_{n-1+|\alpha|}^- \left(\frac{2\sqrt{\bar{w} \cdot u}}{\hbar} \right) \\ &\quad \times K_0(2|w|/\hbar) dV(w). \end{aligned}$$

The conclusion follows from equation (25).

We now prove formula (28). From formula (17), we have

$$\begin{aligned} \sigma_A^{(\hbar)}(\bar{z}, u) &= \frac{(\pi \hbar/2)^{-n}}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \int_{\mathbb{C}^n} \left(\frac{2}{\hbar}\right)^{|\alpha|} w^\alpha \sigma_B^{(\hbar)}(\bar{w}, u) I_{n-1+|\alpha|}^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right) \\ &\quad \times I_{n-1}^- \left(\frac{2\sqrt{\bar{w} \cdot u}}{\hbar} \right) K_0(2|w|/\hbar) dV(w). \end{aligned}$$

Using equation (2), it can be proved by induction that for any multi-index α , we have

$$\left(\frac{\partial}{\partial \bar{z}}\right)^\alpha I_{n-1}^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right) = \left(\frac{2}{\hbar}\right)^{|\alpha|} w^\alpha I_{n-1+|\alpha|}^- \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar} \right). \quad (29)$$

Therefore,

$$\begin{aligned} \sigma_A^{(\hbar)}(\bar{z}, u) &= \frac{(2/\pi)^n \hbar^{-2n}}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \times \cdots \times \left(\frac{2}{\hbar}\right)^{|\alpha|} \left(\frac{\hbar^2}{2}\right)^{|\alpha|} \left(\frac{\partial}{\partial \bar{z}}\right)^\alpha \int_{\mathbb{C}^n} \sigma_B^{(\hbar)}(\bar{w}, u) I_{n-1}^- \\ &\quad \times \left(\frac{2\sqrt{\bar{z} \cdot w}}{\hbar}\right) I_{n-1}^- \left(\frac{2\sqrt{\bar{w} \cdot u}}{\hbar}\right) K_0\left(\frac{2|w|}{\hbar}\right) dV(w) \end{aligned}$$

Following formulas (16) and (17), the integral alone is equal to $(\pi/2\hbar^2)^n [I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar) \times \sigma_B(\bar{z}, u)]$, so that

$$\sigma_A^{(\hbar)}(\bar{z}, u) = \frac{\hbar^{|\alpha|}}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)} \left(\frac{\partial}{\partial \bar{z}}\right)^\alpha \left[I_{n-1}^- \left(\frac{2\sqrt{\bar{z} \cdot u}}{\hbar}\right) \times \sigma_B(\bar{z}, u) \right].$$

Now, using again equation (29) and Leibniz' formula, we deduce the claimed formula. $\blacksquare$

The covariant symbol

$$\left(\frac{2}{\hbar}\right)^{|\alpha|} u^\alpha \frac{I_{n-1+|\alpha|}^-(2\sqrt{\bar{z} \cdot u}/\hbar)}{I_{n-1}^-(2\sqrt{\bar{z} \cdot u}/\hbar)}$$

may be expressed as an asymptotic series of $\hbar$, for $\hbar \rightarrow 0$, as the asymptotic of $I_N^-(\xi)$ is well known. For instance, we have

$$\frac{I_n^-(\omega/\hbar)}{I_{n-1}^-(\omega/\hbar)} \underset{\hbar \rightarrow 0}{\sim} \frac{\hbar}{\omega} \left(1 + (1-2n)\frac{\hbar}{2\omega} + \frac{1}{2}(3-2n)(1-2n) \left(\frac{\hbar}{2\omega}\right)^2 + O(\hbar^3) \right).$$

This gives the asymptotic, as $\hbar \rightarrow 0$, of the covariant symbol of the multiplication by x_j .

4. The star product

According to Berezin's theory, the composition formula (17) will allow us to define a star product. For the standard definition of star products, refer refs. [1, 3, 4, 8, 9].

Let $\delta : z \rightarrow (\bar{z}, z)$ defined on $\mathbb{C}^n \times \mathbb{C}^n$ be the function 'trace on the anti-diagonal.'

DEFINITION 4.1

$$\mathbf{A}_{\hbar} := \{ \sigma_A \circ \delta \text{ with } A \in \mathbf{T}^0(\mathbf{X}) \}.$$

$\mathbf{A}_{\hbar}$ is a sub-algebra of the algebra, with respect to the pointwise product of functions $f \circ \delta$, where $f(\bar{z}, u)$ are functions which are anti-analytic with respect to z and analytic with respect to u . We define the deformation of the point-wise product by the following:

DEFINITION 4.2 For every $f_1, f_2 \in \mathbf{A}_{\hbar}$, we define

$$f_1 *_\hbar f_2(z) = \left(\frac{\pi}{2}\right)^{-n} \hbar^{-2n} \int_{\mathbb{C}^n} \sigma_{A_2}(\bar{z}, w) \sigma_{A_1}(\bar{w}, z) \frac{|I_{n-1}^-(2\sqrt{\bar{z} \cdot w}/\hbar)|^2}{I_{n-1}^-(2|z|/\hbar)} K_0\left(\frac{2|w|}{\hbar}\right) dV(w), \quad (30)$$

where $f_1 := \sigma_{A_1} \circ \delta$ and $f_2 := \sigma_{A_2} \circ \delta$.

THEOREM 4.3 For every $f, f_1, f_2 \in \mathbf{A}_{\hbar}$, the product $f_1 *_{\hbar} f_2$ is associative, satisfies $f *_{\hbar} 1 = 1 *_{\hbar} f = f$ and we have the following asymptotic formula:

$$f_1 *_{\hbar} f_2(z) \stackrel{\hbar \rightarrow 0}{\sim} f_1(z) f_2(z) + \hbar \left[g^{\bar{j}i}(z) \bar{\partial}_{\bar{j}} f_1(z) \partial_i f_2(z) + f_2(z) f_1(z) \left(\frac{n-1}{4|z|} + \frac{R}{2} + g^{\bar{j}i}(z) \bar{\partial}_{\bar{j}} \partial_i \left(\frac{\psi}{g} \right) \right)_{w=z} \right] + O(\hbar^2),$$

where R is the scalar curvature defined by the Kähler metric $ds^2 = g_{i\bar{j}} d\bar{z}_j \wedge dz_i$, and $g^{\bar{j}i}$, ψ and g are functions described subsequently.

Proof The associativity follows from the fact that the composition in $T^0(X)$ is associative. The assertion about the unit element follows from formula (16). To prove the last assertion, the use of the asymptotic expansion of Bessel functions is essential, cf. [5, 5.11.9] [5.11.10]. Indeed, for $z \neq 0$ regarded as a parameter and for every non-negative integer N and every $w \neq 0$ such that $\arg(\bar{z} \cdot w) < \pi/2$, we have

$$\frac{|I_{n-1}^-(2\sqrt{\bar{z} \cdot w}/\hbar)|^2}{I_{n-1}^-(2|z|/\hbar)} K_0 \left(\frac{2|w|}{\hbar} \right) \stackrel{\hbar \rightarrow 0}{\sim} \frac{1}{2} \left(\frac{\hbar}{2} \right)^n \psi(\bar{w}, w) \chi(\bar{w}, w) e^{S(\bar{w}, w)/\hbar} \quad (31)$$

with

$$\begin{aligned} \psi(\bar{w}, w) &= |z|^{n-1/2} (w \cdot \bar{w})^{-1/4} (\bar{z} \cdot w)^{(1/4-n/2)} (z \cdot \bar{w})^{(1/4-n/2)}, \\ S(\bar{w}, w) &= 2(\sqrt{\bar{w} \cdot z} + \sqrt{\bar{z} \cdot w} - |w| - |z|) \end{aligned}$$

and $\chi(\bar{w}, w) = \sum_{m=0}^{N-1} \chi_m(\bar{w}, w) \hbar^m + O(\hbar^N)$, where χ_m is the factor of $\hbar^m$ in the expansion of

$$\begin{aligned} & \left| \sum_{k=0}^{N-1} (-1)^k (n-1, k) \left(\frac{\hbar}{4\sqrt{\bar{z} \cdot w}} \right)^k + O(\hbar^N) \right|^2 \times \left(\sum_{k=0}^{N-1} (-1)^k (n-1, k) \left(\frac{\hbar}{4|z|} \right)^k \right. \\ & \left. + O(\hbar^N) \right)^{-1} \times \left(\sum_{k=0}^{N-1} (0, k) \left(\frac{\hbar}{4|w|} \right)^k + O(\hbar^N) \right) \end{aligned}$$

where

$$(n-1, k) := \frac{(-1)^k}{k!} \left(\frac{3}{2} - n \right)_k \left(n - \frac{1}{2} \right)_k.$$

Particularly, when $m = 0$ or 1 , we have, with $(n-1, 1) = n - 2n + 3/4$,

$$\chi_0(\bar{w}, w) = 1 \quad \text{and} \quad \chi_1(\bar{w}, w) = \frac{1}{4} \left(\frac{(0, 1)}{|w|} + \frac{(n-1, 1)}{|z|} - \frac{(n-1, 1)}{\sqrt{\bar{z} \cdot w}} - \frac{(n-1, 1)}{\sqrt{\bar{w} \cdot z}} \right). \quad (32)$$

Thus, we may use Laplace's method to get the asymptotic expansion of Berezin's transform:

$$\frac{1}{2} (\pi \hbar)^{-n} \int_{\mathbb{C}^n} \sigma(\bar{w}, w) \psi(\bar{w}, w) \chi(\bar{w}, w) e^{S(\bar{w}, w)/\hbar} dV(w), \quad (33)$$

where $\sigma(\bar{w}, w) := F_2(\bar{z}, w) F_1(\bar{w}, z)$.

We are interested in an approximation of $f_1 *_{\hbar} f_2$ up to the order $O(\hbar^2)$:

$$f_1 *_{\hbar} f_2 \stackrel{\hbar \rightarrow 0}{\sim} \frac{1}{2} (\pi \hbar)^{-n} \int_{\mathbb{C}^n} \sigma \psi e^{S/\hbar} dV + \frac{1}{2} (\pi \hbar)^{-n} \hbar \int_{\mathbb{C}^n} \sigma \psi \chi_1 e^{S/\hbar} dV + O(\hbar^2). \quad (34)$$

These integrals will give respectively expressions of the form $A + B\hbar$ and $C\hbar$ plus $O(\hbar^2)$.

If f is any smooth function, Laplace's method gives

$$\int_{\mathbb{C}^n} f e^{S/\hbar} dV \stackrel{\hbar \rightarrow 0}{=} 2(\pi \hbar |z|)^n [f(\bar{z}, z) + O(\hbar)]. \quad (35)$$

Indeed, easy computations show that z is the only critical point for $S(\bar{w}, w)$. Away from this critical point z the integral is $O(\hbar^\infty)$, so that the asymptotic evaluation is actually done in a neighborhood of z . We also have $S(\bar{z}, z) = 0$ and

$$-\bar{\partial}_j \partial_i S(\bar{w}, w) = \frac{1}{|w|} \left(\delta_{ij} - \frac{\bar{w}_i w_j}{2|w|^2} \right),$$

where δ_{ij} is the Kronecker symbol. It is already known that $\det [\delta_{ij} \pm a_i b_j] = 1 \pm \sum_k a_k b_k$ for any complex numbers a_i and b_j , so that

$$\det [-\bar{\partial}_j \partial_i S(\bar{w}, w)]_{i,j=1}^n = \frac{1}{|w|^n} \left(1 - \sum_k \frac{\bar{w}_k w_k}{2|w|^2} \right) = \frac{1}{2|w|^n},$$

which we will denote by $g(w)$.

- Computation of C in equation (34):

We apply equation (35) to $f = \sigma \psi \chi_1$ and deduce that $C = (n - 1/4|z|)\sigma(z)$.

- Computation of A and B in equation (34):

We consider the Kähler potential $\Phi = 2|w|$, its sesquilinear extension, away from the semi-axis of complex numbers with argument π , is $\Phi(\bar{z}, w) := 2\sqrt{\bar{z} \cdot w}$. From it, we derive $S(\bar{w}, w) := \Phi(\bar{z}, w) + \Phi(\bar{w}, z) - \Phi(w) - \Phi(z)$ and $g(w) = \det [g_{i\bar{j}}(w)]$, where $g_{i\bar{j}} := \partial_i \bar{\partial}_j \Phi = -\partial_i \bar{\partial}_j S$. Following Theorem 3 in ref. [8], we have

$$\int_{\mathbb{C}^n} \sigma \frac{\psi}{g} e^{S/\hbar} g dV \stackrel{\hbar \rightarrow 0}{=} (\pi \hbar)^n \left[\frac{\sigma \psi}{g} + \hbar \left(L_1 \left(\frac{\sigma \psi}{g} \right) + \frac{R \sigma \psi}{2g} \right) \right]_{w=z}, \quad (36)$$

where $L_1 := g^{i\bar{j}} \bar{\partial}_j \partial_i$ is the Laplace–Beltrami operator, $g^{i\bar{j}}(w)$ are the coefficients of the inverse matrix of $[g_{i\bar{j}}(w)]$ and $R := L_1(\log g)$ is the scalar curvature.

The first term in the brackets of equation (36) is $\sigma(z)\psi(z)/g(z) = 2\sigma(z)$, so that $A = \sigma(z)$. Now,

$$L_1 \left(\frac{\sigma \psi}{g} \right) = g^{i\bar{j}} \left[\bar{\partial}_j \partial_i \sigma \cdot \left(\frac{\psi}{g} \right) + \bar{\partial}_j \sigma \cdot \partial_i \left(\frac{\psi}{g} \right) + \partial_i \sigma \cdot \bar{\partial}_j \left(\frac{\psi}{g} \right) + \sigma \cdot \bar{\partial}_j \partial_i \left(\frac{\psi}{g} \right) \right]. \quad (37)$$

We only have to show that

$$\partial_i \left(\frac{\psi}{g} \right) (z) = \bar{\partial}_j \left(\frac{\psi}{g} \right) (z) = 0.$$

Indeed, we have

$$\partial_i \left(\frac{\psi}{g} \right) = \frac{\partial_i \psi g - \psi \partial_i g}{g^2}.$$

However,

$$\begin{aligned} \partial_i \psi(w) &= |z|^{n-1/2} (z \cdot \bar{w})^{(1/4-n/2)} \left[-\frac{1}{4} \bar{w}_i (w \cdot \bar{w})^{-5/4} (\bar{z} \cdot w)^{(1/4-n/2)} \right. \\ &\quad \left. + \left(\frac{1}{4} - \frac{n}{2} \right) \bar{z}_i (w \cdot \bar{w})^{-1/4} (\bar{z} \cdot w)^{-(3/4-n/2)} \right] \end{aligned}$$

so that

$$\partial_i \psi(z) = -\frac{n}{2} \bar{z}_i |z|^{-n-2},$$

We also have

$$\partial_i g(w) = -\frac{n}{4} \bar{w}_i |w|^{-n-2},$$

so that we get $\partial_i \psi(z) = 2\partial_i g(z)$.

In the same way, we have $\bar{\partial}_j \psi(z) = 2\bar{\partial}_j g(z)$.

Therefore, ψ and $2g$ are tangents at the critical point z , and we have $\partial_i(\psi/g)(z) = 0$.

We may deduce

$$B = g^{i\bar{j}} \left[\bar{\partial}_j \partial_i \sigma + \frac{\sigma}{2} \cdot \left(\bar{\partial}_j \partial_i \left(\frac{\psi}{g} \right) + R \right) \right]_{w=z}.$$

■

Endowed with the product $*_{\hbar}$, $\mathbf{A}_{\hbar}$ is a non-commutative $*$ -algebra called ‘the algebra of covariant symbols for $H_{\hbar}$ ’. We may use Berezin’s procedure in the same fashion as in refs. [2, 8, 10] to build a star product on the direct sum of $\mathbf{A}_{\hbar}$ s for $\hbar = 1/m$ and $m = 1, 2, 3, \dots$

This article calls for at least two generalizations:

1. The sphere can be replaced by the boundary of a bounded strictly convex domain in $\mathbb{C}^n$.
2. Such a symbolic calculus may be extended to Toeplitz operators of all orders.

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Logarithms of Laplace–Stieltjes integral and maximum of the integrand

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Let $f(x) \geq 0$, $I(\sigma) = \int_0^\infty f(x)e^{x\sigma} dF(x)$, $\mu(\sigma) = \sup \{f(x)e^{x\sigma} : x \geq 0\}$, $\sigma \in \mathbb{R}$, and let Φ be a positive function on $(-\infty, +\infty)$ such that $\Phi'(\sigma) \uparrow +\infty$ as $\sigma \rightarrow +\infty$. We investigate the conditions on F , under which the asymptotical inequality $\ln \mu(\sigma) \leq (1 + o(1))\Phi(\sigma)$ implies the asymptotical inequality $\ln I(\sigma) \leq (1 + o(1))\Phi(\sigma)$ as $\sigma \rightarrow +\infty$.

Keywords: Laplace–Stieltjes integral; Maximum of the integrand

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1. Introduction and auxiliary results

Let V be a class of nonnegative nondecreasing unbounded continuous on the right functions F on $[0, +\infty)$. We say that $F \in V_1$ if $F \in V$ and $F(x) - F(x-0) \leq l < +\infty$ for all $x > 0$.

We assume that a positive function f on $[0, +\infty)$ is such that the Lebesgue–Stieltjes integral $\int_0^A f(x)e^{x\sigma} dF(x)$ exists for every $\sigma \in \mathbb{R}$ and $A \in [0, +\infty)$, and the integral

$$I(\sigma) = \int_0^\infty f(x)e^{x\sigma} dF(x), \quad \sigma \in \mathbb{R}, \quad (1)$$

is called of Laplace–Stieltjes. Integral (1) is a direct generalization of the Laplace integral and of the Dirichlet series. If $F(x) \equiv x$, then $I(\sigma) = \int_0^\infty f(x)e^{x\sigma} dx$ is an ordinary Laplace integral. If (λ_n) is a sequence of nonnegative numbers ($\lambda_0 = 0$) increasing to $+\infty$, $F(x) = n(x) = \sum_{\lambda_n \leq x} 1$ is a counting function of the sequence (λ_n) and f is a nonnegative function on $[0, +\infty)$ such that $f(\lambda_n) = a_n \geq 0$ for all $n \geq 0$, then

$$I(\sigma) = \sum_{n=0}^{\infty} a_n e^{\lambda_n \sigma} \quad (2)$$

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is the Dirichlet series with nonnegative coefficients and exponents. Let

$$\mu(\sigma) = \max \{f(x)e^{x\sigma} : x \geq 0\}, \quad \sigma \in \mathbb{R},$$

be the maximum of the integrand. Then either $\mu(\sigma) < +\infty$ for all $\sigma \in \mathbb{R}$ or $\mu(\sigma) = +\infty$ for all $\sigma \in \mathbb{R}$ or there exists a number σ_μ such that $\mu(\sigma) < +\infty$ for all $\sigma < \sigma_\mu$ and $\mu(\sigma) = +\infty$ for all $\sigma > \sigma_\mu$. It is not difficult to show that

$$\sigma_\mu = \liminf_{x \rightarrow +\infty} \frac{1}{x} \ln \frac{1}{f(x)}.$$

It is clear that if $f(x) = 0$ for all $x \geq x_0$, then $\sigma_\mu = +\infty$. Further, we exclude this case from the consideration, that is, we suppose that $f(x) \not\equiv 0$ on each interval $[x_0, +\infty)$.

Let $-\infty < A \leq +\infty$ and $\Omega(A)$ be a class of positive unbounded on $(-\infty, A)$ functions Φ such that the derivative Φ' is positive, continuously differentiable and increasing to $+\infty$ on $(-\infty, A)$. We denote by φ the inverse function to Φ' , and let

$$\Psi(x) = x - \frac{\Phi(x)}{\Phi'(x)}$$

be the function associated with Φ in the sense of Newton. Then [1, 2], the function φ is continuously differentiable and increasing to A on $(0, +\infty)$, and the function Ψ is continuously differentiable and increasing to A on $(-\infty, A)$.

The relation between the growth of $\ln \mu(\sigma)$ and the behaviour of $\ln f(x)$ in terms of a comparative function $\Phi \in \Omega(A)$ is stated in the following proposition [1, 2].

PROPOSITION 1 *Let $\sigma_\mu = A \in (-\infty, +\infty]$ and $\Phi \in \Omega(A)$. In order that $\ln \mu(\sigma) \leq \Phi(\sigma)$ for all $\sigma_0 \leq \sigma < A$, it is necessary and sufficient that $\ln f(x) \leq -x\Psi(\varphi(x))$ for all $x \geq x_0$.*

By $LS_A(F)$, we denote a class of integrals (1) with $\sigma_\mu = A$ and with given function F . We say that $I \in LS_\infty(F, \Phi)$ if $I \in LS_\infty(F)$ and $\ln \mu(\sigma) \leq \Phi(\sigma)$ for all $\sigma \geq \sigma_0$.

In general class $LS_\infty(F)$, the following result is true: *in order that for every integral $I \in LS_\infty(F)$, the asymptotical inequality $\ln I(\sigma) \leq (1 + o(1)) \ln \mu(\sigma)$ holds as $\sigma \rightarrow +\infty$ outside some set $E \in [0, +\infty)$ of finite measures, it is necessary and sufficient that $\int_{x_0}^{\infty} x^{-2} \ln F(x) dx < +\infty$. The sufficiency is proved in ref. [3]. The proof of the necessity is as in ref. [4].*

Here, we investigate the conditions on $F \in V$ by which the asymptotical inequality $\ln \mu(\sigma) \leq (1 + o(1))\Phi(\sigma)$ implies the asymptotical inequality $\ln I(\sigma) \leq (1 + o(1))\Phi(\sigma)$ as $\sigma \uparrow A = \sigma_\mu$. For the Dirichlet series (2), such a problem is solved in refs. [5–7].

We also need the following proposition.

PROPOSITION 2 *Let $F \in V_1$ and γ be a nonnegative continuous and increasing to $+\infty$ function on $[0, +\infty)$ such that*

$$\limsup_{x \rightarrow +\infty} \frac{\ln F(x)}{\gamma(x)} > 1. \quad (3)$$

Then there exists a function $G \in V$ satisfying the following conditions:

- (1) $G(x) \leq e^{\gamma(x)} + l$ for all $x > 0$;

- (2) *there exist increasing to $+\infty$ sequences (τ_k) and (x_k) , $x_0 < \tau_1 < x_1 < \tau_2 < \dots < x_k < \tau_{k+1} < x_{k+1} < \dots$, such that*

$$G(x_k) \geq e^{\gamma(x_k)}, \quad G(x_k) \geq 2G(\tau_k),$$

and

$$G(x) = \begin{cases} b_k, & x_k \leq x \leq \tau_{k+1} \\ F(x) - c_k, & \tau_k \leq x \leq x_k, \end{cases}$$

where b_k and δ_k are nonnegative constants;

- (3) *if*

$$f(x) = \begin{cases} 0, & x \in [x_k, \tau_{k+1}], \quad (k \geq 0) \\ g(x) > 0, & x \in (\tau_k, x_k), \quad (k \geq 1), \end{cases} \quad (4)$$

then

$$\int_0^\infty f(x) e^{\sigma x} dG(x) = \int_0^\infty f(x) e^{\sigma x} dF(x).$$

Proof We remark that condition (3) implies $\limsup_{x \rightarrow +\infty} F(x)/e^{\gamma(x)} = +\infty$. For simplicity, we assume that $F(0) < 1$.

Let $x_0 = \min \{x > 0: F(x) \geq e^{\gamma(x)}\}$ (as the function γ is continuous and the function F is continuous on the right, in view of (3), such point x_0 exists). If we put $G(x) = F(x)$ for $0 \leq x \leq x_0$, then $G(x_0) \geq e^{\gamma(x_0)}$ and in view of the condition $F(x) - F(x-0) \leq l < +\infty$, we have $G(x) \leq e^{\gamma(x)} + l$ for $0 \leq x \leq x_0$.

Let $s_1(\geq x_0)$ be the abscissa of the intersection point of the curves $y = e^{\gamma(x)}$ and $y = F(x_0) = G(x_0)$, and let $\tau_1 \geq s_1 + 1$ be an arbitrary point, which we choose later. We put $G(x) = F(x_0) = G(x_0)$ on $[x_0, \tau_1]$, that is, on this interval, we have again $G(x) \leq e^{\gamma(x)} + l$.

On $[\tau_1, +\infty)$, we consider the function

$$F_1(x) = F(x) - (F(\tau_1) - F(x_0)) = F(x) - (F(\tau_1) - G(\tau_1)).$$

Then $F_1(\tau_1) = F(x_0)$ and by condition (3), there exists the point $x_1 = \min \{x > \tau_1: F_1(x) \geq e^{\gamma(x)}\}$. Hence, one can see that the point τ_1 may be chosen to be so large such that $F_1(x_1) \geq 2F_1(\tau_1)$.

We put $G(x) = F_1(x)$ on $[\tau_1, x_1]$. Then, on this interval $G(x) \leq e^{\gamma(x)} + l$, $G(x) = F(x) - (F(\tau_1) - F(x_0)) = F(x) - c_1$ and $G(x_1) \geq e^{\gamma(x_1)}$.

Let $s_2(\geq x_1)$ be the abscissa of the intersection point of the curves $y = e^{\gamma(x)}$ $y = F_1(x_1)$, and $\tau_2 \geq s_2 + 1$ be an arbitrary point, which we choose later. We put $G(x) = F_1(x_1)$ on $[x_1, \tau_2]$, that is, on this interval, we have $G(x) \leq e^{\gamma(x)} + l$.

On $[\tau_2, +\infty)$, we consider the function

$$F_2(x) = F_1(x) - (F(\tau_2) - F_1(x_1)).$$

Then $F_2(\tau_2) = F_1(x_1)$, and by condition (3), there exists the point $x_2 = \min \{x > \tau_2: F_2(x) \geq e^{\gamma(x)}\}$. The point τ_2 may be chosen so large that $F_2(x_2) \geq 2F_2(\tau_2)$.

We put $G(x) = F_2(x)$ on $[\tau_2, x_2]$. Then, on this interval, $G(x) \leq e^{\gamma(x)} + l$, $G(x) = F(x) - c_2$ and $G(x_2) \geq e^{\gamma(x_2)}$.

Continuing the process, we construct a function G on $[0, +\infty)$ satisfying conditions (1) and (2).

Finally, for the function (4), we have

$$\begin{aligned}
 \int_0^\infty f(x)e^{\sigma x} dG(x) &= \int_0^\infty f(x)e^{\sigma x} dG(x) + \sum_{k=1}^\infty \int_{\tau_k}^{x_k} f(x)e^{\sigma x} dG(x) \\
 &= \int_0^\infty f(x)e^{\sigma x} dF(x) + \sum_{k=1}^\infty \int_{\tau_k}^{x_k} f(x)e^{\sigma x} d(F(x) - c_k) \\
 &= \int_0^\infty f(x)e^{\sigma x} dF(x) + \sum_{k=1}^\infty \int_{\tau_k}^{x_k} f(x)e^{\sigma x} dF(x) \\
 &= \int_0^\infty f(x)e^{\sigma x} dF(x).
 \end{aligned}$$

Proposition 2 is proved. ■

Remark From the inequality $G(x_k) \geq 2G(\tau_k)$, it follows that there exists $t_k \in (\tau_k, x_k)$ such that

$$G(x_k) - \sqrt{G(x_k)} - l \leq G(t_k) \leq G(x_k) - \sqrt{G(x_k)}. \quad (5)$$

2. Main theorem

We will distinguish only the cases $\sigma_\mu = +\infty$ and $\sigma_\mu = 0$, because the case $\sigma_\mu = A \in (-\infty, +\infty)$ reduces to case $\sigma_\mu = 0$, replacing σ by $\sigma + A$.

THEOREM 1 *Let $F \in V$, $A = +\infty$ or $A = 0$ and the function $\Phi \in \Omega(A)$ be such that the function Φ'/Φ is nondecreasing. In order that for every integral $I \in LS_A(F)$ the asymptotical inequality*

$$\ln \mu(\sigma) \leq (1 + o(1))\Phi(\sigma), \quad \sigma \uparrow A, \quad (6)$$

implies the asymptotical inequality

$$\ln I(\sigma) \leq (1 + o(1))\Phi(\sigma), \quad \sigma \uparrow A, \quad (7)$$

it is sufficient and if, moreover, $F \in V_1$ and $\ln \Phi'(\sigma) = o(\Phi(\sigma))$, $\sigma \uparrow A$, it is necessary that

$$\ln F(x) = o(\Phi(\Psi(\varphi(x))), \quad x \rightarrow +\infty. \quad (8)$$

Proof First we prove the sufficiency. From (6), for every $\varepsilon > 0$ and all $\sigma \in (\sigma_0(\varepsilon), A)$, we have $\ln \mu(\sigma) \leq (1 + \varepsilon)\Phi(\sigma)$ and, therefore, by Proposition 1, for $x \geq x_0(\varepsilon)$,

$$\ln F(x) \leq -x\Psi\left(\varphi\left(\frac{x}{1+\varepsilon}\right)\right). \quad (9)$$

We put $\beta(\sigma) = \Phi(\sigma)/\Phi'(\Psi^{-1}(\sigma))$. Then, $\sigma + \beta(\sigma) < A$ because the inequality in the case $A = +\infty$ is obvious, and if $A = 0$, then

$$\sigma + \beta(\sigma) \leq \sigma + \frac{\Phi(\Psi^{-1}(\sigma))}{\Phi'(\Psi^{-1}(\sigma))} = \sigma + \Psi^{-1}(\sigma) + \Psi(\Psi^{-1}(\sigma)) = \Psi^{-1}(\sigma) < 0.$$

From the inequality $\sigma + \beta(\sigma) < A$, it follows that the function $\Psi^{-1}(\sigma + \beta(\sigma))$ is defined on $(-\infty, A)$.

As $\Psi(\sigma) < \sigma$ $\Psi'(\sigma) = \Phi(\sigma)\Phi''(\sigma)/(\Phi'(\sigma))^2$, we have in view of the nondecrease of the function $\Phi'(\sigma)/\Phi'(\sigma)$,

$$\begin{aligned} \left(\frac{\Phi(\Psi(\sigma))}{\Phi'(\sigma)} \right)' &= \frac{\Phi'(\Psi(\sigma))\Phi(\sigma)\Phi''(\sigma)/\Phi'(\sigma) - \Phi''(\sigma)\Phi'(\Psi(\sigma))}{(\Phi'(\sigma))^2} \\ &= \frac{\Phi(\sigma)\Phi''(\sigma)\Phi(\Psi(\sigma))}{(\Phi'(\sigma))^3} \left(\frac{\Phi'(\Psi(\sigma))}{\Phi(\Psi(\sigma))} - \frac{\Phi'(\sigma)}{\Phi(\sigma)} \right) \leq 0, \end{aligned}$$

that is, the functions $\Phi(\Psi(\sigma))/\Phi'(\sigma)$ and $\beta(\sigma)$ are nonincreasing. Hence, in particular, it follows that $\beta(\sigma) \geq \Phi(\sigma + \beta(\sigma))/\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))$, and in view of (6), $\ln F(x) = o(x)$, $x \rightarrow +\infty$.

For simplicity, we assume that $F(0) = 0$ and put $g(\sigma) = (1 + \varepsilon)\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))$, $0 < \varepsilon < 1$. Then, in view of (9) and the nondecrease of $\Phi(\Psi(\varphi(x)))/x$, we have for $\sigma \in [\sigma_0(\varepsilon), A)$,

$$\begin{aligned} I(\sigma) &= \left(\int_0^{g(\sigma)} + \int_{g(\sigma)}^\infty \right) f(x) e^{x\sigma} dF(x) \\ &\leq \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^\infty \exp \left\{ -x \left(\Psi \left(\varphi \left(\frac{x}{1 + \varepsilon} \right) \right) - \sigma \right) \right\} dF(x) \\ &\leq \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^\infty \exp \left\{ -x \left(\Psi \left(\varphi \left(\frac{g(\sigma)}{1 + \varepsilon} \right) \right) - \sigma \right) \right\} dF(x) \\ &= \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^\infty e^{-x\beta(\sigma)} dF(x) \leq \mu(\sigma)F(g(\sigma)) + \beta(\sigma) \int_{g(\sigma)}^\infty F(x) e^{-x\beta(\sigma)} dx \\ &\leq \mu(\sigma)F(g(\sigma)) + \beta(\sigma) \int_{g(\sigma)}^\infty \exp \{ -x\beta(\sigma) + \varepsilon\Phi(\Psi(\varphi(x))) \} dx \\ &\leq \mu(\sigma)F(g(\sigma)) + \beta(\sigma) \int_{\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))}^\infty \exp \left\{ -x \left(\beta(\sigma) - \varepsilon \frac{\Phi(\Psi(\varphi(x)))}{x} \right) \right\} dx \\ &\leq \mu(\sigma)F(g(\sigma)) + \beta(\sigma) \int_{\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))}^\infty \exp \left\{ -x \left(\beta(\sigma) - \varepsilon \frac{\Phi(\sigma + \beta(\sigma))}{\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))} \right) \right\} dx \\ &\leq \mu(\sigma)F(g(\sigma)) + \beta(\sigma) \int_{\Phi'(\Psi^{-1}(\sigma + \beta(\sigma)))}^\infty \exp \{ -x(1 - \varepsilon)\beta(\sigma) \} dx \\ &\leq \mu(\sigma)F(g(\sigma)) + \frac{1}{1 - \varepsilon}, \end{aligned}$$

that is, $\ln I(\sigma) \leq \ln \mu(\sigma) + \ln F(g(\sigma)) + o(1)$, $\sigma \uparrow A$, and to obtain (8), it is enough to show that $\ln F(g(\sigma)) = o(\Phi(\sigma))$, $\sigma \uparrow A$.

The nonincrease of $\Phi(\Psi(\varphi(x)))/x$ implies the inequality $\Phi(\Psi(\varphi(x/(1 + \varepsilon)))) \geq \Phi(\Psi(\varphi(x)))/(1 + \varepsilon)$, and in view of the nondecrease of Φ'/Φ , we have for some $\xi = \xi(\sigma) \in (\Psi(\sigma), \sigma)$,

$$\ln \Phi(\sigma) - \ln \Phi(\Psi(\sigma)) = \frac{\Phi'(\xi)}{\Phi(\xi)}(\sigma - \Psi(\sigma)) = \frac{\Phi'(\xi)}{\Phi(\xi)} \frac{\Phi(\sigma)}{\Phi'(\sigma)} \leq 1,$$

that is, $\Phi(\sigma) \leq e\Phi(\Psi(\sigma))$, and as $\beta(\sigma) \leq \Psi^{-1}(\sigma)$, we have $\Phi(\sigma + \beta(\sigma)) \leq e\Phi(\sigma)$. Hence and from (9), we obtain

$$\begin{aligned}
 \limsup_{\sigma \uparrow A} \frac{\ln F(g(\sigma))}{\Phi(\sigma)} &= \limsup_{\sigma \uparrow A} \frac{\ln F((1 + \varepsilon)\Phi'(\Psi^{-1}(\sigma + \beta(\sigma))))}{\Phi(\sigma)} \\
 &\leq e \limsup_{\sigma \uparrow A} \frac{\ln F((1 + \varepsilon)\Phi'(\Psi^{-1}(\sigma + \beta(\sigma))))}{\Phi(\sigma + \beta(\sigma))} \\
 &\leq e \limsup_{\sigma \uparrow A} \frac{\ln F((1 + \varepsilon)\Phi'(\Psi^{-1}(\sigma)))}{\Phi(\sigma)} \\
 &\leq e \limsup_{\sigma \uparrow A} \frac{\ln F(t)}{\Phi(\Psi(\varphi(t/(1 + \varepsilon))))} \\
 &\leq \frac{e}{1 + \varepsilon} \limsup_{\sigma \uparrow A} \frac{\ln F(t)}{\Phi(\Psi(\varphi(t)))} = 0.
 \end{aligned}$$

The sufficiency of condition (9) is proved.

We prove the necessity, on the contrary, and assume that the condition (8) does not hold, that is,

$$\limsup_{x \rightarrow +\infty} \frac{\ln F(x)}{\Phi(\Psi(\varphi(x)))} > \eta.$$

We put $\gamma(x) = \eta\Phi(\Psi(\varphi(x)))$ $g(x) = \exp\{-x\Psi(\varphi(x))\}$. Then, by Proposition 1, there exists a function $G \in V$ satisfying the following conditions:

- (1) $G(x) \leq \exp\{\eta\Phi(\Psi(\varphi(x)))\} + l$, for all $x > 0$;
- (2) there exist increasing to $+\infty$ sequences (τ_k) and (x_k) , $x_0 < \tau_1 < x_1 < \tau_2 < \dots < x_k < \tau_{k+1} < x_{k+1} < \dots$, such that

$$G(x_k) \geq \exp\{\eta\Phi(\Psi(\varphi(x_k)))\}, \quad G(x_k) \geq 2G(\tau_k);$$

- (3) if

$$f(x) = \begin{cases} 0, & x \in [x_k, \tau_{k+1}], \quad (k \geq 0) \\ \exp\{-x\Psi(\varphi(x))\}, & x \in (\tau_k, x_k), \quad (k \geq 1), \end{cases} \quad (10)$$

then

$$I(\sigma) = \int_0^\infty f(x)e^{\sigma x} dF(x) = \int_0^\infty f(x)e^{\sigma x} dG(x).$$

From equation (10), the equality $\sigma_\mu = A$ follows and by Proposition 1, $\ln \mu(\sigma, I) \leq \Phi(\sigma)$, that is, the asymptotical inequality (6) holds.

We need to prove that asymptotical inequality (7) does not hold.

Let $t_k \in (\tau_k, x_k)$ be such that inequalities (5) hold. Then for all $\sigma < A$ and $k \geq 1$,

$$I(\sigma) \geq \int_{t_k}^{x_k} \exp\{-x\Psi(\varphi(x)) + \sigma x\} dG(x). \quad (11)$$

We assume at first that $A = +\infty$ and choose $\sigma_k = \varphi(x_k)$. Then from (11) and (5), we obtain

$$\begin{aligned}
 I(\sigma_k) &\geq \int_{t_k}^{x_k} \exp \{-x \Psi(\varphi(x)) + x \sigma_k\} dG(x) \\
 &\geq \exp \{-x_k \Psi(\varphi(x_k)) + t_k \sigma_k\} \int_{t_k}^{x_k} dG(x) \\
 &= \exp \{-x_k \Psi(\varphi(x_k)) + x_k \varphi(x_k) - (x_k - t_k) \varphi(x_k)\} (G(x_k) - G(t_k)) \\
 &\geq \exp \{\Phi(\varphi(x_k))\} \sqrt{G(x_k)} \exp \{-(x_k - t_k) \varphi(x_k)\}.
 \end{aligned} \tag{12}$$

In view of properties (1), (2) and inequality (5), we have for $k \geq k_0$,

$$\begin{aligned}
 t_k &\geq \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(t_k) - l}{\eta} \right) \right) \right) \\
 &\geq \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(x_k) - \sqrt{G(x_k)} - 2l}{\eta} \right) \right) \right) \\
 &\geq \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(x_k)}{\eta} - \frac{1}{\eta} \ln \left(1 - \frac{\sqrt{G(x_k)} - 2l}{G(x_k)} \right) \right) \right) \right) \\
 &\geq \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(x_k)}{\eta} - \frac{2}{\eta \sqrt{G(x_k)}} \right) \right) \right) = x_k - \delta_k,
 \end{aligned}$$

where

$$\begin{aligned}
 \delta_k &= \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(x_k)}{\eta} \right) \right) \right) - \Phi' \left(\Psi^{-1} \left(\Phi^{-1} \left(\frac{\ln G(x_k)}{\eta} - \frac{2}{\eta \sqrt{G(x_k)}} \right) \right) \right) \\
 &= \frac{\Phi''(\Psi^{-1}(\Phi^{-1}(\xi_k)))}{\Psi'(\Psi^{-1}(\Phi^{-1}(\xi_k))) \Phi'(\Phi^{-1}(\xi_k))} \frac{2}{\eta \sqrt{G(x_k)}} \\
 &= \frac{(\Phi'(\Psi^{-1}(\Phi^{-1}(\xi_k))))^2}{\Phi(\Psi^{-1}(\Phi^{-1}(\xi_k))) \Phi'(\Phi^{-1}(\xi_k))} \frac{2}{\eta \sqrt{G(x_k)}},
 \end{aligned}$$

and

$$\frac{\ln G(x_k)}{\eta} - \frac{2}{\eta \sqrt{G(x_k)}} \leq \xi_k \leq \frac{\ln G(x_k)}{\eta},$$

i.e.,

$$\begin{aligned}
 \delta_k &\leq \frac{2}{\eta} \frac{(\Phi'(\Psi^{-1}(\Phi^{-1}(\xi_k))))^2}{\Phi(\Psi^{-1}(\Phi^{-1}(\xi_k))) \Phi'(\Phi^{-1}(\xi_k))} \exp \left\{ -\frac{\ln G(x_k)}{2} \right\} \\
 &\leq \frac{2}{\eta} \frac{(\Phi'(\Psi^{-1}(\Phi^{-1}(\xi_k))))^2}{\Phi(\Psi^{-1}(\Phi^{-1}(\xi_k))) \Phi'(\Phi^{-1}(\xi_k))} e^{-\eta \xi_k / 2}
 \end{aligned}$$

and in view of the inequality $\Phi(\sigma) \leq e\Phi(\Psi(\sigma))$ and the condition $\ln \Phi'(\sigma) = o(\Phi(\sigma))$ ($\sigma \rightarrow +\infty$),

$$\begin{aligned}
 \limsup_{k \rightarrow \infty} \delta_k &\leq \frac{2}{\eta} \limsup_{\xi \rightarrow \infty} \frac{(\Phi'(\Psi^{-1}(\Phi^{-1}(\xi))))^2}{\Phi(\Psi^{-1}(\Phi^{-1}(\xi))) \Phi'(\Phi^{-1}(\xi))} e^{-\eta \xi / 2} \\
 &\leq \limsup_{\xi \rightarrow \infty} (\Phi'(\Psi^{-1}(\Phi^{-1}(\xi))))^2 e^{-\eta \xi / 2}
 \end{aligned}$$

$$\begin{aligned}
&= \limsup_{\sigma \rightarrow \infty} \exp \left\{ 2 \ln \Phi'(\sigma) - \frac{\eta \Phi(\Psi(\sigma))}{2} \right\} \\
&\leq \limsup_{\sigma \rightarrow \infty} \exp \left\{ 2 \ln \Phi'(\sigma) - \frac{\eta \Phi(\sigma)}{2e} \right\} = 0.
\end{aligned}$$

Therefore, from equation (12), we obtain

$$\begin{aligned}
\ln I(\sigma_k) &\geq \Phi(\varphi(x_k)) + \frac{\eta}{2} \Phi(\Psi(\varphi(x_k))) + o(\varphi(x_k)) \\
&\geq \Phi(\sigma_k) + \frac{\eta}{2e} \Phi(\sigma_k) + o(\sigma_k), \quad k \rightarrow \infty,
\end{aligned}$$

that is, the asymptotical inequality (7) does not hold. The necessity of condition (8) is proved in the case $A = +\infty$.

Now let $A = 0$. We choose $\sigma_k = \varphi(t_k)$. Then we have instead of equation (12), the following inequality:

$$\begin{aligned}
I(\sigma_k) &\geq \exp \{-t_k \Psi(\varphi(t_k)) + x_k \sigma_k\} \sqrt{G(x_k)} \\
&= \exp \{-t_k \Psi(\varphi(t_k)) + t_k \varphi(t_k) - (x_k - t_k) |\varphi(x_k)|\} \sqrt{G(x_k)} \\
&= \exp \{\Phi(\sigma_k) - (x_k - t_k) |\sigma_k|\} \sqrt{G(x_k)}.
\end{aligned} \tag{13}$$

As above, $x_k - t_k = \delta_k \rightarrow 0$, $x_k \rightarrow \infty$, and

$$\begin{aligned}
\sqrt{G(x_k)} &\geq \exp \left\{ \frac{\eta}{2} \Phi(\Psi(\varphi(x_k))) \right\} > \exp \left\{ \frac{\eta}{2} \Phi(\Psi(\varphi(t_k))) \right\} \\
&\geq \exp \left\{ \frac{\eta}{2e} \Phi(\varphi(t_k)) \right\} = \exp \left\{ \frac{\eta}{2} \Phi(\sigma_k) \right\}.
\end{aligned}$$

Therefore, in view of equation (13), the asymptotical inequality (7) does not hold, that is, the necessity of condition (8) is proved in the case $A = 0$. The proof of Theorem 1 is complete. ■

3. Remarks and additions

In the case $A = 0$, the condition of nondecrease of Φ'/Φ is not laborious in Theorem 1. Indeed, if $\Phi(\sigma) = B(1/|\sigma|)$, $\sigma < 0$, then $\Phi'(\sigma)/\Phi(\sigma) = B'(1/|\sigma|)/B(1/|\sigma|)|\sigma|^2$ and the condition of nondecrease of Φ'/Φ is equivalent to the condition of nondecrease of the function $x^2 B'(x)/B(x)$, which is in turn a condition on the smoothness but not the rate of growth.

The situation is different if $A = +\infty$. An exponential function $\Phi(\sigma) = Te^{\varrho\sigma}$ ($T > 0$, $\varrho > 0$) satisfies the condition of nondecrease of the function Φ'/Φ , and a power function $\Phi(\sigma) = T\sigma^p$ ($p > 1$), $\sigma \geq \sigma_0$, does not satisfy the condition. However, the following theorem is true.

THEOREM 2 *Let $F \in V$ and the function $\Phi \in \Omega(+\infty)$ be such that*

$$1 < h \leq \frac{\sigma \Phi'(\sigma)}{\Phi(\sigma)} \leq H < +\infty, \quad \sigma \in [\sigma_0, +\infty). \tag{14}$$

In order that for every integral $I \in LS_{+\infty}(F)$, the asymptotical inequality (6) implies the asymptotical inequality (7), it is sufficient and if, moreover, $F \in V_1$, it is necessary that condition (8) holds.

Proof Using condition (14) and Lagrange theorem, it is easy to prove that $\Phi(\sigma) \leq \Phi(\Psi(\sigma)) \exp\{H/(h-1)\}$, $\Phi(\Psi(\varphi(2x))) \leq \Phi(\Psi(\varphi(x))) \exp\{H/(h-1)\}$ and $\Phi(2\sigma) \leq \Phi(\sigma) \exp\{H\}$.

We prove the sufficiency of condition (8). As $\Phi(\Psi(\sigma)) \leq \Psi(\sigma)\Phi'(\Psi(\sigma))/h \leq \Psi(\sigma)\Phi'(\sigma)/h$, we have $\Phi(\Psi(\varphi(x))) \leq 1/h \exp\{H/(h-1) - \ln h\} x \Psi(\varphi(x/2))$ and condition (8) implies $\ln F(x) = o(x \Psi(\varphi(x/2)))$, $x \rightarrow +\infty$. Therefore, if we put $g(\sigma) = 2\Phi'(\Psi^{-1}(2\sigma))$, then in view of equation (9) with $e = 1$, we have for all σ large enough,

$$\begin{aligned} I(\sigma) &\leq \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^{\infty} \exp\left\{-x\left(\Psi\left(\varphi\left(\frac{x}{2}\right)\right) - \sigma\right)\right\} dF(x) \\ &\leq \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^{\infty} \exp\left\{-\frac{x}{2}\Psi\left(\varphi\left(\frac{x}{2}\right)\right)\right\} dF(x) \\ &\leq \mu(\sigma)F(g(\sigma)) + 1, \end{aligned}$$

and to obtain equation (8), as in the proof of Theorem 1, it is enough to prove that $\ln F(g(\sigma)) = o(\Phi(\sigma))$, $\sigma \rightarrow +\infty$. But

$$\begin{aligned} \limsup_{\sigma \rightarrow +\infty} \frac{\ln F(g(\sigma))}{\Phi(\sigma)} &= \limsup_{\sigma \rightarrow +\infty} \frac{\ln F(2\Phi'(\Psi^{-1}(2\sigma)))}{\Phi(\sigma)} \\ &\leq \limsup_{x \rightarrow +\infty} \frac{\ln F(x)}{\Phi(1/2)\Psi(\varphi(x/2))} \\ &\leq e^H \exp\left\{\frac{H}{h-1}\right\} \limsup_{x \rightarrow +\infty} \frac{\ln F(x)}{\Phi(\Psi(\varphi(x)))} = 0, \end{aligned}$$

that is, the sufficiency of condition (8) is proved.

Now we prove its necessity. As in the proof of the necessity of condition (8) in Theorem 1 (case $A = +\infty$), we choose the function f in the form (19) and obtain estimate (12), that is,

$$I(\varphi(x_k)) \geq \exp\{\Phi(\varphi(x_k))\} \sqrt{G(x_k)} \exp\{-\delta_k \varphi(x_k)\}.$$

Thus, we need to prove that $\delta_k \rightarrow 0$, $k \rightarrow \infty$, and to estimate $\sqrt{G(x_k)}$ from the following.

Using condition (14), we have (see proof of Theorem 1)

$$\begin{aligned} \limsup_{k \rightarrow \infty} \delta_k &\leq \frac{2}{\eta} \limsup_{\xi \rightarrow \infty} \frac{(\Phi'(\Psi^{-1}(\Phi^{-1}(\xi))))^2}{\Phi(\Psi^{-1}(\Phi^{-1}(\xi)))\Phi'(\Phi^{-1}(\xi))} e^{-\eta\xi/2} \\ &= \frac{2}{\eta} \limsup_{\sigma \rightarrow \infty} \frac{(\Phi'(\Psi^{-1}(\sigma)))^2}{\Phi(\Psi^{-1}(\sigma))\Phi'(\sigma)} \exp\left\{-\frac{\eta}{2}\Phi(\sigma)\right\} \\ &\leq \frac{2H}{\eta} \limsup_{\sigma \rightarrow \infty} \frac{\Phi(\Psi^{-1}(\sigma))}{(\Psi^{-1}(\sigma))^2\Phi'(\sigma)} \exp\left\{-\frac{\eta}{2}\Phi(\sigma)\right\} \\ &\leq \frac{2H}{\eta} \exp\left\{\frac{H}{h-1}\right\} \limsup_{\sigma \rightarrow \infty} \frac{\Phi(\sigma)}{(\Psi^{-1}(\sigma))^2\Phi'(\sigma)} \exp\left\{-\frac{\eta}{2}\Phi(\sigma)\right\} = 0, \end{aligned}$$

and

$$\ln \sqrt{G(x_k)} \geq \frac{\eta}{2} \Phi(\Psi(\varphi(x_k))) \geq \frac{\eta}{2} \exp\left\{-\frac{H}{h-1}\right\} \Phi(\varphi(x_k)).$$

Therefore, as mentioned earlier, the asymptotical inequality (7) does not hold. The proof of Theorem 2 is complete. ■

We remark that, as condition (14) implies the relation $\Phi(\Psi(\varphi(x))) \asymp \Phi(\varphi(x))$, $x \rightarrow +\infty$, we can replace in Theorem 2 condition (14) by the condition $\ln F(x) = o(\Phi(\varphi(x)))$, $x \rightarrow +\infty$.

We also remark that condition (14) does not hold if the function $\Phi \in \Omega(+\infty)$ is such that $\Phi(\sigma) = \sigma\alpha(\sigma)$ for $\sigma \geq \sigma_0$, where α is a slowly increasing function. In the latter case, the following theorem is true.

THEOREM 3 *Let $F \in V$ and the function $\Phi \in \Omega(+\infty)$ be such that $\Phi(\sigma) = \sigma\alpha(\sigma)$ for $\sigma \geq \sigma_0$, where α is a positive, twice continuous differentiable and slowly increasing to $+\infty$ function on $[\sigma_0, +\infty)$ such that*

$$\alpha(\sigma) \sim \alpha(\sigma \ln \sigma) \sim \alpha(\sigma\alpha(\sigma)) \sim \alpha(\sigma^2\alpha'(\sigma)) \sim \alpha\left(\frac{\sigma^2\alpha'(\sigma)}{\alpha(\sigma)}\right), \quad \sigma \rightarrow +\infty. \quad (15)$$

In order that for every integral $I \in LS_{+\infty}(F)$ the asymptotical inequality (6) implies the asymptotical inequality (7), it is sufficient that

$$\limsup_{x \rightarrow +\infty} \frac{\alpha(\ln F(x))}{x} < 1, \quad (16)$$

and if, moreover, $F \in V_1$ and

$$\frac{\sigma\alpha'(\sigma)}{\alpha(\sigma)} \ln \alpha(\sigma) \rightarrow 0, \quad \sigma \rightarrow +\infty,$$

it is necessary that

$$\limsup_{x \rightarrow +\infty} \frac{\alpha(\ln F(x))}{x} \leq 1. \quad (17)$$

Proof As $\Phi(\sigma) = \sigma\alpha(\sigma)$ because $\sigma \geq \sigma_0$ and the function α is slowly increasing, we have $\Phi(\sigma) = (1 + o(1))\alpha(\sigma)$ and $\Psi(\sigma) = (1 + o(1))\sigma^2\alpha'(\sigma)/\alpha(\sigma)$ as $\sigma \rightarrow +\infty$.

Now, we assume that condition (16) holds. Then, there exists a number $q \in (0, 1)$ such that $\ln F(x) \geq \alpha^{-1}(qx)$, $x \geq x_0$. We put $\varepsilon = (1 - q)/2q$ and show that

$$\alpha^{-1}(qx) = o\left(x\Psi\left(\varphi\left(\frac{x}{1+\varepsilon}\right)\right)\right), \quad x \rightarrow +\infty.$$

Indeed, if there exist a number $\eta > 0$ and an increasing to $+\infty$ sequence (x_k) such that $\alpha^{-1}(qx_k) \geq \eta x\Psi(\varphi(x_k/(1+\varepsilon)))$, then for $\sigma_k = \varphi(x_k/(1+\varepsilon))$, we have $q(1+\varepsilon)\Phi'(\sigma_k) \geq \alpha(\eta\Phi'(\sigma_k)\Psi(\sigma_k))$, that is, in view of equation (15),

$$\frac{1+q}{2}\alpha(\sigma_k) \geq (1+o(1))\alpha(\sigma_k^2\alpha'(\sigma_k)) = (1+o(1))\alpha(\sigma_k), \quad k \rightarrow \infty,$$

and we obtain a contradiction, because $q < 1$.

Hence, it follows that for $\varepsilon = (1 - q)/2q$,

$$\ln F(x) = o\left(x\Psi\left(\varphi\left(\frac{x}{1+\varepsilon}\right)\right)\right), \quad x \rightarrow +\infty.$$

Therefore, if we put

$$g(\sigma) = (1 + \varepsilon)\Phi'(\Psi^{-1}(2\sigma)) = \frac{1+q}{2q}\Phi'(\Psi^{-1}(2\sigma)),$$

then in view of (9), as in the proof of Theorem 2, we have for large enough σ ,

$$I(\sigma) \leq \mu(\sigma)F(g(\sigma)) + \int_{g(\sigma)}^{\infty} \exp\left\{-\frac{x}{2}\left(\Psi\left(\varphi\left(\frac{x}{1+\varepsilon}\right)\right)\right)\right\} dF(x) \leq \mu(\sigma)F(g(\sigma)) + 1,$$

and we need, as above, to prove that $\ln F(g(\sigma)) = o(\Phi(\sigma))$, $\sigma \rightarrow +\infty$.

At first, we show that

$$\alpha^{-1}\left(\frac{1+q}{2}x\right) = o(\Phi(\Psi\varphi(x))), \quad x \rightarrow +\infty.$$

Indeed, if there exists a number $\eta > 0$ and an increasing to $+\infty$ sequence (x_k) such that $\alpha^{-1}((1+q)x_k/2) \geq \eta\Phi(\Psi\varphi(x_k))$, then for $\sigma_k = \varphi(x_k/(1+\varepsilon))$, as above, we have

$$\begin{aligned} \frac{1+q}{2}\alpha(\sigma_k) &\geq (1+o(1))\alpha(\Phi(\Psi(\sigma_k))) = (1+o(1))\alpha(\Psi(\sigma_k))\alpha(\Psi(\sigma_k)) \\ &= (1+o(1))\alpha(\Psi(\sigma_k)) = (1+o(1))\alpha\left(\frac{\sigma_k^2\alpha'(\sigma_k)}{\alpha(\sigma_k)}\right) \\ &= (1+o(1))\alpha(\sigma_k), \quad k \rightarrow \infty, \end{aligned}$$

that is, we obtain the contradiction.

Using the proved relation, we have

$$\limsup_{\sigma \rightarrow +\infty} \frac{\ln F(g(\sigma))}{\Phi(\sigma)} \leq \limsup_{\sigma \rightarrow +\infty} \frac{\alpha'(qg(\sigma))}{\Phi(\sigma)} = 2 \limsup_{x \rightarrow +\infty} \frac{\alpha'((1+q)x/2)}{\Phi(\Psi(\varphi(x)))} = 0.$$

The sufficiency of condition (16) is proved.

Now we assume that condition (17) does not hold, that is, $\limsup_{x \rightarrow +\infty} \alpha(\ln F(x))/x > 1$. Then there exists a number $q > 1$ such that $\limsup_{x \rightarrow +\infty} \ln F(x)/\alpha^{-1}(qx) > 1$, because $\alpha^{-1}(q_1x) = o(\alpha^{-1}(q_2x))$, $x \rightarrow +\infty$, if $q_1 < q_2$. We put $\gamma(x) = \alpha^{-1}(qx)$ and $g(x) = \exp\{-\int_0^x \alpha^{-1}(t) dt\}$. Then by Proposition 2, there exists a function $G \in V_1$, which satisfies the following conditions:

- (1) $G(x) \leq \exp\{\alpha^{-1}(qx)\} + l$ for all $x > 0$;
- (2) there exist an increasing to $+\infty$ sequences (τ_k) and (x_k) , $x_0 < \tau_1 < x_1 < \tau_2 < \dots < x_k < \tau_{k+1} < x_{k+1} < \dots$, such that

$$G(x_k) \geq \exp\{\alpha^{-1}(qx_k)\}, \quad G(x_k) \geq 2G(\tau_k);$$

- (3) if

$$f(x) = \begin{cases} 0, & x \in [x_k, \tau_{k+1}], \quad (k \geq 0) \\ \exp\{-\int_0^x \alpha^{-1}(t) dt\}, & x \in (\tau_k, x_k), \quad (k \geq 1), \end{cases} \quad (18)$$

then

$$I(\sigma) = \int_0^{\infty} f(x)e^{\sigma x} dF(x) = \int_0^{\infty} f(x)e^{\sigma x} dG(x).$$

From equation (18), we have for arbitrary $\sigma < +\infty$,

$$\begin{aligned}\ln \mu(\sigma) &\leq \max \left\{ -\int_0^x \alpha^{-1}(t) dt + \sigma x : x \geq 0 \right\} = -\int_0^{\alpha(\sigma)} \alpha^{-1}(t) dt + \sigma \alpha(\sigma) \\ &= -\int_{\alpha^{-1}(t)}^{\sigma} \alpha^{-1}(t) dt + \sigma \alpha(\sigma) \\ &= \int_{\alpha^{-1}(t)}^{\sigma} \alpha(x) dx = (1 + o(1))\sigma \alpha(\sigma), \quad \sigma \rightarrow +\infty,\end{aligned}$$

that is, $\sigma_\mu = +\infty$ and the asymptotical inequality (6) holds.

Now we prove that the asymptotical inequality (7) does not hold.

Let $t_k \in (\tau_k, x_k)$ such that inequalities (5) hold. Then for $\sigma_k = \alpha^{-1}(x_k)$ and $k \geq 1$,

$$\begin{aligned}I(\sigma_k) &\geq \int_{t_k}^{x_k} \exp \left\{ -\int_0^x \alpha^{-1}(t) dt + \sigma_k x \right\} dG(x) \\ &\geq \exp \left\{ -\int_0^{x_k} \alpha^{-1}(t) dt + \sigma_k t_k \right\} \sqrt{G(x_k)} \\ &\geq \exp \left\{ -\int_0^{x_k} \alpha^{-1}(t) dt + x_k \alpha^{-1}(x_k) - (x_k - t_k) \sigma_k \right\} \exp \left\{ \frac{1}{2} \alpha^{-1}(qx_k) \right\} \\ &\geq \exp \{-\delta_k \sigma_k\} \exp \left\{ \frac{1}{2} \alpha^{-1}(qx_k) \right\} = \exp \{-\delta_k \sigma_k\} \exp \left\{ \frac{1}{2} \alpha^{-1}(q\alpha(\sigma_k)) \right\}, \quad (19)\end{aligned}$$

where $\delta_k = x_k - t_k$. Because, as in the proof of Theorem 1, in view of the slow increase of α ,

$$\delta_k \leq x_k - \frac{1}{q} \alpha(\ln G(t_k) - l) \leq \frac{1}{q} \alpha(\ln G(x_k)) - \frac{1}{q} \alpha \left(\ln G(x_k) - \frac{2}{\sqrt{G(x_k)}} \right) \rightarrow 0$$

as $k \rightarrow \infty$, we obtain from equation (19),

$$\liminf_{k \rightarrow \infty} \frac{\ln I(\sigma_k)}{\sigma_k \alpha(\sigma_k)} \geq \frac{1}{2} \liminf_{k \rightarrow \infty} \frac{\alpha^{-1}(q\alpha(\sigma_k))}{\sigma_k \alpha(\sigma_k)} \geq \frac{1}{2} \liminf_{x \rightarrow +\infty} \frac{\alpha^{-1}(qx)}{x \alpha^{-1}(x)} = +\infty,$$

because for some $\xi \in (x, qx)$,

$$\ln \frac{\alpha^{-1}(qx)}{x \alpha^{-1}(x)} = \frac{\alpha^{-1}(\xi)'}{\alpha^{-1}(\xi)} (q-1)x - \ln x \geq \frac{\alpha^{-1}(\xi)'}{\alpha^{-1}(\xi)} (q-1)\xi - \ln \xi - \ln q,$$

and in view of the condition $(\sigma \alpha'(\sigma)/\alpha(\sigma)) \ln \alpha(\sigma) \rightarrow 0, \sigma \rightarrow +\infty$,

$$\lim_{\xi \rightarrow +\infty} \frac{d \ln \alpha^{-1}(x)}{\ln x d \ln x} = \lim_{\sigma \rightarrow +\infty} \frac{d \ln \sigma}{\ln \alpha(\sigma) d \ln \alpha(\sigma)} = +\infty.$$

The proof of Theorem 3 is complete. ■

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The Hankel transform of the sum of consecutive generalized Catalan numbers

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We discuss the properties of the Hankel transformation of a sequence whose elements are the sums of consecutive generalized Catalan numbers and find their values in the closed form.

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1. Introduction

The *Hankel transform* of a given sequence $A = \{a_0, a_1, a_2, \dots\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ (see [7–10]) where $h_n = |a_{i+j-2}|_{i,j=1}^n$, *i.e.*

$$A = \{a_n\}_{n \in \mathbb{N}_0} \longrightarrow h = \{h_n\}_{n \in \mathbb{N}_0} : h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & & a_{n+1} \\ \vdots & & \ddots & \\ a_n & a_{n+1} & & a_{2n} \end{vmatrix}. \quad (1)$$

In this paper, we will consider the sequence of the sums of two adjacent generalized Catalan numbers with parameter L :

$$a_0 = L + 1, \quad a_n = a_n(L) = c(n; L) + c(n + 1; L) \quad (n \in \mathbb{N}), \quad (2)$$

where

$$c(n; L) = T(2n, n; L) - T(2n, n - 1; L) \quad (3)$$

with

$$T(n, k; L) = \sum_{j=0}^{n-k} \binom{k}{j} \binom{n-k}{j} L^j. \quad (4)$$

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The properties of $\{T(n, k; L)\}$ were studied by P. Barry in [1].

Example 1.1 Let $L = 1$. Vandermonde's convolution identity implies that

$$\binom{n}{k} = \sum_j \binom{k}{j} \binom{n-k}{j}.$$

Hence

$$T(2n, n; 1) = \binom{2n}{n}, \quad T(2n, n-1; 1) = \binom{2n}{n-1},$$

wherefrom we get Catalan numbers

$$c(n) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}$$

and

$$a_n = c(n) + c(n+1) = \frac{(2n)!(5n+4)}{n!(n+2)!} \quad (n = 0, 1, 2, \dots).$$

In paper [2], Cvetković *et al.* have proved that the Hankel transform of a_n equals sequence of Fibonacci numbers with odd indices

$$h_n = F_{2n+1} = \frac{1}{\sqrt{5} 2^{n+1}} \left\{ (\sqrt{5} + 1)(3 + \sqrt{5})^n + (\sqrt{5} - 1)(3 - \sqrt{5})^n \right\}.$$

Example 1.2 For $L = 2$ we get like $a_n(2)$ the next numbers

$$3, 8, 28, 112, 484, \dots$$

and the Hankel transform h_n :

$$3, 20, 272, 7424, 405504, \dots$$

One of us, Barry conjectured that

$$h_n(2) = 2^{((n^2-n)/2)-2} \left\{ (2 + \sqrt{2})^{n+1} + (2 - \sqrt{2})^{n+1} \right\}.$$

In general, Barry made the conjecture, which we will prove through this paper.

THEOREM 1.3 (The main result) *For the generalized Pascal triangle associated to the sequence $n \mapsto L^n$, the Hankel transform of the sequence*

$$c(n; L) + c(n+1; L)$$

is given by

$$\begin{aligned} h_n = \frac{L^{(n^2-n)/2}}{2^{n+1} \sqrt{L^2+4}} \cdot \left\{ (\sqrt{L^2+4} + L)(\sqrt{L^2+4} + L + 2)^n \right. \\ \left. + (\sqrt{L^2+4} - L)(L + 2 - \sqrt{L^2+4})^n \right\}. \end{aligned} \quad (5)$$

From now till the end, let us denote by

$$\xi = \sqrt{L^2 + 4}, \quad t_1 = L + 2 + \xi, \quad t_2 = L + 2 - \xi. \quad (6)$$

Now, we can write

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot ((\xi + L)t_1^n + (\xi - L)t_2^n).$$

Or, introducing

$$\varphi_n = t_1^n + t_2^n, \quad \psi_n = t_1^n - t_2^n \quad (n \in \mathbb{N}_0), \quad (7)$$

the final statement can be expressed by

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1}\xi} \cdot (L\psi_n + \xi\varphi_n). \quad (8)$$

LEMMA 1.4 *The values φ_n and ψ_n satisfy the next relations:*

$$\varphi_j \cdot \varphi_k = \varphi_{j+k} + (4L)^j \varphi_{k-j}, \quad \psi_j \cdot \psi_k = \varphi_{j+k} - (4L)^j \varphi_{k-j} \quad (0 \leq j \leq k), \quad (9)$$

$$\varphi_j \cdot \psi_k = \psi_{j+k} + (4L)^j \psi_{k-j}, \quad \psi_j \cdot \varphi_k = \psi_{j+k} - (4L)^j \psi_{k-j} \quad (0 \leq j \leq k). \quad (10)$$

COROLLARY 1.5 *Assuming that the main theorem is true, the function $h_n = h_n(L)$ is the next polynomial*

$$h_n(L) = 2^{-n} L^{n(n-1)/2} \cdot \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L+2)^{n-2i-1} (L^2+4)^i + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L+2)^{n-2i} (L^2+4)^i \right\}.$$

Proof By previous notation, we can write

$$\begin{aligned} & (L + \xi)(L + 2 + \xi)^n - (L - \xi)(L + 2 - \xi)^n \\ &= (L + \xi) \sum_{k=0}^n \binom{n}{k} (L + 2)^{n-k} \xi^k - (L - \xi) \sum_{k=0}^n (-1)^k \binom{n}{k} (L + 2)^{n-k} \xi^k \\ &= \sum_{k=0}^n (1 - (-1)^k) \binom{n}{k} L(L + 2)^{n-k} \xi^k + \sum_{k=0}^n (1 + (-1)^k) \binom{n}{k} (L + 2)^{n-k} \xi^{k+1} \\ &= 2 \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i+1} + 2 \sum_{i=0}^{[n/2]} \binom{n}{2i} (L + 2)^{n-2i} \xi^{2i+1} \\ &= 2\xi \left\{ \sum_{i=0}^{[(n-1)/2]} \binom{n}{2i+1} L(L + 2)^{n-2i-1} \xi^{2i} + \sum_{i=0}^{[n/2]} \binom{n}{2i} (L + 2)^{n-2i} \xi^{2i} \right\}, \end{aligned}$$

wherefrom immediately follows the polynomial expression for h_n . ■

2. The generating function for the sequences of numbers and orthogonal polynomials

The Jacobi polynomials are given by

$$P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} (x-1)^{n-k} (x+1)^k \quad (a, b > -1).$$

Also, they can be written in the form

$$P_n^{(a,b)}(x) = \left(\frac{x-1}{2}\right)^n \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} \left(\frac{x+1}{x-1}\right)^k.$$

From the fact

$$L = \frac{x+1}{x-1} \iff x = \frac{L+1}{L-1} \quad (x \neq 1, L \neq 1),$$

we conclude that:

$$T(2n, n; L) = (L-1)^n \cdot P_n^{(0,0)}\left(\frac{L+1}{L-1}\right),$$

$$T(2n+2, n; L) = (L-1)^n \cdot P_n^{(2,0)}\left(\frac{L+1}{L-1}\right).$$

The generating function $G(x, t)$ for the Jacobi polynomials is

$$G^{(a,b)}(x, t) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x) t^n = \frac{2^{a+b}}{\phi \cdot (1-t+\phi)^a \cdot (1+t+\phi)^b}, \quad (11)$$

where

$$\phi = \phi(x, t) = \sqrt{1-2xt+t^2}.$$

Now,

$$\sum_{n=0}^{\infty} T(2n, n; L) t^n = \sum_{n=0}^{\infty} P_n^{(0,0)}\left(\frac{L+1}{L-1}\right) ((L-1)t)^n = G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right),$$

$$\sum_{n=0}^{\infty} T(2n+2, n; L) t^n = \sum_{n=0}^{\infty} P_n^{(2,0)}\left(\frac{L+1}{L-1}\right) ((L-1)t)^n = G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right).$$

Also,

$$\begin{aligned} \sum_{n=0}^{\infty} T(2n, n-1; L) t^n &= t \cdot \left\{ G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - 1 \right\}, \\ \sum_{n=0}^{\infty} T(2n+2, n+1; L) t^n &= \frac{1}{t} \cdot \left\{ G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - 1 \right\}. \end{aligned}$$

The generating function $\mathcal{G}(t; L)$ for the sequence $\{a_n\}_{n \geq 0}$ is given by

$$\begin{aligned} \mathcal{G}(t; L) &= \sum_{n=0}^{\infty} a_n t^n = \frac{t+1}{t} G^{(0,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) \\ &\quad - (t+1) G^{(2,0)}\left(\frac{L+1}{L-1}, (L-1)t\right) - \frac{1}{t}. \end{aligned} \quad (12)$$

After some computation, we prove the following theorem.

THEOREM 2.1 The generating function $\mathcal{G}(t; L)$ for the sequence $\{a_n\}_{n \geq 0}$ is

$$\mathcal{G}(t; L) = \frac{t+1}{\rho(t; L)} \left\{ \frac{1}{t} - \frac{4}{(1 - (L-1)t + \rho(t; L))^2} \right\} - \frac{1}{t}, \quad (13)$$

where

$$\rho(t; L) = \phi \left(\frac{L+1}{L-1}, (L-1)t \right) = \sqrt{1 - 2(L+1)t + (L-1)^2 t^2}. \quad (14)$$

The function $\rho(t; L)$ has domain

$$D_\rho = \left(-\infty, \frac{1 - 2\sqrt{L} + L}{1 - 2L + L^2} \right) \cup \left(\frac{1 + 2\sqrt{L} + L}{1 - 2L + L^2}, +\infty \right) \quad (L \neq 1)$$

and

$$D_\rho = \left(-\infty, \frac{1}{4} \right) \quad (L = 1).$$

Example 2.2 For $L = 1$, we get

$$\mathcal{G}(t; 1) = \sum_{n=0}^{\infty} a_n(1) t^n = \frac{1}{t} \left(\frac{(1 - \sqrt{1-4t})(1+t)}{2t} - 1 \right) \quad (15)$$

and for $L = 2$, we find

$$\mathcal{G}(t; 2) = \sum_{n=0}^{\infty} a_n(2) t^n = -\frac{1}{t} + \frac{t+1}{\sqrt{t^2 - 6t + 1}} \left\{ \frac{1}{t} - \frac{4}{(1 - t + \sqrt{t^2 - 6t + 1})^2} \right\}. \quad (16)$$

3. The weight function corresponding to the functional

It is known (for example, see Krattenthaler [3]) that the Hankel determinant h_n of order n of the sequence $\{a_n\}_{n \geq 0}$ equals

$$h_n = a_0^n \beta_1^{n-1} \beta_2^{n-2} \cdots \beta_{n-2}^2 \beta_{n-1}, \quad (17)$$

where $\{\beta_n\}_{n \geq 1}$ is the sequence given by:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{a_0}{1 + \alpha_0 x - (\beta_1 x^2 / (1 + \alpha_1 x - (\beta_2 x^2 / (1 + \alpha_2 x - \cdots))))}. \quad (18)$$

The sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\beta_n\}_{n \geq 1}$ are the coefficients in the recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n) Q_n(x) - \beta_n Q_{n-1}(x), \quad (19)$$

where $\{Q_n(x)\}_{n \geq 0}$ is the monic polynomial sequence orthogonal with respect to the functional $\mathcal{U}$ determined by

$$\mathcal{U}[x^n] = a_n \quad (n = 0, 1, 2, \dots). \quad (20)$$

In this section, the functional will be constructed for the sum of consecutive generalized Catalan numbers. We would like to express $\mathcal{U}[f]$ in the form:

$$\mathcal{U}[f(x)] = \int_R f(x) d\psi(x),$$

where $\psi(x)$ is a distribution, or, even more, to find the weight function $w(x)$ such that $w(x) = \psi'(x)$.

Denote by $F(z; L)$ the function

$$F(z; L) = \sum_{k=0}^{\infty} a_k z^{-k-1}.$$

From the generating function (13), we have:

$$F(z; L) = z^{-1} \mathcal{G}(z^{-1}; L) \quad (21)$$

and after some simplifications we obtain that

$$\begin{aligned} F(z; L) &= -1 + \frac{2(z+1)}{L-1+z+\sqrt{L^2+(z-1)^2-2L(z+1)}} \\ &= -1 + \frac{2(z+1)}{L-1+z(1+z\rho(\frac{1}{z}, L))} \end{aligned}$$

Example 3.1 From (15) and (16), we yield:

$$\begin{aligned} F(z; 1) &= z^{-1} \mathcal{G}(z^{-1}; 1) = \frac{1}{2} \left\{ z - 1 - (z+1) \sqrt{1 - \frac{4}{z}} \right\}, \\ F(z; 2) &= \frac{-1}{2z} \left\{ 1 + z \left(2 - z + (z+1) \sqrt{1 - \frac{6}{z} + \frac{1}{z^2}} \right) \right\}. \end{aligned}$$

Notice that

$$\begin{aligned} \int F(z; 2) dz &= z + \frac{1}{4} z(z-1) \rho\left(\frac{1}{z}, 2\right) + \log(z) \\ &\quad - \frac{1}{2} \log\left(1 + z \left(\rho\left(\frac{1}{z}, 2\right) - 3\right)\right) - \frac{7}{2} \log\left(z - 3 + z\rho\left(\frac{1}{z}, 2\right)\right). \end{aligned}$$

It will be the impulse for further discussion.

Denote by

$$R(z; L) = z\rho\left(\frac{1}{z}, L\right) = \sqrt{L^2 + (z-1)^2 - 2L(z+1)}.$$

From the theory of distribution functions (see Chihara [4]), especially by the Stieltjes inversion formula

$$\psi(t) - \psi(0) = -\frac{1}{\pi} \lim_{y \rightarrow 0^+} \int_0^t \Im F(x + iy; L) dx, \quad (22)$$

we conclude that holds

$$\mathcal{F}(z; L) = \int F(z; L) dz = \frac{1}{4} [z^2 - 2Lz - (z - L + 1)R(z; L) - l_1(z) + l_2(z)], \quad (23)$$

where

$$l_1(z) = 2(3L + 1) \log[z - (L + 1) + R(z; L)],$$

$$l_2(z) = 2(L - 1) \log \left[\frac{-(L - 1)R(z; L) - (L - 1)^2 + z(L + 1)}{z^2(L - 1)^3} \right].$$

Rewriting the function $R(z; L)$ in the form

$$R(z; L) = \sqrt{(z - L - 1)^2 - 4L}$$

and replacing $z = x + iy$, we have

$$R(x; L) = \lim_{y \rightarrow 0^+} R(x + iy; L) = \begin{cases} i\sqrt{4L - (x - L - 1)^2}, & x \in (a, b), \\ \sqrt{(x - L - 1)^2 - 4L}, & \text{otherwise,} \end{cases}$$

where

$$a = (\sqrt{L} - 1)^2, \quad b = (\sqrt{L} + 1)^2. \quad (24)$$

In the case when $x \notin ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2)$, value $R(x; L)$ is real. Therefore we can calculate imaginary part of $\mathcal{F}(x; L) = \lim_{y \rightarrow 0^+} \mathcal{F}(x + iy; L)$:

$$\Im \mathcal{F}(x; L) = \Im[l_2(x) - l_1(x)] = 0.$$

Otherwise, if $x \in ((\sqrt{L} - 1)^2, (\sqrt{L} + 1)^2)$ we have that:

$$l_1(x) = 2(3L + 1) \log \left[x - (L + 1) \pm i\sqrt{4L - (x - L - 1)^2} \right],$$

$$\Im l_1(x) = \begin{cases} 2(3L + 1) \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)}, & x \geq L + 1, \\ 2(3L + 1) \left(\pi + \arctan \frac{\sqrt{4L - (x - L - 1)^2}}{x - (L + 1)} \right), & x < L + 1, \end{cases}$$

$$l_2(x) = 2(L - 1) \log \left[\frac{-(L - 1)^2 + 2x(L + 1) - i(L - 1)\sqrt{4L - (x - L - 1)^2}}{x^2(L - 1)^3} \right],$$

$$\Im l_2(x) = \begin{cases} 2(L - 1) \left(2\pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x \geq \frac{(L - 1)^2}{L + 1}, \\ 2(L - 1) \left(\pi + \arctan \frac{x(L + 1) - (L - 1)^2}{\sqrt{4L - (x - L - 1)^2}} \right), & x < \frac{(L - 1)^2}{L + 1}. \end{cases}$$

After substituting all considered cases in (23), we finally obtain the value

$$\Im \mathcal{F}(x; L) = \lim_{y \rightarrow 0^+} \Im \mathcal{F}(x + iy; L) = \Im l_2(x) - \Im l_1(x) - (x - L + 1)\sqrt{4L - (x - L - 1)^2}.$$

From the relation (22), we conclude that

$$\omega(x; L) = \psi'(x; L) = -\frac{1}{\pi} \frac{d}{dx} \Im \mathcal{F}(x; L) \quad (25)$$

and finally, we obtain

$$\begin{aligned}\omega(x; L) &= \frac{1}{2\pi} \left(1 + \frac{1}{x}\right) \sqrt{4L - (x - L - 1)^2} \\ &= \frac{\sqrt{L}}{\pi} \left(1 + \frac{1}{x}\right) \sqrt{1 - \left(\frac{x - L - 1}{2\sqrt{L}}\right)^2}.\end{aligned}\tag{26}$$

The previous formula holds for $x \in (a, b)$, and otherwise is $\omega(x; L) = 0$.

4. Determining the three-term recurrence relation

The crucial moment in our proof of the conjecture is to determine the sequence of polynomials $\{Q_n(x)\}$ orthogonal with respect to the weight $w(x; L)$ given by (26) on the interval (a, b) and to find the sequences $\{\alpha_n\}$ $\{\beta_n\}$ in the three-term recurrence relation.

Example 4.1 For $L = 4$, we can find the first members:

$$\begin{aligned}Q_0(x) &= 1, & \|Q_0\|^2 &= 5, \\ Q_1(x) &= x - \frac{24}{5}, & \|Q_1\|^2 &= \frac{104}{5}, \\ Q_2(x) &= x^2 - \frac{127}{13}x + \frac{256}{13}, & \|Q_2\|^2 &= \frac{1088}{13}, \\ Q_3(x) &= x^3 - \frac{541}{17}x^2 + \frac{1096}{17}x - \frac{1344}{17}, & \|Q_3\|^2 &= \frac{5696}{17},\end{aligned}$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169}.$$

Hence

$$h_1 = a_0 = 5, \quad h_2 = a_0^2 \beta_1 = 104, \quad h_3 = a_0^3 \beta_1^2 \beta_2 = 5^3 \left(\frac{104}{25}\right)^2 \frac{680}{169} = 8704.$$

At the beginning, we will notice that in the definition of the weight function appears the square root member.

That is why, let us consider the monic orthogonal polynomials $\{S_n(x)\}$ with respect to the $p^{(1/2, 1/2)}(x) = \sqrt{1 - x^2}$ on the interval $(-1, 1)$. These polynomials are monic Chebyshev polynomials of the second kind:

$$S_n(x) = \frac{\sin((n+1) \arccos x)}{2^n \cdot \sqrt{1 - x^2}}.$$

They satisfy the three-term recurrence relation (Chihara [4]):

$$S_{n+1}(x) = (x - \alpha_n^*) S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \dots),\tag{27}$$

with initial values

$$S_{-1}(x) = 0, \quad S_0(x) = 1,$$

where

$$\alpha_n^* = 0 \quad (n \geq 0) \quad \text{and} \quad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \quad (n \geq 1).$$

If we use the weight function $\hat{w}(x) = (x - c) p^{(1/2, 1/2)}(x)$, then the corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows (see, for example, Gautschi [5]):

$$\begin{aligned} \lambda_n &= S_n(c), \\ \hat{\alpha}_n &= c - \frac{\lambda_{n+1}}{\lambda_n} - \beta_{n+1}^* \frac{\lambda_n}{\lambda_{n+1}}, \\ \hat{\beta}_n &= \beta_n^* \frac{\lambda_{n-1} \lambda_{n+1}}{\lambda_n^2} \quad (n \in \mathbb{N}_0). \end{aligned} \quad (28)$$

From the relation (27), we conclude that the sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ satisfies the following recurrence relation:

$$4\lambda_{n+1} - 4c\lambda_n + \lambda_{n-1} = 0 \quad (\lambda_{-1} = 0; \lambda_0 = 1). \quad (29)$$

The characteristic equation

$$4z^2 - 4cz + 1 = 0$$

has the solutions

$$z_{1,2} = \frac{1}{2}(c \pm \sqrt{c^2 - 1})$$

and the integral solution of (29) is

$$\lambda_n = E_1 z_1^n + E_2 z_2^n \quad (n \in \mathbb{N}).$$

We evaluate values E_1 and E_2 from the initial conditions $(\lambda_{-1} = 0; \lambda_0 = 1)$.

In order to solve our problem, we will choose $c = -(L + 2/\sqrt{L})$. Hence

$$z_k = \frac{-t_k}{4\sqrt{L}} \quad (k = 1, 2), \quad \text{where} \quad t_{1,2} = L + 2 \pm \sqrt{L^2 + 4}.$$

Finally, we obtain:

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2} \sqrt{L^2 + 4}} (t_1^{n+1} - t_2^{n+1}) \quad (\lambda = -1, 0, 1, \dots),$$

i.e.,

$$\lambda_n = \frac{(-1)^n}{2 \cdot 4^n L^{n/2} \xi} \psi_{n+1} \quad (\lambda = -1, 0, 1, \dots).$$

After replacing in (28), we obtain:

$$\hat{\alpha}_n = -\frac{L+2}{2\sqrt{L}} + \frac{1}{4\sqrt{L}} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + \sqrt{L} \cdot \frac{\psi_{n+1}}{\psi_{n+2}}, \quad (30)$$

$$\hat{\beta}_n = \frac{\psi_n \psi_{n+2}}{4\psi_{n+1}^2}. \quad (31)$$

If a new weight function $\tilde{w}(x)$ is introduced by

$$\tilde{w}(x) = \hat{w}(ax + b),$$

then we have

$$\tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \quad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \quad (n \geq 0).$$

Now, by using $x \mapsto (x - L - 1)/2\sqrt{L}$, i.e., $a = 1/2\sqrt{L}$ and $b = -(L + 1/2\sqrt{L})$, we have the weight function

$$\tilde{w}(x) = \hat{w}\left(\frac{x - L - 1}{2\sqrt{L}}\right) = \frac{1}{2} \left(\frac{x - L - 1}{2\sqrt{L}} + \frac{L + 2}{2\sqrt{L}} \right) \sqrt{1 - \left(\frac{x - L - 1}{2\sqrt{L}} \right)^2}.$$

Thus

$$\tilde{\alpha}_n = -1 + \frac{1}{2} \cdot \frac{\psi_{n+2}}{\psi_{n+1}} + 2L \cdot \frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0) \quad (32)$$

and

$$\tilde{\beta}_0 = (L + 2)\frac{\pi}{2}, \quad \tilde{\beta}_n = L \frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} \quad (n \in \mathbb{N}). \quad (33)$$

Example 4.2 For $L = 4$, we get

$$\begin{aligned} P_0(x) &= 1, & \|P_0\|^2 &= 3\pi, \\ P_1(x) &= x - \frac{17}{3}, & \|P_1\|^2 &= \frac{32\pi}{3}, \\ P_2(x) &= x^2 - \frac{43}{4}x + \frac{101}{4}, & \|P_2\|^2 &= 42\pi, \\ P_3(x) &= x^3 - \frac{331}{21}x^2 + \frac{1579}{21}x - \frac{2189}{21}, & \|P_3\|^2 &= \frac{3520\pi}{21}, \end{aligned}$$

wherefrom

$$\tilde{\alpha}_0 = \frac{17}{3}, \quad \tilde{\beta}_0 = 3\pi, \quad \tilde{\alpha}_1 = \frac{61}{12}, \quad \tilde{\beta}_1 = \frac{32}{9}, \quad \tilde{\alpha}_2 = \frac{421}{84}, \quad \tilde{\beta}_2 = \frac{63}{16}.$$

Introducing the weight

$$\check{w}(x) = \frac{2L}{\pi} \tilde{w}(x)$$

will not change the monic polynomials and their recurrence relations, only it will multiply the norms by the factor $2L/\pi$, i.e.

$$\begin{aligned} \check{P}_k(x) &\equiv P_k(x), \quad \|\check{P}_k\|_{\check{w}}^2 = \int_a^b \check{P}_k(x) \check{w}(x) dx = \frac{2L}{\pi} \|P_k\|^2 \quad (k \in \mathbb{N}_0), \\ \check{\beta}_0 &= L(L + 2), \quad \check{\beta}_k = \tilde{\beta}_k \quad (k \in \mathbb{N}), \quad \check{\alpha}_k = \tilde{\alpha}_k \quad (k \in \mathbb{N}_0). \end{aligned}$$

Here is

$$\check{\beta}_0 \check{\beta}_1 \cdots \check{\beta}_{n-1} = \frac{L^n}{2} \cdot \frac{\psi_{n+1}}{\psi_n}. \quad (34)$$

In [6], Gautschi has treated the next problem: If we know all about the MOPS orthogonal with respect to $\check{w}(x)$ what can we say about the sequence $\{Q_n(x)\}$ orthogonal with respect to

a weight

$$w_d(x) = \frac{\check{w}(x)}{x-d} \quad (d \notin \text{support}(\check{w})) ?$$

Gautshi has proved that, by the auxiliary sequence

$$r_{-1} = - \int_{\mathbb{R}} w_d(x) dx, \quad r_n = d - \check{\alpha}_n - \frac{\check{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \dots),$$

it can be determined

$$\alpha_{d,0} = \check{\alpha}_0 + r_0, \quad \alpha_{d,k} = \check{\alpha}_k + r_k - r_{k-1}, \quad \beta_{d,0} = -r_{-1}, \quad \beta_{d,k} = \check{\beta}_{k-1} \frac{r_{k-1}}{r_{k-2}} \quad (k \in \mathbb{N}).$$

In our case it is enough to take $d = 0$ to get the final weight

$$w(x) = \frac{\check{w}(x)}{x}.$$

Hence

$$r_{-1} = -(L+1), \quad r_n = - \left(\check{\alpha}_n + \frac{\check{\beta}_n}{r_{n-1}} \right) \quad (n = 0, 1, \dots). \quad (35)$$

LEMMA 4.3 *The parameters r_n have the explicit form*

$$r_n = - \frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi\varphi_{n+2}}{L\psi_{n+1} + \xi\varphi_{n+1}} \quad (n \in \mathbb{N}_0). \quad (36)$$

Proof We will use the mathematical induction. For $n = 0$, we really get the expected value

$$r_0 = - \frac{L^2 + 2L + 2}{(L+1)(L+2)}.$$

Suppose that it is true for $k = n$. Now, by the properties for φ_n and ψ_n , we have

$$\check{\alpha}_{n+1} \cdot r_n + \check{\beta}_{n+1} = - \frac{\psi_{n+1}}{\psi_{n+3}} \cdot \frac{L\psi_{n+3} + \xi\varphi_{n+3}}{L\psi_{n+1} + \xi\varphi_{n+1}}.$$

Dividing with r_n , we conclude that the formula is valid for r_{n+1} . ■

Example 4.4 For $L = 4$, we get

$$r_{-1} = -5, \quad r_0 = -\frac{13}{15}, \quad r_1 = -\frac{51}{52}, \quad r_2 = -\frac{356}{357},$$

wherefrom

$$\alpha_0 = \frac{24}{5}, \quad \beta_0 = 5, \quad \alpha_1 = \frac{323}{65}, \quad \beta_1 = \frac{104}{25}, \quad \alpha_2 = \frac{1104}{221}, \quad \beta_2 = \frac{680}{169},$$

just the same as in Example 4.1.

Proof of the main result Krattenthaler's formula (17) can also be written in the form

$$h_1 = a_0, \quad h_n = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \cdot h_{n-1}. \quad (37)$$

From the theory of orthogonal polynomials, it is known that

$$\|Q_{n-1}\|^2 = \beta_0 \beta_1 \beta_2 \cdots \beta_{n-2} \beta_{n-1} \quad (n = 2, 3, \dots), \quad (38)$$

wherefrom

$$h_1 = a_0, \quad h_n = \|Q_{n-1}\|^2 \cdot h_{n-1} \quad (n = 2, 3, \dots). \quad (39)$$

■

Here,

$$\|Q_{n-1}\|^2 = \beta_0 \frac{r_{n-2}}{r_{-1}} \prod_{k=0}^{n-2} \check{\beta}_k = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}}. \quad (40)$$

We will apply the mathematical induction again. The formula for h_n is true for $n = 1$. Suppose that it is valid for $k = n - 1$. Then

$$h_n = \frac{L^{n-1}}{2} \cdot \frac{L\psi_n + \xi\varphi_n}{L\psi_{n-1} + \xi\varphi_{n-1}} \cdot \frac{L^{(n-1)(n-2)/2}}{2^n \xi} \cdot (L\psi_{n-1} + \xi\varphi_{n-1}),$$

wherefrom it follows that the final statement

$$h_n = \frac{L^{n(n-1)/2}}{2^{n+1} \xi} \cdot (L\psi_n + \xi\varphi_n) \quad (n \in \mathbb{N})$$

is true.

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On the Green function of the $(\oplus + m^2)^k$ operator

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In this article, we study the Green function of the operator $(\oplus + m^2)^k$ which is iterated k -times and is defined by

$$(\oplus + m^2)^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 + m^2 \right]^k,$$

where m is a positive real number and $p + q = n$ is the dimension of the n -dimensional Euclidean space $\mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. At first, we study the elementary solution or Green function of the operator $(\oplus + m^2)^k$. Moreover, the operator $(\oplus + m^2)^k$ can be related to the ultra-hyperbolic Klein–Gordon operator $(\square + m^2)^k$, the Helmholtz operator $(\Delta + m^2)^k$ and the diamond operator of the form $(\diamond + m^2)^k$, and also we obtain the elementary solutions of such operators. We also apply such a Green function to obtain the solution of the equation $(\oplus + m^2)^k U(x) = f(x)$, where f is a generalized function and $U(x)$ is an unknown function for $x \in \mathbb{R}^n$.

Keywords: Ultra-hyperbolic Klein–Gordon operator; Elementary solution; Helmholtz operator; Diamond operator

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1. Introduction

The operator $\oplus$ has been studied first by Kananthai, Suantai and Longani [1] and is defined by

$$\begin{aligned} \oplus^k &= \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \times \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k \\ &\times \left[\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right]^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^4 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^4 \right]^k, \end{aligned} \quad (1)$$

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where $p + q = n$ is the dimension of $\mathbb{R}^n$, $i = \sqrt{-1}$ and k is a nonnegative integer. The diamond operator is denoted by

$$\diamond = \left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2. \quad (2)$$

The operators L_1 and L_2 are defined by

$$L_1 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} + i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \quad (3)$$

and

$$L_2 = \sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} - i \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}. \quad (4)$$

Thus equation (1) can be written as

$$\oplus^k = \diamond^k L_1^k L_2^k.$$

Otherwise, the operator $\diamond$ can also be expressed in the form $\diamond = \square \Delta = \Delta \square$, where $\square$ is the ultra-hyperbolic operator defined by

$$\square = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}, \quad (5)$$

where $p + q = n$ and Δ is the Laplacian defined by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \quad (6)$$

In 1988, Trione [2] studied the elementary solution of the ultra-hyperbolic Klein–Gordon operator, which iterates k -times, and is defined by

$$(\square + m^2)^k = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2 \right]^k. \quad (7)$$

We obtain the elementary solution $W_{2k}^H(u, m)$ defined by

$$W_{2k}^H(u, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r! \Gamma(k)} (m^2)^r R_{2k+2r}^H(u), \quad (8)$$

where $R_{\alpha+2r}^H(u)$ is defined by equation (14). See, also ref. [3, pp. 154–156]. Kananthai [4] obtained the function

$$G(x) = [W_{2k}^H(u, m) * W_{2k}^e(v, m)] * (S^{*k}(x))^{-1}, \quad (9)$$

which is a Green function for the operator

$$(\diamond + m^2)^k = \left(\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 + m^2 \right)^k, \quad (10)$$

and where $W_{2k}^e(v, m)$ is defined by

$$W_{2k}^e(v, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r! \Gamma(k)} (m^2)^r (-1)^{k+r} R_{2k+2r}^e(v), \quad (11)$$

in which $R_{2k+2r}^e(v)$ is defined by equation (15) where $\beta = 2k + 2r$, m is a nonnegative real number. Now,

$$S(x) = \delta - m^2 (W_2^H(u, m) * W_2^e(v, m)) * (R_{-2}^H(u) + R_{-2}^e(v)). \quad (12)$$

$S^{*k}(x)$ denotes the convolution of S itself k -times, $(S^{*k}(x))^{*-1}$ is the inverse of $S^{*k}(x)$ in the convolution algebra.

Now, the purpose of this work is to find the elementary solution or Green function of the operator $(\oplus + m^2)^k$, that is,

$$(\oplus + m^2)^k U(x) = \delta(x), \quad (13)$$

where $U(x)$ is the Green function, δ is the Dirac-delta distribution, k is a nonnegative integer and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. We then find the solution of the equation $(\oplus + m^2)^k U(x) = f(x)$, where f is a given generalized function and $U(x)$ is an unknown function.

2. Preliminaries

DEFINITION 2.1 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$u = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2, \quad p + q = n.$$

Denote by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ the interior of forward cone and $\overline{\Gamma}_+$ denote its closure.

For any complex number α , we define the function

$$R_\alpha^H(u) = \begin{cases} \frac{u^{(\alpha-n)/2}}{K_n(\alpha)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \quad (14)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2+\alpha-n)/2) \Gamma((1-\alpha)/2) \Gamma(\alpha)}{\Gamma((2+\alpha-p)/2) \Gamma((p-\alpha)/2)}.$$

Let $\text{supp } R_\alpha^H(u) \subset \overline{\Gamma}_+$, where $\text{supp } R_\alpha^H(u)$ denotes the support of $R_\alpha^H(u)$. The function R_α^H is first introduced by Nozaki [5, p. 72] and is called the ultra-hyperbolic kernel of Marcel Riesz. $R_\alpha^H(u)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and a distribution of α if $\text{Re}(\alpha) < n$.

DEFINITION 2.2 Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and write

$$v = x_1^2 + x_2^2 + \dots + x_n^2.$$

For any complex number β , we define the function

$$R_\beta^e(v) = 2^{-\beta} \pi^{-n/2} \Gamma\left(\frac{n-\beta}{2}\right) \frac{v^{(\beta-n)/2}}{\Gamma(\beta/2)}. \quad (15)$$

The function $R_\beta^e(v)$ is called the elliptic kernel of Marcel Riesz. It is an ordinary function if $\operatorname{Re}(\beta) \geq n$ and a distribution of β if $\operatorname{Re}(\beta) < n$.

DEFINITION 2.3 Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$ and we write

$$w = x_1^2 + x_2^2 + \dots + x_p^2 - i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2)$$

and

$$z = x_1^2 + x_2^2 + \dots + x_p^2 + i(x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2), \quad p + q = n.$$

For any complex number γ and v , we define

$$S_\gamma(w) = 2^{-\gamma} \pi^{-n/2} \Gamma\left(\frac{n-\gamma}{2}\right) \frac{w^{(\gamma-n)/2}}{\Gamma(\gamma/2)} \quad (16)$$

and

$$T_v(z) = 2^{-v} \pi^{-n/2} \Gamma\left(\frac{n-v}{2}\right) \frac{z^{(v-n)/2}}{\Gamma(v/2)}. \quad (17)$$

DEFINITION 2.4 Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$, and the function $Y_{\alpha, \beta, \gamma, v}(u, v, w, z, m)$ is defined by

$$Y_{\alpha, \beta, \gamma, v}(u, v, w, z, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(\eta/2 + r)}{r! \Gamma(\eta/2)} (m^2)^r K_{\alpha+2r, \beta+2r, \gamma+2r, v+2r}(u, v, w, z), \quad (18)$$

where $K_{\alpha+2r, \beta+2r, \gamma+2r, v+2r}(u, v, w, z)$ is defined by

$$K_{\alpha+2r, \beta+2r, \gamma+2r, v+2r}(u, v, w, z) = (-1)^{\beta/2+r} R_{\alpha+2r}^H(u) * R_{\beta+2r}^e(v) * S_{\gamma+2r}(w) * T_{v+2r}(z), \quad (19)$$

η is any complex number and m is a nonnegative real number.

LEMMA 2.1 The functions $R_{2k}^H(u)$ and $(-1)^k R_{2k}^e(v)$ are the elementary solutions of the operators $\square^k$ and $\triangle^k$, respectively, where $\square^k$ and $\triangle^k$ are the operators iterated k -times defined by equations (5) and (6), respectively; $R_{2k}^H(u)$ is defined by equation (14) replacing with $\alpha = 2k$ and $R_{2k}^e(v)$ defined by equation (15) replacing with $\beta = 2k$.

Proof See refs. [8, p. 147; 6, p. 31]. ■

LEMMA 2.2 The convolution $R_{2k}^H(u) * (-1)^k R_{2k}^e(v)$ is an elementary solution of the operator $\diamond^k$ iterated k -times and is defined by equation (2).

Proof See ref. [6, p. 33]. ■

LEMMA 2.3 *The function $S_{2k}(w) * T_{2k}(z)$ is the elementary solution of the operator $L^k = L_1^k L_2^k = L_2^k L_1^k$, denoted by*

$$L^k = \left[\left(\sum_{r=1}^p \frac{\partial^2}{\partial x_r^2} \right)^2 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k, \quad (20)$$

where $S_{2k}(w)$ and $T_{2k}(z)$ are defined by equations (16) and (17), respectively, with $\gamma = \nu = 2k$. The operators L_1^k and L_2^k are defined by equations (3) and (4), respectively.

Proof We need to show that

$$L^k (S_{2k}(w) * T_{2k}(z)) = L_1^k ((-1)^k (-i)^{q/2} S_{2k}(w)) * L_2^k ((-1)^k (i)^{q/2} T_{2k}(z)) = \delta.$$

To prove this, see ref. [1, p. 223]. ■

3. Main results

THEOREM 3.1 *Given the equation*

$$(\oplus + m^2)^k G(x) = \delta(x), \quad (21)$$

where $(\oplus + m^2)^k$ is the operator iterated k -times defined by equation (1), δ is the Dirac-delta distribution, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer, we obtain $G(x) = Y_{2k, 2k, 2k, 2k}(u, v, w, z, m)$,

$$Y_{2k, 2k, 2k, 2k}(u, v, w, z, m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(k+r)}{r! \Gamma(k)} (m^2)^r K_{2k+2r, 2k+2r, 2k+2r, 2k+2r}(u, v, w, z), \quad (22)$$

as an elementary solution of equation (21), where m is a nonnegative real number and $K_{2k+2r, 2k+2r, 2k+2r, 2k+2r}(u, v, w, z)$ is defined by equation (19) with $\alpha = \beta = \gamma = \nu = \eta = 2k$.

Proof At first, the following formula is valid [7, formula (2), p. 3]:

$$\Gamma\left(\frac{\eta}{2} + r\right) = \frac{\eta}{2} \left(\frac{\eta}{2} + 1\right) \cdots \left(\frac{\eta}{2} + r - 1\right) \Gamma\left(\frac{\eta}{2}\right).$$

Equivalently,

$$\begin{aligned} (-1)^r \frac{1}{r!} \Gamma\left(\frac{\eta}{2} + r\right) &= \frac{(-1)^r (\eta/2)(\eta/2 + 1) \cdots (\eta/2 + r - 1) \Gamma(\eta/2)}{r!} \\ &= \frac{(-\eta/2)(-\eta/2 - 1) \cdots [-(\eta/2 + r - 1)]}{r!} \Gamma\left(\frac{\eta}{2}\right). \end{aligned}$$

Now we put, by definition,

$$(-1)^r \frac{1}{r!} \Gamma\left(\frac{\eta}{2} + r\right) = \left(-\frac{\eta}{2}\right)_r \Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $Y_{\alpha,\beta,\gamma,v}(u, v, w, z, m)$ which is defined by equation (18) and becomes

$$Y_{\alpha,\beta,\gamma,v}(u, v, w, z, m) = \sum_{r=0}^{\infty} \binom{\frac{\eta}{2}}{r} (m^2)^r K_{\alpha+2r,\beta+2r,\gamma+2r,v+2r}(u, v, w, z). \quad (23)$$

Putting $\eta = 2k$ and $\alpha = \beta = \gamma = v = 2k$ in equation (23), we have

$$\begin{aligned} Y_{2k,2k,2k,2k}(u, v, w, z, m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r K_{2k+2r,2k+2r,2k+2r,2k+2r}(u, v, w, z) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R_{2k+2r}^H(u) * R_{2k+2r}^e(v) \\ &\quad * S_{2k+2r}(w) * T_{2k+2r}(z). \end{aligned} \quad (24)$$

Since the operators $\diamond, L_1, L_2, \square$ and $\triangle$ defined by equations (2)–(6), respectively, are linearly continuous and have 1–1 mapping, all of them possess their own inverses. From Lemma 2.1–2.3, we obtain

$$\begin{aligned} Y_{2k,2k,2k,2k}(u, v, w, z, m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \square^{-k-r} \delta * \triangle^{-k-r} \delta * L_1^{-k-r} \delta * L_2^{-k-r} \delta \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \diamond^{-k-r} L^{-k-r} \delta \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \oplus^{-k-r} \delta \\ &= (\oplus + m^2)^{-k} \delta. \end{aligned} \quad (25)$$

By applying the operator $(\oplus + m^2)^k$ to both sides of equation (25), we obtain

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(u, v, w, z, m) = (\oplus + m^2)^k \cdot (\oplus + m^2)^{-k} \delta. \quad (26)$$

Thus,

$$(\oplus + m^2)^k Y_{2k,2k,2k,2k}(u, v, w, z, m) = \delta.$$

Moreover, by putting $\beta = \gamma = v = -2r$ in equation (19), we obtain

$$\begin{aligned} K_{\alpha+2r,0,0,0}(u, v, w, z) &= R_{\alpha+2r}^H(u) * R_0^e(v) * S_0(w) * T_0(z) \\ &= R_{\alpha+2r}^H(u) * \delta * \delta * \delta = R_{\alpha+2r}^H(u). \end{aligned}$$

Then equation (23) becomes

$$Y_{\alpha,-2r,-2r,-2r}(u, v, w, z, m) = \sum_{r=0}^{\infty} \binom{\frac{\eta}{2}}{r} (m^2)^r R_{\alpha+2r}^H(u). \quad (27)$$

Now, putting $\alpha = \eta = 2k$, we obtain

$$\begin{aligned} Y_{2k, -2r, -2r, -2r}(u, v, w, z, m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r R_{2k+2r}^H(u) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \square^{-k-r} \delta \\ &= (\square + m^2)^{-k} \delta. \end{aligned} \quad (28)$$

By applying the operator $(\square + m^2)^k$ to both sides of equation (28), we obtain

$$(\square + m^2)^k Y_{2k, -2r, -2r, -2r}(u, v, w, z, m) = (\square + m^2)^k (\square + m^2)^{-k} \delta = \delta. \quad (29)$$

Then $Y_{2k, -2r, -2r, -2r}(u, v, w, z, m) = W_{2k}^H(u, m)$ as an elementary solution of the ultra-hyperbolic Klein–Gordon operator iterated k -times defined by equation (8). In particular, putting $\alpha = \gamma = \nu = -2r$ and $\beta = \eta = 2k$ in equation (23), we obtain

$$\begin{aligned} Y_{-2r, 2k, -2r, -2r}(u, v, w, z, m) &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R_{2k+2r}^e(v) \\ &= \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r \Delta^{-k-r} \delta \\ &= (\Delta + m^2)^{-k} \delta. \end{aligned} \quad (30)$$

By applying the operator $(\Delta + m^2)^k$ to both sides of equation (30), we obtain that $Y_{-2r, 2k, -2r, -2r}(u, v, w, z, m)$ is an elementary solution of Helmholtz operator iterated k -times. Similarly,

$$Y_{2k, 2k, -2r, -2r} = \sum_{r=0}^{\infty} \binom{-k}{r} (m^2)^r (-1)^{k+r} R_{\alpha+2r}^H(u) * R_{2k+2r}^e(v)$$

is the Green function of the operator $(\diamond + m^2)^k$. ■

THEOREM 3.2 *Given the equation*

$$(\oplus + m^2)^k U(x) = f(x), \quad (31)$$

where f is a given generalized function and $U(x)$ is an unknown function, we obtain the solution

$$U(x) = Y_{2k, 2k, 2k, 2k}(u, v, w, z, m) * f(x), \quad (32)$$

where $Y_{2k, 2k, 2k, 2k}(u, v, w, z, m)$ is defined by equation (24).

Proof Convolving both sides of equation (31) by $Y_{2k, 2k, 2k, 2k}(u, v, w, z, m)$ and applying Theorem 3.1, we obtain equation (32) as required. ■

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On the theorems of Hardy and Miyachi for the Jacobi–Dunkl transform

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We generalize theorems of Hardy and Miyachi for the Fourier transform on real line to the Jacobi–Dunkl transform. More precisely in the first part of this paper we give another proof of the heart of Hardy’s theorem (we mean $\alpha\beta = 1/4$), which has been obtained recently by Chouchane, Mili, and Trimèche. In the second part, we get some analog of Miyachi’s theorem for the Jacobi–Dunkl transform.

Keywords: Hardy’s theorem; Jacobi–Dunkl transform; Heat kernel

1. Introduction

A famous theorem of Hardy ([1], [2]) asserts that a measurable function f on $\mathbb{R}$ and its Fourier transform $\hat{f}$ cannot both be very rapidly decreasing. More precisely, assume that $|f(t)| \leq Ce^{-\alpha|t|^2}$ and $|\hat{f}(\lambda)| \leq Ce^{-\beta|\lambda|^2}$ where C , α and β are positive constants. Hardy’s theorem [2] states that if

- i) $\alpha\beta > 1/4$, then $f(t) = 0$,
- ii) $\alpha\beta = 1/4$, then $f(t) = \text{const } e^{-\alpha t^2}$,
- iii) $\alpha\beta < 1/4$, then there are infinitely many such f that are lineary independent and satisfy the above conditions.

We note that (ii) is the heart of the theorem because it implies (i) and (iii). This central part of Hardy’s theorem can be reformulated in terms of heat kernel $h_t(x) = (4\pi t)^{-(1/2)}e^{-(x^2/4t)}$, $t > 0$. We note that $\hat{h}_t(x) = e^{-tx^2}$, and thus the only functions satisfying (ii) are constant multiples of h_β , with $\beta = (1/4\alpha)$. An analog of this celebrated result for the Fourier transform on semi-simple Lie groups and Riemannian symmetric spaces of the non-compact type have been the object of interest in several recent papers [3–10]. The interested reader may consult the Thangavelu’s book [10] and the references therein. An $\mathbb{L}^p$ version of this theorem was, furthermore, given by Cowling and Price [11]. Gallardo and Trimèche have obtained an $\mathbb{L}^p$ version

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of Hardy's theorem for the Dunkl transform on $\mathbb{R}^n$ [12]. Chouchane *et al.* have formulated in [13] the Hardy's theorem in terms of the heat kernel of the Jacobi–Dunkl operator.

In the first part of this paper, applying the same method as in [14], we give another proof of the generalization of the analogue of the Hardy's theorem for the Jacobi–Dunkl transform obtained already by Chouchane *et al.* [13]. Their approach is different from ours. Their proof is based on Hardy's theorem for the usual Fourier transform and the properties of the Jacobi–Dunkl intertwining operators and its dual. In the second part, we show that Miyachi's theorem can also be reformulated in terms of the heat kernel of the Jacobi–Dunkl operator.

The contents of this paper are as follows:

In section 2, we review some main results concerning the harmonic analysis associated with the Jacobi–Dunkl operator. In section 3, we recall two crucial lemmas of the complex variables theory, which are a version of the Phragmen–Lindelöf theorem. Section 4 is devoted to giving another proof of Hardy's theorem associated with the Jacobi–Dunkl transform.

In the last section, an analog of Miyachi's theorem is obtained for the Jacobi–Dunkl transform.

The proof of these results requires many tools introduced in ref. [13, 15].

2. The harmonic analysis associated with the Jacobi–Dunkl operator

In this section, we collect relevant material from the harmonic analysis associated with the Jacobi–Dunkl operator, which was developed recently in [13] and [15].

2.1 The Jacobi–Dunkl kernel

In this subsection, we consider the differential-difference operator $\Lambda_{\alpha,\beta}$ on $\mathbb{R}$, given by

$$\Lambda_{\alpha,\beta}f(x) = \frac{d}{dx}f(x) + [(2\alpha + 1)\coth(x) + (2\beta + 1)\tanh(x)]\frac{f(x) - f(-x)}{2}$$

where $\alpha \geq \beta \geq (-1/2)$; $\alpha \neq (-1/2)$.

The Jacobi–Dunkl kernel $\Psi_{\lambda}^{(\alpha,\beta)}$ is the unique C^∞ -solution on $\mathbb{R}$ of the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}u = -i\lambda u \\ u(0) = 1, \end{cases} \quad \lambda \in \mathbb{C}. \quad (1)$$

It has the the Laplace integral representation

$$\forall \lambda \in \mathbb{C}, \quad \forall x \in \mathbb{R} \setminus \{0\}, \quad \Psi_{\lambda}^{(\alpha,\beta)}(x) = \int_{-|x|}^{|x|} K(x, y)e^{-i\lambda y} dy, \quad (2)$$

where $K(x, \cdot)$ is a positive function on $\mathbb{R}$, continuous on $(-|x|, |x|)$ and satisfies

$$\forall x \in \mathbb{R} \setminus \{0\}, \quad \int_{\mathbb{R}} K(x, y) dy = 1. \quad (3)$$

From (2) and (3) we deduce that the function $\Psi_{\lambda}^{(\alpha,\beta)}$ satisfies the following elementary estimate

LEMMA 1 Assume that $\alpha \geq \beta \geq (-1/2)$; $\alpha \neq (-1/2)$, $\lambda \in \mathbb{C}$. Then

$$\left| \Psi_{\lambda}^{(\alpha, \beta)} \right| \leq \Psi_{i\operatorname{Im}(\lambda)}^{(\alpha, \beta)} \quad (4)$$

and

$$\forall x \in \mathbb{R}, \quad \left| \Psi_{\lambda}^{(\alpha, \beta)}(x) \right| \leq e^{|\operatorname{Im}(\lambda)||x|}. \quad (5)$$

2.2 The Jacobi–Dunkl transform

We denote by $\rho = \alpha + \beta + 1$, where $\alpha \geq \beta \geq (-1/2)$; $\alpha \neq (-1/2)$, $\mathcal{D}(\mathbb{R})$ the space of compactly supported C^∞ -functions on $\mathbb{R}$, $\mathcal{S}(\mathbb{R})$ the space of C^∞ -functions g on $\mathbb{R}$ which are rapidly decreasing together with their derivatives and by $\mathcal{S}^r(\mathbb{R})$, $0 < r \leq 1$, the generalized Schwartz space defined by $\mathcal{S}^r(\mathbb{R}) = (\cosh(x))^{(-2\rho/r)} \mathcal{S}(\mathbb{R})$.

The Jacobi–Dunkl transform of function $f \in \mathcal{D}(\mathbb{R})$ is defined by

$$\forall \lambda \in \mathbb{C}, \quad \mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{\lambda}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx, \quad (6)$$

where

$$\Delta_{\alpha, \beta}(x) = (2 \sinh(x))^{2\alpha+1} (2 \cosh(x))^{2\beta+1}$$

This transform has the following inversion formula [13, 15].

LEMMA 2 For $\alpha \geq \beta \geq (-1/2)$ and $\alpha \neq (-1/2)$, $f \in \mathcal{S}^r(\mathbb{R})$, ($0 < r \leq 1$), $x \in \mathbb{R}$,

$$f(x) = \int_{\mathbb{R}} \mathcal{F}f(\lambda) \Psi_{-\lambda}^{(\alpha, \beta)}(x) d\sigma(\lambda), \quad (7)$$

where $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi(\lambda^2 - \rho^2)^{(1/2)} \left| \mathbf{c}((\lambda^2 - \rho^2)^{(1/2)}) \right|^2} \chi_{\mathbb{R} \setminus (-\rho, \rho)}(\lambda) d\lambda, \quad (8)$$

where $\chi_{\mathbb{R} \setminus (-\rho, \rho)}(\lambda)$ is the characteristic function of $\mathbb{R} \setminus (-\rho, \rho)$ and

$$\mathbf{c}(\mu) = \frac{2^{\rho-i\mu} \Gamma(\alpha+1) \Gamma(i\mu)}{\Gamma(((1/2)(\rho+i\mu)) \Gamma(1/2)(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C} \setminus \{i\mathbb{N}\}. \quad (9)$$

2.3 The heat kernel

DEFINITION 3 Let $t > 0$. The heat kernel E_t associated with the Jacobi–Dunkl operator is defined by

$$\forall x \in \mathbb{R}, \quad E_t(x) = \mathcal{F}^{-1}(e^{-t\lambda^2})(x). \quad (10)$$

This heat kernel E_t has the following properties [13] and [15]

(P₁) For all $t > 0$, E_t is an even positive C^∞ -function on $\mathbb{R}$.

(P₂) $\forall t > 0, \forall \lambda \in \mathbb{R}, \mathcal{F}E_t(\lambda) = e^{-t\lambda^2}$.

2.4 Version of the Phragmen-Lindelöff theorem

The proof of the main results of the following section depends on the two complex variable lemmas, which will be presented in this section.

LEMMA 4 *Let h be an entire function on $\mathbb{C}$ such that:*

$$\forall z \in \mathbb{C}, \quad |h(z)| \leq C e^{-a|z|^2}, \quad (11)$$

and

$$\forall t \in \mathbb{R}, \quad |h(t)| \leq C e^{at^2}, \quad (12)$$

for some positive constants a and C . Then $h(z) = \text{const } e^{-az^2}$, $z \in \mathbb{C}$.

Proof (See ref. 7, Lemma 2.1) ■

As usual, let us define $\log^+(x) = \log(x)$ if $x > 1$, and $\log^+(x) = 0$ otherwise. We also need the following lemma

LEMMA 5 *Suppose g is an entire function and suppose there exist constants $A, B > 0$ such that*

$$\text{For all } z \in \mathbb{C}, \quad |g(z)| \leq A e^{B(\text{Re}(z))^2}. \quad (13)$$

Also suppose

$$\int_{-\infty}^{+\infty} \log^+ |g(z)| < \infty. \quad (14)$$

Then g is a constant function.

Proof (See [16, Lemma 4]) ■

3. Main results

In this section we state and prove analogues of Hardy's and Miyachi's theorems.

3.1 An analog of Hardy's theorem

THEOREM 6 *Let f be a measurable function on $\mathbb{R}$ such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \leq M E_a(x) \quad (15)$$

and

$$\forall \lambda \in \mathbb{R}, \quad |\mathcal{F}f(\lambda)| \leq M e^{-a\lambda^2} \quad (16)$$

for some constant $a > 0$, and $M > 0$. Then the function f is a constant multiple of the heat kernel E_a .

Proof First, since by the condition (15), $\mathcal{F}f(\lambda)$ is well defined for all λ , and $\mathcal{F}f$ is an entire function on $\mathbb{C}$. Moreover, using the estimate (4) and (15) for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} |\mathcal{F}f(\lambda)| &\leq \int_{\mathbb{R}} |f(x)| \left| \Psi_{\lambda}^{(\alpha, \beta)}(x) \right| \Delta_{\alpha, \beta}(x) dx \\ &\leq M \int_{\mathbb{R}} E_a(x) \Psi_{i\text{Im}(\lambda)}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx \\ &= M\mathcal{F}(E_a)(i\text{Im}(\lambda)). \end{aligned}$$

From (10) we obtain

$$|\mathcal{F}f(\lambda)| \leq Me^{a(\text{Im}(\lambda))^2}. \quad (17)$$

Or $(\text{Im}(\lambda))^2 \leq |\lambda|^2$ then we have

$$|\mathcal{F}f(\lambda)| \leq Me^{a|\lambda|^2}, \quad \text{for all } \lambda \in \mathbb{C}. \quad (18)$$

We also have by assumption

$$|\mathcal{F}f(\lambda)| \leq Me^{-a|\lambda|^2}, \quad \text{for all } \lambda \in \mathbb{R}.$$

So by Lemma 4, we have

$$\mathcal{F}f(\lambda) = \text{const} \cdot e^{-a\lambda^2}, \quad \text{for } \lambda \in \mathbb{C}.$$

Using (10) we deduce that

$$f(x) = \text{const} \cdot E_a(x).$$

This proves the theorem. ■

3.2 An analogue of Miyachi's theorem

We denote by

- $\mathbb{L}_{\alpha, \beta}^p(\mathbb{R})$, $p \in (1, \infty)$, the space of measurable function f on $\mathbb{R}$ such that

$$\|f\|_{1, \alpha, \beta} = \int_{\mathbb{R}} |f(x)| \Delta_{\alpha, \beta}(x) dx < \infty$$

and

$$\|f\|_{\infty, \alpha, \beta} = \text{ess sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

- $\mathbb{L}^p(\mathbb{R})$, $p \in (1, \infty)$ is defined in the obvious way.

Our principal result reads as follows:

THEOREM 7 *Let $a > 0$. Suppose f is a function on $\mathbb{R}$ such that*

$$E_a^{-1}(x)f(x) \in (\mathbb{L}_{\alpha, \beta}^1 + \mathbb{L}_{\alpha, \beta}^{\infty})(\mathbb{R}_x)$$

and

$$\int_{-\infty}^{+\infty} \log^+ \left(\frac{|\mathcal{F}f(\lambda)e^{a\lambda^2}|}{\xi} \right) d\lambda < \infty$$

for some ξ , $0 < \xi < \infty$. Then f is a constant multiple of the heat kernel E_a .

Proof We consider

$$\mathcal{F}f(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{\lambda}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx, \quad \text{for } \lambda \in \mathbb{C}$$

Using estimate (5) in Lemma 1, we get the following estimate on $\mathcal{F}f(\lambda)$ (for different positive constants C)

$$|\mathcal{F}f(\lambda)| \leq \int_{\mathbb{R}} |f(x)| \Psi_{i\text{Im}(\lambda)}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx.$$

The integrand of the above integral can be written as follows:

$$|E_a^{-1}(x)f(x)| E_a(x) \Psi_{i\text{Im}(\lambda)}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x).$$

The first factor of which belongs to $(\mathbb{L}_{\alpha, \beta}^1 + \mathbb{L}_{\alpha, \beta}^\infty)(\mathbb{R}_x)$, by assumption and the second belongs to $(\mathbb{L}_{\alpha, \beta}^1 \cap \mathbb{L}_{\alpha, \beta}^\infty)(\mathbb{R}_x)$.

Hence, $\mathcal{F}f(\lambda)$ is well defined for all $\lambda \in \mathbb{C}$, and $\mathcal{F}f$ is an entire function on $\mathbb{C}$.

Moreover,

$$\begin{aligned} |\mathcal{F}f(\lambda)| &\leq \int_{\mathbb{R}} |E_a^{-1}(x)f(x)| \cdot E_a(x) \Psi_{i\text{Im}(\lambda)}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx \\ &\leq C \int_{\mathbb{R}} E_a(x) \Psi_{i\text{Im}(\lambda)}^{(\alpha, \beta)}(x) \Delta_{\alpha, \beta}(x) dx \\ &\leq C \mathcal{F}(E_a)(i\text{Im}(\lambda)) \\ &\leq C e^{a(\text{Im}(\lambda))^2} \end{aligned}$$

with a constant C , independent of λ .

Let $g(\lambda) = e^{a\lambda^2} \mathcal{F}f(\lambda)$, this is also an entire function.

Using (18), we have

$$|g(\lambda)| \leq C e^{a(\text{Re}(\lambda))^2}, \quad \text{for } \lambda \in \mathbb{C}$$

We also have, by assumption,

$$\int_{-\infty}^{+\infty} \log^+ \left(\frac{|g(\lambda)|}{\xi} \right) d\lambda < \infty.$$

Hence, applying the crucial Lemma 5 to the function $g(\lambda)/\xi$, we see that $g(\lambda) = \text{constant} = K$, or equivalently

$$\mathcal{F}f(\lambda) = K e^{-a\lambda^2}.$$

From (10), by Fourier inversion we deduce that the function f satisfies

$$f(x) = K E_a(x).$$

The theorem is then proved. ■

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On Mahler's inequality for the sum of polynomials

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Suppose that $m \geq 2$ is a positive integer. The upper and lower bounds on the best possible constant $\lambda_{m,n}$ are provided for which the inequality $M(f_1 + \dots + f_m) \leq \lambda_{m,n} m^{-1} (M(f_1) + \dots + M(f_m))$ holds for any polynomials $f_1, \dots, f_m \in \mathbb{C}[z]$ of degree at most n each. In particular, for $n > 2m^2$ and $m \rightarrow \infty$, it is proved that $2 - \pi^2/(12m^2) + o(m^{-2}) \leq \lambda_{m,n} \leq 2 - \pi/(2em^2) + o(m^{-2})$. For $m = 5$ and $n \geq 43$, our result yields the bounds $1.967 < \lambda_{5,n} < 1.977$.

Keywords: Mahler's measure; Complex polynomial; Gamma function

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1. Introduction

Recall that, for any polynomial $f(z) = a(z - z_1) \cdots (z - z_n) \in \mathbb{C}[z]$, its Mahler's measure is defined as $M(f) = |a| \prod_{j=1}^n \max\{1, |z_j|\}$. The Mahler measure of an algebraic number, which is, by definition, the Mahler measure of its minimal (irreducible) integer polynomial, is one of the subjects studied in number theory. It is one of the heights of an algebraic number and has applications in certain diophantine problems. By Jensen's formula, Mahler's measure may be also written as

$$M(f) = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| dt \right).$$

This shows that the Mahler measure of f can be obtained as a limit of its L^p norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p dt \right)^{1/p},$$

namely, $M(f) = \lim_{p \rightarrow 0+} \|f\|_p$. This explains the notation $M(f) = \|f\|_0$, which is sometimes more familiar to those working in analysis rather than in number theory. At the other

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end of the interval $(0, \infty)$, namely, as $p \rightarrow \infty$, $\|f\|_\infty = \max_{|z|=1} |f(z)|$. Below, we shall use the fact that $p \mapsto \|f\|_p$ is a nondecreasing function in p .

Let $m \geq 2$ be a fixed positive integer, and let $f_1(z), \dots, f_m(z)$ be polynomials with complex coefficients, of degree at most n each. With the above notation, using $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$, we get $M(f_1 + \dots + f_m) \leq \|f_1 + \dots + f_m\|_\infty \leq \|f_1\|_\infty + \dots + \|f_m\|_\infty$. Thus, for each $j = 1, \dots, m$, using $\|f_j\|_\infty \leq L(f_j) \leq 2^n M(f_j)$, where $L(f_j)$ is the length of f_j [1], we deduce that

$$M(f_1 + \dots + f_m) \leq 2^n (M(f_1) + \dots + M(f_m)).$$

This upper bound was proved by Mahler himself. See [2] and [3] for the first two of his papers on the quantity $M(f)$ which was named Mahler's measure after him. Mahler asked for an improvement of the constant 2^n by a smaller constant. Duncan [4] replaced 2^n by $\binom{2n}{n}^{1/2} \sim 2^n (\pi n)^{-1/4}$. Arestov [5] replaced 2^n by $(1/2)40^{n/6}$ in the case when $m = 2$ and $n \geq 6$. In section 2, by a standard method, we shall prove that

$$M(f_1 + \dots + f_m) \leq \frac{2^n \Gamma((n+1)/2)}{\sqrt{\pi} \Gamma((n+2)/2)} (M(f_1) + \dots + M(f_m)). \quad (1)$$

The constant $v(n) = 2^n \Gamma((n+1)/2) / (\sqrt{\pi} \Gamma((n+2)/2))$ is slightly smaller than $\binom{2n}{n}^{1/2}$. Observe that $v(n) = \binom{n}{n/2}$ for n even. It is remarked that the inequality for the difference of two Mahler measures $|M(f_1)^{1/n} - M(f_2)^{1/n}| \leq 2L(f_1 - f_2)^{1/n}$, where f_1, f_2 are complex polynomials of degree $\leq n$, was recently obtained by Chern and Vaaler [6].

However, when m is fixed and n is large, for an improvement of the constant $v(n)$ in (1), it is more convenient to search for the best constant c in $M(f_1 + \dots + f_m) \leq c^n m^{-1} (M(f_1) + \dots + M(f_m))$. We thus define $\lambda_{m,n}$ as the infimum over all positive numbers c for which the inequality $M(f_1 + \dots + f_m) \leq c^n m^{-1} (M(f_1) + \dots + M(f_m))$ holds for any complex polynomials $f_1, \dots, f_m$ of degree at most n each, so that

$$M(f_1 + \dots + f_m) \leq \lambda_{m,n}^n m^{-1} (M(f_1) + \dots + M(f_m))$$

with the smallest possible positive number $\lambda_{m,n}$.

Set

$$c_m = \exp \left(\frac{m}{\pi} \int_0^{\pi/m} \log(2 \cos(t/2)) dt \right) \quad (2)$$

and

$$b_m = 2 \min_{y \in (0, \infty)} \left(\frac{m \Gamma(y + 1/2)}{\sqrt{\pi} \Gamma(y + 1)} \right)^{1/(2y)}. \quad (3)$$

Let also y_m be the point where the minimum of the function

$$B_m(y) = 2 \left(\frac{m \Gamma(y + 1/2)}{\sqrt{\pi} \Gamma(y + 1)} \right)^{1/(2y)} \quad (4)$$

in $(0, \infty)$ is attained, so that $b_m = B_m(y_m)$. [We have $b_m < 2$, because $B_m(y) \rightarrow \infty$ as $y \rightarrow 0+$ and $B_m(y) \rightarrow 2$ as $y \rightarrow \infty$, where $B_m(y)$ is increasing for y large enough.]

In this note, the following bounds are proved on $\lambda_{m,n}$.

THEOREM 1 For each $m \geq 2$ and $n \geq [2y_m] + 1$, we have

$$M(f_1 + \dots + f_m) \leq b_m^n m^{-1} (M(f_1) + \dots + M(f_m)),$$

where $f_1, \dots, f_m$ are complex polynomials of degree at most n each, so that $\lambda_{m,n} \leq b_m$.

Moreover, there exist $f_1, \dots, f_m \in \mathbb{C}[z]$ of even degree $n \geq 2$ each such that

$$M(f_1 + \dots + f_m) \geq c_m^n m^{-1} (M(f_1) + \dots + M(f_m)),$$

so that $\lambda_{m,n} \geq c_m$. [Here, c_m and b_m are given in (2) and (3).]

For $m \leq 10$, the values of c_m , b_m and $2y_m$ truncated at the fifth decimal place are given in the following table.

m	c_m	b_m	$2y_m$
2	1.79162	1.84926	5.84239
3	1.90814	1.93467	14.54109
4	1.94845	1.96354	26.66969
5	1.96704	1.97674	42.25106
6	1.97712	1.98388	61.29031
7	1.98320	1.98817	83.78914
8	1.98714	1.99095	109.74822
9	1.98984	1.99285	139.16788
10	1.99177	1.99421	172.04829

The gap between c_m and b_m tends to zero, because both c_m and b_m tend to 2 as $m \rightarrow \infty$. For $m \rightarrow \infty$, it is known [7] that

$$c_m = 2 - \frac{\pi^2}{12m^2} + o(m^{-2}).$$

In section 4, we will show that

$$b_m = 2 - \frac{\pi}{2em^2} + o(m^{-2})$$

for $m \rightarrow \infty$. The numerical values $\pi^2/12 = 0.822467033 \dots$ and $\pi/(2e) = 0.577863675 \dots$ show the size of a gap between these two asymptotical bounds.

In the proof of the upper bound $\lambda_{m,n} \leq b_m$ we shall use the method of Arestov [5]. For $m = 2$, a small improvement from $\lambda_{2,n} \leq 40^{1/6} = 1.84931 \dots$ [5] to $\lambda_{2,n} \leq b_2 = 1.84926 \dots$ comes simply by setting $y = y_2 = 2^{-1} \cdot 5.84239 \dots$ from the table into $B_m(y)$ rather than $y = 3$ as in [5]. The proof of the upper bound for $M(f_1 + \dots + f_m)$ with the required constant $b_m^n m^{-1}$ will be given in the next section. The example producing the constant $c_m^n m^{-1}$, which will complete the proof of the theorem, will be given in section 3.

We conjecture that the constant c_m is (asymptotically) best possible, namely, $\lambda_{m,n} \rightarrow c_m$ as $n \rightarrow \infty$.

The constants c_m given in (2) have already appeared in the literature on several occasions. It is known that

$$c_2 = e^{2G/\pi} = M(1 + z_1 + z_2 - z_1 z_2),$$

where $G = 1 - 1/3^2 + 1/5^2 - 1/7^2 + \dots$ is Catalan's constant and $M(1 + z_1 + z_2 - z_1 z_2)$ is the Mahler measure of the polynomial in two variables $1 + z_1 + z_2 - z_1 z_2$ [8]. There is a

relation between c_m given in (2) and Clausen's integral $Cl_2(\theta) = -\int_0^\theta \log(2 \sin(t/2)) dt$ (see, e.g., pp. 1005–1006 in ref. [9])

$$\frac{\pi}{m} \log c_m = -\int_0^{(m-1)\pi/m} \log(2 \sin(t/2)) dt = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \sin(k\pi/m)}{k^2}.$$

In [7], Boyd showed that, for each $m \geq 2$, the number c_m is the best possible constant (as $n \rightarrow \infty$) for which the inequality

$$\|f_1\|_\infty \cdots \|f_m\|_\infty \leq c_m^n \|f_1 \cdots f_m\|_\infty$$

holds for complex polynomials $f_1, \dots, f_m$ of degree at most n . See also [11], where a similar inequality was obtained for polynomials in several variables.

2. Upper bound

We shall use the following inequality for two norms (L^p norm versus L^0 norm) of a polynomial f of degree $\leq n$ due to Arestov [5, 10]

$$\|f\|_p \leq N(p, n) M(f), \quad (5)$$

where

$$N(p, n) = \|(z+1)^n\|_p = 2^n \left(\frac{\Gamma((np+1)/2)}{\sqrt{\pi} \Gamma((np+2)/2)} \right)^{1/p}. \quad (6)$$

Taking $p = 1$ and using (5) and (6) we obtain

$$\begin{aligned} M(f_1 + \cdots + f_m) &\leq \|f_1 + \cdots + f_m\|_1 \leq \|f_1\|_1 + \cdots + \|f_m\|_1 \\ &\leq N(1, n)(M(f_1) + \cdots + M(f_m)) \\ &= \frac{2^n \Gamma((n+1)/2)}{\sqrt{\pi} \Gamma((n+2)/2)} (M(f_1) + \cdots + M(f_m)), \end{aligned}$$

which proves (1).

To deduce an upper bound for $M(f_1 + \cdots + f_m)$, let us start with the observation that this quantity is less than or equal to $\|f_1 + \cdots + f_m\|_p$, where $p = 2y_m/n$. For $n \geq 2y_m$, we have $p \in (0, 1]$. Thus, using $|a+b|^p \leq |a|^p + |b|^p$, which holds for any real numbers a and b , we derive that

$$\|f_1 + \cdots + f_m\|_p \leq (\|f_1\|_p^p + \cdots + \|f_m\|_p^p)^{1/p}.$$

Next, the inequality

$$\left(\frac{a_1 + \cdots + a_m}{m} \right)^{1/p} \leq \frac{a_1^{1/p} + \cdots + a_m^{1/p}}{m},$$

which holds for any non-negative numbers $a_1, \dots, a_m$, and $p \in (0, 1]$, implies that $(\|f_1\|_p^p + \cdots + \|f_m\|_p^p)^{1/p} \leq m^{1/p-1}(\|f_1\|_p + \cdots + \|f_m\|_p)$. By Arestov's inequality (5), this is further

bounded from above by $m^{1/p-1}N(p, n)(M(f_1) + \cdots + M(f_m))$. Finally, from (6)

$$m^{1/p-1}N(p, n) = m^{n/(2y_m)-1}N(2y_m/n, n) = 2^n m^{n/(2y_m)-1} \left(\frac{\Gamma(y_m + 1/2)}{\sqrt{\pi}\Gamma(y_m + 1)} \right)^{n/(2y_m)},$$

which is equal to $b_m^n m^{-1}$, by (4) and (3). Hence,

$$M(f_1 + \cdots + f_m) \leq b_m^n m^{-1}(M(f_1) + \cdots + M(f_m)),$$

as claimed.

3. Lower bound

For the lower bound, let us consider the following example. Set $\mu_j = e^{2\pi i j/m}$ and, for each $j = 1, \dots, m$, put

$$f_j(z) = e^{-\pi i(j-1)n/m}(z + \mu_{j-1})^n.$$

Clearly, $M(f_j) = 1$ for each $j = 1, \dots, m$, so $M(f_1) + \cdots + M(f_m) = m$.

It remains to evaluate the Mahler measure of the polynomial

$$f(z) = f_1(z) + \cdots + f_m(z) = (z + 1)^n + \sum_{j=2}^m e^{-\pi i(j-1)n/m}(z + \mu_{j-1})^n.$$

Setting $z = e^{it}$, we obtain $f_j(e^{it}) = e^{itn/2}(2 \cos(t/2 - \pi(j-1)/m))^n$, thus

$$f(e^{it}) = e^{itn/2} \left((2 \cos(t/2))^n + \sum_{j=1}^{m-1} (2 \cos(t/2 - \pi j/m))^n \right). \quad (7)$$

Recall that n is even. Then the function

$$|f(e^{it})| = 2^n (|\cos(t/2)|^n + |\cos(t/2 - \pi/m)|^n + \cdots + |\cos(t/2 - \pi(m-1)/m)|^n) \quad (8)$$

is periodic with period $2\pi/m$. It follows that

$$\log M(f) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| dt = \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} \log |f(e^{it})| dt.$$

Clearly, (8) yields the inequality $|f(e^{it})| \geq |2 \cos(t/2)|^n$. Thus, using (2) we obtain

$$\begin{aligned} \log M(f) &\geq \frac{m}{2\pi} \int_{-\pi/m}^{\pi/m} \log |2 \cos(t/2)|^n dt = \frac{mn}{2\pi} \int_{-\pi/m}^{\pi/m} \log |2 \cos(t/2)| dt \\ &= \frac{mn}{\pi} \int_0^{\pi/m} \log(2 \cos(t/2)) dt = n \log c_m. \end{aligned}$$

Hence $M(f) \geq c_m^n$. From this, we conclude that

$$M(f_1 + \cdots + f_m)/(M(f_1) + \cdots + M(f_m)) = M(f)m^{-1} \geq c_m^n m^{-1},$$

as claimed. This proves the theorem.

It is easy to see that, by (7) for any (even or odd) n , the same example with $f = f_1 + \cdots + f_m$ gives $\log M(f) = n \log c_m + o(n)$ as $n \rightarrow \infty$. So, independently of the parity of n , we deduce that $(M(f_1 + \cdots + f_m)/(M(f_1) + \cdots + M(f_m)))^{1/n} \rightarrow c_m$ as $n \rightarrow \infty$.

4. Evaluation of the minimum

We shall use Stirling's formula for the gamma function

$$\log \Gamma(y) = y \log y - y + \frac{1}{2} \log(2\pi/y) + \varepsilon(y), \quad (9)$$

where $\varepsilon(y) \rightarrow 0$ as $y \rightarrow \infty$.

It follows from (9) that

$$\log \Gamma(y + 1/2) - \log \Gamma(y + 1) = 1/2 + y \log(y + 1/2) - (y + 1/2) \log(y + 1) + \varepsilon_1(y),$$

where $\varepsilon_1(y) \rightarrow 0$ as $y \rightarrow \infty$. Hence, by (4) we find that

$$\begin{aligned} \log B_m(y) &= \log 2 + (1/2y)(\log(m\Gamma(y + 1/2)) - \log(\sqrt{\pi}\Gamma(y + 1))) \\ &= \log 2 + (1/4y)(\log(em^2/\pi) + 2y \log(y + 1/2) \\ &\quad - (2y + 1) \log(y + 1) + 2\varepsilon_1(y)). \end{aligned}$$

The derivative of the function $y^{-1}(\log(em^2/\pi) + 2y \log(y + 1/2) - (2y + 1) \log(y + 1))$ is equal to $y^{-2}(\log((y + 1)\pi/em^2) - y/(y + 1)(2y + 1))$. Hence, for m large enough, $B_m(y)$ attains its minimum when $y + 1$ is 'approximately' equal to em^2/π . More precisely, $y_m = em^2/\pi + o(m^2)$ for $m \rightarrow \infty$. So in evaluating $b_m = B_m(y_m)$ as $m \rightarrow \infty$, we can simply select, for instance, $y = em^2/\pi$. Putting this value into $\log B_m(y)$, we find easily that

$$\log B_m(em^2/\pi) = \log 2 - \pi/(4em^2) + o(m^{-2}).$$

Hence, for $m \rightarrow \infty$, we have $\log b_m = \log B_m(y_m) = \log 2 - \pi/(4em^2) + o(m^{-2})$. It follows that $b_m = 2 - \pi/(2em^2) + o(m^{-2})$, as claimed.

It is clear that the inequality $n > 2y_m$ holds in the case when m is large enough and $n > 2m^2 > 2em^2/\pi$, so the theorem can be applied for $n > 2m^2$ and m large enough. This explains the condition $n > 2m^2$ imposed on n in the abstract.

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L^p -Poisson integral representations of solutions of the Hua system on the Lie ball in $\mathbb{C}^n$

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Let $D = \{z \in \mathbb{C}^n / 1 - 2\bar{z}z^t + |zz^t|^2 > 0 \text{ and } |zz^t| < 1\}$ be the Lie ball in $\mathbb{C}^n$ and let $\mathcal{H}_q$ be the associated Hua operator. Then, for a complex number λ such that $\operatorname{Re}(i\lambda) > n/2 - 1$, we establish that every function F satisfying the following Hua system of differential equations:

$$\mathcal{H}_q F(z) = -\left(\lambda^2 + \frac{n^2}{4}\right) F(z) \cdot e$$

is the Poisson transform of an L^p -function ($1 < p < +\infty$) over the Shilov boundary S of D if and only if it satisfies the following growth condition of Hardy type:

$$\|F\|_{\lambda,p} = \sup_{0 \leq r < 1} (1 - r^2)^{-(n/2 - \operatorname{Re}(i\lambda))} \left[\int_S |F(ru)|^p du \right]^{1/p} < \infty.$$

Keywords: Lie ball; Shilov boundary; Hua operator; Hyperfunctions

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1. Introduction

Let $X = G/K$ be an irreducible Hermitian symmetric space of non-compact type and let $\mathcal{H}_q$ be the associated Hua operator [1]. Then it is well known that the Poisson transform P_λ is a G -isomorphism from the space $A'(S)$ of all hyperfunctions on the Shilov boundary S of X onto an eigenspace of the Hua operator for certain complex parameter λ , see ref. [2]. In ref. [3], Boussejra characterizes the L^2 -range of the Poisson transform on the Shilov boundary of the matrix ball $SU(n, n)/S(U(n) \times U(n))$.

In this article, we characterize the L^p -range ($1 < p < \infty$) of the Poisson transform in the Lie ball $SO(2, n)/S(O(2) \times O(n))$. This extends our result in ref. [4].

For exact statement of the main result of this article, we fix some needed notations.

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Let

$$D = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n / 1 - 2\bar{z}z^t + |zz^t|^2 > 0 \text{ and } |zz^t| < 1\}$$

be the Lie ball in $\mathbb{C}^n$, and let $P(z, u)$ be its Poisson kernel with respect to the Shilov boundary S of D , given by [5]

$$P(z, u) = \frac{(1 - 2\bar{z}z^t + |zz^t|^2)^{n/2}}{|(z - u)(z - u)^t|^n}.$$

Let du be the normalized invariant measure on S . For $\lambda \in \mathbb{C}$, the Poisson transform P_λ is defined for $f \in A'(S)$ by

$$[P_\lambda f](z) = \int_S P_\lambda(z, u) f(u) du,$$

where

$$P_\lambda(z, u) = (P(z, u))^{(i\lambda + n/2n)}.$$

It is clear that the function $[P_\lambda f](z)$ satisfies the Hua system of differential equations:

$$\mathcal{H}_q F(z) = -\left(\lambda^2 + \frac{n^2}{4}\right) F(z) \cdot e \quad (1)$$

for f running $A'(S)$. Now, to characterize the L^p -range, we introduce the following space $E_{\lambda,p}^*(D, \mathcal{H}_q)$ consisting of the $\mathbb{C}$ -valued function F on D satisfying equation (1), such that

$$\|F\|_{\lambda,p} = \sup_{0 \leq r < 1} (1 - r^2)^{-(n/2 - \operatorname{Re}(i\lambda))} \left[\int_S |F(ru)|^p du \right]^{1/p} < \infty.$$

The main result of this article is the following theorem.

THEOREM 1.1 *If $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(i\lambda) > n/2 - 1$, then the Poisson transform P_λ is a topological isomorphism from $L^p(S)$ onto the Banach space $E_{\lambda,p}^*(D, \mathcal{H}_q)$.*

Moreover, there exists a positive constant $\gamma(\lambda)$ such that for every function $f \in L^p(S)$, we have

$$|C(\lambda)| \|f\|_{L^p(S)} \leq \|P_\lambda f\|_{\lambda,p} \leq \gamma(\lambda) \|f\|_{L^p(S)}, \quad (2)$$

where the constant $C(\lambda)$ is given explicitly by equation (4).

As an immediate consequence of this theorem, we get a characterization of the Hua harmonic functions on D (i.e. $\mathcal{H}_q F = 0$) that are Poisson integrals of L^p -functions on S .

More precisely, let $H^p(D)$ denote the Hardy space of all Hua harmonic functions F on D . That is,

$$H^p(D) = \{F: D \longrightarrow \mathbb{C}; \quad \mathcal{H}_q F = 0 \text{ and } \|F\|_p < \infty\},$$

where $\|F\|_p = \sup_{0 \leq r < 1} (\int_S |F(ru)|^p du)^{1/p}$. Then, we get

COROLLARY 1.1 *The Poisson transform $P_{(-in)/2}$ is a topological isomorphism from $L^p(S)$ onto the Hardy space $H^p(D) = E_{(-in)/2,p}^*(D, \mathcal{H}_q)$.*

The main difficulty in proving Theorem 1.1 is to show that for $F \in E_{\lambda,p}^*(D, \mathcal{H}_q)$, there exists $f \in L^p(S)$ such that $P_\lambda f = F$. Indeed, as $\operatorname{Re}(i\lambda) > n/2 - 1$, we know by Shimeno

result that there exists f in $A'(S)$ such that $P_\lambda f = F$. Next, expanding f into its K -type and using the asymptotic of the Hua type integral,

$$\Phi_{\lambda, m}^n(r) = \int_S P_\lambda(z, u) \Delta_m(u) ddu.$$

In the above equation, $\Delta_m, m \in \wedge = \{(m_1, m_2) \in \mathbb{Z}^2 / m_1 \geq m_2\}$, is the conical polynomial defined in ref. [6].

The article is organized as follows:

Preliminaries and notations are given in section 2,

the precise action of the Poisson transform P_λ on $L^2(S)$ is given in section 3 and

Proof of the main result is given in section 4.

2. Preliminary results

In this section, we review some well-known results of harmonic analysis on the Lie ball D in $\mathbb{C}^n$ (referring to ref. [5] for more details on this subject).

For any matrix a , we denote by a^t and $\bar{a}$ the transpose and the conjugate of a , respectively. Let $G = SO(2, n)$ be the group of all matrices g in $SL(n+2, \mathbb{R})$ such that ${}^t g J g = J$, where $J = \begin{pmatrix} I_2 & 0 \\ 0 & -I_n \end{pmatrix}$.

Let $D = \{z = (z_1, z_2, \dots, z_{n-1}, z_n) \in \mathbb{C}^n / 1 - 2\bar{z}z^t + |zz^t|^2 > 0 \text{ and } |zz^t| < 1\}$ be the Lie ball in $\mathbb{C}^n$; here, z is considered as $1 \times n$ matrix and $|w|^2 = w\bar{w}$ for any $w \in \mathbb{C}^n$. The Shilov boundary of D is given by

$$S = \{u = e^{i\theta} x \in \mathbb{C}^n; \quad 0 \leq \theta < 2\pi, \quad x \in S^{n-1}\},$$

with

$$S^{n-1} = \{(x_1, x_2, \dots, x_{n-1}, x_n) \in \mathbb{R}^n; \quad x_1^2 + x_2^2 + \dots + x_{n-1}^2 + x_n^2 = 1\}.$$

The group $G = SO(2, n)$ acts transitively on D by [5]

$$g: z \mapsto g \cdot z = \left\{ \left[\left(\left(\frac{zz^t + 1}{2} \right), i \left(\frac{zz^t - 1}{2} \right) \right) A^t + z B^t \right] \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}^{-1} \\ \times \left\{ \left(\left(\frac{zz^t + 1}{2} \right), i \left(\frac{zz^t - 1}{2} \right) \right) C^t + z D^t \right\}.$$

for $g = \begin{pmatrix} ccA & B \\ C & D \end{pmatrix} \in G = SO(2, n)$. Thus as homogeneous space, we have the identification $D = G/K$, where K is the stabilizer in G of 0 given by $K = S(O(2) \times O(n))$. The action of G extends naturally to $\bar{D}$ and under this action, the group K acts transitively on the Shilov boundary S and we have $S = K/SO(n-1)$. Let $L^2(S)$ be the space of all square integrable $\mathbb{C}$ -valued functions on S with respect to the measure du . Then the group K acts on $L^2(S)$ by

$$f \longrightarrow \pi(k)f = f \circ k^{-1}, \quad k \in K,$$

and under this action, the space $L^2(S)$ has the following Peter-Weyl decomposition [6, 7]:

$$L^2(S) = \bigoplus_{m \in \wedge} V_m,$$

where $\wedge$ is the set of all two-tuple $m = (m_1, m_2) \in \mathbb{Z}^2$ with $m_1 \geq m_2$.

For $m_2 \geq 0$, the K -irreducible component V_m is the finite linear span $\{\Delta_m \circ k, k \in K\}$. For $m_2 < 0$, putting $m' = (m_1 - m_2, 0)$, $V_m = \{\Delta^{m_2} \cdot P / P \in V_{m'}\}$, where Δ is the Koecher's determinant and Δ_m is the conical polynomial defined in ref. [6] by

$$\Delta_m(z_1, \dots, z_n) = (z_1 - iz_n)^{m_1 - m_2} (z_1^2 + \dots + z_n^2)^{m_2}.$$

3. The precise action of the Poisson transform P_λ on $L^2(S)$

In this section, we give the explicit form of the Poisson transform P_λ on each K -type of $L^2(S)$.

PROPOSITION 3.1 *Let $m = (m_1, m_2) \in \wedge$ and let $f \in V_m$. Then, we have*

$$(P_\lambda f)(ru) = \Phi_{\lambda, m}^n(r) f(u).$$

Proof For every fixed $r \in [0, 1]$, we introduce the operator

$$\begin{aligned} P_\lambda^r: V_m &\longrightarrow L^2(S), \\ f &\longmapsto P_\lambda^r f. \end{aligned}$$

where

$$(P_\lambda^r f)(u) = \int_S P_\lambda(ru, v) f(v) dv.$$

Let $T_{m'}$ be the orthogonal projection from $L^2(S)$ onto $V_{m'}$. Then, by the K -invariance property of the Poisson kernel $P_\lambda(ru, v)$, we get $\pi_{m'} \circ (T_{m'} \circ P_\lambda^r) = (T_{m'} \circ P_\lambda^r) \circ \pi_m$, where $\pi_{m'}$ (respectively π_m) is the restriction of π to the subspace $V_{m'}$ (respectively V_m). Therefore, by Schur lemma, we have $T_{m'} \circ P_\lambda^r \equiv 0$ or $T_{m'} \circ P_\lambda^r$ is invertible linear operator. The fact that the representations $\pi_{m'}$ and π_m are irreducibles and inequivalent, we get $T_{m'} \circ P_\lambda^r \equiv 0$ for every $m' \in \wedge$ such that $m' \neq m$. Therefore, the operator P_λ^r acts from V_m onto V_m . Since P_λ^r commutes with the K -irreducible representation π_m on V_m , it follows by Schur lemma that the operator P_λ^r is scalar on each component V_m . Hence, there exists a constant $\Phi_{\lambda, m}^n(r)$ such that

$$P_\lambda^r = \Phi_{\lambda, m}^n(r) \cdot I \quad \text{on } V_m, \quad (3)$$

where I is identity operator on V_m . Taking Δ_m as test functions in equation (3), we get

$$\Phi_{\lambda, m}^n(r) = \int_S P_\lambda(re, v) \Delta_m(v) dv.$$

We can combine the result of Shimeno [2] and Proposition 3.1 to get the explicit series expansion of every eigenfunction of $\mathcal{H}_q$ at $x = ru$. Namely, we have the following corollary. ■

COROLLARY 3.1 *Let $\lambda \in \mathbb{C}$ such that $-i\lambda - j(n - 2/2) \notin \{1, 2, \dots\}$ for $j = 0$ or 1 and let F be a $\mathbb{C}$ -valued function on D solution of the Hua system*

$$(\mathcal{H}_q F)(z) = -\left(\lambda^2 + \frac{n^2}{4}\right) F(z) \cdot e.$$

Then, there exists a sequence of spherical harmonics $(f_m)_{m \in \wedge}$ such that F has the following expansion:

$$F(ru) = \sum_{m \in \wedge} \Phi_{\lambda, m}^n(r) f_m(u)$$

in $C^\infty([0, 1] \times S)$.

We finish this section by recalling a well-known result on the asymptotic behaviour of the generalized spherical function $\Phi_{\lambda,m}^n(r)$.

LEMMA 3.1 [8] *For $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(i\lambda) > n/2 - 1$, we have*

$$\lim_{r \rightarrow 1^-} \Phi_{\lambda,m}^n(r) = C(\lambda)(1 - r^2)^{n/2 - i\lambda},$$

as r goes to 1^- , uniformly in $m \in \Lambda$.

Here, $C(\lambda)$ is the Harish-Chandra C -function given by

$$C(\lambda) = \int_{N^-} e^{-2(\eta + \lambda) \log n} dn, \quad (4)$$

where 2η is the genus of the Lie ball in $\mathbb{C}^n$, and N^- is a subgroup of $SO(n, 2)$, see ref. [8].

4. Proof of the main result (Theorem 1.1)

For the necessary condition, let $f \in L^p(S)$. Then the Poisson–Shilov transform P_λ can be written as a convolution over the compact group K :

$$(P_\lambda f)(ru) = (f * P_\lambda^r)(k), \quad u = k \cdot e.$$

By the Young–Hausdorff inequality, we have

$$\|P_\lambda f\|_{L^p(S)} \leq \|f\|_{L^p(S)} \|P_\lambda^r\|_{L^1(S)}.$$

Next, using the Forelli–Rudin generalized inequality [4], we conclude that there exists a positive constant $\gamma(\lambda)$ such that

$$\int_K |(re - u)(re - u)^t|^{-(\operatorname{Re}(i\lambda) + n/2)} du \leq \gamma(\lambda)(1 - r^2)^{-2\operatorname{Re}(i\lambda)}, \quad \text{whenever } \operatorname{Re}(i\lambda) > \frac{n}{2} - 1.$$

Therefore, we obtain

$$\sup_{0 \leq r < 1} (1 - r^2)^{-(n/2 - \operatorname{Re}(i\lambda))} \left[\int_S |F(ru)|^p du \right]^{1/p} \leq \gamma(\lambda) \|f\|_{L^p(S)}.$$

This finishes the proof of the first part of this theorem.

The sufficiency condition:

For $p = 2$, let F be in $E_\lambda^*(D, \mathcal{H}_q)$. As $\operatorname{Re}(i\lambda) > n/2 - 1$, there exists a hyperfunction $f \in \mathcal{A}'(S)$ such that $F = P_\lambda f$. Let $f = \sum_{m \in \Lambda} f_m$ be its K -type decomposition, then using Proposition 3.1, we get

$$F(ru) = \sum_{m \in \Lambda} \Phi_{\lambda,m}^n(r) f_m(u) \quad \text{in } C^\infty([0, 1] \times S).$$

As $\|F\|_{\lambda,2} < \infty$, we get

$$(1 - r^2)^{-(n/2 - \operatorname{Re}(i\lambda))} \left\{ \sum_{m \in \Lambda} |\Phi_{\lambda,m}^n(r)|^2 \|f_m\|_{L^2}^2 \right\}^{1/2} < \infty,$$

for every $r \in [0, 1[$.

Let $\wedge_\circ$ be a finite subset of $\wedge$. Then we have

$$(1 - r^2)^{-(n/2 - \operatorname{Re}(i\lambda))} \left\{ \sum_{m \in \wedge_\circ} |\Phi_{\lambda,m}^n(r)|^2 \|f_m\|_{L^2}^2 \right\}^{1/2} \leq \|F\|_{\lambda,2} < \infty,$$

for every $r \in [0, 1[$.

Next, using the asymptotic behaviour of $\Phi_{\lambda,m}^n$ given by Lemma 3.1, we obtain

$$|C(\lambda)|^2 \sum_{m \in \wedge_\circ} \|f_m\|_{L^2}^2 \leq \|F\|_{\lambda,2}^2 < \infty,$$

from which we deduce that the left hand side of the estimate (2) in Theorem 1.1 holds. For the case $1 < p < \infty$, let $F \in E_\lambda^*(D, \mathcal{H}_q)$. Then, as $\operatorname{Re}(i\lambda) > n/2 - 1$, we can use Shimeno result [2] to see that there exists a hyperfunction f on the Shilov boundary S , such that $F = P_\lambda f$. Let $f = \sum_{m \in \wedge} a_m f_m$ be its Fourier series expansion, where $(f_m)_{m \in \wedge}$ is the basis of $L^2(S)$ constituted of the spherical harmonic functions f_m . Hence applying Proposition 3.1, the element $F \in E_{\lambda,p}^*(D, \mathcal{H}_q)$ can be written as

$$F_r(r) = F(ru) = \sum_{m \in \wedge} \Phi_{\lambda,m}^n(r) a_m f_m(u) \quad \text{in } C^\infty([0, 1[) \times S. \quad (5)$$

As

$$\sup_{0 \leq r < 1} \|(1 - r^2)^{-(n/2 - i\lambda)} F_r\|_p = \|F\|_{\lambda,p} < \infty,$$

we see that the family of functions $r \mapsto (1 - r^2)^{-(n/2 - i\lambda)} F_r$ is uniformly bounded in the space $L^p(S)$. Then, by the weak compactness of the unit ball in $L^p(S)$, there is a sequence $0 \leq r_j < 1$ with $\lim_{j \rightarrow \infty} r_j = 1$ and a $\mathbb{C}$ -valued function g in $L^p(S)$ such that $g = \lim_{j \rightarrow \infty} (1 - r_j^2)^{-(n/2 - i\lambda)} F_{r_j}$ weakly in $L^p(S)$. Moreover, we have

$$\|g\|_p \leq \liminf_{j \rightarrow +\infty} \|(1 - r_j^2)^{-(n/2 - i\lambda)} F_{r_j}\|_p. \quad (6)$$

Applying this weak convergence, in particular to the Poisson–Shilov kernel $P_\lambda(ru, v)$, we get

$$\int_S P_\lambda(ru, v) g(v) dv = \lim_{k \rightarrow \infty} \int_S (1 - r_k^2)^{-(n/2 - i\lambda)} F_{r_k}(v) P_\lambda(ru, v) dv$$

Then, by substituting $F_{r_k}(v)$ by its series expansion given in equation (5), we get

$$\begin{aligned} (P_\lambda g)(ru) &= \lim_{k \rightarrow \infty} \int_S (1 - r_k^2)^{-(n/2 - i\lambda)} P_\lambda(ru, v) \sum_{m \in \wedge} \Phi_{\lambda,m}^n(r_k) a_m f_m(v) dv. \\ &= \lim_{k \rightarrow \infty} \sum_{m \in \wedge} (1 - r_k^2)^{-(n/2 - i\lambda)} \Phi_{\lambda,m}^n(r_k) \int_S P_\lambda(ru, v) a_m f_m(v) dv. \end{aligned}$$

Therefore,

$$(P_\lambda g)(ru) = \lim_{k \rightarrow \infty} \sum_{m \in \wedge} (1 - r_k^2)^{-(n/2 - i\lambda)} \Phi_{\lambda,m}^n(r_k) \Phi_{\lambda,m}^n(r) a_m f_m(u).$$

Now, by using the fact that for every $m \in \wedge$,

$$\sup_{0 \leq r < 1} |(1 - r^2)^{-(n/2 - i\lambda)} \Phi_{\lambda,m}^n(r_k)| \leq \gamma(\lambda),$$

and the fact that the series $\sum_{m \in \wedge} \Phi_{\lambda,m}^n(r) f_m(u)$ is absolutely convergent on $[0, 1[\times S$.

Thus the result on the asymptotic behaviour of $\Phi_{\lambda,m}^n(r)$ is given by

$$\lim_{k \rightarrow \infty} (1 - r_k^2)^{-((n/2)-i\lambda)} \Phi_{\lambda,m}(r_k) = C(\lambda), \quad m \in \wedge,$$

we deduce that for $\lambda \in \mathbb{C}$ satisfying $\operatorname{Re}(i\lambda) > n/2 - 1$, we have

$$(P_\lambda g)(ru) = C(\lambda) \sum_{m \in \wedge} \Phi_{\lambda,m}^n(r) a_m f_m(u) = C(\lambda) (P_\lambda f)(ru).$$

Hence, by the injectivity of the Poisson–Shilov transform P_λ , we have $f = 1/C(\lambda)g$. Then $f \in L^p(S)$, and from equation (6), we have

$$|C(\lambda)| \|f\|_p \leq \|F\|_{\lambda,p}.$$

Thus, the desired result is established.

Remark 4.1 Note that, to prove Theorem 1.1 in the case n , one can proceed also by computing explicitly $\Phi_{\lambda,m}^n(r)$ and its asymptotic behaviour.

$$\begin{aligned} \Phi_{\lambda,m}^2(r) &= 4\pi^2 (1 - r^2)^{i\lambda+1} \varphi_{\lambda,|m_1|}(r) \varphi_{\lambda,|m_2|}(r) \\ \Phi_{\lambda,m}^4(r) &= \frac{(2\pi)^3 (1 - r^2)^{i\lambda+2}}{(m_1 - m_2 + 1)} \begin{vmatrix} \varphi_{\lambda,|m_1|}(r) & \varphi_{\lambda,|m_1+1|}(r) \\ \varphi_{\lambda,|m_2-1|}(r) & \varphi_{\lambda,|m_2|}(r) \end{vmatrix} \\ \Phi_{\lambda,m}^n(r) &= (-1)^{(n-2)/2} 2\pi^2 (1 - r^2)^{i\lambda+n/2} c(n) \prod_{j=2}^{(n-2)/2} (n-2j) \\ &\quad \times \sum_{k=0}^{(n-6)/2} A_{k+1}^{n-4/2}(m) \left(\frac{-1}{2}\right)^k \sum_{p=0}^k \binom{k}{p} \\ &\quad \times \left[\frac{a_{n-k-4}}{2}(m) \begin{vmatrix} \varphi_{\lambda,|m_1+((n/2)-p-3)|}(r) & \varphi_{\lambda,|m_1+((n/2)-p-1)|}(r) \\ \varphi_{\lambda,|m_2-((n/2)-p-1)|}(r) & \varphi_{\lambda,|m_2-((n/2)-p-3)|}(r) \end{vmatrix} \right. \\ &\quad \left. - a_{n-k-3}(m) \begin{vmatrix} \varphi_{\lambda,|m_1+((n/2)-p-2)|}(r) & \varphi_{\lambda,|m_1+((n/2)-p-1)|}(r) \\ \varphi_{\lambda,|m_2-((n/2)-p-1)|}(r) & \varphi_{\lambda,|m_2-((n/2)-p-2)|}(r) \end{vmatrix} \right], \quad n \geq 6, \quad (7) \end{aligned}$$

with

$$\begin{aligned} c(n) &= \int_0^\pi \sin^{n-4} \theta_3 d\theta_3 \dots \int_0^\pi \sin \theta_{n-3} d\theta_{n-3} \int_0^{2\pi} d\theta_{n-2} \\ \varphi_{\lambda,k}(r) &= r^k \frac{((i\lambda + n/2)2)_k}{(1)_k} F\left(\frac{i\lambda + n/2}{2}, \frac{i\lambda + n/2}{2} + k, k+1; r^2\right), \quad \lambda \in \mathbb{C} \text{ and } k \in \mathbb{Z}^+, \end{aligned}$$

where $(a)_k = a(a+1)(a+2) \dots (a+k-1)$ is the Pochhammer symbol and $F(a, b, c; x)$ is the usual Gauss hypergeometric function.

The constants

$$\left\{ \begin{array}{l} a_i(m) = \frac{1}{m_1 - m_2 + i}, \quad i \in \{1, 2, \dots\} \\ A_1^1(m) = a_1(m) \\ A_1^j(m) = A_1^{j-1}(m)a_{2j-1}(m) \\ A_i^j(m) = (A_i^{j-1}(m) + A_{i-1}^{j-1}(m))a_{2j-i}(m), \quad 1 < i < j \\ A_j^j(m) = A_{j-1}^{j-1}(m)a_j(m) \end{array} \right. \quad (8)$$

which we can write also in the following form:

$$A_{k+1}^j(m) = \sum_{i_{k+1}=j > i_k > i_{k-1} > \dots > i_2 > i_1 \geq i_0=1} \frac{\prod_{l=0}^k ((m_1 - m_2 - l)/2 + i_l)_{(i_{l+1}-i_l)}}{(m_1 - m_2 + 1)_{(2j-(k+1))}}. \quad (9)$$

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Some properties of the Mittag-Leffler functions

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This article deals with a study of some properties of the Mittag-Leffler functions including a definite integral and a recurrence relation. Several examples are given to illustrate the validity of theorems which have been proved in this work.

1. Introduction

In 1903, Mittag-Leffler [1] introduced a function defined by an infinite series

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0 \quad (1)$$

and investigated the properties of this function. This is an example of entire functions of order $\rho = 1/\alpha$. In fact, given any arbitrary but finite numbers $0 < \rho, \sigma < \infty$, we can construct an entire function of order ρ and type σ as

$$f(z) = \sum_{n=0}^{\infty} \frac{(\sigma z)^n}{\Gamma(n/\rho + \beta)}. \quad (2)$$

This is perhaps one of the simplest entire functions having the specified order and type. When $\alpha = 1$, $E_{\alpha}(z)$ reduces to e^z . In fact, $E_1(z) = e^z$ and $E_2(z^2) = \cosh z$. Another function having similar properties to those of the Mittag-Leffler function is the function given by

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0, \beta > 0, \quad (3)$$

which reduces to $E_{\alpha}(z)$ when $\beta = 1$. This may be called the generalized Mittag-Leffler function. It can also be defined by the complex integral

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_C \frac{t^{\alpha-1} e^t}{t^{\alpha} - z} dt, \quad (4)$$

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where the integral is taken along a loop C which starts and ends at $-\infty$ and encircles the disk $|t| \leq |z|^{1/\alpha}$ in the positive sense. Moreover, it satisfies the following relations:

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + z E_{\alpha,\alpha+\beta}(z), \quad (5)$$

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^\alpha z) dt = \frac{1}{1-z}, \quad (6)$$

and

$$\left(\frac{d}{dz}\right)^n [z^{\beta-1} E_{\alpha,\beta}(z^\alpha)] = z^{\beta-n-1} E_{\alpha,\beta-n}(z^\alpha), \quad (7)$$

$$L[t^{m\alpha+\beta-1} E_{\alpha,\beta}^{(m)}(\pm at^\alpha)] = \frac{m! s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}, \quad (8)$$

where L is the Laplace transformation and

$$E_{\alpha,\beta}^{(m)}(z) = \frac{d^m}{dz^m} E_{\alpha,\beta}(z). \quad (9)$$

Result (8) was utilized by many authors including Debnath [2], Debnath and Bhatta [3] and Podlubny [4] in finding the solution of the initial value problems of a class of fractional partial differential equations.

The purpose of this article is to study some new properties of the Mittag-Leffler function including a definite integral and a recurrence relation. Some examples are provided.

2. Integral formula and recurrence relation

We replace β by $\beta + r$, where $r \in \mathbb{R}$ and $\beta + r > 0$, to obtain

$$E_{\alpha,\beta+r}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta + r)}. \quad (10)$$

Similarly, we find

$$E_{\alpha,\beta+r+1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta + r + 1)} = \sum_{m=0}^{\infty} \frac{z^m}{(\alpha m + \beta + r) \Gamma(\alpha m + \beta + r)} \quad (11)$$

and

$$\begin{aligned} E_{\alpha,\beta+r+2}(z) &= \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta + r + 2)} \\ &= \sum_{m=0}^{\infty} \frac{z^m}{(\alpha m + \beta + r)(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)}. \end{aligned} \quad (12)$$

Result (12) can be rewritten as

$$\begin{aligned} E_{\alpha,\beta+r+2}(z) &= \sum_{m=0}^{\infty} \left[\frac{1}{\alpha m + \beta + r} - \frac{1}{\alpha m + \beta + r + 1} \right] \frac{z^m}{\Gamma(\alpha m + \beta + r)} \\ &= E_{\alpha,\beta+r+1}(z) - \sum_{m=0}^{\infty} \frac{z^m}{(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)}. \end{aligned} \quad (13)$$

Putting $z = 1$ in equation (13) gives

$$E_{\alpha, \beta+r+2}(1) = E_{\alpha, \beta+r+1}(1) - \sum_{m=0}^{\infty} \frac{1}{(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)}. \quad (14)$$

We next derive the integral representation of equation (14) by observing

$$z^{\beta+r} E_{\alpha, \beta+r}(z^\alpha) = \sum_{m=0}^{\infty} \frac{z^{\alpha m + \beta + r}}{\Gamma(\alpha m + \beta + r)}. \quad (15)$$

Integrating both sides gives

$$\begin{aligned} \int_0^z z^{\beta+r} E_{\alpha, \beta+r}(z^\alpha) dz &= \sum_{m=0}^{\infty} \int_0^z \frac{z^{\alpha m + \beta + r}}{\Gamma(\alpha m + \beta + r)} dz \\ &= \sum_{m=0}^{\infty} \frac{z^{\alpha m + \beta + r + 1}}{(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)}. \end{aligned} \quad (16)$$

When $z = 1$ in equation (16), we obtain

$$\int_0^1 t^{\beta+r} E_{\alpha, \beta+r}(t^\alpha) dt = \sum_{m=0}^{\infty} \frac{1}{(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)}. \quad (17)$$

Using equation (17), it follows from equation (14) that

$$E_{\alpha, \beta+r+2}(1) = E_{\alpha, \beta+r+1}(1) - \int_0^1 t^{\beta+r} E_{\alpha, \beta+r}(t^\alpha) dt$$

or equivalently,

$$\int_0^1 t^{\beta+r} E_{\alpha, \beta+r}(t^\alpha) dt = E_{\alpha, \beta+r+1}(1) - E_{\alpha, \beta+r+2}(1). \quad (18)$$

For any $\beta > 0$, there exists $r \in \mathbb{R}$ such that $(\beta + r) = n$, when n is a positive integer. Consequently, we obtain from equation (18) that

$$\int_0^1 t^n E_{\alpha, n}(t^\alpha) dt = E_{\alpha, n+1}(1) - E_{\alpha, n+2}(1). \quad (19)$$

As an example, we set $n = 1$, $\alpha = 1$ and use the fact $E_{1,1}(z) = e^z$ to obtain

$$\int_0^1 t e^t dt = E_{1,2}(1) - E_{1,3}(1). \quad (20)$$

Next, denoting the series in equation (13) by S , equation (13) can be written in an equivalent form

$$S = \sum_{m=0}^{\infty} \frac{z^m}{(\alpha m + \beta + r + 1) \Gamma(\alpha m + \beta + r)} = E_{\alpha, \beta+r+1}(z) - E_{\alpha, \beta+r+2}(z). \quad (21)$$

We now consider the identity

$$\frac{z^m}{(\alpha m + \beta + r + 1)\Gamma(\alpha m + \beta + r)} = \frac{z^m}{(\alpha m + \beta + r + 2)(\alpha m + \beta + r + 1)\Gamma(\alpha m + \beta + r)} + \frac{z^m}{(\alpha m + \beta + r + 2)\Gamma(\alpha m + \beta + r)}$$

and then the sum defined by

$$\begin{aligned} S &= \sum_{m=0}^{\infty} \frac{z^m (\alpha m + \beta + r)}{\Gamma(\alpha m + \beta + r + 3)} + \sum_{m=0}^{\infty} \frac{z^m (\alpha m + \beta + r + 1) (\alpha m + \beta + r)}{\Gamma(\alpha m + \beta + r + 3)}, \\ S &= \left\{ \alpha \sum_{m=0}^{\infty} \frac{m z^m}{\Gamma(\alpha m + \beta + r + 3)} + (\beta + r) \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta + r + 3)} \right\} \\ &\quad + \left\{ \alpha^2 \sum_{m=0}^{\infty} \frac{m^2 z^m}{\Gamma(\alpha m + \beta + r + 3)} + p \sum_{m=0}^{\infty} \frac{m z^m}{\Gamma(\alpha m + \beta + r + 3)} \right. \\ &\quad \left. + q \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta + r + 3)} \right\}, \end{aligned} \quad (22)$$

where $p = \alpha(2\beta + 2r + 1)$ and $q = (\beta + r)^2 + (\beta + r)$.

We now evaluate each term of the right hand side of equation (22) to obtain S . We observe

$$z^2 E_{\alpha, \beta+r+3}(z) = \sum_{m=0}^{\infty} \frac{z^{m+2}}{\Gamma(\alpha m + \beta + r + 3)}. \quad (23)$$

Thus,

$$\frac{d^2}{dz^2} [z^2 E_{\alpha, \beta+r+3}(z)] = \sum_{m=0}^{\infty} \frac{(m+2)(m+1)z^m}{\Gamma(\alpha m + \beta + r + 3)}$$

or

$$\begin{aligned} &2E_{\alpha, \beta+r+3}(z) + 4zE'_{\alpha, \beta+r+3}(z) + z^2E''_{\alpha, \beta+r+3}(z) \\ &= \sum_{m=0}^{\infty} \frac{m^2 z^m}{\Gamma(\alpha m + \beta + r + 3)} + 3 \sum_{m=0}^{\infty} \frac{m z^m}{\Gamma(\alpha m + \beta + r + 3)} + 2E_{\alpha, \beta+r+3}(z) \end{aligned}$$

or equivalently

$$\sum_{m=0}^{\infty} \frac{m^2 z^m}{\Gamma(\alpha m + \beta + r + 3)} = 4zE'_{\alpha, \beta+r+3}(z) + z^2E''_{\alpha, \beta+r+3}(z) - 3 \sum_{m=0}^{\infty} \frac{m z^m}{\Gamma(\alpha m + \beta + r + 3)} \quad (24)$$

Again,

$$zE_{\alpha, \beta+r+3}(z) = \sum_{m=0}^{\infty} \frac{z^{m+1}}{\Gamma(\alpha m + \beta + r + 3)}.$$

Differentiating both sides gives

$$\frac{d}{dz} [zE_{\alpha, \beta+r+3}(z)] = \sum_{m=0}^{\infty} \frac{(m+1)z^m}{\Gamma(\alpha m + \beta + r + 3)}$$

or

$$E_{\alpha, \beta+r+3}(z) + zE'_{\alpha, \beta+r+3}(z) = \sum_{m=0}^{\infty} \frac{mz^m}{\Gamma(\alpha m + \beta + r + 3)} + E_{\alpha, \beta+r+3}(z).$$

Hence,

$$\sum_{m=0}^{\infty} \frac{mz^m}{\Gamma(\alpha m + \beta + r + 3)} = zE'_{\alpha, \beta+r+3}(z). \quad (25)$$

Now, from equation (24) using equation (25) we get

$$\sum_{m=0}^{\infty} \frac{m^2 z^m}{\Gamma(\alpha m + \beta + r + 3)} = zE'_{\alpha, \beta+r+3}(z) + z^2 E''_{\alpha, \beta+r+3}(z). \quad (26)$$

Now, from equation (22) using equations (24) and (26) we get

$$S = (\beta + r + q) E_{\alpha, \beta+r+3}(z) + z(\alpha^2 + \alpha + p) E'_{\alpha, \beta+r+3}(z) + z^2 E''_{\alpha, \beta+r+3}(z). \quad (27)$$

Substituting the value of p and q in equation (27) yields

$$\begin{aligned} S &= (\beta + r)(\beta + r + 2)E_{\alpha, \beta+r+3}(z) + z\alpha[\alpha + 2 + 2(\beta + r)]E'_{\alpha, \beta+r+3}(z) \\ &\quad + z^2 E''_{\alpha, \beta+r+3}(z). \end{aligned} \quad (28)$$

It follows from equation (21) using the value of S in equation (28) that

$$\begin{aligned} &(\beta + r)(\beta + r + 2) E_{\alpha, \beta+r+3}(z) + z\alpha[\alpha + 2 + 2(\beta + r)] E'_{\alpha, \beta+r+3}(z) + z^2 E''_{\alpha, \beta+r+3}(z) \\ &= E_{\alpha, \beta+r+1}(z) - E_{\alpha, \beta+r+2}(z). \end{aligned} \quad (29)$$

For any $\beta > 0$, there exists $r \in \mathbb{R}$ such that $\beta + r = n$ where $n = 1, 2, 3, \dots$. We then obtain from equation (29) that

$$n(n + 2) E_{\alpha, n+3}(z) + z\alpha[\alpha + 2 + 2n] E'_{\alpha, n+3}(z) + z^2 E''_{\alpha, n+3}(z) = E_{\alpha, n+1}(z) - E_{\alpha, n+2}(z)$$

or equivalently

$$\begin{aligned} E_{\alpha, n+1}(z) &= n(n + 2)E_{\alpha, n+3}(z) + \alpha z[\alpha + 2(n + 1)]E'_{\alpha, n+3}(z) \\ &\quad + z^2 E''_{\alpha, n+3}(z) + E_{\alpha, n+2}(z) \end{aligned} \quad (30)$$

Result (30) represents a recurrence relation for the generalized Mittag-Leffler function.

Example When $\alpha = 1$ and $n = 1$, equation (30) becomes

$$E_{1,2}(z) = 3E_{1,4}(z) + 5zE'_{1,4}(z) + z^2 E''_{1,4}(z) + E_{1,3}(z). \quad (31)$$

It is easy to verify this result from direct substitution of $E_{1,4}(z) = \sum_{m=0}^{\infty} z^m / \Gamma(m + 4)$, $E'_{1,4}(z) = \sum_{m=1}^{\infty} m z^{m-1} / \Gamma(m + 4)$, $E''_{1,4}(z) = \sum_{m=2}^{\infty} m(m - 1) z^{m-2} / \Gamma(m + 4)$ and $E_{1,3}(z) = \sum_{m=0}^{\infty} z^m / \Gamma(m + 3)$.

Now,

$$\begin{aligned}
 & 3E_{1,4}(z) + 5zE'_{1,4}(z) + z^2E''_{1,4}(z) + E_{1,3}(z) \\
 &= 3 \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+4)} + 5 \sum_{m=1}^{\infty} \frac{mz^m}{\Gamma(m+4)} + \sum_{m=2}^{\infty} \frac{m(m-1)z^m}{\Gamma(m+4)} + \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+3)} \\
 &= \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+2)} = E_{1,2}(z)
 \end{aligned}$$

All these results lead to equation (31).

3. Differential equation for the Mittag-Leffler function

To obtain the differential equation, we need the Lacroix formula for the fractional order derivative given by (see Debnath [5])

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad (32)$$

where $\alpha, \beta \in \mathbb{R}$, $D^\alpha \equiv d^\alpha/dx^\alpha$ and

$$D^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}. \quad (33)$$

THEOREM 3.1 *The function $E_n(z^n)$ ($n = 1, 2, 3, \dots$) satisfies the n th order ordinary differential equation*

$$\frac{d^n}{dz^n} [E_n(z^n)] = E_n(z^n). \quad (34)$$

Proof The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \quad \alpha > 0.$$

Replacing z by z^α in the above expression we get

$$E_\alpha(z^\alpha) = 1 + \frac{z^\alpha}{\Gamma(\alpha+1)} + \frac{z^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{z^{3\alpha}}{\Gamma(3\alpha+1)} + \dots \quad (35)$$

Taking derivative of order α (>0) on both sides of equation (35) and using equations (32) and (33), we obtain

$$\begin{aligned}
 D^\alpha [E_\alpha(z^\alpha)] &= \frac{\Gamma(1)}{\Gamma(1-\alpha)} z^{-\alpha} + \frac{\Gamma(\alpha+1)}{\Gamma(1)} \cdot \frac{1}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} \\
 &\quad \cdot \frac{z^\alpha}{\Gamma(2\alpha+1)} + \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)} \cdot \frac{z^{2\alpha}}{\Gamma(3\alpha+1)} + \dots \\
 &= \frac{z^{-\alpha}}{\Gamma(1-\alpha)} + 1 + \frac{z^\alpha}{\Gamma(\alpha+1)} + \frac{z^{2\alpha}}{\Gamma(2\alpha+1)} + \dots \quad (36)
 \end{aligned}$$

Now, we know, $1/\Gamma(1-\alpha) = 0$, when $\alpha = n$ with $n = 1, 2, 3, \dots$

So, we have

$$\begin{aligned} D^n [E_n(z^n)] &= 1 + \frac{z^n}{\Gamma(n+1)} + \frac{z^{2n}}{\Gamma(2n+1)} + \cdots \\ &= E_n(z^n) \end{aligned}$$

Thus,

$$\frac{d^n}{dz^n} [E_n(z^n)] = E_n(z^n), \quad n = 1, 2, 3, \dots$$

This completes the proof. ■

Note: According to the theory of n th order ordinary differential equation, equation (34) must have n linearly independent solutions. We next formulate the initial value in the following theorem.

THEOREM 3.2 *The function $E_n(z^n)$ is the unique solution of the ordinary differential equation*

$$\frac{d^n w(z)}{dz^n} = w(z) \quad (37)$$

with the initial conditions

$$w(0) = 1, \quad w'(0) = w''(0) = w'''(0) = \cdots = w^{(n-1)}(0) = 0. \quad (38)$$

Proof We observe

$$E_n(z^n) = 1 + \frac{z^n}{n!} + \frac{z^{2n}}{(2n)!} + \frac{z^{3n}}{(3n)!} + \cdots \quad (39)$$

Differentiating both sides n times with respect to z gives

$$\frac{d^n}{dz^n} [E_n(z^n)] = 1 + \frac{z^n}{n!} + \frac{z^{2n}}{(2n)!} + \cdots = E_n(z^n).$$

Thus, $E_n(z^n)$ satisfies equation (37).

Also it is easy to verify that $E_n(z^n)$ satisfies all the initial conditions (38).

Also, we know $E_n(z^n)$ is a continuous and differentiable function for all $z \in \mathbb{C}$.

Therefore, from the uniqueness of the solution of ordinary differential equations, $E_n(z^n)$ is the unique solution of differential equation (37), with initial condition (38).

We verify the truth of the theorem by the following examples. ■

Example 1 When $n = 1$, $E_1(z) = e^z$ is the unique solution of

$$\frac{dw}{dz} = w$$

with the initial condition $w(0) = 1$.

Example 2 When $n = 2$, $w(z) = E_2(z^2) = \cosh z$ is the unique solution of

$$\frac{d^2}{dz^2} w(z) = w(z)$$

with the initial conditions $w(0) = 1$ and $w'(0) = 0$.

Example 3 When $n = 3$, $w(z) = E_3(z^3)$ is the unique solution of

$$\frac{d^3}{dz^3} w(z) = w(z)$$

with the initial conditions $w(0) = 1$ and $w'(0) = w''(0) = 0$.

In fact, the solution is given by

$$\begin{aligned} w(z) &= \frac{1}{3} e^{-z/2} \left\{ e^{3z/2} + 2 \cos\left(\frac{\sqrt{3}z}{2}\right) \right\} \\ &= 1 + \frac{z^3}{3!} + \frac{z^6}{6!} + \cdots + \infty \\ &= E_3(z^3). \end{aligned}$$

Example 4 When $n = 4$, $w(z) = E_4(z^4)$ is the unique solution of the equation

$$\frac{d^4 w(z)}{dz^4} = w(z)$$

with the initial conditions $w(0) = 1$ and $w'(0) = w''(0) = w'''(0) = 0$.

The solution $w(z)$ is given by

$$\begin{aligned} w(z) &= \frac{1}{4} e^{-z} (1 + e^{2z} + 2e^z \cos z) \\ &= 1 + \frac{z^4}{4!} + \frac{z^8}{8!} + \cdots + \infty \\ &= E_4(z^4). \end{aligned}$$

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On the non-commutative neutrix product of Dirac-delta distribution with an infinitely differentiable function

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In this article, we first consider the non-commutative neutrix product of $f_+(x)$ and $\delta^{(r)}(x)$ and also the product of $f_-(x)$ and $\delta^{(r)}(x)$ for $r = 0, 1, 2, \dots$, where f is an infinitely differentiable function and $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$, H denoting Heaviside's function. Then we generalize some results obtained by Fisher [Fisher, B., 1971, The product of distributions. *Quarterly Journal of Mathematics Oxford*, **22**, 291–298 and Fisher, B., 1982, A non-commutative neutrix product of distributions. *Mathematische Nachrichten*, **108**, 117–127.].

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1. Introduction

The technique of neglecting appropriately defined infinite quantities was devised by J. Hadamard and the resulting finite value extracted from the divergent integral is usually referred to as the Hadamard Finite Part. In fact, Hadamard's method can be regarded as a particular application of the neutrix calculus developed by van der Corput [1]. This is a very general principle for the discarding of unwanted infinite quantities from asymptotic expansions and has been widely exploited in the context of distributions by Fisher in connection with the problem of distributional multiplication [2–4].

In this article, using the definition of neutrix product of distributions given by Fisher, we evaluate the neutrix product of the Dirac-delta distribution with $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$, where $f(x)$ is an infinitely differentiable function.

To define the neutrix product of distributions, we shall first of all let ρ be a fixed infinitely differentiable function having the properties:

- (i) $\rho(x) = 0$ for $|x| \geq 1$,
- (ii) $\rho(x) \geq 0$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$.

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We now define the function $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$. It is obvious that $\{\delta_n\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac-delta function $\delta(x)$. Now let f be an arbitrary distribution and define the function f_n by

$$f_n(x) = (f * \delta_n) = \langle f(x-t), \delta_n(t) \rangle.$$

Then $\{f_n\}$ is a sequence of infinitely differentiable functions converging to the distribution f . Let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ be the space of distributions.

The following definition was given in ref. [3].

DEFINITION 1 Let f and g be arbitrary distributions and let $g_n = g * \delta_n$. We will say that the neutrix product $f \circ g$ of f and g exists and is equal to h on the open interval (a, b) ($-\infty \leq a < b \leq \infty$) if

$$\lim_{n \rightarrow \infty} \langle f g_n, \varphi \rangle = \langle h, \varphi \rangle \quad (1)$$

for all $\varphi \in \mathcal{D}$, where N is the neutrix having domain $N' = \{1, 2, \dots, n, \dots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \dots)$$

and all functions which converge to zero in the usual sense as n tends to infinity [2, 3, 5].

In ref. [6], the product of two distributions f and g was defined to be the limit of the sequence $\{f g_n\}$ in the usual sense, provided this limit exists. It is obvious that if the product $f g$ exists by this definition, then it will exist by the new definition and will define the same distribution.

The following two theorems were given in refs. [3, 5].

THEOREM 1 Let f and g be distributions for which the product exists. Then the neutrix product $f \circ g$ exists and defines the same distribution.

THEOREM 2 Let f and g be distributions and suppose that the neutrix product $f \circ g$ and $f \circ g'$ exist on the open interval (a, b) . Then the neutrix product $f' \circ g$ exists on the interval (a, b) and

$$(f \circ g)' = f' \circ g + f \circ g' \quad (2)$$

on the interval (a, b) .

Gelfand and Shilov define the distributions $F(x_+, -r)$ and $F(x_-, -s)$ corresponding to the functions x_+^{-r} and x_-^{-s} , respectively, by

$$\begin{aligned} & \langle F(x_+, -r), \varphi(x) \rangle \\ &= \int_0^\infty x^{-r} \left[\varphi(x) - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!} x^i - \frac{\varphi^{(r-1)}(0)}{(r-1)!} H(1-x) x^{r-1} \right] dx. \end{aligned} \quad (3)$$

and

$$\begin{aligned} & \langle F(x_-, -s), \varphi(x) \rangle \\ &= \int_0^\infty x^{-s} \left[\varphi(-x) - \sum_{i=0}^{s-2} \frac{\varphi^{(i)}(0)}{i!} (-x)^i - \frac{\varphi^{(s-1)}(0)}{(s-1)!} H(1-x) (-x)^{s-1} \right] dx \end{aligned} \quad (4)$$

where $H(x)$ denotes the Heaviside function [7].

The distributions x_+^{-r} and x_-^{-s} are defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)} \quad x_-^{-s} = -\frac{1}{(s-1)!} (\ln x_-)^{(s)},$$

which is distinct from the definition given by Gelfand and Shilov.

Further, we have [8]

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r}{(r-1)!} \psi(r-1) \delta^{(r-1)}(x) \quad (5)$$

and

$$x_-^{-s} = F(x_-, -s) - \frac{1}{(s-1)!} \psi(s-1) \delta^{(s-1)}(x) \quad (6)$$

for $r, s = 1, 2, \dots$, where

$$\psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r i^{-1}, & r \geq 1, \end{cases}$$

Before giving our main result, we prove the following theorem.

THEOREM 3 *Let $f(x)$ be an infinitely differentiable function on the open interval (a, b) , where $-\infty \leq a < 0 < b \leq \infty$. Then the neutrix products $f_+(x) \circ \delta^{(r)}(x)$, $f_-(x) \circ \delta^{(r)}(x)$, $\delta^{(r)}(x) \circ f_+(x)$ and $\delta^{(r)}(x) \circ f_-(x)$ exist*

$$f_+(x) \circ \delta^{(r)}(x) = \sum_{k=0}^r \binom{r}{k} \frac{(-1)^k}{2} f^{(k)}(0) \delta^{(r-k)}(x) \quad (7)$$

$$= \delta^{(r)}(x) \circ f_+(x) \quad (8)$$

$$= f_-(x) \circ \delta^{(r)}(x) \quad (9)$$

$$= \delta^{(r)}(x) \circ f_-(x) \quad (10)$$

for $r = 0, 1, 2, \dots$.

Proof We have on the interval $[0, 1/n]$, the intersection of the supports of $f_+(x)$ and $[\delta^{(r)}(x)]_n$ that

$$\langle f_+(x) [\delta^{(r)}(x)]_n, x^k \rangle = \int_0^{1/n} f(x) \delta_n^{(r)}(x) x^k dx$$

for $k = 0, 1, 2, \dots, r$ and making the substitution $nx = u$

$$\langle f_+(x) [\delta^{(r)}(x)]_n, x^k \rangle = n^{r-k} \int_0^1 f\left(\frac{u}{n}\right) \rho^{(r)}(u) u^k du. \quad (11)$$

Using Taylor's theorem

$$f\left(\frac{u}{n}\right) = \sum_{i=0}^{r-k} \frac{f^{(i)}(0)}{i! n^i} u^i + \frac{u^{r-k+1}}{(r-k+1)! n^{r-k+1}} f^{(r-k+1)}\left(\frac{\xi u}{n}\right),$$

then

$$\begin{aligned}\langle f_+(x)\delta_n^{(r)}(x), x^k \rangle &= n^{r-k} \sum_{i=0}^{r-k} \frac{f^{(i)}(0)}{n^i i!} \int_0^1 u^{k+i} \rho^{(r)}(u) du \\ &\quad + \frac{f^{(r-k+1)}(\xi u/n)}{n(r-k+1)!} \int_0^1 u^{r+1} \rho^{(r)}(u) du.\end{aligned}$$

As

$$\left| \frac{f^{(r-k+1)}(\xi u/n)}{n(r-k+1)!} \int_0^1 u^{r+1} \rho^{(r)}(u) du \right| \longrightarrow 0$$

as $n \rightarrow \infty$, it follows that

$$\begin{aligned}\langle f_+(x)\delta_n^{(r)}(x), x^k \rangle &= O(n) + \frac{f^{(r-k)}(0)}{(r-k)!} \int_0^1 u^r \rho^{(r)}(u) du + o\left(\frac{1}{n}\right) \\ &= O(n) + \frac{1}{2}(-1)^r \frac{r!}{(r-k)!} f^{(r-k)}(0) + o\left(\frac{1}{n}\right)\end{aligned}$$

where

$$\int_0^1 u^r \rho^{(r)}(u) du = \frac{1}{2}(-1)^r r!.$$

Thus

$$\text{N-}\lim_{n \rightarrow \infty} \langle f_+(x)\delta_n^{(r)}(x), x^k \rangle = \frac{1}{2}(-1)^r \frac{r!}{(r-k)!} f^{(r-k)}(0). \quad (12)$$

Let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$. Then

$$\varphi(x) = \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{r+1}}{(r+1)!} \varphi(\xi x)$$

and so

$$\begin{aligned}\langle f_+(x)\delta_n^{(r)}(x), \varphi(x) \rangle &= \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} \langle f_+(x)\delta_n^{(r)}(x), x^k \rangle \\ &\quad + \frac{1}{(r+1)!} \langle f_+(x)\delta_n^{(r)}(x), x^{r+1} \varphi^{(r+1)}(\xi x) \rangle.\end{aligned}$$

As

$$\begin{aligned}&\left| \int_{-\infty}^{\infty} f_+(x)\delta_n^{(r)}(x) x^{r+1} \varphi^{(r+1)}(\xi x) dx \right| \\ &\leq \sup_{0 \leq x \leq 1/n} \{ |\varphi^{(r+1)}(x)| \} \int_{-\infty}^{\infty} |f_+(x)\delta_n^{(r)}(x) x^{r+1}| dx \longrightarrow 0\end{aligned}$$

as $n \rightarrow \infty$, it now follows from equations (11) and (12) that

$$\begin{aligned}
 \text{N-}\lim_{n \rightarrow \infty} \langle f_+(x) \delta_n^{(r)}(x), \varphi(x) \rangle &= \langle f_+(x) \circ \delta^{(r)}(x), \varphi(x) \rangle \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^r f^{(r-k)}(0) \varphi^{(k)}(0) \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^r f^{(k)}(0) \varphi^{(r-k)}(0) \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^k f^{(k)}(0) \langle \delta^{(r-k)}(x), \varphi(x) \rangle.
 \end{aligned}$$

Equation (7) follows.

Now writing

$$[f_+(x)]_n = f_+(x) * \delta_n(x) = \int_{-1/n}^x f(x-t) \delta_n(t) dt,$$

we have

$$\begin{aligned}
 \langle \delta^{(r)}(x) [f_+(x)]_n, \varphi(x) \rangle &= (-1)^r \langle \delta^{(r)}(x), [[f_+(x)]_n \varphi(x)]^{(r)} \rangle \\
 &= (-1)^r \sum_{k=0}^r \binom{r}{k} [f_+(x)]_n^{(k)} \varphi^{(r-k)}(x) \Big|_{x=0} \\
 &= (-1)^r \sum_{k=0}^r \binom{r}{k} \int_{-1/n}^0 f(-t) \delta_n^{(k)}(t) dt, \tag{13}
 \end{aligned}$$

where

$$[[f_+(x)]_n^{(k)}]_{x=0} = \int_{-1/n}^0 f(-t) \delta_n^{(k)}(t) dt.$$

Making the substitution $-nt = u$

$$f\left(\frac{u}{n}\right) = \sum_{i=0}^k \frac{f^{(i)}(0)}{i!} \frac{u^i}{n^i} + \frac{u^{k+1}}{n^{k+1}(k+1)!} f^{(k+1)}\left(\frac{\xi u}{n}\right),$$

we have

$$\begin{aligned}
 [f_+(x)]_n^{(k)} \Big|_{x=0} &= (-1)^k \sum_{i=0}^k \frac{n^{k-i} f^{(i)}(0)}{i!} \int_0^1 u^i \rho^{(k)}(u) du \\
 &\quad + \frac{(-1)^{k+1}}{n(k+1)!} \int_0^1 u^{k+1} f^{(k+1)}\left(\frac{\xi u}{n}\right) \rho^{(k)}(u) du \\
 &= \frac{(-1)^k f^{(k)}(0)}{k!} \int_0^1 u^k \rho^{(k)}(u) du + o\left(\frac{1}{n}\right) \\
 &= \frac{1}{2} f^{(k)}(0) + o\left(\frac{1}{n}\right) \tag{14}
 \end{aligned}$$

where

$$\int_0^1 u^i \rho^{(k)}(u) du = \begin{cases} 0, & i < k, \\ \frac{1}{2}(-1)^k k!, & i = k. \end{cases}$$

and

$$\left| \frac{(-1)^{k+1}}{n(k+1)!} \int_0^1 u^{k+1} f^{(k+1)}\left(\frac{\xi u}{n}\right) \rho^{(k)} du \right| \longrightarrow 0$$

as n tends to infinity. Thus

$$\lim_{n \rightarrow \infty} [f_+(x)]_n^{(k)} \big|_{x=0} = \frac{1}{2} f^{(k)}(0). \quad (15)$$

It now follows from equations (13), (14) and (15) that

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle \delta^{(r)}(x) [f_+(x)]_n, \varphi(x) \rangle &= \langle \delta^{(r)}(x) \circ f_+(x), \varphi(x) \rangle \\ &= (-1)^r \sum_{k=0}^r \binom{r}{k} [f_+(x)]_n^{(k)} \varphi^{(r-k)}(x) \big|_{x=0} \\ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^k f^{(k)}(0) \langle \delta^{(r-k)}(x), \varphi(x) \rangle \end{aligned}$$

and so equation (8) follows.

Next we will consider the neutrix product of $f_-(x)$ and $\delta^{(r)}(x)$.

Making the substitution $-nx = u$, then we have on the interval $[-1/n, 0]$ that

$$\begin{aligned} \langle f_-(x) \delta_n^{(r)}(x), x^k \rangle &= \int_{-1/n}^0 f(x) \delta_n^{(r)}(x) x^k dx \\ &= (-1)^{r-k} n^{r-k} \int_0^1 f\left(\frac{-u}{n}\right) \rho^{(r)}(u) u^k du. \end{aligned} \quad (16)$$

Similarly using the Taylor's theorem

$$f\left(\frac{-u}{n}\right) = \sum_{i=0}^{r-k} \frac{f^{(i)}(0)}{i!} \frac{(-u)^i}{n^i} + \frac{(-u)^{r-k+1}}{n^{r-k+1}(r-k+1)!} f^{(r-k+1)}\left(\frac{-\xi u}{n}\right),$$

then

$$\begin{aligned} \langle f_-(x) \delta_n^{(r)}(x), x^k \rangle &= (-1)^{r-k} n^{r-k} \sum_{i=0}^{r-k} \frac{(-1)^i f^{(i)}(0)}{n^i i!} \int_0^1 u^{k+i} \rho^{(r)}(u) du \\ &\quad + \frac{(-1)^{r-k+1}}{n^{r-k+1}(r-k+1)!} \int_0^1 u^{r+1} f^{(r-k+1)}\left(\frac{-\xi u}{n}\right) \rho^{(r)}(u) du. \end{aligned}$$

As

$$\left| \frac{f^{(r-k+1)}(-\xi u/n)}{n^{r-k+1}(r-k+1)!} \int_0^1 u^{r+1} \rho^{(r)}(u) du \right| \longrightarrow 0$$

as n tends to infinity, we have

$$\begin{aligned}\langle f_-(x)\delta_n^{(r)}(x), x^k \rangle &= O(n) + \frac{f^{(r-k)}(0)}{(r-k)!} \int_0^1 u^r \rho^{(r)}(u) du + o\left(\frac{1}{n}\right) \\ &= O(n) + \frac{1}{2}(-1)^r \frac{r!}{(r-k)!} f^{(r-k)}(0) + o\left(\frac{1}{n}\right).\end{aligned}$$

Thus

$$\text{N-}\lim_{n \rightarrow \infty} \langle f_-(x)\delta_n^{(r)}(x), x^k \rangle = \frac{1}{2}(-1)^r \frac{r!}{(r-k)!} f^{(r-k)}(0) \quad (17)$$

for $k = 0, 1, 2, \dots, r$.

Let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$. Then

$$\varphi(x) = \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} x^k + \frac{x^{r+1}}{(r+1)!} \varphi(\xi x)$$

and so

$$\begin{aligned}\langle f_-(x)\delta_n^{(r)}(x), \varphi(x) \rangle &= \sum_{k=0}^r \frac{\varphi^{(k)}(0)}{k!} \langle f_-(x)\delta_n^{(r)}(x), x^k \rangle \\ &\quad + \frac{1}{(r+1)!} \langle f_-(x)\delta_n^{(r)}(x), x^{r+1} \varphi^{(r+1)}(\xi x) \rangle.\end{aligned}$$

As

$$\begin{aligned}&\left| \int_{-\infty}^{\infty} f_-(x)\delta_n^{(r)}(x) x^{r+1} \varphi^{(r+1)}(\xi x) dx \right| \\ &\leq \sup_{-1/n \leq x \leq 0} \{|\varphi^{(r+1)}(x)|\} \int_{-\infty}^0 |f_-(x)\delta_n^{(r)}(x) x^{r+1}| dx \longrightarrow 0\end{aligned}$$

as $n \rightarrow \infty$, it now follows from equations (16) and (17) that

$$\begin{aligned}\text{N-}\lim_{n \rightarrow \infty} \langle f_-(x)\delta_n^{(r)}(x), \varphi(x) \rangle &= \langle f_-(x) \circ \delta^{(r)}(x), \varphi(x) \rangle \\ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^r f^{(r-k)}(0) \varphi^{(k)}(0) \\ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^r f^{(k)}(0) \varphi^{(r-k)}(0) \\ &= \sum_{k=0}^r \binom{r}{k} \frac{1}{2} (-1)^k f^{(k)}(0) \langle \delta^{(r-k)}(x), \varphi(x) \rangle.\end{aligned}$$

Equation (9) follows.

Finally writing

$$[f_-(x)]_n = f_-(x) * \delta_n(x) = \int_x^{1/n} f(x-t) \delta_n(t) dt,$$

we have

$$\begin{aligned} \langle [f_-(x)]_n \delta^{(r)}(x), \varphi(x) \rangle &= \langle \delta^{(r)}(x), [f_-(x)]_n \varphi(x) \rangle \\ &= (-1)^r \langle \delta(x), [[f_-(x)]_n \varphi(x)]^{(r)} \rangle \\ &= (-1)^r \sum_{k=0}^r \binom{r}{k} [f_-(x)]_n^{(k)} \varphi^{(r-k)}(x) \Big|_{x=0} \\ &= (-1)^r \sum_{k=0}^r \binom{r}{k} \int_0^{1/n} f(-t) \delta_n^{(k)}(t) dt \cdot \varphi^{(r-k)}(0) \end{aligned} \quad (18)$$

for $r = 0, 1, 2, \dots$, where

$$[f_-(x)]_n^{(k)} \Big|_{x=0} = \int_0^{1/n} f(-t) \delta_n^{(k)}(t) dt.$$

Making the substitution $nt = u$

$$[f_-(x)]_n^{(k)} \Big|_{x=0} = n^k \int_0^1 f\left(\frac{-u}{n}\right) \rho^{(k)}(u) du$$

and by the Taylor's theorem

$$f\left(\frac{-u}{n}\right) = \sum_{i=0}^k \frac{f^{(i)}(0)}{i!} \frac{(-u)^i}{n^i} + \frac{(-u)^{k+1}}{n^{k+1}(k+1)!} f^{(k+1)}\left(\frac{-\xi u}{n}\right)$$

it follows that

$$\begin{aligned} [f_-(x)]_n^{(k)} \Big|_{x=0} &= \sum_{i=0}^k \frac{(-1)^i f^{(i)}(0)}{i!} n^{k-i} \int_0^1 u^i \rho^{(k)}(u) du \\ &\quad + \frac{(-1)^{k+1}}{n(k+1)!} \int_0^1 u^{k+1} f^{(k+1)}\left(\frac{-\xi u}{n}\right) \rho^{(k)}(u) du \\ &= \frac{(-1)^k f^{(k)}(0)}{k!} \int_0^1 u^k \rho^{(k)}(u) du + o\left(\frac{1}{n}\right) \\ &= \frac{1}{2} f^{(k)}(0) + o\left(\frac{1}{n}\right) \end{aligned}$$

where

$$\left| \frac{(-1)^{k+1}}{n(k+1)!} \int_0^1 u^{k+1} f^{(k+1)}\left(\frac{-\xi u}{n}\right) \rho^{(k)}(u) du \right| \longrightarrow 0$$

as n tends to infinity. Thus

$$[f_-(x)]_n^{(k)} \Big|_{x=0} = \frac{1}{2} f^{(k)}(0). \quad (19)$$

It follows from equations (18) and (19) that

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \langle [f_-(x)]_n \delta^{(r)}(x), \varphi(x) \rangle &= \langle \delta^{(r)}(x) \circ f_-(x), \varphi(x) \rangle \\
 &= \lim_{n \rightarrow \infty} (-1)^r \sum_{k=0}^r \binom{r}{k} [f_-(x)]_n^{(k)} \varphi^{(r-k)}(x) \Big|_{x=0} \\
 &= \sum_{k=0}^r \binom{r}{k} \frac{(-1)^k}{2} f^{(k)}(0) \langle \delta^{(r-k)}(x), \varphi(x) \rangle.
 \end{aligned}$$

This completes the proof of the theorem. ■

COROLLARY 1 *Let $f(x)$ be an infinitely differentiable function on the open interval (a, b) , where $-\infty \leq a < b \leq \infty$. Then the products $f(x) \circ \delta^{(r)}(x)$ and $\delta^{(r)}(x) \circ f(x)$ exist and*

$$\begin{aligned}
 \langle f(x) \circ \delta^{(r)}(x), \varphi(x) \rangle &= \sum_{k=0}^r \binom{r}{k} (-1)^k f^{(k)}(0) \langle \delta^{(r-k)}(x), \varphi(x) \rangle \\
 &= \langle \delta^{(r)}(x) \circ f(x), \varphi(x) \rangle
 \end{aligned} \tag{20}$$

Proof The proof of the theorem follows immediately on noting that $f = f_+ + f_-$. ■

Example 1 Let $f(x) = x^r$, then $f_+(x) = x_+^r$ and $f_-(x) = (-1)^r x_-^r$ and we have

$$\begin{aligned}
 x^r \circ \delta^{(r)}(x) &= (-1)^r r! \delta(x) \\
 x_+^r \circ \delta^{(r)}(x) &= \frac{(-1)^r}{2} r! \delta(x) \\
 x_-^r \circ \delta^{(r)}(x) &= \frac{1}{2} r! \delta(x).
 \end{aligned}$$

These results are also obtained in refs. [3, 9].

Example 2 Let $f(x) = \sin x$, then

$$\begin{aligned}
 \sin x \circ \delta^{(r)}(x) &= - \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r}{2k+1} (-1)^k \delta^{(r-2k+1)}(x) \\
 \sin_+ x \circ \delta^{(r)}(x) &= - \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r}{2k+1} \frac{(-1)^k}{2} \delta^{(r-2k+1)}(x) \\
 &= \sin_- x \circ \delta^{(r)}(x).
 \end{aligned}$$

Example 3 Now let $f(x) = 1$, then $f_+(x) = H(x)$ and we have following from Theorem 3 that

$$H(x) \circ \delta^{(r)}(x) = \frac{1}{2} \delta^{(r)}(x).$$

This result is also obtained in ref. [2].

Now we need the following theorems which were proved in ref. [10].

THEOREM 4 *The neutrix products $x_-^{-r} \circ x_+^s$ and $x_+^s \circ x_-^{-r}$ exist*

$$x_-^{-r} \circ x_+^s = x_+^s \circ x_-^{-r} = 0 \quad (21)$$

for $r = 1, 2, \dots$ and $s = r, r + 1, \dots$ and

$$x_-^{-r} \circ x_+^s = \sum_{i=s+1}^r \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!} \left[c(\rho) + \frac{1}{2} \psi(i-s-1) \right] \delta^{(r-s-1)}(x) \quad (22)$$

$$x_+^s \circ x_-^{-r} = \sum_{i=s+1}^r \binom{r}{i} \frac{(-1)^{i-1} s!}{(r-1)!} c(\rho) \delta^{(r-s-1)}(x) \quad (23)$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots, r-1$, where

$$c(\rho) = \int_0^1 \ln t \rho(t) dt.$$

THEOREM 5 *The neutrix products $x_-^{-r} \circ x_-^s$ and $x_-^s \circ x_-^{-r}$ exist and*

$$x_-^{-r} \circ x_-^s = x_-^s \circ x_-^{-r} = 0 \quad (24)$$

for $r = 1, 2, \dots$ and $s = r, r + 1, \dots$ and

$$\begin{aligned} x_-^{-r} \circ x_-^s &= x_-^{-r+s} + \frac{1}{(r-s-1)!} [\psi(r-1) + \psi(r-s-1)] \delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^r \binom{r}{i} \frac{(-1)^{s+i} s!}{(r-1)!} [c(\rho) + \frac{1}{2} \psi(i-s-1)] \delta^{(r-s-1)}(x) \end{aligned} \quad (25)$$

$$\begin{aligned} x_-^s \circ x_-^{-r} &= x_-^{-r+s} + \frac{1}{(r-s-1)!} [\psi(r-1) + \psi(r-s-1)] \delta^{(r-s-1)}(x) \\ &+ \sum_{i=s+1}^r \binom{r}{i} \frac{(-1)^{s+i} s!}{(r-1)!} c(\rho) \delta^{(r-s-1)}(x) \end{aligned} \quad (26)$$

for $r = 1, 2, \dots$ and $s = 0, 1, 2, \dots, r-1$.

THEOREM 6 *Let $f(x)$ be an infinitely differentiable function on the open interval (a, b) , where $-\infty \leq a < 0 < b \leq \infty$. Then the non-commutative products $f_+(x) \circ x_-^{-r}$ and $x_-^{-r} \circ f_+(x)$ exist and*

$$\langle f_+(x) \circ x_-^{-r}, \varphi(x) \rangle = \sum_{k=0}^{r-1} \frac{S_{k,r} f^{(k)}(0)}{k!(r-1)!} \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle \quad (27)$$

$$\langle x_-^{-r} \circ f_+(x), \varphi(x) \rangle = \sum_{k=0}^{r-1} \frac{S_{k,r}^* f^{(k)}(0)}{k!(r-1)!} \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle \quad (28)$$

for $\varphi \in \mathcal{D}$ and $r = 1, 2, \dots$, where

$$S_{k,r} = \sum_{i=k+1}^r \binom{r}{i} (-1)^{i-1} k! c(\rho)$$

and

$$S_{k,r}^* = \sum_{i=k+1}^r \binom{r}{i} (-1)^{i-1} k! [c(\rho) + \frac{1}{2} \psi(i-k-1)].$$

Proof As $f(x)$ is an infinitely differentiable function, by Taylor's theorem we have

$$f(x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{x^r f^{(r)}(\xi x)}{r!}$$

and

$$f_+(x) = \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} x_+^k + \frac{x_+^r f^{(r)}(\xi x)}{r!}.$$

Let $\varphi(x)$ be an arbitrary function in $\mathcal{D}$. Then

$$\begin{aligned} \langle f_+(x)(x_-^{-r})_n, \varphi(x) \rangle &= \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} \langle x_+^k (x_-^{-r})_n, \varphi(x) \rangle \\ &\quad + \frac{1}{r!} \langle x_+^r f^{(r)}(\xi x) (x_-^{-r})_n, \varphi(x) \rangle. \end{aligned} \quad (29)$$

Now using Theorem 4, we have

$$\text{N-lim}_{n \rightarrow \infty} \langle x_+^k (x_-^{-r})_n, \varphi(x) \rangle = \langle x_+^k \circ x_-^{-r}, \varphi(x) \rangle = 0$$

for $r = 1, 2, \dots$ and $k = r, r+1, \dots$ and

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} \langle x_+^k (x_-^{-r})_n, \varphi(x) \rangle &= \langle x_+^k \circ x_-^{-r}, \varphi(x) \rangle \\ &= \sum_{i=k+1}^r \binom{r}{i} \frac{(-1)^{i-1} k!}{(r-1)!} c(\rho) \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle \end{aligned}$$

for $r = 1, 2, \dots$ and $k = 0, 1, 2, \dots, r-1$.

Further we have

$$|\langle x_+^r f^{(r)}(\xi x) (x_-^{-r})_n, \varphi(x) \rangle| \leq \sup_{0 \leq x \leq 1/n} \{|f^{(r)}(x)|\} |\langle x_+^r (x_-^{-r})_n, \varphi(x) \rangle| \longrightarrow 0$$

as n tends to infinity [10].

It follows from equation (29) that

$$\begin{aligned} \text{N-lim}_{n \rightarrow \infty} \langle f_+(x)(x_-^{-r})_n, \varphi(x) \rangle &= \langle f_+(x) \circ x_-^{-r}, \varphi(x) \rangle \\ &= \sum_{k=0}^{r-1} \frac{S_{k,r} f^{(k)}(0)}{k!(r-1)!} \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle. \end{aligned}$$

Thus equation (27) follows.

Next writing

$$[f_+(x)]_n = \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} (x_+^k)_n + \frac{1}{r!} [x_+^r f^{(r)}(\xi x)]_n$$

we have

$$\begin{aligned} \langle x_-^{-r} [f_+(x)]_n, \varphi(x) \rangle &= \sum_{k=0}^{r-1} \frac{f^{(k)}(0)}{k!} \langle x_-^{-r} (x_+^k)_n, \varphi(x) \rangle \\ &\quad + \frac{1}{r!} \langle x_-^{-r} [x_+^r f^{(r)}(\xi x)]_n, \varphi(x) \rangle \end{aligned}$$

for an arbitrary $\varphi \in \mathcal{D}$.

Thus again using Theorem 4, we have

$$\begin{aligned} \mathbf{N}\text{-}\lim_{n \rightarrow \infty} \langle x_-^{-r} (x_+^k)_n, \varphi(x) \rangle &= \langle x_-^{-r} \circ x_+^k, \varphi(x) \rangle \\ &= \sum_{i=k+1}^r \binom{r}{i} \frac{(-1)^{i-1} k!}{(r-1)!} \left[c(\rho) + \frac{1}{2} \psi(i-k-1) \right] \\ &\quad \times \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle \end{aligned}$$

for $r = 1, 2, \dots$ and $k = 0, 1, 2, \dots, r-1$.

Also we have

$$\left| \langle x_-^{-r} [x_+^r f^{(r)}(\xi x)]_n, \varphi(x) \rangle \right| \longrightarrow 0$$

as $n \rightarrow \infty$.

It follows from what we just proved that

$$\begin{aligned} \mathbf{N}\text{-}\lim_{n \rightarrow \infty} \langle x_-^{-r} [f_+(x)]_n, \varphi(x) \rangle &= \langle x_-^{-r} \circ f_+(x), \varphi(x) \rangle \\ &= \sum_{k=0}^{r-1} \frac{S_{k,r}^* f^{(k)}(0)}{k!(r-1)!} \langle \delta^{(r-k-1)}(x), \varphi(x) \rangle \end{aligned}$$

This completes the proof of the theorem. ■

COROLLARY 2 *Let $f(x)$ be an infinitely differentiable function on the interval (a, b) , where $-\infty \leq a < 0 < b \leq \infty$. Then the non-commutative products $f_-(x) \circ x_-^{-r}$ and $x_-^{-r} \circ f_-(x)$ exist and*

$$f_-(x) \circ x_-^{-r} = \sum_{k=0}^{r-1} \frac{(-1)^k f^{(k)}(0)}{k!} [x_-^{-r+k} + M_{k,r} \delta^{(r-k-1)}(x)] \quad (30)$$

$$x_-^{-r} \circ f_-(x) = \sum_{k=0}^{r-1} \frac{(-1)^k f^{(k)}(0)}{k!} [x_-^{-r+k} + M_{k,r}^* \delta^{(r-k-1)}(x)] \quad (31)$$

where

$$\begin{aligned} M_{k,r} &= \frac{1}{(r-k-1)!} [\psi(r-1) + \psi(r-k-1)] \delta^{(r-k-1)}(x) \\ &\quad + \sum_{i=k+1}^r \binom{r}{i} \frac{(-1)^{k+i} k!}{(r-1)!} c(\rho) \delta^{(r-k-1)}(x) \end{aligned}$$

and

$$M_{k,r}^* = M_{k,r} + \sum_{i=k+1}^r \binom{r}{i} \frac{(-1)^{k+i} k!}{2(r-1)!} \psi(r-k-1) \delta^{(r-k-1)}(x).$$

Proof The proof is similar to the proof of Theorem 6. The equations (30) and (31) follow from Theorem 5 and on noting that

$$f_{-}(x) = \sum_{k=0}^{r-1} \frac{(-1)^k f^{(k)}(0)}{k!} x_{-}^k + \frac{(-1)^r x_{-}^r f^{(r)}(\xi x)}{r!}$$

and

$$[f_{-}(x)]_n = \sum_{k=0}^{r-1} \frac{(-1)^k f^{(k)}(0)}{k!} (x_{-}^k)_n + \frac{(-1)^r}{r!} [x_{-}^r f^{(r)}(\xi x)]_n.$$

■

Remark If $f(x) = x^s$, then the results given in Theorem 6 and Corollary 2 are seen to be in agreement with those given in Theorems 4 and 5.

Example 4 With $f(x) = e^x \sin x$ and $f_{+}(x) = e^x \sin_{+} x$, we have

$$(e^x \sin x) \circ x_{-}^{-r} = \sum_{k=0}^{r-1} \frac{S_{k,r} 2^{k/2} \sin[k\pi/4]}{(k!)^2 (r-1)!} \delta^{(r-k-1)}(x)$$

for $r = 0, 1, 2, \dots$, where

$$e^x \sin x = \sum_{k=0}^{\infty} \frac{2^{k/2} \sin((1/4)k\pi)}{k!} x^k.$$

Example 5 With $f(x) = \cos x$ and $f_{+}(x) = \cos_{+} x$, we have

$$\begin{aligned} \cos_{+} x \circ x_{-}^{-r} &= \sum_{k=0}^{r-1} \frac{S_{k,r} \cos(k\pi/2)}{k!(r-1)!} \delta^{(r-k-1)}(x), \\ x_{-}^{-r} \circ \cos_{+} x &= \sum_{k=0}^{r-1} \frac{S_{k,r}^{*} \cos(k\pi/2)}{k!(r-1)!} \delta^{(r-k-1)}(x). \end{aligned}$$

Further the non-commutative neutrix products such as $f_{+}(x) \circ F(x_{-}, -r)$, $f_{-}(x) \circ F(x_{-}, -r)$, $F(x_{-}, -r) \circ f_{+}(x)$ and $F(x_{-}, -r) \circ f_{-}(x)$ can be easily obtained by using equation (6).

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Some improvements over Love's inequality for the Laguerre function

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Dedicated to the Memories of Eric Russell Love (1912–2001) and Yudell Leo Luke (1918–1983)

In the present article, Love's bounding inequalities for the Laguerre function $L_v^{(\mu)}(x)$ are significantly improved by two different methods in the real domain. The first method is based upon Luke's exponential inequalities for the confluent hypergeometric function $\Phi \equiv {}_1F_1$, while the second approach explores the upper bounds for the first-kind Bessel function $J_\mu(x)$ by Landau, and Olenko's recently derived bound for the same Bessel function. Finally, we deduce a bounding function for $L_v^{(\mu)}(x)$ combining Krasikov's uniform bound for the Bessel functions with Olenko's result cited above.

Keywords: Bessel function of the first kind; Confluent hypergeometric function; Krasikov's uniform bound for the Bessel function; Laguerre function; Landau's inequalities for the Bessel function; Love's inequalities; Luke's bound for the confluent hypergeometric function; Olenko's bound for the Bessel function

AMS Subject Classification: Primary 26D15, 33C45; Secondary 33C15

1. Introduction

The classical *Laguerre function* $L_v^{(\mu)}(z)$ is an entire function defined by (see, for example, [1, chapter 6, section 6.6])

$$L_v^{(\mu)}(z) = \frac{\Gamma(\mu + v + 1)}{\Gamma(\mu + 1)\Gamma(v + 1)} \Phi(-v; \mu + 1; z) \quad (1)$$

$$(z, v \in \mathbb{C}; \mu \in \mathbb{C} \setminus \mathbb{Z}^-; \mathbb{Z}^- := \{-1, -2, -3, \dots\}),$$

where

$$\Phi(a; c; z) \equiv {}_1F_1\left(\begin{matrix} a \\ c \end{matrix} \middle| z\right) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \quad (a, z \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}^-)$$

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is the familiar confluent hypergeometric function and $(\kappa)_n$ denotes the Pochhammer symbol (or the *shifted* factorial) defined by

$$(\kappa)_n = \kappa(\kappa + 1) \cdots (\kappa + n - 1) = \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\})$$

in terms of Euler's Gamma function:

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad (\Re(s) > 0).$$

In his article [2], Love (1912–2001) proved the following bounding inequality for the Laguerre function $L_v^{(\mu)}(x)$:

$$|L_v^{(\mu)}(x)| \leq \frac{\Gamma(\Re(\mu + v) + 1) \Gamma(\Re(\mu) + (1/2))}{|\Gamma(v + 1) \Gamma(\Re(\mu) + 1) \Gamma(\mu + 1/2)|} e^x =: R_v^\mu(x) \quad (2)$$

$$\left(x > 0; \Re(\mu) > -\frac{1}{2}; \Re(\mu + v) > -1 \right).$$

In proving his result (2), Love [2] exploited the following well-known integral representation [2, equation (12)]:

$$L_v^{(\mu)}(x) = \frac{e^x x^{-(\mu/2)}}{\Gamma(v + 1)} \int_0^\infty e^{-t} t^{v+(\mu/2)} J_\mu(2\sqrt{xt}) dt \quad (x \geq 0; \Re(\mu + v) > -1) \quad (3)$$

involving the first-kind Bessel function $J_\nu(z)$ of order ν , defined by

$$J_\nu(z) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(v + n + 1)} \left(\frac{x}{2}\right)^{v+2n} \quad (z \in \mathbb{C} \setminus (-\infty, 0); \nu \in \mathbb{C}).$$

The integral formula (3) will be one of our starting points in our present investigation aiming at improving the bound in Love's inequality (2).

2. Inequalities derivable by bounding the confluent hypergeometric function

Several interesting inequalities involving the confluent hypergeometric function $\Phi \equiv {}_1F_1$ were proven by Yudell Leo Luke (1918–1983) [3, sections 5 and 6]. Since Love's inequality (2) also contains e^x in the bound, we choose to recall an inequality from Luke's work in the following (slightly modified) form [3, p. 62, equation (6.21)]:

$$\Phi(a; c; x) \leq \frac{(c-1)e^x}{(c-1-a)(1+x)} \quad (x \geq 0; c-1 > a \geq 0), \quad (4)$$

where the equality *obviously* holds true when $x = a = 0$. Now, by applying Luke's inequality (4) to the confluent hypergeometric function Φ in equation (1), we arrive at our first result asserted by Theorem 1 below.

THEOREM 1 *The following inequality holds true:*

$$|L_v^{(\mu)}(x)| \leq \frac{(\mu + v)e^x}{v(\mu - |v|)B(\mu, v)(1 + x)} =: \varrho_v^\mu(x) \quad (x \geq 0; \mu > |v|; v > -1), \quad (5)$$

where

$$B(\alpha, \beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\min\{\Re(\alpha), \Re(\beta)\} > 0)$$

denotes the familiar Beta function. Moreover, for all

$$x \geq \max \left\{ 0, \frac{|v|}{\mu - |v|} \right\},$$

the upper bound in equation (5) is sharper than Love's upper bound in equation (2).

Proof At the outset, we assume that

$$x \geq 0 \quad \text{and} \quad \min\{\mu, v, \mu + v\} > -1.$$

Then, in light of definition (1) and Luke's inequality (4), it is easily observed that

$$\begin{aligned} |L_v^{(\mu)}(x)| &= \frac{\Gamma(\mu + v + 1)}{\Gamma(\mu + 1)\Gamma(v + 1)} |\Phi(-v; \mu + 1; x)| \\ &\leq \frac{\Gamma(\mu + v + 1)}{\Gamma(\mu + 1)\Gamma(v + 1)} |\Phi(|v|; \mu + 1; x)| \\ &\leq \frac{\Gamma(\mu + v + 1)}{\Gamma(\mu + 1)\Gamma(v + 1)} \cdot \frac{\mu e^x}{(\mu - |v|)(1 + x)} \\ &= \frac{(\mu + v)e^x}{v(\mu - |v|)B(\mu, v)(1 + x)} \quad (x \geq 0; \mu > |v|; v > -1), \end{aligned}$$

which proves inequality (5) asserted by Theorem 1.

In order to demonstrate the second assertion of Theorem 1, we simply concentrate upon the comparison of the upper bounds $R_v^\mu(x)$ and $\varrho_v^\mu(x)$ when the parameters $\mu, v \in \mathbb{R}$ are assumed to satisfy the hypothesis of Theorem 1. Under this assumption, $R_v^\mu(x)$ reduces immediately to

$$\frac{\Gamma(\mu + v + 1)e^x}{\Gamma(\mu + 1)\Gamma(v + 1)},$$

so that

$$\frac{R_v^\mu(x)}{\varrho_v^\mu(x)} = \left(1 - \frac{|v|}{\mu}\right)(1 + x) \geq 1$$

under the *additional* hypothesis of Theorem 1. The proof of Theorem 1 is thus completed. ■

Remark 1 Since

$$\varrho_v^\mu(x) = \mathcal{O}(x^{-1}e^x) \quad (x \rightarrow \infty)$$

and

$$R_v^\mu(x) = \mathcal{O}(e^x) \quad (x \rightarrow \infty),$$

the upper bound $\varrho_v^\mu(x)$ asserted by Theorem 1 would dominate the upper bound $R_v^\mu(x)$ in Love's inequality (2) for large $x \in \mathbb{R}^+$ at least under the hypotheses of Theorem 1.

3. Inequalities obtained by bounding the Bessel function

Returning now to the integral representation (3) of the Laguerre function $L_v^{(\mu)}(x)$, we find that

$$|L_v^{(\mu)}(x)| \leq \frac{e^x x^{-(\mu/2)}}{\Gamma(v+1)} \int_0^\infty e^{-t} t^{v+(\mu/2)} |J_\mu(2\sqrt{xt})| dt \quad (\min\{\mu+v, v\} > -1). \quad (6)$$

We are thus faced with the problem of bounding the Bessel function $|J_\mu(2\sqrt{xt})|$ by a sharp bound on the positive real half-axis, which can provide a refinement of Love's inequality (2) obtained by using Poisson's integral for the Bessel function [2, equation (11)]. Fortunately, in order to get efficient tools to accomplish this goal, we have the recent investigations by Landau [4] and Olenko [5]. In particular, Landau [4] gave the following (in a sense the best possible) bounds for the first-kind Bessel function $J_\nu(x)$ with respect to ν and x :

$$|J_\mu(x)| \leq b_L \mu^{-(1/3)} \left(b_L := \sqrt[3]{2} \sup_{x \in \mathbb{R}^+} \{\text{Ai}(x)\} \right) \quad (7)$$

and

$$|J_\mu(x)| \leq c_L |x|^{-(1/3)} \left(c_L := \sup_{x \in \mathbb{R}^+} \{x^{1/3} J_0(x)\} \right), \quad (8)$$

where $\text{Ai}(x)$ stands for the familiar Airy function defined by

$$\text{Ai}(x) := \frac{\pi}{3} \sqrt{\frac{x}{3}} \left[J_{-(1/3)} \left(2 \left(\frac{x}{3} \right)^{3/2} \right) + J_{1/3} \left(2 \left(\frac{x}{3} \right)^{3/2} \right) \right].$$

In fact, Krasikov [6] pointed out that these inequalities are sharp only in the transition region, that is, for x around $j_{\mu,1}$, which denotes the least positive zero of $J_\mu(x)$. We will be back to Krasikov's bounds for the Bessel function in the discussion part (section 4) of this article.

First of all, we recall the following sharp upper bound for the Bessel function, which was established recently by Olenko [5, p. 456, Theorem 1]:

$$\sup_{x \geq 0} \sqrt{x} |J_\mu(x)| \leq b_L \sqrt{\mu^{1/3} + \frac{\alpha_1}{\mu^{1/3}} + \frac{3\alpha_1^2}{10\mu}} =: d_O \quad (\mu > 0), \quad (9)$$

where α_1 is the smallest positive zero of the Airy function $\text{Ai}(x)$ and b_L is Landau's constant in his inequality (7) (see, for details, Olenko [5, section 3]).

We now possess the necessary mathematical tools to obtain the main improvements over Love's inequality (2).

THEOREM 2 *The following bounding inequality holds true for the Laguerre function $L_v^{(\mu)}(x)$:*

$$|L_v^{(\mu)}(x)| \leq m_v^\mu(x) \frac{e^x x^{-(\mu/2)}}{\Gamma(v+1)} \quad \left(x > 0; \mu > 0; v > -1; \mu + 2v > -\frac{3}{2} \right), \quad (10)$$

where

$$m_v^\mu(x) := \min_{x, \mu, v} \left\{ \frac{b_L \Gamma((\mu/2) + v + 1)}{\mu^{1/3}}, \frac{c_L \Gamma((\mu/2) + v + (5/6))}{\sqrt[3]{2} x^{1/6}}, \frac{d_O \Gamma((\mu/2) + v + (3/4))}{\sqrt{2} x^{1/4}} \right\}. \quad (11)$$

Proof Applying Landau's first inequality (7) to the Bessel function appearing in the integrand of equation (6), we clearly see that

$$|L_v^{(\mu)}(x)| \leq \frac{b_L e^x x^{-(\mu/2)}}{\mu^{1/3} \Gamma(v+1)} \int_0^\infty e^{-t} t^{(\mu/2)+v} dt = \frac{b_L \Gamma((\mu/2) + v + 1)}{\mu^{1/3} \Gamma(v+1)} e^x x^{-\mu/2}, \quad (12)$$

where we have also evaluated the resulting Eulerian integral for the Gamma function.

Estimating now the Bessel function in the integrand of equation (6) by Landau's second bound (8), we similarly get

$$\begin{aligned} |L_v^{(\mu)}(x)| &\leq \frac{c_L e^x x^{-(\mu/2)-(1/6)}}{\sqrt[3]{2} \Gamma(v+1)} \int_0^\infty e^{-t} t^{(\mu/2)+v-(1/6)} dt \\ &= \frac{c_L \Gamma((\mu/2) + v + (5/6))}{\sqrt[3]{2} \Gamma(v+1)} e^x x^{-(\mu/2)-(1/6)}. \end{aligned} \quad (13)$$

Finally, Olenko's bound (9), when used in evaluation of the same integral (6), would result in the following estimate:

$$\begin{aligned} |L_v^{(\mu)}(x)| &\leq \frac{d_O e^x x^{-(\mu/2)-(1/4)}}{\sqrt{2} \Gamma(v+1)} \int_0^\infty e^{-t} t^{(\mu/2)+v-(1/4)} dt \\ &= \frac{d_O \Gamma((\mu/2) + v + (3/4))}{\sqrt{2} \Gamma(v+1)} e^x x^{-(\mu/2)-(1/4)}. \end{aligned} \quad (14)$$

Collecting all three estimates given by equations (12) to (14), we easily arrive at inequality (10) asserted by Theorem 2. ■

Remark 2 Estimate (12) is of the order of magnitude

$$\mathcal{O}(x^{-(\mu/2)} e^x) \quad (x \rightarrow \infty),$$

the second estimate (13) and the third estimate (14) are of the orders of magnitude given by

$$\mathcal{O}(x^{-(\mu/2)-(1/6)} e^x) \quad (x \rightarrow \infty) \quad \text{and} \quad \mathcal{O}(x^{-(\mu/2)-(1/4)} e^x) \quad (x \rightarrow \infty),$$

respectively. Consequently, since $\mu > 0$, each of our bounding inequalities (12) to (14) dominates Love's inequality (2) rather significantly for sufficiently large $x \in \mathbb{R}^+$. This is true especially for our bounding inequality in equation (12) that was obtained by applying Olenko's result (9).

4. Further remarks and observations

Recently, Krasikov [6] obtained the following uniform bound for the first-kind Bessel function $J_\mu(x)$:

$$\begin{aligned} [J_\mu(x)]^2 &\leq \frac{4[4x^2 - (2\mu + 1)(2\mu + 5)]}{\pi[(4x^2 - \lambda)^{3/2} - \lambda]} \\ &\left(x > \frac{\sqrt{\lambda + \lambda^{2/3}}}{2}; \lambda := (2\mu + 1)(2\mu + 3); \mu > -\frac{1}{2} \right). \end{aligned} \quad (15)$$

The estimate in equation (15) is sharp in a certain sense (see, for details, [6, Theorem 2]). Krasikov [6] also noted that equation (15) provides sharp bound in the whole oscillatory

region. However, in the transition region, this estimate becomes very poor and should be replaced by another more appropriate estimate. Keeping Remark 2 in mind, we propose to combine Krasikov's bound (15) with Olenko's bound (9), since the bounding function in equation (9) possesses the order of magnitude $\mathcal{O}(x^{-(1/2)})$ for sufficiently large $x \in \mathbb{R}^+$.

Let us introduce a parameter $r \in (0, 2)$ in the integrand of the integral on the right-hand side of equation (6) in the following manner:

$$\mathcal{I} := \int_0^\infty e^{-(1-(r/2))t} t^{\mu/2+\nu-(1/4)} \cdot e^{-(rt/2)} t^{1/4} |J_\mu(2\sqrt{xt})| dt. \quad (16)$$

We now apply the Cauchy–Bunyakovsky–Schwarz inequality to the integral (16). We thus find that

$$\begin{aligned} \mathcal{I}^2 &\leq \int_0^\infty e^{-(2-r)t} t^{\mu+2\nu-(1/2)} dt \int_0^\infty e^{-rt} t^{1/2} |J_\mu(2\sqrt{xt})|^2 dt \\ &= \frac{\Gamma(\mu+2\nu+1/2)}{(2-r)^{\mu+2\nu+1/2}} \left[\left(\int_0^A + \int_A^\infty \right) e^{-rt} t^{1/2} |J_\mu(2\sqrt{xt})|^2 dt \right], \end{aligned} \quad (17)$$

where the parameter A will be chosen conveniently. Denote by $\mathfrak{I}_j$ ($j = 1, 2$) the integrals in equation (17), respectively. For obvious reasons, $|J_\mu(\cdot)|$ will be estimated by Olenko's bound (9) in $\mathfrak{I}_1$, while Krasikov's estimate (15) will be used in $\mathfrak{I}_2$. We thus find that

$$\mathfrak{I}_1 = \int_0^A e^{-rt} t^{1/2} |J_\mu(2\sqrt{xt})|^2 dt \leq \frac{d_0^2}{2\sqrt{x}} \int_0^A e^{-rt} dt = \frac{d_0^2}{2r\sqrt{x}} (1 - e^{-rA}). \quad (18)$$

Furthermore, since

$$\phi_x(t) := \frac{16xt - (2\mu+1)(2\mu+5)}{(16xt - \lambda)^{3/2} - \lambda} \leq \phi_x(A) \quad (x > 0),$$

$\phi_x(t)$ being monotonously increasing for

$$t \in (A, \infty) \quad \left(A > \frac{\lambda + \lambda^{2/3}}{16x} \right).$$

Therefore, for simplicity, we take

$$A \equiv \frac{\lambda + (\lambda + 1)^{2/3}}{16x},$$

so that

$$\begin{aligned} \mathfrak{I}_2 &= \int_A^\infty e^{-rt} t^{1/2} |J_\mu(2\sqrt{xt})|^2 dt \\ &\leq \frac{4}{\pi} \int_A^\infty e^{-rt} t^{1/2} \frac{16xt - (2\mu+1)(2\mu+5)}{(16xt - \lambda)^{3/2} - \lambda} dt \\ &\leq \frac{4}{\pi} \int_{[\lambda+(\lambda+1)^{2/3}]/(16x)}^\infty e^{-rt} t^{1/2} [16xt - (2\mu+1)(2\mu+5)] dt \\ &\leq \frac{4\mathcal{K}_\mu}{\pi r^{3/2}} \Gamma\left(\frac{3}{2}, \frac{r}{16x} [\lambda + (\lambda+1)^{2/3}]\right), \end{aligned} \quad (19)$$

where

$$\mathcal{K}_\mu := [(2\mu+1)(2\mu+3) + 1]^{2/3} - 2(2\mu+1) > 0$$

and

$$\Gamma(s, A) := \int_A^\infty t^{s-1} e^{-t} dt$$

denotes the *incomplete Gamma function of the second kind*.

We now make the upper bound for $\mathfrak{J}_1$ as precise as possible for the above-assigned value of the parameter A , so that equation (18) becomes

$$\mathfrak{J}_1 \leq \frac{d_O^2}{2r\sqrt{x}} \left[1 - \exp\left(-\frac{r}{16x}[\lambda + (\lambda + 1)^{2/3}]\right) \right]. \quad (20)$$

We thus have found that

$$\mathfrak{J}_1 = \mathcal{O}(x^{-3/2}) \quad (x \rightarrow \infty) \quad \text{and} \quad \mathfrak{J}_2 \leq \frac{2\mathcal{K}_\mu}{\sqrt{\pi}r^{3/2}} \quad (x \rightarrow \infty; \mu \text{ fixed}; r \in (0, 2)). \quad (21)$$

Indeed, since

$$e^{-t} \geq 1 - t \quad (t \in \mathbb{R}),$$

we get

$$\begin{aligned} \mathfrak{J}_1 &\leq \frac{d_O^2}{2r\sqrt{x}} \left[1 - \exp\left(-\frac{r}{16x}[\lambda + (\lambda + 1)^{2/3}]\right) \right] \\ &\leq \frac{d_O^2}{2\sqrt{x}} \frac{\lambda + (\lambda + 1)^{2/3}}{16x} = \frac{d_O^2(\lambda + (\lambda + 1)^{2/3})}{32x^{3/2}}. \end{aligned}$$

In view of the obvious relationship:

$$\Gamma(s, 0) \equiv \Gamma(s),$$

the second claim in equation (21) follows from our observation (19) by letting $A \rightarrow 0$ for sufficiently large $x \in \mathbb{R}^+$ and for fixed μ and $r \in (0, 2)$.

Finally, by collecting the estimates in equations (17), (19) to (21), together with the explanations around them, we complete the proof of the desired inequality for the Laguerre function $L_v^{(\mu)}(x)$ with the aid of Krasikov's bound (15) and Olenko's bound (9).

THEOREM 3 *The following bounding inequality holds true for the Laguerre function $L_v^{(\mu)}(x)$ when $x > 0$, $\mu > 0$ and $r \in (0, 2)$:*

$$\begin{aligned} |L_v^{(\mu)}(x)| &\leq \frac{\sqrt{\Gamma(\mu + 2v + 1/2)}e^x x^{-(\mu/2)}}{\Gamma(v + 1)(2 - r)^{((\mu/2)+v+(1/4))}} \left(\frac{d_O^2}{2\sqrt{x}} \left[1 - \exp\left(-\frac{r}{16x}[\lambda + (\lambda + 1)^{2/3}]\right) \right] \right. \\ &\quad \left. + \frac{4\mathcal{K}_\mu}{\pi r^{3/2}} \Gamma\left(\frac{3}{2}, \frac{r}{16x}[\lambda + (\lambda + 1)^{2/3}]\right) \right)^{1/2}, \end{aligned} \quad (22)$$

where

$$\lambda = (2\mu + 1)(2\mu + 3) \quad \text{and} \quad \mathcal{K}_\mu = [(2\mu + 1)(2\mu + 3) + 1]^{2/3} - 2(2\mu + 1).$$

Remark 3 The asymptotic behavior of the upper bound in equation (22) is given by

$$|L_v^{(\mu)}(x)| = \mathcal{O}(e^x x^{-(\mu/2)}) \quad x \rightarrow \infty; \mu \text{ fixed}; r \in (0, 2) \text{ fixed}. \quad (23)$$

In fact, the asymptotic behavior of the sum $\mathfrak{J}_1 + \mathfrak{J}_2$ is determined by the *slower* growth rate of the $\mathfrak{J}_2$, which is caused by Krasikov's bound (15).

We conclude this investigation by pointing out that we cannot improve the estimates and asymptotics by taking any more sophisticated evaluation method than the one based above upon the use of the Cauchy–Bunyakovsky–Schwarz inequality (e.g., the Hölder inequality), since $\mathcal{J}_2$ does not possess a vanishing upper bound as $x \rightarrow \infty$. Therefore, the most powerful upper bound remains (14) with the following order of magnitude:

$$\mathcal{O}(e^x x^{-(\mu/2)-(1/4)}) \quad (x \rightarrow \infty).$$

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Tauberian theorems for distributional wavelet transform

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In this paper we investigated the asymptotic behaviour at 0 and infinity of the distributional wavelet transform. Assuming that the wavelet transform $\mathcal{W}_g f(b, a)$ has the ordinary asymptotic behaviour at 0 (resp. at infinity) with respect to both variables (resp. to the variable b), we obtained the result for the quasiasymptotic behaviour (resp. the S-asymptotics) at 0 (resp. at infinity) of the distribution $f \in \mathcal{S}'(\mathbb{R})$. Additionally, we proved that the distribution $f \in \mathcal{D}'_{L^2}(\mathbb{R})$ has the S-asymptotics at infinity equal to zero if its wavelet transform $\mathcal{W}_g f(b, a)$ has the S-asymptotics at infinity with respect to the variable b .

Keywords: Wavelet transform; Quasiasymptotic behaviour; S-asymptotics

AMS Subject Classification: 46F12; 42C40

1. Introduction

The quasiasymptotic behaviour of distributions at infinity and zero was introduced and analysed by Russian mathematicians Zavialov, Vladimirov and Drožinov, which resulted from their investigations in the quantum field theory. They proved many general Abelian and Tauberian theorems for the distributional Laplace transform by using the notion of the quasiasymptotics [1]. Pilipović and his coworkers have also made a great contribution towards the development of this theory, proving many properties of the quasiasymptotics over the real line and applying it to the asymptotic analysis at zero and infinity of the Stieltjes transform [2, 3].

Another definition of the asymptotic behaviour at infinity of distributions (so called 'asymptotics by translation') was introduced by Brychkov and Shirokov (see [4] and references therein). Pilipović and Stanković modified this definition by introducing the S-asymptotics, which was used to prove the theorems of Abelian and Tauberian type for the Weierstrass transform of distributions [2, 3, 5, 6].

Dependence of the localization properties of the wavelet transform $\mathcal{W}_g f(b, a)$ from the localization of both the analyzing function $f \in L^2(\mathbb{R})$ and the wavelet $g \in L^2(\mathbb{R})$ is investigated by Holschneider [7]. Working in Fourier space, Pathak derived Abelian theorems at 0 and infinity for the wavelet transform of L^2 functions and tempered distributions [8].

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Contrary to the approaches in refs. [7, 8] that are based on classical estimations, we use the theory of asymptotic behaviour of distributions in the analysis of the asymptotic behaviour of the wavelet transform. We obtained more general results for the dependence of the localization properties of the wavelet transform only from localization of the analyzing function for an arbitrary analyzing wavelet. Several Abelian theorems for the behaviour at 0 of the wavelet transform are obtained in refs. [9, 10] by using the notion of the quasiasymptotics of distributions.

In this paper, we proved Tauberian type results for distributional wavelet transform. We defined the conditions which combine with the ordinary asymptotic behaviour at 0 of the wavelet transform $\mathcal{W}_g f(b, a)$ with respect to b and a , yield to the quasiasymptotic behaviour at 0 of the distribution $f \in \mathcal{S}'(\mathbb{R})$ (Theorem 3.1). The conditions under which the ordinary asymptotic behaviour at infinity of the wavelet transform $\mathcal{W}_g f(b, a)$ with respect to the variable b determines the S-asymptotics at infinity of the distribution $f \in \mathcal{S}'(\mathbb{R})$ are also given (Theorem 3.2). Additionally, the inversion formula for the wavelet transform of a function from $L^2(\mathbb{R})$ is extended over the distribution space $\mathcal{D}'_{L^2}(\mathbb{R})$ and used to obtain the next result. We proved that the distribution $f \in \mathcal{D}'_{L^2}(\mathbb{R})$ has the S-asymptotics at infinity equal to zero if its wavelet transform $\mathcal{W}_g f(b, a)$ has the S-asymptotics at infinity with respect to the variable b (Theorem 3.3).

2. Definitions and known results

As usual, the sets of positive integers and real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The Schwartz spaces of test functions and distributions on the real line are denoted by $\mathcal{D}(\mathbb{R})$ and $\mathcal{D}'(\mathbb{R})$, respectively. By $\mathcal{S}(\mathbb{R})$ is denoted the space of rapidly decreasing smooth functions defined on the real line, supplied with the usual topology. Its dual is well-known space of tempered distributions $\mathcal{S}'(\mathbb{R})$. We refer to ref. [11] for the properties of these spaces.

The spaces $S_+(\mathbb{R})$, $S_-(\mathbb{R})$, $S_0(\mathbb{R})$ and $S(H)$ of highly localized functions over the real line $\mathbb{R}$ and the half-plane $H = \{(b, a) : b \in \mathbb{R}, a > 0\}$ are described in ref. [7, chapter 1, section 13]. It is proved that they are closed subspaces of $\mathcal{S}(\mathbb{R})$; $S_+(\mathbb{R})$ [resp. $S_-(\mathbb{R})$] consists of those functions in $\mathcal{S}(\mathbb{R})$ whose Fourier transforms are supported by the positive (resp. negative) frequencies only, and $S_0(\mathbb{R})$ consists of functions from $\mathcal{S}(\mathbb{R})$ for which all the moments vanish [7, chapter 1, Theorem 19.1.3].

Recall that $\mathcal{D}_{L^2}(\mathbb{R})$ is the space of smooth functions φ on $\mathbb{R}$ for which $\varphi^{(\alpha)} \in L^2(\mathbb{R})$, $\forall \alpha \in \mathbb{N}_0$. Its topology is defined by the norms

$$\|\varphi\|_k = \left(\sum_{\alpha \leq k} \|\varphi^{(\alpha)}\|_{L^2(\mathbb{R})}^2 \right)^{1/2}, \quad k \in \mathbb{N}_0.$$

By $\mathcal{D}'_{L^2}(\mathbb{R})$ is denoted the dual space of $\mathcal{D}_{L^2}(\mathbb{R})$; $\mathcal{D}'_{L^2}(\mathbb{R})$ is the subspace of $\mathcal{S}'(\mathbb{R})$ [12]. We will give the theorem for representation of elements of the space $\mathcal{D}'_{L^2}(\mathbb{R})$ [see ref. 13]:

THEOREM 2.1 *Distribution $f \in \mathcal{D}'_{L^2}(\mathbb{R})$ if and only if there exist integer $m = m(f) > 0$ and $f_\alpha \in L^2(\mathbb{R})$, $\alpha = 0, 1, \dots, m$ such that $f = \sum_{\alpha=0}^m f_\alpha^{(\alpha)}$.*

We refer to ref. [7] for the definition and properties of the wavelet transform over the spaces $L^2(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$. The wavelet transform of $f \in L^2(\mathbb{R})$ with respect to the wavelet $g \in S_0(\mathbb{R})$

is defined by

$$\mathcal{W}_g f(b, a) = \int_{-\infty}^{+\infty} f(t) \frac{1}{a} \bar{g}\left(\frac{t-b}{a}\right) dt, \quad b \in \mathbb{R}, \quad a > 0. \quad (1)$$

Let g be a strictly admissible wavelet, *i.e.* it holds

$$c_g = \int_0^{+\infty} \frac{da}{a} |\hat{g}(ak)|^2 = 1, \quad \forall k \in \mathbb{R} \setminus \{0\},$$

where $\hat{g}$ is the Fourier transform of g . Then, the reconstruction formula for (1) is given by

$$f(x) = \int_0^{+\infty} \frac{da}{a} \int_{-\infty}^{+\infty} \mathcal{W}_g f(b, a) \frac{1}{a} \bar{g}\left(\frac{x-b}{a}\right) db \quad (2)$$

(for more details see [7, chapter 1, sections 14 and 23].

In Theorem 2.2 Pathak proved the differentiability of $\mathcal{W}_g f(b, a)$ with respect to both variables:

THEOREM 2.2 [8, Theorem 3.1] *The function $\mathcal{W}_g f(b, a)$ is differentiable, and for each $i \in \mathbb{N}_0$, $j \in \mathbb{N}_0$ we have*

$$\left(\frac{\partial}{\partial a}\right)^i \left(\frac{\partial}{\partial b}\right)^j \mathcal{W}_g f(b, a) = \frac{1}{2\pi} \left\langle \hat{f}(\omega), (i\omega)^j e^{ib\omega} \left(\frac{\partial}{\partial a}\right)^i \bar{\hat{g}}(a\omega) \right\rangle.$$

The wavelet transform of the distribution $f \in \mathcal{S}'(\mathbb{R})$ with respect to the wavelet $g \in S_0(\mathbb{R})$ is given by

$$\mathcal{W}_g f(b, a) = \langle f(t), \bar{g}_{b,a}(t) \rangle, \quad g_{b,a}(t) = \frac{1}{a} g\left(\frac{t-b}{a}\right), \quad b \in \mathbb{R}, \quad a > 0, \quad t \in \mathbb{R}$$

(see [7, chapter 1, section 25]).

For every $a > 0$, $\mathcal{W}_g f(\cdot, a)$ is a smooth function of polynomial growth since $\mathcal{W}_g f(b, a) = (f * \check{\check{g}}_a)(b)$, $b \in \mathbb{R}$, where $\check{\check{g}}_a(t) = \bar{g}(-t/a)/a$, $t \in \mathbb{R}$ is a rapidly decreasing function.

Recall the definition of the quasiasymptotic behaviour of distributions from $\mathcal{S}'(\mathbb{R})$ at 0 [3]. Many properties of the quasiasymptotic behaviour can be found in refs. [2, 3].

A function $c : (0, a) \rightarrow \mathbb{R}$ (resp. $c : (a, +\infty) \rightarrow \mathbb{R}$), $a > 0$ is called a ‘regularly varying’ at 0 (resp. at infinity) if it is a positive continuous function and there exists $\alpha \in \mathbb{R}$ such that

$$\lim_{\epsilon \rightarrow 0^+} \frac{c(\epsilon x)}{c(\epsilon)} = x^\alpha \quad \left(\text{resp.} \quad \lim_{h \rightarrow +\infty} \frac{c(hx)}{c(h)} = x^\alpha \right)$$

for every $x > 0$. If $\alpha = 0$ then the function c is called ‘slowly varying’ at 0 (resp. at infinity) and we denote it by L .

Let $f \in \mathcal{S}'(\mathbb{R})$ and $c(\epsilon)$, $\epsilon \in (0, a)$, $a > 0$ be a continuous positive function. It is said that f has the quasiasymptotic behaviour at 0 related to $c(\epsilon)$ if there exists $h \in \mathcal{S}'(\mathbb{R})$, $h \neq 0$

such that

$$\lim_{\epsilon \rightarrow 0^+} \left\langle \frac{f(\epsilon x)}{c(\epsilon)}, \varphi(x) \right\rangle = \langle h(x), \varphi(x) \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Then, we write $f \sim^q h$ at 0 related to $c(\epsilon)$ in $\mathcal{S}'(\mathbb{R})$. It is known that

$$h(x) = C_+ f_{\alpha+1}(x) + C_- f_{\alpha+1}(-x), \quad x \in \mathbb{R}, \quad \text{and} \quad c(\epsilon) = \epsilon^\alpha L(\epsilon), \quad \epsilon \in (0, a), \quad a > 0,$$

where L is a slowly varying function at 0, $\alpha \in \mathbb{R}$, $(C_+, C_-) \neq (0, 0)$ and

$$f_{\alpha+1}(x) = \begin{cases} H(x)x^\alpha / \Gamma(\alpha + 1), & \alpha > -1 \\ D^k f_{\alpha+k+1}(x), & \alpha \leq -1, \quad \alpha + k > -1, \quad k \in \mathbb{N}, \quad x \in \mathbb{R}, \end{cases}$$

where D is a derivative in a sense of distributions, Γ is the gamma function and H is the Heaviside's function [1].

We use the definition of the S-asymptotics at infinity of distributions from $\mathcal{S}'(\mathbb{R})$ from ref. [5]. We refer to refs. [2, 3, 5, 6] for its properties.

Let $f \in \mathcal{S}'(\mathbb{R})$ and $c(h)$, $h > h_0 > 0$ be a continuous positive function. f has the S-asymptotics at infinity related to $c(h)$ if there exists $u \in \mathcal{S}'(\mathbb{R})$, $u \neq 0$ such that

$$\lim_{h \rightarrow +\infty} \left\langle \frac{f(x+h)}{c(h)}, \varphi(x) \right\rangle = \langle u(x), \varphi(x) \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}). \quad (3)$$

Then, we write $f \sim^s u$ related to $c(h)$ in $\mathcal{S}'(\mathbb{R})$. It is known that

$$u(x) = C, \quad x \in \mathbb{R} \quad \text{and} \quad c(h) = L(e^h), \quad h > h_0 > 0$$

for some constant $C \neq 0$ and some slowly varying function L at infinity.

If $f \in \mathcal{D}'(\mathbb{R})$ and the limit in (3) exists for every $\varphi \in \mathcal{D}(\mathbb{R})$, then f has the S-asymptotics related to $c(h)$ in $\mathcal{D}'(\mathbb{R})$. We will give the next proposition:

PROPOSITION 2.3 [2, Proposition 10.3] *Let $f \in \mathcal{S}'(\mathbb{R})$ and $c(h)$, $h > h_0 > 0$ be a continuous positive function. If $f \sim^s u$ related to $c(h)$ in $\mathcal{D}'(\mathbb{R})$, then $f \sim^s u$ related to $c(h)$ in $\mathcal{S}'(\mathbb{R})$, too.*

We will also use the following results.

THEOREM 2.4 [14, Theorem 4] *Let c be a regularly varying function at 0, $f \in \mathcal{S}'(\mathbb{R})$, $g_0 \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} g_0(x) dx \neq 0$, and let these conditions be satisfied:*

(i)

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a} g_0 \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle = M_{b,a}, \quad (b, a) \in \mathbb{R} \times (0, 1];$$

(ii) $\exists N, k$ such that

$$\sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} g_0 \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| \leq \frac{N}{a^k}, \quad a^2 + b^2 = 1.$$

Then f has the quasiasymptotic behaviour at 0 related to $c(\epsilon)$.

THEOREM 2.5 [14, Theorem 5] *Let c be a regularly varying function at infinity, $f \in \mathcal{S}'(\mathbb{R})$, $g_0 \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{+\infty} g_0(x) dx \neq 0$, and let these conditions be satisfied:*

(i)

$$\lim_{h \rightarrow +\infty} \frac{1}{c(h)} \left\langle f(x+h), \frac{1}{a} g_0\left(\frac{x}{a}\right) \right\rangle = M_a, \quad a \in (0, 1];$$

(ii) $\exists N, k$ such that

$$\sup_{h \geq 1} \frac{1}{c(h)} \left| \left\langle f(x+h), \frac{1}{a} g_0\left(\frac{x}{a}\right) \right\rangle \right| \leq \frac{N}{a^k}, \quad a \in (0, 1).$$

Then f has the S -asymptotics at infinity related to $c(h)$.

3. Tauberian theorems for wavelet transform

In the following theorem we will fix $b \in \mathbb{R}$ and $a \in (0, 1]$ and consider the behaviour of $f \in \mathcal{S}'(\mathbb{R})$ at 0 assuming that $\mathcal{W}_g f(b, a)$ has the ordinary asymptotic behaviour at 0 with respect to the both variables.

THEOREM 3.1 *Let $f \in \mathcal{S}'(\mathbb{R})$ be of finite order $k \in \mathbb{N}_0$, $c(\epsilon), \epsilon \in (0, \epsilon_0), \epsilon_0 > 0$ be a regularly varying function at 0 such that $c(\epsilon) \epsilon^{k+1} \geq A > 0, 0 < \epsilon \leq 1$, and $g \in \mathcal{S}_0(\mathbb{R})$. If*

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a \sqrt{\pi}} e^{-(x-\epsilon b/\epsilon a)^2} \right\rangle = M'_{b,a} \quad (4)$$

and

$$\lim_{\epsilon \rightarrow 0^+} \frac{\mathcal{W}_g f(\epsilon b, \epsilon a)}{c(\epsilon)} = M''_{b,a} \quad (5)$$

for $b \in \mathbb{R}$ and $a \in (0, 1]$, then f has the quasiasymptotic behaviour at 0 related to $c(\epsilon)$.

Proof We will prove that the assumptions of this theorem imply those of Theorem 2.4. So, the assertion of the theorem follows by Theorem 2.4.

Let $g \in \mathcal{S}_0(\mathbb{R})$. Then $g \in \mathcal{S}(\mathbb{R})$ and $\int_{-\infty}^{+\infty} x^m g(x) dx = 0, \forall m \in \mathbb{N}_0$. We put

$$g(x) = g(x) + \frac{e^{-x^2}}{\sqrt{\pi}} - \frac{e^{-x^2}}{\sqrt{\pi}}. \quad (6)$$

Let $g_0(x) = g(x) + e^{-x^2}/\sqrt{\pi}$. Since $g \in \mathcal{S}(\mathbb{R})$ and $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ it follows that $g_0 \in \mathcal{S}(\mathbb{R})$. Clearly $\int_{-\infty}^{+\infty} g_0(x) dx = 1 \neq 0$.

Let $f \in \mathcal{S}'(\mathbb{R})$. By (6) we have

$$\begin{aligned} \left\langle f(x), \frac{1}{a} \bar{g}\left(\frac{x-b}{a}\right) \right\rangle &= \left\langle f(x), \frac{1}{a} \bar{g}_0\left(\frac{x-b}{a}\right) \right\rangle - \left\langle f(x), \frac{1}{a \sqrt{\pi}} e^{-(x-b/a)^2} \right\rangle \quad \text{and} \\ \left\langle f(x), \frac{1}{\epsilon a} \bar{g}\left(\frac{x-\epsilon b}{\epsilon a}\right) \right\rangle &= \left\langle f(x), \frac{1}{\epsilon a} \bar{g}_0\left(\frac{x-\epsilon b}{\epsilon a}\right) \right\rangle - \left\langle f(x), \frac{1}{\epsilon a \sqrt{\pi}} e^{-(x-\epsilon b/\epsilon a)^2} \right\rangle. \end{aligned} \quad (7)$$

Since

$$\frac{\mathcal{W}_g f(\epsilon b, \epsilon a)}{c(\epsilon)} = \frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a} \bar{g}\left(\frac{x-\epsilon b}{\epsilon a}\right) \right\rangle$$

it follows from (7) that

$$\frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a} \bar{g}_0\left(\frac{x-\epsilon b}{\epsilon a}\right) \right\rangle = \frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a \sqrt{\pi}} e^{-(x-\epsilon b/\epsilon a)^2} \right\rangle + \frac{\mathcal{W}_g f(\epsilon b, \epsilon a)}{c(\epsilon)}. \quad (8)$$

Using (8) and the assumptions (4) and (5), we get

$$\lim_{\epsilon \rightarrow 0^+} \frac{1}{c(\epsilon)} \left\langle f(x), \frac{1}{\epsilon a} \bar{g}_0 \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle = M'_{b,a} + M''_{b,a}, \quad b \in \mathbb{R}, a \in (0, 1].$$

Hence, the condition (i) of Theorem 2.4 is satisfied.

Let $f \in \mathcal{S}'(\mathbb{R})$ be of finite order $k \in \mathbb{N}_0$, and $\varphi \in \mathcal{S}(\mathbb{R})$. Then there exists $B > 0$ such that

$$\begin{aligned} \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} \varphi \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| &\leq B \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left\| \frac{1}{\epsilon a} \varphi \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\|_k \\ &= B \sup_{0 < \epsilon \leq 1} \sup_{\alpha \leq k, x \in \mathbb{R}} \frac{1}{c(\epsilon) \epsilon a} \left(1 + \left| \frac{x - \epsilon b}{\epsilon a} \right|^2 \right)^{k/2} \left| \left(\varphi \left(\frac{x - \epsilon b}{\epsilon a} \right) \right)^{(\alpha)} \right| \\ &= B \sup_{0 < \epsilon \leq 1} \sup_{\alpha \leq k, x \in \mathbb{R}} \frac{1}{c(\epsilon) \epsilon^{\alpha+1} a^{\alpha+1}} \left(1 + \left| \frac{x - \epsilon b}{\epsilon a} \right|^2 \right)^{k/2} \left| \varphi^{(\alpha)} \left(\frac{x - \epsilon b}{\epsilon a} \right) \right|. \end{aligned}$$

Since $0 < \epsilon \leq 1$ and $0 < a \leq 1$ it holds $(1/\epsilon^{\alpha+1}) \leq (1/\epsilon^{k+1})$ and $(1/a^{\alpha+1}) \leq (1/a^{k+1})$ for $\alpha \leq k$. Then

$$\begin{aligned} \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} \varphi \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| &\leq B \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon) \epsilon^{k+1} a^{k+1}} \sup_{\alpha \leq k, x \in \mathbb{R}} \left(1 + \left| \frac{x - \epsilon b}{\epsilon a} \right|^2 \right)^{k/2} \left| \varphi^{(\alpha)} \left(\frac{x - \epsilon b}{\epsilon a} \right) \right| \\ &= \frac{B \|\varphi\|_k}{a^{k+1}} \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon) \epsilon^{k+1}}. \end{aligned}$$

From the assumption that $c(\epsilon)$, $\epsilon \in (0, \epsilon_0)$, $\epsilon_0 > 0$ is a regularly varying function at 0 such that $c(\epsilon)\epsilon^{k+1} \geq A > 0$, $0 < \epsilon \leq 1$, we obtain

$$\sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} \varphi \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| \leq \frac{B \|\varphi\|_k}{A a^{k+1}} = \frac{N_\varphi}{a^{k+1}}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}). \quad (9)$$

Since $g \in \mathcal{S}(\mathbb{R})$ and $e^{-x^2} \in \mathcal{S}(\mathbb{R})$ it follows from (9) that there exist $N' > 0$ and $N'' > 0$ such that

$$\sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} |\mathcal{W}_g f(\epsilon b, \epsilon a)| = \sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} \bar{g} \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| \leq \frac{N'}{a^{k+1}} \quad (10)$$

and

$$\sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a \sqrt{\pi}} e^{-(x - \epsilon b/\epsilon a)^2} \right\rangle \right| \leq \frac{N''}{a^{k+1}}. \quad (11)$$

By (8) and the estimations (10) and (11), it follows that

$$\sup_{0 < \epsilon \leq 1} \frac{1}{c(\epsilon)} \left| \left\langle f(x), \frac{1}{\epsilon a} \bar{g}_0 \left(\frac{x - \epsilon b}{\epsilon a} \right) \right\rangle \right| \leq \frac{N}{a^{k+1}},$$

where $N = N' + N''$. Hence, the condition (ii) of Theorem 2.4 is satisfied, too. ■

Analogously, using the Theorem 2.5 we obtained the next result for the S-asymptotics of $f \in \mathcal{S}'(\mathbb{R})$ at infinity assuming that $\mathcal{W}_g f(b, a)$ has the ordinary asymptotic behaviour at infinity with respect to the variable b .

THEOREM 3.2 *Let $f \in \mathcal{S}'(\mathbb{R})$ be of finite order, $c(h)$, $h > h_0 > 0$ be a regularly varying function at infinity such that $c(h) \geq A > 0$, $h \geq 1$, and $g \in \mathcal{S}_0(\mathbb{R})$. If*

$$\lim_{h \rightarrow +\infty} \frac{1}{c(h)} \left\langle f(x+h), \frac{1}{a\sqrt{\pi}} e^{-(x/a)^2} \right\rangle = M'_a$$

and

$$\lim_{h \rightarrow +\infty} \frac{\mathcal{W}_g f(h, a)}{c(h)} = M''_a$$

for $a \in (0, 1]$, then f has the S-asymptotics at infinity related to $c(h)$.

Now, we will extend the inversion formula (2) over the distribution space $\mathcal{D}'_{L^2}(\mathbb{R})$.

Let $f \in \mathcal{D}'_{L^2}(\mathbb{R})$ and $g \in \mathcal{S}_0(\mathbb{R})$ is a strictly admissible wavelet. By Theorem 2.1 it follows that there exist integer $m > 0$ and $F_\alpha \in L^2(\mathbb{R})$, $\alpha = 0, 1, \dots, m$ such that $f = \sum_{\alpha=0}^m F_\alpha^{(\alpha)}$. Then, for $\varphi \in \mathcal{D}_{L^2}(\mathbb{R})$ we have

$$\langle f(x), \varphi(x) \rangle = \left\langle \sum_{\alpha=0}^m F_\alpha^{(\alpha)}(x), \varphi(x) \right\rangle = \sum_{\alpha=0}^m (-1)^\alpha \langle F_\alpha(x), \varphi^{(\alpha)}(x) \rangle.$$

Since $F_\alpha \in L^2(\mathbb{R})$, $\alpha = 0, 1, \dots, m$ the formula (2) holds for every F_α . Hence, we get

$$\begin{aligned} \langle f(x), \varphi(x) \rangle &= \sum_{\alpha=0}^m (-1)^\alpha \left\langle \left\langle \mathcal{W}_g F_\alpha(b, a), \frac{1}{a^2} \bar{g}\left(\frac{x-b}{a}\right) \right\rangle, \varphi^{(\alpha)}(x) \right\rangle \\ &= \sum_{\alpha=0}^m \left\langle \left\langle \mathcal{W}_g F_\alpha(b, a), \frac{1}{a^{2+\alpha}} \left(\frac{\partial}{\partial x}\right)^\alpha \bar{g}\left(\frac{x-b}{a}\right) \right\rangle, \varphi(x) \right\rangle \\ &= \sum_{\alpha=0}^m (-1)^\alpha \left\langle \left\langle \mathcal{W}_g F_\alpha(b, a), \frac{1}{a^{2+\alpha}} \left(\frac{\partial}{\partial b}\right)^\alpha \bar{g}\left(\frac{x-b}{a}\right) \right\rangle, \varphi(x) \right\rangle \\ &= \sum_{\alpha=0}^m \left\langle \left\langle \left(\frac{\partial}{\partial b}\right)^\alpha \mathcal{W}_g F_\alpha(b, a), \frac{1}{a^2} \bar{g}\left(\frac{x-b}{a}\right) \right\rangle, \varphi(x) \right\rangle \\ &= \left\langle \left\langle \sum_{\alpha=0}^m \left(\frac{\partial}{\partial b}\right)^\alpha \mathcal{W}_g F_\alpha(b, a), \frac{1}{a^2} \bar{g}\left(\frac{x-b}{a}\right) \right\rangle, \varphi(x) \right\rangle. \end{aligned} \quad (12)$$

By Theorem 2.2 it follows that

$$\begin{aligned} \sum_{\alpha=0}^m \left(\frac{\partial}{\partial b}\right)^\alpha \mathcal{W}_g F_\alpha(b, a) &= \sum_{\alpha=0}^m \frac{1}{2\pi} \left\langle \hat{F}_\alpha(\omega), (i\omega)^\alpha e^{ib\omega} \bar{\hat{g}}(a\omega) \right\rangle \\ &= \sum_{\alpha=0}^m \frac{1}{2\pi} \left\langle \widehat{F}_\alpha(\omega), \overline{(-i\omega)^\alpha e^{-ib\omega} \hat{g}(a\omega)} \right\rangle \\ &= \sum_{\alpha=0}^m \frac{(-1)^\alpha}{2\pi} \left\langle \widehat{F}_\alpha(\omega), \overline{(i\omega)^\alpha \hat{g}_{b,a}(\omega)} \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha=0}^m \frac{(-1)^\alpha}{2\pi} \left\langle \widehat{F_\alpha}(\omega), \overline{\left(\left(\frac{\partial}{\partial x} \right)^\alpha g_{b,a} \right)(\omega)} \right\rangle \\
&= \sum_{\alpha=0}^m (-1)^\alpha \left\langle F_\alpha(x), \left(\frac{\partial}{\partial x} \right)^\alpha \bar{g}_{b,a}(x) \right\rangle \\
&= \sum_{\alpha=0}^m \langle F_\alpha^{(\alpha)}(x), \bar{g}_{b,a}(x) \rangle \\
&= \mathcal{W}_g f(b, a).
\end{aligned} \tag{13}$$

By (12) and (13) we obtain

$$\langle f(x), \varphi(x) \rangle = \left\langle \left\langle \mathcal{W}_g f(b, a), \frac{1}{a^2} \bar{g} \left(\frac{x-b}{a} \right) \right\rangle, \varphi(x) \right\rangle, \quad \forall \varphi \in \mathcal{D}_{L^2}(\mathbb{R}). \tag{14}$$

Using (14) and assuming that for every $a > 0$ the wavelet transform $\mathcal{W}_g f(b, a)$ has the S-asymptotics at infinity with respect to the variable b , we proved that the distribution $f \in \mathcal{D}'_{L^2}(\mathbb{R})$ has the S-asymptotics at infinity equal to zero.

THEOREM 3.3 *Let $f \in \mathcal{D}'_{L^2}(\mathbb{R})$, $g \in \mathcal{S}_0(\mathbb{R})$, $c(h)$, $h > h_0 > 0$ be a continuous positive function and $\mathcal{W}_g f(b, a)$ satisfies these conditions:*

- (i) $\lim_{h \rightarrow +\infty} \langle \mathcal{W}_g f(b+h, a)/c(h), \theta(b) \rangle = \langle M_a, \theta(b) \rangle$, $\forall \theta \in \mathcal{S}(\mathbb{R})$, $\forall a > 0$ and
- (ii) $|\mathcal{W}_g f(b+h, a)|/c(h) \leq Q(a)P(b)$, $h > h_0 > 0$, $b \in \mathbb{R}$, $a > 0$,
where $P(b)$ is polynomial of order s and $Q(a)(1+a)^s/a \in L^1((0, +\infty))$.

Then

$$\lim_{h \rightarrow +\infty} \left\langle \frac{f(x+h)}{c(h)}, \varphi(x) \right\rangle = 0, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}).$$

Proof Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then from (14) we have

$$\begin{aligned}
\left\langle \frac{f(x+h)}{c(h)}, \varphi(x) \right\rangle &= \left\langle \left\langle \frac{\mathcal{W}_g f(b, a)}{c(h)}, \frac{1}{a^2} \bar{g} \left(\frac{x-b-h}{a} \right) \right\rangle, \varphi(x) \right\rangle \\
&= \left\langle \left\langle \frac{\mathcal{W}_g f(b+h, a)}{c(h)}, \frac{1}{a^2} \bar{g} \left(\frac{x-b}{a} \right) \right\rangle, \varphi(x) \right\rangle \\
&= \int_0^{+\infty} \frac{da}{a} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathcal{W}_g f(b+h, a)}{c(h)} \frac{1}{a} \bar{g} \left(\frac{x-b}{a} \right) \varphi(x) dx db.
\end{aligned}$$

Let

$$R_h(a) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathcal{W}_g f(b+h, a)}{c(h)} \frac{1}{a} \bar{g} \left(\frac{x-b}{a} \right) \varphi(x) dx db.$$

Since

$$\lim_{h \rightarrow +\infty} \left\langle \frac{\mathcal{W}_g f(b+h, a)}{c(h)}, \theta(b) \right\rangle = \langle M_a, \theta(b) \rangle, \quad \forall \theta \in \mathcal{S}(\mathbb{R}),$$

we have

$$\lim_{h \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\mathcal{W}_g f(b+h, a)}{c(h)} \frac{1}{a} \bar{g} \left(\frac{x-b}{a} \right) db = \int_{-\infty}^{+\infty} M_a \frac{1}{a} \bar{g} \left(\frac{x-b}{a} \right) db.$$

If we put $(x - b)/a = t$ in the last integral, we get

$$\lim_{h \rightarrow +\infty} \int_{-\infty}^{+\infty} \frac{\mathcal{W}_g f(b + h, a)}{c(h)} \frac{1}{a} \bar{g} \left(\frac{x - b}{a} \right) db = M_a \int_{-\infty}^{+\infty} \bar{g}(t) dt = 0.$$

Lebesgue's dominated convergence theorem implies

$$\lim_{h \rightarrow +\infty} R_h(a) = \lim_{h \rightarrow +\infty} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \frac{\mathcal{W}_g f(b + h, a)}{c(h)} \frac{1}{a} \bar{g} \left(\frac{x - b}{a} \right) db \right) \varphi(x) dx = 0.$$

Let us estimate $a \mapsto R_h(a)$, $a > 0$. Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $\text{supp } \varphi \subset [-r, r]$, $r > 0$.

$$\begin{aligned} |R_h(a)| &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|\mathcal{W}_g f(b + h, a)|}{c(h)} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx db \\ &= \int_{|b| \leq 2r} \frac{|\mathcal{W}_g f(b + h, a)|}{c(h)} \left(\int_{-\infty}^{+\infty} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx \right) db \\ &\quad + \int_{|b| > 2r} \frac{|\mathcal{W}_g f(b + h, a)|}{c(h)} \left(\int_{-\infty}^{+\infty} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx \right) db. \end{aligned}$$

By the assumption (ii) we get

$$\begin{aligned} |R_h(a)| &\leq Q(a) \left[\int_{|b| \leq 2r} P(b) \left(\int_{-\infty}^{+\infty} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx \right) db \right. \\ &\quad \left. + \int_{|b| > 2r} P(b) \left(\int_{-\infty}^{+\infty} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx \right) db \right] \\ &= Q(a)[I(a) + II(a)]. \end{aligned} \tag{15}$$

It is clear that there exists constant A such that

$$I(a) = \int_{|b| \leq 2r} P(b) \left(\int_{-\infty}^{+\infty} \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \varphi(x) \right| dx \right) db \leq A, \quad a > 0. \tag{16}$$

We now estimate

$$II(a) = \int_{-\infty}^{+\infty} \left(\int_{|b| > 2r} P(b) \frac{1}{a} \left| \bar{g} \left(\frac{x - b}{a} \right) \right| db \right) |\varphi(x)| dx.$$

If we change the variable in the last integral by $(x - b)/a = t$, we obtain

$$II(a) \leq \int_{|x| \leq r} \left(\int_{-\infty}^{+\infty} P(x - at) |\bar{g}(t)| dt \right) |\varphi(x)| dx.$$

Let $a < 1$. Since $P(x - at) \leq B_1 P(at)$, we have

$$II(a) \leq B_1 \int_{|x| \leq r} \left(\int_{-\infty}^{+\infty} P(at) |\bar{g}(t)| dt \right) |\varphi(x)| dx \leq B \tag{17}$$

for some constant B .

Let now $a \geq 1$. Since $P(x - at) \leq C_1(P(x) + P(at))$, we get

$$\begin{aligned}
 II(a) &\leq C_1 \int_{|x| \leq r} \left(\int_{-\infty}^{+\infty} (P(x) + P(at)) |\bar{g}(t)| dt \right) |\varphi(x)| dx \\
 &= C_1 \int_{|x| \leq r} \left(\int_{-\infty}^{+\infty} P(x) |\bar{g}(t)| dt \right) |\varphi(x)| dx \\
 &\quad + C_1 \int_{|x| \leq r} \left(\int_{-\infty}^{+\infty} P(at) |\bar{g}(t)| dt \right) |\varphi(x)| dx \\
 &\leq C_2 + C_3 \int_{|x| \leq r} \int_{-\infty}^{+\infty} a^s t^s |\bar{g}(t)| \varphi(x) dt dx \\
 &\leq C_2 + C_4 a^s.
 \end{aligned} \tag{18}$$

By (15)–(18) it follows that there exist constants D and C such that

$$|R_h(a)| \leq DQ(a), \quad a < 1 \quad \text{and} \quad |R_h(a)| \leq CQ(a)(1 + a^s) \leq CQ(a)(1 + a)^s, \quad a \geq 1.$$

By the assumptions it follows that the distribution f has the S -asymptotics at infinity related to $c(h)$ in $\mathcal{D}'(\mathbb{R})$ with the limit zero. By Proposition 2.3 we have the assertion of the theorem. ■

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Uncertainty principles for the Hankel transform

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Hardy, Beurling, Cowling–Price, Gelfand–Shilov, and Donoho–Stark uncertainty principles are obtained for the Hankel transform.

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1. Introduction

The Hankel transform of order $\nu > -1/2$ is defined as follows

$$f_\nu(x) := \int_0^\infty \sqrt{xy} J_\nu(xy) f(y) dy, \quad \nu > -1/2.$$

Here

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+\nu+1)k!}$$

is the Bessel function of the first kind [1]. If $f \in L_2(\mathbb{R}_+)$, then $f_\nu \in L_2(\mathbb{R}_+)$, and the Parseval formula $\|f\|_{L_2(\mathbb{R}_+)} = \|f_\nu\|_{L_2(\mathbb{R}_+)}$ holds. The inverse Hankel transform has the symmetric form

$$f(x) := \int_0^\infty \sqrt{xy} J_\nu(xy) f_\nu(y) dy.$$

The Hankel transform arises from the eigenvalues expansion of a Schrödinger operator. The multidimensional Fourier transform of radial functions also reduces to the Hankel transform. The following inequality for the Hankel transform

$$\|xf(x)\|_{L_2(\mathbb{R}_+)} \cdot \|xf_\nu(x)\|_{L_2(\mathbb{R}_+)} \geq (\nu+1) \|f\|_{L_2(\mathbb{R}_+)}^2$$

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has been established in ref. [2]. It is an analogue of the classical Heisenberg–Weyl uncertainty principle for the Fourier transform on $\mathbb{R}$ stating that for $f \in L_2(\mathbb{R})$

$$\|(x - x_0)f(x)\|_{L_2(\mathbb{R})} \cdot \|(\xi - \xi_0)\hat{f}(\xi)\|_{L_2(\mathbb{R})} \geq \frac{1}{4}\|f\|_{L_2(\mathbb{R})}^2,$$

where

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x) dx$$

is the Fourier transform of f . Roughly speaking, the uncertainty principle for the Fourier transform says that a function and its Fourier transform cannot both be sharply localized. That is, it is impossible for a non-zero function and its Fourier transform to be simultaneously small. ‘Smallness’ had taken different interpretations in different contexts. Hardy [3], Cowling and Price [4], and Beurling [5, 6] for example, showed such impossibility when ‘smallness’ is interpreted as sharp pointwise or integrable decay. Benedicks [7], Slepian *et al.* [8, 9], and Donoho and Stark [10] gave qualitative uncertainty principles for the Fourier transforms.

The classical Heisenberg uncertainty principle has been extended to other transforms such as the Y -transform [11], and the Dunkl transform [12]. The Donoho–Stark uncertainty principle has been extended to integral operators with a bounded kernel for which there is a Plancherel theorem [13]. The purpose of this paper is to obtain uncertainty principles similar to Hardy’s, Beurling’s, Cowling–Price’s, Gelfand–Shilov’s, and Donoho–Stark’s principles for the Hankel transform.

The structure of the paper is as follows: In section 2 we prove a Hardy’s theorem for the Hankel transform. A similar result is obtained for integral transforms arising from singular Sturm–Liouville problems. In section 3 we state and prove a Beurling type theorem. This theorem is used in section 4 to establish a Cowling–Price type and a Gelfand–Shilov type theorems. In section 5 a qualitative study (Donoho–Stark’s theorem) and its application to a recovery problem are conducted.

2. Hardy uncertainty principle

Hardy theorem [3] was perhaps the first result on pointwise decay rate of f and its Fourier transform $\hat{f}$ at infinity. It says that if $|f(x)| \leq Ce^{-a^2x^2}$, and $|\hat{f}(\lambda)| \leq Ce^{-\lambda^2/(4a^2)}$, then f is a multiple of $e^{-a^2x^2}$.

In this section, we will obtain a Hardy uncertainty principle for the Hankel transform.

THEOREM 1 *Let $|f(y)| \leq Cy^{v+1/2}e^{-a^2y^2}$ and $|f_v(x)| \leq Cx^{v+1/2}e^{-(x^2/4a^2)}$. Then $f(y)$ is a multiple of $y^{v+1/2}e^{-a^2y^2}$.*

Here and afterwards C is a universal constant, that can be different in distinct places.

Proof We have

$$x^{-v-(1/2)}f_v(x) = \int_0^\infty (xy)^{-v}J_v(xy)y^{v+1/2}f(y)dy.$$

Since $(xy)^{-v}J_v(xy)$ is an entire, even, and bounded function in variable x , and $y^{v+1/2}f(y)$ is an integrable function, then $x^{-v-1/2}f_v(x)$ is an entire and even function. Moreover, for any

complex variable z ,

$$\begin{aligned}
 |z^{-\nu-(1/2)} f_\nu(z)| &\leq \int_0^\infty |y^{\nu+1/2} f(y)| |(zy)^{-\nu} J_\nu(zy)| dy \\
 &\leq C \int_0^\infty y^{2\nu+1} e^{-a^2 y^2} \sum_{k=0}^\infty \frac{(|z|y/2)^{2k}}{\Gamma(\nu+k+1)k!} dy \\
 &= C \sum_{k=0}^\infty \frac{(|z|/2)^{2k}}{\Gamma(\nu+k+1)k!} \int_0^\infty y^{2\nu+2k+1} e^{-a^2 y^2} dy \\
 &= C \sum_{k=0}^\infty \frac{(|z|/2)^{2k}}{\Gamma(\nu+k+1)k!} \frac{\Gamma(\nu+k+1)}{2a^{2\nu+2k+2}} \\
 &= C e^{(|z|^2/4a^2)}.
 \end{aligned}$$

Thus $z^{-(\nu/2)-(1/4)} f_\nu(z^{(1/2)})$ is an entire function, of the order $e^{(|z|/4a^2)}$ on the whole complex plane, and $e^{-(|z|/4a^2)}$ on the real line. According to ref. [14] $z^{-(\nu/2)-(1/4)} f_\nu(z^{(1/2)})$ must be a multiple of $e^{-(z/4a^2)}$. Hence $f(y) = C y^{\nu+1/2} e^{-a^2 y^2}$. Thus, the Theorem is proved. ■

As a consequence we can state now a Hardy uncertainty principle for the Hankel transform.

COROLLARY 1 *One cannot have both $|f(y)| \leq C y^{\nu+1/2} e^{-\alpha y^2}$, $\alpha > 0$, and $|f_\nu(x)| \leq C x^{\nu+1/2} e^{-\beta x^2}$, $\beta > 0$, where $\alpha\beta > (1/4)$ unless $f(y) = 0$.*

The Hardy uncertainty principle can be extended to more general integral transforms arising from singular Sturm–Liouville eigenfunction expansions.

Consider a singular Sturm–Liouville problem:

$$\begin{aligned}
 Ly := \frac{d^2 y}{dx^2} - q(x)y &= -\lambda y, \quad 0 < x < \infty, \\
 y(0, \lambda) \sin \alpha + y'(0, \lambda) \cos \alpha &= 0,
 \end{aligned} \tag{1}$$

where q is a real-valued function such that

$$\int_0^\infty |q(x)| dx < \infty. \tag{2}$$

Under condition (2) L is a self-adjoint operator. The spectrum of L consists of $[0, \infty]$ and possibly a finite number of negative eigenvalues. Let $\phi(x, \lambda)$ be the solution of the Sturm–Liouville problem (1) such that

$$\phi(a, \lambda) = \cos \alpha, \quad \phi'(a, \lambda) = -\sin \alpha.$$

It is well-known [15] that there exists a non-decreasing (spectral) function $\rho(\lambda)$ such that if $f \in L_2(\mathbb{R}_+)$, then

$$g(\lambda) = \int_0^\infty f(x) \phi(x, \lambda) dx, \tag{3}$$

belongs to $L_2(\mathbb{R}, d\rho)$. The Parseval equality

$$\|f\|_{L_2(\mathbb{R}_+)} = \|g\|_{L_2(\mathbb{R}, d\rho)},$$

and the expansion

$$f(x) = \int_{\mathbb{R}} \phi(x, \lambda) g(\lambda) d\rho(\lambda), \quad (4)$$

also hold. The latter integral converges in the mean in $L_2(\mathbb{R}_+)$. Conversely, if $g \in L_2(\mathbb{R}, d\rho)$, then the integral (4) converges in the mean in $L_2(\mathbb{R}_+)$ to a function f and formula (3) holds. This is the Plancherel theorem for the pair of Sturm–Liouville transforms (3) and (4).

THEOREM 2 *Let $|f(x)| \leq Ce^{-a^2x^2}$ and $|g(\lambda)| \leq Ce^{-(\lambda/4a^2)}$. Then $g(\lambda) = Ce^{-(\lambda/4a^2)}$.*

In other words, both $f(x)$ and $g(\lambda)$ cannot decay too fast at infinity.

Proof Under the condition (2) $\phi(x, \lambda) = O(e^{x\sqrt{|\lambda|}})$, uniformly in $0 \leq x < \infty$, $|\lambda| \geq \epsilon > 0$ [15]. Because $\phi(x, \lambda)$ is an entire function with respect to variable λ , of the order $O(e^{x\sqrt{|\lambda|}})$, and $f(x) = O(e^{-a^2x^2})$, the integral

$$g(\lambda) = \int_0^\infty \phi(x, \lambda) f(x) dx,$$

converges uniformly in any compact domain of λ . Therefore, $g(\lambda)$ is an entire function, and for $|\lambda| \geq \epsilon > 0$ we have

$$\begin{aligned} |g(\lambda)| &\leq \int_0^\infty |\phi(x, \lambda)| |f(x)| dx \\ &\leq C \int_0^\infty e^{x\sqrt{|\lambda|} - a^2x^2} dx \\ &\leq Ce^{(|\lambda|/4a^2)} \int_{-\infty}^\infty e^{-(\sqrt{|\lambda|}/(2a) - ax)^2} dx \\ &= Ce^{(|\lambda|/4a^2)} \int_{-\infty}^\infty e^{-a^2x^2} dx \\ &= C \frac{\sqrt{\pi}}{a} e^{(|\lambda|/4a^2)}. \end{aligned}$$

Thus, $g(\lambda)$ is an entire function, of the order $O(e^{(|\lambda|/4a^2)})$ at infinity and $O(e^{-(\lambda/4a^2)})$ as $\lambda \rightarrow \infty$. Then according to ref. [14] $g(\lambda)$ must be a multiple of $e^{-(\lambda/4a^2)}$. Hence, the Theorem is proved. ■

3. Beurling uncertainty principle

The following integral variant of the Hardy theorem for the Fourier transform has been discovered by Beurling [6, 16]. Suppose that f and its Fourier transform $\hat{f}$ belong to $L_1(\mathbb{R})$, and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x) \hat{f}(\lambda)| e^{|x\lambda|} dx d\lambda < \infty.$$

Then $f = 0$.

In this section, we will prove a Beurling uncertainty principle for the Hankel transform.

THEOREM 3 Let $y^{v+(1/2)}f(y) \in L(\mathbb{R}_+)$, $x^{v+(1/2)}f_v(x) \in L(\mathbb{R}_+)$, and

$$\int_0^\infty \int_0^\infty (xy)^{v+(1/2)} e^{xy} |f(y)f_v(x)| dx dy < \infty.$$

Then $f = 0$.

Proof Recall the Hankel convolution [17]

$$(f * g)(y) := \frac{2^{v-1}}{\sqrt{\pi}\Gamma(v+1/2)} \int_0^\infty \int_0^\infty (yuv)^{(1/2)-v} \Delta^{2v-1} f(u)g(v) du dv,$$

where Δ is the area of the triangle with sides y , u , and v ($\Delta = 0$ if y , u , and v are not the sides of a triangle). Then

$$(f * g)_v(x) = x^{-v-(1/2)} f_v(x) g_v(x).$$

Define

$$h(y) := f(y) * (y^{v+(1/2)} e^{-y^2/2}).$$

Since $(y^{v+(1/2)} e^{-y^2/2})_v = x^{v+(1/2)} e^{-x^2/2}$ (see ref. [1]), then

$$h_v(x) = x^{-v-(1/2)} f_v(x) (y^{v+(1/2)} e^{-y^2/2})_v = e^{-x^2/2} f_v(x).$$

If $y^{v+(1/2)}f(y) \in L(\mathbb{R}_+)$, then $z^{-v-(1/2)}f_v(z)$ is an entire even function of order 1. Therefore, $z^{-v-1/2}h_v(z) = z^{-v-1/2}e^{-z^2/2}f_v(z)$ is an entire even function of order 2. Because $x^{v+(1/2)}h_v(x)$ is integrable, $y^{-v-(1/2)}h(y)$ is an even entire function of order 2. Let $g(z) = z^{-v-(1/2)}h(z)$. Consider the integral

$$\begin{aligned} I &:= \int_0^\infty |g(y)| \sup_{|z|=y} |g(z)| dy \\ &\leq \int_0^\infty y^{-v-1/2} |h(y)| \sup_{|z|=y} \int_0^\infty |(zx)^{-v} J_v(zx) x^{v+1/2} h_v(x)| dx dy. \end{aligned}$$

Using the integral representation [1]

$$J_v(z) = \frac{(z/2)^v}{\sqrt{\pi}\Gamma(v+1/2)} \int_{-1}^1 (1-t^2)^{v-(1/2)} \cos(zt) dt, \quad (5)$$

and the inequality $|\cos(zt)| \leq e^{|z|}$ for $-1 \leq t \leq 1$ we get

$$\begin{aligned} |J_v(z)| &\leq \frac{(|z|/2)^v e^{|z|}}{\sqrt{\pi}\Gamma(v+1/2)} \int_{-1}^1 (1-t^2)^{v-1/2} dt \\ &= \frac{(|z|/2)^v e^{|z|}}{\sqrt{\pi}\Gamma(v+1/2)} \frac{2^{2v}\Gamma^2(v+1/2)}{\Gamma(2v+1)} \\ &= \frac{(2|z|)^v e^{|z|}\Gamma(v+1/2)}{\sqrt{\pi}\Gamma(2v+1)}. \end{aligned}$$

Applying the double angle formula for the Gamma function [1]

$$\sqrt{\pi}\Gamma(2v+1) = 2^{2v}\Gamma(v+1/2)\Gamma(v+1),$$

we arrive at

$$|J_\nu(z)| \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{|z|}{2}\right)^\nu e^{|z|}.$$

Consequently $|(zx)^{-\nu} J_\nu(zx)| \leq \frac{e^{|z|x}}{2^\nu \Gamma(\nu+1)}$, and therefore,

$$I \leq \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^\infty y^{-\nu-(1/2)} |h(y)| \int_0^\infty e^{xy} x^{\nu+(1/2)} |h_\nu(x)| dx dy.$$

Replacing $h_\nu(x) = e^{-x^2/2} f_\nu(x)$ and using the definition of h

$$h(y) := \frac{2^{\nu-1}}{\sqrt{\pi} \Gamma(\nu+1/2)} \int_0^\infty \int_0^\infty (yuv)^{(1/2)-\nu} \Delta^{2\nu-1} f(u) v^{\nu+(1/2)} e^{-v^2/2} du dv,$$

we arrive at

$$\begin{aligned} I &\leq \frac{1}{2\sqrt{\pi} \Gamma(\nu+1/2) \Gamma(\nu+1)} \int_0^\infty \int_0^\infty |f(u)| |f_\nu(x)| u^{(1/2)-\nu} x^{\nu+(1/2)} e^{-x^2/2} \\ &\quad \times \int_0^\infty \int_0^\infty y^{-2\nu} v \Delta^{2\nu-1} e^{xy-v^2/2} dv dy du dx. \end{aligned}$$

Using the formula for area of a triangle with sides y , u , and v

$$\Delta^2 = \frac{1}{4} (v^2 - (y-u)^2)_+ ((y+u)^2 - v^2)_+$$

where $\varphi_+(x)$ is a cut-off function. The inner integral can be estimated as follows

$$\begin{aligned} &\int_0^\infty \int_0^\infty y^{-2\nu} v \Delta^{2\nu-1} e^{xy-v^2/2} dv dy \\ &= \frac{1}{2^{2\nu-1}} \int_0^\infty y^{-2\nu} e^{xy} \int_{|y-u|}^{y+u} v (v^2 - (y-u)^2)^{\nu-(1/2)} ((y+u)^2 - v^2)^{\nu-(1/2)} e^{-v^2/2} dv dy \\ &\leq \frac{1}{2^{2\nu-1}} \int_0^\infty y^{-2\nu} e^{xy} \int_{|y-u|}^{y+u} v (v^2 - (y-u)^2)^{\nu-(1/2)} ((y+u)^2 - v^2)^{\nu-(1/2)} e^{-(y-u)^2/2} dv dy \\ &= \frac{1}{2^{2\nu}} \int_0^\infty y^{-2\nu} e^{xy} \int_{(y-u)^2}^{(y+u)^2} (w - (y-u)^2)^{\nu-(1/2)} ((y+u)^2 - w)^{\nu-(1/2)} e^{-(y-u)^2/2} dw dy \\ &= \frac{1}{2^{2\nu}} \int_0^\infty y^{-2\nu} e^{xy-(y-u)^2/2} \frac{\Gamma^2(\nu+1/2)}{\Gamma(2\nu+1)} [(y+u)^2 - (y-u)^2]^{2\nu} dy \\ &= \frac{4^\nu \Gamma^2(\nu+1/2)}{\Gamma(2\nu+1)} u^{2\nu} e^{x^2/2+xu} \int_0^\infty e^{-(x-y-u)^2/2} dy \\ &\leq \frac{4^\nu \Gamma^2(\nu+1/2)}{\Gamma(2\nu+1)} u^{2\nu} e^{x^2/2+xu} \int_{-\infty}^\infty e^{-y^2/2} dy \\ &= \frac{\sqrt{2\pi} 4^\nu \Gamma^2(\nu+1/2)}{\Gamma(2\nu+1)} u^{2\nu} e^{x^2/2+xu}. \end{aligned}$$

Consequently,

$$I \leq \frac{\sqrt{\pi} 2^{2\nu-(1/2)} \Gamma(\nu+1/2)}{\Gamma(\nu+1) \Gamma(2\nu+1)} \int_0^\infty \int_0^\infty (ux)^{\nu+(1/2)} e^{ux} |f(u) f_\nu(x)| du dx = \tilde{C} < \infty. \quad (6)$$

The rest of the proof is similar to ref. [5] and is included here just for completeness. One claims that the entire function

$$G(z) := \int_0^z g(y)g(iy) dy$$

is bounded by $\tilde{C}$. It is bounded by $\tilde{C}$ on the real line by equation (6). To bound G in the first quadrant, let $\rho \in [0, \pi/2)$, $\alpha \in (0, \rho)$, and set

$$G_\alpha(z) := \int_0^z g(e^{-i\alpha}y)g(iy) dy.$$

Then G_α is entire of order 2 and bounded by $\tilde{C}$ on the imaginary axis and on the half-line with angle α . By the Phragmen-Lindelöf theorem it is bounded by $\tilde{C}$ in the intervening angle as well. Boundedness in the first quadrant follows by continuity and independence of the bound on ρ . A similar argument yields boundedness in the other quadrants. Therefore, G is constant. Thus $g(y) = 0$, and therefore $f(y) = 0$. Hence, the Theorem is proved. ■

4. Gelfand–Shilov and Cowling–Price uncertainty principles

Gelfand and Shilov [18] have shown that if

$$\int_{\mathbb{R}} |f(x)|e^{|ax|^{1/p}} dx < \infty, \quad \int_{\mathbb{R}} |\hat{f}(\lambda)|e^{|b\lambda|^{1/q}} d\lambda < \infty,$$

with $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$, and $ab > 1/4$, then $f = 0$. Cowling and Price [4] have shown that if $p, q \in [1, \infty]$ and $\min(p, q) < \infty$, then from

$$\|e^{ax^2}f(x)\|_{L_p(\mathbb{R})} + \|e^{b\lambda^2}\hat{f}(\lambda)\|_{L_q(\mathbb{R})} < \infty,$$

with $ab > 1/4$ it follows $f = 0$. In this section, we will obtain similar results for the Hankel transform.

Let M and M^* be functions satisfying

$$xy \leq M(y) + M^*(x). \quad (7)$$

If

$$\int_0^\infty y^{v+(1/2)} e^{M(y)} |f(y)| dy < \infty, \quad \int_0^\infty x^{v+(1/2)} e^{M^*(x)} |f_v(x)| dx < \infty,$$

then

$$\begin{aligned} & \int_0^\infty \int_0^\infty (xy)^{v+(1/2)} e^{xy} |f(y)f_v(x)| dx dy \\ & \leq \int_0^\infty \int_0^\infty (xy)^{v+(1/2)} e^{M(y)+M^*(x)} |f(y)f_v(x)| dx dy \\ & = \int_0^\infty y^{v+(1/2)} e^{M(y)} |f(y)| dy \int_0^\infty x^{v+(1/2)} e^{M^*(x)} |f_v(x)| dx < \infty. \end{aligned}$$

Consequently, Beurling Theorem 3 implies that $f(y) = 0$. In particular, if $M(y) = y^p/p$, and $M^*(x) = x^q/q$, where p, q are conjugate exponents $1/p + 1/q = 1$, then the pair M, M^*

satisfies the condition (7). Thus, we obtain an analogue of the Gelfand–Shilov uncertainty principle for the Hankel transform.

COROLLARY 2 *If*

$$\int_0^\infty y^{v+(1/2)} e^{y^p/p} |f(y)| dy < \infty, \quad \int_0^\infty x^{v+(1/2)} e^{x^q/q} |f_v(x)| dx < \infty,$$

then $f = 0$.

Another example of the pair M, M^* comes from Young functions. Let m be a function positive and increasing on $(0, \infty)$. Define $M(y) = \int_0^y m(t) dt$ and $M^*(x) = \int_0^x m^{-1}(t) dt$, where m^{-1} is the inverse of m . The convex M is called a Young function with Young dual M^* . The Young pair satisfies inequality (7).

Let

$$\|y^{v+(1/2)} e^{ay^2} f(y)\|_{L_p(\mathbb{R}_+)} < \infty, \quad \|x^{v+(1/2)} e^{bx^2} f_v(x)\|_{L_q(\mathbb{R}_+)} < \infty,$$

where $1 \leq p, q < \infty$, and $ab > 1/4$. Choose a_0, b_0 such that $0 < a_0 < a, 0 < b_0 < b, a_0 b_0 > 1/4$. The Hölder inequality gives

$$\begin{aligned} \int_0^\infty y^{v+(1/2)} e^{a_0 y^2} |f(y)| dy &= \int_0^\infty y^{v+(1/2)} e^{ay^2} |f(y)| e^{-(a-a_0)y^2} dy \\ &\leq \|y^{v+(1/2)} e^{ay^2} f(y)\|_{L_p(\mathbb{R}_+)} \|e^{-(a-a_0)y^2}\|_{L_{p'}(\mathbb{R}_+)} < \infty, \end{aligned}$$

where p' is the conjugate exponent of p ($p^{-1} + p'^{-1} = 1$). Similarly,

$$\int_0^\infty x^{v+1/2} e^{b_0 x^2} |f_v(x)| dx < \infty.$$

Because $a_0 y^2 + b_0 x^2 \geq 2\sqrt{a_0 b_0} xy \geq xy$, the inequality (7) holds. The Beurling theorem therefore yields $f = 0$. Hence, we have proved an analogue of the Cowling–Price theorem

COROLLARY 3 *If*

$$\|y^{v+(1/2)} e^{ay^2} f(y)\|_{L_p(\mathbb{R}_+)} < \infty, \quad \|x^{v+(1/2)} e^{bx^2} f_v(x)\|_{L_q(\mathbb{R}_+)} < \infty,$$

where $1 \leq p, q < \infty$ and $ab > 1/4$ then $f = 0$.

5. Donoho–Stark uncertainty principle

We say that f is ϵ -concentrated on a measurable set $\mathbb{E}$ if

$$\|f - \chi_{\mathbb{E}} f\|_2 < \epsilon.$$

Donoho and Stark [10] show that if f of unit L_2 norm is $\epsilon_{\mathbb{T}}$ concentrated on a measurable set $\mathbb{T}$ and its Fourier transform $\hat{f}$ is $\epsilon_{\mathbb{W}}$ on a measurable set $\mathbb{W}$, then

$$|\mathbb{W}| \cdot |\mathbb{T}| \geq (1 - \epsilon_{\mathbb{T}} - \epsilon_{\mathbb{W}})^2.$$

Here, $|\mathbb{T}|$ is the Lebesgue measure of the set $\mathbb{T}$. This inequality has been slightly improved in ref. [16] to

$$|\mathbb{W}| \cdot |\mathbb{T}| \geq (1 - (\epsilon_{\mathbb{T}}^2 + \epsilon_{\mathbb{W}}^2)^{1/2})^2. \quad (8)$$

In this section, we will extend the Donoho–Stark uncertainty principle to the Hankel transform.

Let $P_{\mathbb{E}}$ denote the time-limiting operator

$$(P_{\mathbb{E}}f)(x) = \begin{cases} f(x), & x \in \mathbb{E} \\ 0, & x \notin \mathbb{E} \end{cases}.$$

This operator cuts off the part of f outside $\mathbb{E}$. So f is ϵ -concentrated on a set $\mathbb{E}$ if, and only if

$$\|f - P_{\mathbb{E}}f\|_{L_2(\mathbb{R}_+)} \leq \epsilon.$$

For simplicity, we will use P_X to denote $P_{[0,X]}$. Clearly $\|P_{\mathbb{E}}\| = 1$. The second operator is the frequency-limiting operator

$$(Q_{\mathbb{E}}f)(x) := \int_{\mathbb{E}} \sqrt{xy} J_v(xy) f_v(y) dy.$$

$Q_{\mathbb{E}}f$ is a partial reconstruction of f using only frequency of f_v in $\mathbb{E}$. If $g = Q_{\mathbb{E}}f$, then g_v vanishes outside $\mathbb{E}$, and f_v is ϵ -concentrated on $\mathbb{E}$ if, and only if,

$$\|f - Q_{\mathbb{E}}f\|_{L_2(\mathbb{R}_+)} \leq \epsilon.$$

Using the Parseval formula $\|f\|_{L_2(\mathbb{R}_+)} = \|f_v\|_{L_2(\mathbb{R}_+)}$ one can show that $\|Q_{\mathbb{E}}\| = 1$. For simplicity let $Q_Y = Q_{[0,Y]}$. We have

$$\begin{aligned} (P_X Q_Y f)(x) &= P_X \int_0^Y \sqrt{xy} J_v(xy) \int_0^\infty \sqrt{yt} J_v(yt) f(t) dt dy \\ &= P_X \int_0^\infty f(t) \int_0^Y \sqrt{xy} J_v(xy) \sqrt{yt} J_v(yt) dy dt \\ &= \int_0^\infty q(x, t) f(t) dt, \end{aligned}$$

where

$$q(x, t) := \begin{cases} \int_0^Y \sqrt{xy} J_v(xy) \sqrt{yt} J_v(yt) dy, & x < X \\ 0, & x \geq X \end{cases}.$$

The Hilbert-Schmidt norm of $P_X Q_Y$ is $(\int_0^\infty \int_0^\infty |q(x, t)|^2 dt dx)^{1/2}$. The norm $\|P_X Q_Y\|$ does not exceed the Hilbert-Schmidt norm of $P_X Q_Y$, therefore,

$$\begin{aligned} \|P_X Q_Y\|^2 &\leq \int_0^\infty \int_0^\infty |q(x, t)|^2 dt dx \\ &= \int_0^X \int_0^\infty |q(x, t)|^2 dx dt. \end{aligned}$$

Notice that

$$q(x, t) = (\sqrt{xy} J_v(xy) \chi_{[0,Y]})_v,$$

the Hankel transform of $\sqrt{xy} J_v(xy) \chi_{[0,Y]}$ with respect to y , where $\chi_{\mathbb{E}}$ is the characteristic function of a set $\mathbb{E}$. Thus, the Parseval identity for the Hankel transform yields

$$\int_0^\infty |q(x, t)|^2 dt = \int_0^Y |\sqrt{xy} J_v(xy)|^2 dy.$$

Consequently,

$$\|P_X Q_Y\|^2 \leq \int_0^X \int_0^Y |\sqrt{xy} J_\nu(xy)|^2 dy dx.$$

Using the integral representation [1]

$$J_\nu(x) = \frac{(x/2)^\nu}{\sqrt{\pi} \Gamma(\nu + (1/2))} \int_{-1}^1 (1-t^2)^{\nu-(1/2)} \cos(xt) dt$$

for $x > 0$ we get

$$\begin{aligned} |J_\nu(x)| &\leq \frac{(x/2)^\nu}{\sqrt{\pi} \Gamma(\nu + (1/2))} \int_{-1}^1 (1-t^2)^{\nu-(1/2)} dt \\ &= \frac{(x/2)^\nu}{\sqrt{\pi} \Gamma(\nu + (1/2))} \frac{2^{2\nu} \Gamma^2(\nu + (1/2))}{\Gamma(2\nu + 1)} \\ &= \frac{(2x)^\nu \Gamma(\nu + (1/2))}{\sqrt{\pi} \Gamma(2\nu + 1)}. \end{aligned}$$

Applying again the double angle formula for the Gamma function [1]

$$\sqrt{\pi} \Gamma(2\nu + 1) = 2^{2\nu} \Gamma(\nu + 1/2) \Gamma(\nu + 1),$$

we arrive at

$$|J_\nu(x)| \leq \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu.$$

Hence

$$\int_0^X \int_0^Y |\sqrt{xy} J_\nu(xy)|^2 dy dx \leq \frac{1}{2^{2\nu} \Gamma^2(\nu + 1)} \int_0^X \int_0^Y (xy)^{2\nu+1} dy dx = \frac{(XY)^{2\nu+2}}{2^{2\nu+2} \Gamma^2(\nu + 2)}.$$

We arrive at

$$\|P_X Q_Y\| \leq \frac{(XY)^{\nu+1}}{2^{\nu+1} \Gamma(\nu + 2)}.$$

Let $XY < 2\Gamma^{(1/(\nu+1))}(\nu + 2)$. Then $\|P_X Q_Y\| < 1$, and, therefore, $I - P_X Q_Y$ is invertible with

$$\begin{aligned} \|(I - P_X Q_Y)^{-1}\| &\leq \sum_{k=0}^{\infty} \|P_X Q_Y\|^k \\ &\leq \sum_{k=0}^{\infty} \left[\frac{(XY)^{\nu+1}}{2^{\nu+1} \Gamma(\nu + 2)} \right]^k \\ &= \frac{2^{\nu+1} \Gamma(\nu + 2)}{2^{\nu+1} \Gamma(\nu + 2) - (XY)^{\nu+1}}. \end{aligned}$$

We have

$$I = P_X + P_{(X,\infty)} = P_X Q_Y + P_X Q_{(Y,\infty)} + P_{(X,\infty)}.$$

The orthogonality of P_X and $P_{(X,\infty)}$ gives

$$\|P_X Q_{(Y,\infty)} f\|_{L_2(\mathbb{R}_+)}^2 + \|P_{(X,\infty)} f\|_{L_2(\mathbb{R}_+)}^2 = \|P_X Q_{(Y,\infty)} f + P_{(X,\infty)} f\|_{L_2(\mathbb{R}_+)}^2.$$

Together with $\|P_X\| = 1$ it yields

$$\begin{aligned} \|f\|_{L_2(\mathbb{R}_+)}^2 &\leq \|(I - P_X Q_Y)^{-1}\|^2 \|(I - P_X Q_Y)f\|_{L_2(\mathbb{R}_+)}^2 \\ &\leq \left(\frac{2^{\nu+1}\Gamma(\nu+2)}{2^{\nu+1}\Gamma(\nu+2) - (XY)^{\nu+1}} \right)^2 [\|P_X Q_{(Y,\infty)} f\|_{L_2(\mathbb{R}_+)}^2 + \|P_{(X,\infty)} f\|_{L_2(\mathbb{R}_+)}^2] \\ &\leq \left(\frac{2^{\nu+1}\Gamma(\nu+2)}{2^{\nu+1}\Gamma(\nu+2) - (XY)^{\nu+1}} \right)^2 [\|Q_{(Y,\infty)} f\|_{L_2(\mathbb{R}_+)}^2 + \|P_{(X,\infty)} f\|_{L_2(\mathbb{R}_+)}^2]. \end{aligned}$$

If f of unit norm is ϵ_X -timelimited on $[0, X]$, then $\|P_{(X,\infty)} f\|_{L_2(\mathbb{R}_+)} \leq \epsilon_X$. If f of unit norm is ϵ_Y -bandlimited on $[0, Y]$, then $\|Q_{(Y,\infty)} f\|_{L_2(\mathbb{R}_+)} \leq \epsilon_Y$. Then if f of unit norm is both ϵ_X -timelimited and ϵ_Y -bandlimited,

$$1 \leq \left(\frac{2^{\nu+1}\Gamma(\nu+2)}{2^{\nu+1}\Gamma(\nu+2) - (XY)^{\nu+1}} \right)^2 (\epsilon_X^2 + \epsilon_Y^2),$$

or

$$XY \geq 2\Gamma^{(1/(\nu+1))}(\nu+2)(1 - (\epsilon_X^2 + \epsilon_Y^2)^{1/2})^{(1/(\nu+1))}.$$

We arrive at the Donoho–Stark uncertainty principle for the Hankel transform.

THEOREM 4 *Let a unit norm signal f be ϵ_X -timelimited on $[0, X]$ and ϵ_Y -bandlimited on $[0, Y]$. Then*

$$XY \geq 2\Gamma^{(1/(\nu+1))}(\nu+2)(1 - (\epsilon_X^2 + \epsilon_Y^2)^{1/2})^{(1/(\nu+1))}.$$

Let f be an $L_2(\mathbb{R}_+)$ function, that is bandlimited to the interval $[0, Y]$ (i.e., f_ν vanishes off $[0, Y]$). Suppose that f is corrupted by both noise $n \in L_2(\mathbb{R}_+)$ and unregistered values on $[0, X]$. Thus, the observable function r satisfies

$$r(x) = \begin{cases} f(x) + n(x), & x > X \\ 0, & x \leq X \end{cases}.$$

Here, we have assumed without loss of generality that $n = 0$ on $[0, X]$. Equivalently,

$$r = (I - P_X)f + n.$$

We say that f can be stably reconstructed from r , if there exists a linear operator K and a constant C such that

$$\|f - Kr\|_{L_2(\mathbb{R}_+)} \leq C\|n\|_{L_2(\mathbb{R}_+)}. \quad (9)$$

COROLLARY 4 *If*

$$XY < 2\Gamma^{(1/(\nu+1))}(\nu+2),$$

then f can be stably reconstructed from r .

Proof Let $K = (I - P_X Q_Y)^{-1}$. From $XY < 2\Gamma^{1/(v+1)}(v+2)$ it follows that $\|P_X Q_Y\| < 1$, hence the existence of K . Since $(I - P_X)f = (I - P_X Q_Y)f$ for every f , bandlimited to $[0, Y]$,

$$\begin{aligned} f - Kr &= f - K((I - P_X)f + n) \\ &= f - K(I - P_X Q_Y)f - Kn \\ &= f - (I - P_X Q_Y)^{-1}(I - P_X Q_Y)f - Kn \\ &= 0 - Kn. \end{aligned}$$

So

$$\begin{aligned} \|f - Kr\|_{L_2(\mathbb{R}_+)} &= \|Kn\|_{L_2(\mathbb{R}_+)} \\ &\leq \|(I - P_X Q_Y)^{-1}\| \|n\|_{L_2(\mathbb{R}_+)} \\ &\leq (1 - \|P_X Q_Y\|)^{-1} \|n\|_{L_2(\mathbb{R}_+)} \\ &\leq \frac{2^{v+1}\Gamma(v+2)}{2^{v+1}\Gamma(v+2) - (XY)^{v+1}} \|n\|_{L_2(\mathbb{R}_+)}. \end{aligned}$$

The constant C in equation (9) is therefore not larger than

$$\frac{2^{v+1}\Gamma(v+2)}{2^{v+1}\Gamma(v+2) - (XY)^{v+1}}. \quad \blacksquare$$

The identity $K = \sum_{k=0}^{\infty} (P_X Q_Y)^k$ suggests the following algorithm for computing Kr :

$$\begin{aligned} f^{(0)} &= r, \\ f^{(k+1)} &= r + P_X Q_Y f^{(k)}. \end{aligned}$$

Then $f^{(k)} = \sum_{j=0}^k (P_X Q_Y)^j r$ converges to Kr as $k \rightarrow \infty$. The iterate $f^{(k)}$ is the result of bandlimiting to $[0, Y]$ and then timelimiting to $[0, X]$ the previous $f^{(k-1)}$, then adding the result back to the original data r . On $[X, \infty)$ where the data are given, $f^{(k)} = r$ at each iteration k , while on the unobserved interval $[0, X]$ the missing values are filled in by a gradual adjustment, iteration after iteration.

de Jeu [12] extended the inequality (8) to any integral transform with bounded kernel, where there is a Parseval theorem. In a very recent paper [19] by Olenko an upper bound for the kernel $\sqrt{x}J_\nu(x)$ has been obtained, and therefore, the de Jeu result can be applied directly to the Hankel transform.

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Some remarks on a proof of geometrical Paley–Wiener theorems for the Dunkl transform

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We argue that a proof of the geometrical form of the Paley–Wiener theorems for the Dunkl transform in the literature is not correct.

Keywords: Dunkl transform; Paley–Wiener theorem

AMS Subject Classification: Primary: 33C52; Secondary: 33C67

1. Introduction

In ref. [1] a proof of the geometrical form of the Paley–Wiener theorems for the Dunkl transform is presented. In our opinion, however, this proof is not correct.[†]

The material, which was communicated to the author is presented below. It is our opinion that at this moment the geometrical forms of the Paley–Wiener theorems for the Dunkl transform are still unproven.

2. Arguments

In ref. [1] the geometric form of the Paley–Wiener theorem for the Dunkl transform is stated for functions and for distributions, as Theorems 6.2 and 6.3, respectively. The crucial ingredient in the proof of these results is Proposition 6.3. Our arguments concern the proof of this proposition. In formulating them, we will use the notation and definitions of ref. [1].

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[†]Publisher's Note: the authors of ref. [1] have been advised by this author of these remarks and invited to make a response, but at time of writing have not responded.

In the proof of Proposition 6.3, a W -invariant compact convex subset E of $\mathbb{R}^d$ is considered. For $x \in E$ fixed, the function f_x on $\mathbb{R}^d$ is defined as

$$f_x(y) = \frac{e^{-i(x,y)}}{(1 + \|y\|^2)^p} \quad (y \in \mathbb{R}^d), \quad (1)$$

where p is an integer such that $p \geq \gamma + d/2 + 1$. Since the constant γ is assumed to be strictly positive in line 3 on page 29, we see that $p \geq 2$.

The function F_x is defined on $\mathbb{R}^d$ in equation (69) as essentially the inverse Dunkl transform of f_x , namely

$$F_x(t) = \int_{\mathbb{R}^d} f_x(y) K(iy, t) \omega_k(y) dy \quad (t \in \mathbb{R}^d).$$

Following this definition, it is observed that F_x is continuous.

After a computation involving Riemann sums and contour integration, it is then concluded on line 3 of page 31 that F_x has support in the set E . Since E is compact, one sees – if the line of reasoning in [1] is correct – that F_x is a compactly supported continuous function.

But this never holds. Indeed, if F_x were compactly supported, then f_x could be reconstructed from F_x as is stated in the last line on page 31 (an application of the inversion theorem for the Dunkl transform)

$$f_x(y) = \frac{C_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} F_x(t) K(-iy, t) \omega_k(t) dt \quad (y \in \mathbb{R}^d). \quad (2)$$

However, as with the ordinary Fourier transform, the fact that F_x is a continuous function with compact support implies [2, Part 3 of Lemma 4.4] that its Dunkl transform has an entire extension to $\mathbb{C}^d$, *i.e.*, that f_x has an entire extension to $\mathbb{C}^d$. But it follows from equation (1), written as

$$f_x(y) = \frac{e^{-i(x,y)}}{(1 + (y, y))^p},$$

where $(\cdot, \cdot)$ is the holomorphic standard bilinear form on $\mathbb{C}^d$, that f_x has no such extension from $\mathbb{R}^d$ to $\mathbb{C}^d$, since this extension would have a pole along the divisor $\{y \in \mathbb{C}^d \mid (y, y) = -1\}$, as a consequence of the fact that $p \geq 2$. This is a contradiction, so F_x can never have compact support.

The above argument shows that F_x can never be a continuous function with compact support, but apart from that it is also easy to give an elementary counterexample to the claim in ref. [1] that F_x always has these properties, as follows. Consider the W -invariant compact convex set $E = \{0\}$ and choose $x = 0 \in E$. Then the fact that the continuous function F_0 has support in E implies that $F_0 = 0$. But in that case (2) implies that $f_0 = 0$, contradicting equation (1).

The fact that F_x has support in E is the cornerstone of the proof of Proposition 6.3 and, therefore, also of the proof of the geometrical Paley–Wiener theorems in ref. [1]. Since this statement is in fact false, as shown above, we consider the proof of the geometrical Paley–Wiener theorems in ref. [1] to be incorrect.

Remark 2.1 The crucial mistake in the line of reasoning, which leads to the statement that F_x has support in E appears to be the following.

In line 9 of page 31 it is stated that, after shifting the domain of integration over $i\eta$ for $\eta \in \mathbb{R}^d$, one has

$$F_x(t) = \frac{1}{(i\pi)^d} \int_{\mathbb{R}^d} e^{-i(x, u+i\eta)} K(i(u+i\eta), t) m_k(u+i\eta) du \quad (t \in \mathbb{R}^d). \quad (3)$$

Following this it is claimed that an application of equation (20) in ref. [1], *i.e.*, an application of the estimate

$$|K(x, z)| \leq \exp(\|x\| \|\operatorname{Re} z\|) \quad (x \in \mathbb{R}^d, z \in \mathbb{C}^d), \quad (4)$$

implies – here one also uses the symmetry of the Dunkl kernel – that

$$|F_x(t)| \leq \frac{e^{(I_E(\eta) - (t, \eta))}}{\pi^d} \int_{\mathbb{R}^d} |m_k(u+i\eta)| du \quad (t \in \mathbb{R}^d), \quad (5)$$

where $I_E(\eta) = \sup_{x \in E} (x, \eta)$. Next it is argued that equation (5) implies the crucial fact that $F_x(t) = 0$ if $t \notin E$, by shifting the domain to infinity in a direction depending on t .

However, the estimate in equation (5) does not follow from equation (3) and (4). One can only conclude that

$$|F_x(t)| \leq \frac{e^{(I_E(\eta) + \|t\| \|\eta\|)}}{\pi^d} \int_{\mathbb{R}^d} |m_k(u+i\eta)| du \quad (t \in \mathbb{R}^d). \quad (6)$$

But then shifting the domain of integration no longer works. If one chooses, e.g., for E a ball with radius R centered at the origin, then a factor $e^{(R + \|t\|)\|\eta\|}$ appears before the integral in equation (6), and this factor has exponential growth in any direction for η , rather than exponential decay.

Remark 2.2 If the multiplicity is strictly positive, then in the one-dimensional case the kernel $K(iy, x)$ has exponential growth if $0 \neq x \in \mathbb{R}$ is fixed and y tends to infinity in the imaginary direction through the upper or lower half plane. The choice of the half plane is immaterial, quite in contrast to the ordinary exponential function. The available estimates for the Dunkl kernel suggest that the same behavior may occur in any dimension. As a consequence, shifting the domain of integration is in our opinion not likely to work as long as the integrand contains the Dunkl kernel as a factor. This technical difficulty was already observed in ref. [3] and the above remarks about the proof of the geometrical Paley–Wiener theorems in ref. [1] seem to underline this observation.

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Some applications of de Bruijn's inequality for power series

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Some inequalities for power series with real coefficients via the de Bruijn inequality are established. Applications for fundamental complex functions such as the exp, cos, sin, cosh, sinh and ln are also given.

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1. Introduction

If we consider an analytic function $f(z)$ defined by the power series $\sum_{n=0}^{\infty} a_n z^n$ with complex coefficients a_n and apply the well-known Cauchy–Bunyakovsky–Schwarz (CBS) inequality

$$\left| \sum_{j=1}^n a_j b_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2, \quad (1)$$

holding for the complex numbers $a_j, b_j, j \in \{1, \dots, n\}$, then we can deduce that

$$|f(z)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n \right|^2 \leq \sum_{n=0}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |z|^{2n} = \frac{1}{1 - |z|^2} \cdot \sum_{n=0}^{\infty} |a_n|^2 \quad (2)$$

for any $z \in D(0, R) \cap D(0, 1)$, where R is the radius of convergence of f .

The above inequality gives some information about the magnitude of the function f provided that numerical series $\sum_{n=0}^{\infty} |a_n|^2$ is convergent and z is not too close to the boundary of the open disk $D(0, 1)$.

It is then natural to ask the question as to whether we can obtain better bounds for the magnitude of f if the coefficients are restricted to be real numbers, a case which is of large

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interest due to the fact that many usual complex functions can be represented as power series with real coefficients.

If we restrict ourselves more and assume that the coefficients in the representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ are nonnegative, and the assumption incorporates various examples of complex functions that will be indicated in the sequel, on utilising the weighted version of the CBS-inequality, namely

$$\left| \sum_{j=1}^n w_j a_j b_j \right|^2 \leq \sum_{j=1}^n w_j |a_j|^2 \sum_{j=1}^n w_j |b_j|^2, \quad (3)$$

where $w_j \geq 0$, while $a_j, b_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$, we can state that

$$|f(zw)|^2 = \left| \sum_{n=0}^{\infty} a_n z^n w^n \right|^2 \leq \sum_{n=0}^{\infty} a_n |z|^{2n} \sum_{n=0}^{\infty} a_n |w|^{2n} = f(|z|^2) f(|w|^2) \quad (4)$$

for any $z, w \in \mathbb{C}$ with $zw, |z|^2, |w|^2 \in D(0, R)$.

The problem of improving (4) for one of the numbers z or w assumed to be real is then also natural to be considered and the obtained result would give interesting examples of inequalities for real and complex numbers.

Motivated by the above questions and utilising a tool that has been available in the literature since 1960 and is known as the de Bruijn inequality, we establish in this paper some inequalities for functions defined by power series which will improve the above results. Particular examples that are related to some fundamental complex functions such as the exp, cos, sin, cosh, sinh and ln are presented.

2. General results via the de Bruijn inequality

In an effort to provide a refinement for the celebrated CBS inequality for complex numbers

$$\left| \sum_{j=1}^n a_j b_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2, \quad (5)$$

de Bruijn established in 1960, [1] (see also [2, p. 48] or [3, p. 89]) the following result:

LEMMA 1 (de Bruijn, 1960) *If $\mathbf{b} = (b_1, \dots, b_n)$ is an n -tuple of real numbers and $z = (z_1, \dots, z_n)$ an n -tuple of complex numbers, then*

$$\left| \sum_{k=1}^n b_k z_k \right|^2 \leq \frac{1}{2} \sum_{k=1}^n b_k^2 \left[\sum_{k=1}^n |z_k|^2 + \left| \sum_{k=1}^n z_k^2 \right| \right]. \quad (6)$$

Equality holds in (6) if and only if for $k \in \{1, \dots, n\}$, $b_k = \operatorname{Re}(\lambda z_k)$, where λ is a complex number such that the quantity $\lambda^2 \sum_{k=1}^n z_k^2$ is a nonnegative real number.

On utilising this result, we can establish some inequalities for power series as follows:

THEOREM 1 Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be an analytic function defined by a power series with nonnegative coefficients a_n , $n \geq 0$ and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If a is a real number and z a complex number such that $az, a^2, z^2, |z|^2 \in D(0, R)$, then:

$$|f(az)|^2 \leq \frac{1}{2} f(a^2) [f(|z|^2) + |f(z^2)|]. \quad (7)$$

Proof First of all, notice that, by de Bruijn's inequality (6) for the choices $b_k = \sqrt{a_k} c_k$, $z_k = \sqrt{a_k} w_k$ with $a_k \geq 0$, $c_k \in \mathbb{R}$ and $w_k \in \mathbb{C}$, $k \in \{0, \dots, n\}$, we can state the weighted inequality:

$$\left| \sum_{k=0}^n a_k c_k w_k \right|^2 \leq \frac{1}{2} \sum_{k=0}^n a_k c_k^2 \left[\sum_{k=0}^n a_k |w_k|^2 + \left| \sum_{k=0}^n a_k w_k^2 \right| \right]. \quad (8)$$

Now, on making use of (8) for the partial sums of the function f defined above, we are able to state that:

$$\left| \sum_{k=0}^n a_k a^k z^k \right|^2 \leq \frac{1}{2} \sum_{k=0}^n a_k a^{2k} \left[\sum_{k=0}^n a_k |z|^{2k} + \left| \sum_{k=0}^n a_k z^{2k} \right| \right] \quad (9)$$

for any $n \geq 0$, $a \in \mathbb{R}$, $z \in \mathbb{C}$ with $az, a^2, |z|^2, z^2 \in D(0, R)$.

Taking the limit as $n \rightarrow \infty$ in (9) and noticing that all the involved series are convergent, we deduce the desired inequality (7). ■

The inequality (7) is a valuable source of particular inequalities for real numbers a and complex numbers z as will be outlined in the following.

(1) If in (7) we choose the function $f(z) = \sum_{n=0}^{\infty} z^n$, $z \in D(0, 1)$, then we can state that

$$\left| \frac{1}{1-az} \right|^2 \leq \frac{1}{2} \cdot \frac{1}{1-a^2} \left[\frac{1}{1-|z|^2} + \frac{1}{|1-z^2|} \right] \quad (10)$$

for any $a \in (-1, 1)$ and $z \in D(0, 1)$. In an equivalent form, (10) can be stated as

$$[1 - |z|^2 + |1 - z^2|] |1 - az|^2 \geq 2(1 - a^2)(1 - |z|^2)|1 - z^2| \quad (11)$$

for any $a \in (-1, 1)$ and $z \in D(0, 1)$.

A direct proof of (11) is, in our opinion, difficult and we leave as an open problem to the reader to find alternative proofs of (11) without the use of the de Bruijn inequality, which has been used as a key point in obtaining the result (7).

(2) The choice of another fundamental power series, $f(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$, will produce via the inequality (7) the following result:

$$|\exp(az)|^2 \leq \frac{1}{2} \exp(a^2) [\exp(|z|^2) + |\exp(z^2)|] \quad (12)$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

The choice $z = i$ in (12) generates the following simple and interesting result, for which the authors were not able to find any reference in the literature:

$$|\exp(ia)|^2 \leq \frac{e^2 + 1}{2e} \exp(a^2), \quad (13)$$

for any $a \in \mathbb{R}$.

- (3) Now, if we choose the power series $f(z) = \ln(1 - z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n$, $z \in D(0, 1)$ and apply Theorem 1, then we get the following inequality for logarithms:

$$|\ln(1 - az)|^2 \leq \frac{1}{2} \ln\left(\frac{1}{1 - a^2}\right) \left[\ln\left(\frac{1}{1 - |z|^2}\right) + |\ln(1 - z^2)| \right] \quad (14)$$

for any $a \in (-1, 1)$ and $z \in D(0, 1)$.

If in (14) we choose $z = \pm ib$ with $b \in (-1, 1)$, then we obtain the simpler result:

$$|\ln(1 \pm iab)|^2 \leq \frac{1}{2} \ln\left(\frac{1}{1 - a^2}\right) \ln\left(\frac{1 + b^2}{1 - b^2}\right) \quad (15)$$

for any $a, b \in (-1, 1)$.

- (4) If we utilise the following function as power series representations with nonnegative coefficients:

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}, \quad z \in \mathbb{C};$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}, \quad z \in \mathbb{C};$$

$$\frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0, 1);$$

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi} (2n+1) n!} z^{2n+1}, \quad z \in D(0, 1);$$

$$\tanh^{-1}(z) = \sum_{n=1}^{\infty} \frac{1}{2n-1} z^{2n-1}, \quad z \in D(0, 1)$$

$${}_2F_1(\alpha, \beta, \gamma, z) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \alpha) \Gamma(n + \beta) \Gamma(\gamma)}{n! \Gamma(\alpha) \Gamma(\beta) \Gamma(n + \gamma)} z^n, \quad \alpha, \beta, \gamma > 0, z \in D(0, 1);$$

where Γ is the Gamma function, then we can state the following inequalities:

$$|\cosh(az)|^2 \leq \frac{1}{2} \cosh(a^2) [\cosh(|z|^2) + |\cosh(z^2)|], \quad a \in \mathbb{R}, z \in \mathbb{C}; \quad (16)$$

$$|\sinh(az)|^2 \leq \frac{1}{2} \sinh(a^2) [\sinh(|z|^2) + |\sinh(z^2)|], \quad a \in \mathbb{R}, z \in \mathbb{C}; \quad (17)$$

$$\left| \ln\left(\frac{1+az}{1-az}\right) \right| \leq \frac{1}{2} \ln\left(\frac{1+a^2}{1-a^2}\right) \left[\ln\left(\frac{1+|z|^2}{1-|z|^2}\right) + \left| \ln\left(\frac{1+z^2}{1-z^2}\right) \right| \right] \\ a \in (-1, 1), \quad z \in D(0, 1); \quad (18)$$

$$|\sin^{-1}(az)|^2 \leq \frac{1}{2} \sin^{-1}(a^2) [\sin^{-1}(|z|^2) + |\sin^{-1}(z^2)|], \\ a \in (-1, 1), \quad z \in D(0, 1); \quad (19)$$

$$|\tanh^{-1}(az)|^2 \leq \frac{1}{2} \tanh^{-1}(a^2) [\tanh^{-1}(|z|^2) + |\tanh^{-1}(z^2)|], \\ a \in (-1, 1), \quad z \in D(0, 1); \quad (20)$$

and

$$|{}_2F_1(\alpha, \beta, \gamma, az)|^2 \leq \frac{1}{2} \cdot {}_2F_1(\alpha, \beta, \gamma, a^2) [{}_2F_1(\alpha, \beta, \gamma, |z|^2) + {}_2F_1(\alpha, \beta, \gamma, z^2)]$$

$$a \in (-1, 1), \quad z \in D(0, 1), \quad \alpha, \beta, \gamma > 0. \quad (21)$$

Now, by the help of the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we can naturally construct another power series which will have as coefficients the absolute values of the coefficient of the original series, namely, $f_A(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series will have the same radius of convergence as the original series.

As some natural examples that are useful for applications, we can point out that, if

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n = \ln \frac{1}{1+z}, \quad z \in D(0, 1);$$

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbb{C};$$

$$h(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbb{C};$$

$$l(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z}, \quad z \in D(0, 1),$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are obviously

$$f_A(z) = \ln \frac{1}{1-z}, \quad g_A(z) = \cosh z,$$

$$h_A(z) = \sinh z \quad \text{and} \quad l_A(z) = \frac{1}{1-z}$$

and they are defined on the same domain as the generating functions.

The following result contains an inequality connecting the function f with its transform f_A .

THEOREM 2 *Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$ be a function defined by a power series with real coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$. If $a \in \mathbb{R}$ and $z \in \mathbb{C}$ are such that $az, a^2, |z|^2, z^2 \in D(0, R)$, then:*

$$|f(az)|^2 \leq \frac{1}{2} f_A(a^2) [f_A(|z|^2) + |f_A(z^2)|]. \quad (22)$$

Proof Firstly, observe that for each $n \geq 0$, $a_n = |a_n| \operatorname{sgn}(a_n)$ where $\operatorname{sgn}(x)$ is the sign function defined to be 1 if $x > 0$, -1 if $x < 0$ and 0 if $x = 0$.

Utilising the de Bruijn inequality with positive weights we have

$$\begin{aligned} |f(az)|^2 &= \left| \sum_{n=0}^{\infty} |a_n| \operatorname{sgn}(a_n) a^n z^n \right|^2 \\ &\leq \frac{1}{2} \sum_{n=0}^{\infty} |a_n| [\operatorname{sgn}(a_n)]^2 a^{2n} \left[\sum_{n=0}^{\infty} |a_n| |z|^{2n} + \sum_{n=0}^{\infty} |a_n| |z^{2n}| \right] \\ &= \frac{1}{2} f_A(a^2) [f_A(|z|^2) + |f_A(z^2)|] \end{aligned}$$

for any $a \in \mathbb{R}$, $z \in \mathbb{C}$ with $az, a^2, |z|^2, z^2 \in D(0, R)$. ■

In the following examples we exemplify how the above inequality may be used to establish some inequalities for real and complex numbers that otherwise would be very difficult to prove.

(1) If we take the function

$$f(z) = \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n+1}, \quad z \in \mathbb{C},$$

then

$$f_A(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \sinh(z) = \frac{1}{2} (e^z - e^{-z}) \quad \text{for } z \in \mathbb{C}.$$

Applying the inequality (22) for this selection will produce the result

$$|\sin(az)|^2 \leq \frac{1}{2} \sinh(a^2) [\sinh(|z|^2) + |\sinh(z^2)|] \quad (23)$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$.

Now if in (23) we choose $z = ib$, with $b \in \mathbb{R}$, then we obtain the inequality

$$|\sin(iab)|^2 \leq \sinh(a^2) \sinh(b^2) \quad (24)$$

for any $a, b \in \mathbb{R}$.

(2) The function

$$f(z) = \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}, \quad z \in \mathbb{C}$$

has the transform

$$f_A(z) = \cosh(z) = \frac{1}{2} (e^z + e^{-z}) \quad \text{for } z \in \mathbb{C}.$$

Utilising the inequality (22) for f as above gives

$$|\cos(az)|^2 \leq \frac{1}{2} \cosh(a^2) [\cosh(|z|^2) + |\cosh(z^2)|] \quad (25)$$

for any $a \in \mathbb{R}$ and $z \in \mathbb{C}$. In particular, we have, with $z = ib$,

$$|\cos(iab)|^2 \leq \cosh(a^2) \cosh(b^2) \quad (26)$$

for each $a, b \in \mathbb{R}$.

From a different perspective, we can state the following result which establishes a connection between two power series, one having positive coefficients.

THEOREM 3 Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be two power series with $b_n \in \mathbb{R}$ and $a_n > 0$ for $n \geq 0$. If g is convergent on $D(0, R_1)$, f is convergent on $D(0, R_2)$ and the numerical series $\sum_{n=0}^{\infty} \frac{b_n^2}{a_n}$ is convergent, then we have the inequality:

$$|g(z)|^2 \leq \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{b_n^2}{a_n} \right) [f(|z|^2) + |f(z^2)|], \quad (27)$$

for any $z \in \mathbb{C}$ with $z \in D(0, R_1)$ and $a, |z|^2 \in D(0, R_2)$.

Proof Utilising the de Bruijn weighted inequality (8) we can state that

$$\begin{aligned} |g(z)|^2 &= \left| \sum_{n=0}^{\infty} \frac{b_n}{a_n} a_n z^n \right|^2 \\ &\leq \frac{1}{2} \sum_{n=0}^{\infty} a_n \left(\frac{b_n}{a_n} \right)^2 \left[\sum_{n=0}^{\infty} a_n |z|^{2n} + \left| \sum_{n=0}^{\infty} a_n z^{2n} \right| \right] \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{b_n^2}{a_n} [f(|z|^2) + |f(z^2)|] \end{aligned}$$

for $z \in \mathbb{C}$ with $z, z^2, |z|^2 \in D(0, R_1) \cap D(0, R_2)$. ■

Remark 1 The above inequality is useful in comparing different functions for which a bound for the numerical series $\sum_{n=0}^{\infty} \frac{b_n^2}{a_n}$ is known.

COROLLARY 1 Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be a power series with real coefficients and convergent on $D(0, R)$. If the numerical series $\sum_{n=0}^{\infty} b_n^2$ is convergent, then

$$|g(z)|^2 \leq \frac{1}{2} \left(\sum_{n=0}^{\infty} b_n^2 \right) \cdot \frac{|1 - z^2| + 1 - |z|^2}{|1 - z^2| (1 - |z|^2)} \quad (28)$$

for any $z \in \mathbb{C}$ with $z \in D(0, R)$ and $z^2, |z|^2 \in D(0, 1)$.

COROLLARY 2 Let $g(z)$ be as in Corollary 1. If the numerical series $\sum_{n=0}^{\infty} (n! b_n^2)$ is convergent, then

$$|g(z)|^2 \leq \frac{1}{2} \sum_{n=0}^{\infty} (n! b_n^2) [\exp |z|^2 + |\exp(z^2)|], \quad (29)$$

for any $z \in D(0, R)$.

Remark 2 If we consider the series expansion

$$\frac{1}{z} \cdot \ln \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{n+1} z^n, \quad z \in D(0, 1) \setminus \{0\},$$

then, on utilising the inequality (28) for the choice $b_n := \frac{1}{n+1}$ and taking into account that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$$

we can state the inequality

$$|\ln(1-z)| \leq \frac{\pi^2}{12} |z| \frac{|1-z^2| + 1 - |z|^2}{|1-z^2|(1-|z|^2)} \quad \text{for any } z \in D(0, 1). \quad (30)$$

3. Some inequalities for the polylogarithm

Before we state our results for the polylogarithm that can be obtained on utilising the de Bruijn inequality, we recall some concepts that will be used in the sequel.

The polylogarithm $Li_n(z)$, also known as the de Jonquière's function is the function defined by

$$Li_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \quad (31)$$

defined in the complex plane over the unit disk $D(0, 1)$.

The special case $z = 1$ reduces to $Li_s(z) = \zeta(s)$, where ζ is the Riemann zeta function.

The polylogarithm of nonnegative integer order arises in the sums of the form

$$\sum_{k=1}^{\infty} k^n r^k = Li_{-n}(r) = \frac{1}{(1-r)^{n+1}} \sum_{i=0}^n E_{n,i} r^{n-i}$$

where $E_{n,i}$ is an Eulerian number, namely, we recall that

$$E_{n,k} := \sum_{j=0}^{k+1} (-1)^j \binom{n+1}{j} (k-j+1)^n.$$

Polylogarithms also arise in sums of generalised harmonic numbers $H_{n,r}$ as

$$\sum_{n=1}^{\infty} H_{n,r} z^n = \frac{Li_r(z)}{1-z} \quad \text{for } z \in D(0, 1),$$

where we recall that

$$H_{n,r} := \sum_{k=1}^n \frac{1}{k^r} \quad \text{and} \quad H_{n,1} := H_n = \sum_{k=1}^n \frac{1}{k}.$$

Special forms of low-order polylogarithms include

$$\begin{aligned} Li_{-2}(z) &= \frac{z(z+1)}{(1-z)^3}, & Li_{-1}(z) &= \frac{z}{(1-z)^2}, \\ Li_0(z) &= \frac{z}{1-z} \quad \text{and} \quad Li_1(z) = -\ln(1-z), & z &\in D(0, 1). \end{aligned}$$

At argument $z = -1$, the general polylogarithms become $Li_n(-1) = -\eta(x)$, where $\eta(x)$ is the Dirichlet eta function.

It is obvious that Li_n being a power series with nonnegative coefficients all the results in the above section hold true. Therefore we have, for instance the inequality:

$$|Li_n(az)|^2 \leq \frac{1}{2} Li_n(a^2) [Li_n(|z|^2) + |Li_n(z^2)|] \quad (32)$$

for any $a \in (-1, 1)$ and $z \in D(0, 1)$, where n is a negative or a positive integer.

In the following we present some results that connect different order polylogarithms:

THEOREM 4 *Let $a \in (-1, 1)$, $z \in D(0, 1)$ and p, q, r integers such that the following series exist. Then*

$$|Li_{r+p+q}(az)|^2 \leq \frac{1}{2} Li_{r+2p}(a^2) [Li_{r+2q}(|z|^2) + |Li_{r+2q}(z^2)|] \quad (33)$$

Proof Utilising the de Bruijn's inequality with positive weights we have:

$$\begin{aligned} |Li_{r+p+q}(az)|^2 &= \left| \sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{1}{k^p} a^k \cdot \frac{1}{k^q} z^k \right|^2 \\ &\leq \frac{1}{2} \left(\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{a^{2k}}{k^{2p}} \right) \left[\sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{|z|^{2k}}{k^{2q}} + \left| \sum_{k=1}^{\infty} \frac{1}{k^r} \cdot \frac{z^{2k}}{k^{2q}} \right| \right] \\ &= \frac{1}{2} Li_{r+2p}(a^2) [Li_{r+2q}(|z|^2) + |Li_{r+2q}(z^2)|] \end{aligned}$$

and the inequality is proved. ■

In particular, we can state the following inequality which incorporates the zeta function:

COROLLARY 3 *Let $z \in D(0, 1)$ and p, q, r integers such that $r + 2p > 1$. Then*

$$|Li_{r+p+q}(z)|^2 \leq \frac{1}{2} \zeta(r+2p) [Li_{r+2q}(|z|^2) + |Li_{r+2q}(z^2)|] \quad (34)$$

The proof follows by Theorem 4 for $a = 1$.

Remark 3 On utilising (34) and taking into account that some particular values of ζ are known, then we can state the following results:

$$|Li_{q+1}(z)|^2 \leq \frac{\pi^2}{12} [Li_{2q}(|z|^2) + |Li_{2q}(z^2)|] \quad (35)$$

and

$$|Li_{q+2}(z)|^2 \leq \frac{\pi^4}{180} [Li_{2q}(|z|^2) + |Li_{2q}(z^2)|] \quad (36)$$

for any $z \in D(0, 1)$ and q an integer.

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Parseval–Goldstein type identities involving the $\mathcal{L}_4$ -transform and the $\mathcal{P}_4$ -transform and their applications

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In the present article the authors introduce several new integral transforms including the $\mathcal{L}_4$ -transform and the $\mathcal{P}_4$ -transform as generalizations of the classical Laplace transform and the classical Stieltjes transform, respectively. It is shown that the second iterate of the $\mathcal{L}_4$ -transform is essentially the $\mathcal{P}_4$ -transform. Using this relationship, a number of new Parseval–Goldstein type identities are obtained for these and many other well-known integral transforms. The identities proven in this article give rise to useful corollaries for evaluating infinite integrals of special functions. Some examples are also given as illustrations of the results presented here.

Keywords: $\mathcal{L}_2$ -transforms; Laplace transforms; Widder potential transforms; $\mathcal{L}_4$ -transforms; $\mathcal{P}_4$ -transforms; Stieltjes transforms; K -transforms; Hankel transforms; Fourier sine transforms; Fourier cosine transforms; Parseval–Goldstein type theorems

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1. Introduction, definitions and preliminaries

Four decades ago, Widder [1] presented a systematic account of the so-called Widder potential transform:

$$\mathcal{P}\{f(x); y\} = \int_0^\infty \frac{x f(x)}{x^2 + y^2} dx, \quad (1)$$

which, by an exponential change of variable, becomes a convolution transform with a kernel belonging to a general class treated by Hirschman and Widder [2]. In fact, as Widder [1] pointed out, the kernel of the integral transform (1) is the same as that appearing in the familiar Poisson integral representation of a function which is harmonic in a half-plane. With this connection in view, Widder [1] gave several inversion formulas for the Widder potential transform (1) and applied his results to harmonic functions. Subsequently, Srivastava and Singh [3] gave

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the following Parseval–Goldstein type identity for the Widder potential transform (1) together with several illustrative examples of its applications (see also [4]):

$$\int_0^\infty y \mathcal{P}\{f(x); y\} g(y) dy = \int_0^\infty x \mathcal{P}\{g(y); x\} f(x) dx. \quad (2)$$

The Widder potential transform (1) is related to the classical Stieltjes transform as follows:

$$\mathcal{P}\{f(x); y\} = \frac{1}{2} \mathcal{S}\{f(x^{1/2}); y^2\}, \quad (3)$$

where the Stieltjes transform is defined by

$$\mathcal{S}\{f(x); y\} = \int_0^\infty \frac{f(x)}{x+y} dx. \quad (4)$$

The following identity similar to (2), which was introduced earlier by Goldstein [5]:

$$\int_0^\infty y \mathcal{L}\{f(x); y\} g(y) dy = \int_0^\infty x \mathcal{L}\{g(y); x\} f(x) dx, \quad (5)$$

is popularly known as the Parseval–Goldstein identity for the classical Laplace transform:

$$\mathcal{L}\{f(x); y\} = \int_0^\infty \exp(-xy) f(x) dx. \quad (6)$$

Yürekli [6, 7] and Sadek and Yürekli [8] gave various Parseval–Goldstein type identities for the $\mathcal{L}_2$ -transform defined by

$$\mathcal{L}_2\{f(x); y\} = \int_0^\infty x \exp(-x^2 y^2) f(x) dx, \quad (7)$$

the Widder potential transform (1), the Fourier sine transform:

$$\mathcal{F}_S\{f(x); y\} = \int_0^\infty \sin(xy) f(x) dx, \quad (8)$$

and the Fourier cosine transform:

$$\mathcal{F}_C\{f(x); y\} = \int_0^\infty \cos(xy) f(x) dx. \quad (9)$$

The $\mathcal{L}_2$ -transform in (7) and the classical Laplace transform (6) are related by means of the following identity:

$$\mathcal{L}_2\{f(x); y\} = \frac{1}{2} \mathcal{L}\{f(x^{1/2}); y^2\}. \quad (10)$$

Some results involving the Hankel transform and the $\mathcal{K}_\nu$ -transform were given by (among others) Srivastava and Yürekli [9]. The Hankel transform is defined by

$$\mathcal{H}_\nu\{f(x); y\} = \int_0^\infty (xy)^{1/2} J_\nu(xy) f(x) dx, \quad (11)$$

where $J_\nu(x)$ denotes the Bessel function of order ν , and the $\mathcal{K}_\nu$ -transform is defined by

$$\mathcal{K}_\nu\{f(x); y\} = \int_0^\infty (xy)^{1/2} K_\nu(xy) f(x) dx, \quad (12)$$

where $K_\nu(x)$ is the modified Bessel function (or the Macdonald function) of order ν .

We now introduce several new integral transforms and establish identities for these integral transforms. The $\mathcal{L}_4$ -transform is defined here as follows:

$$\mathcal{L}_4\{f(x); y\} = \int_0^\infty x^3 \exp(-x^4 y^4) f(x) dx, \quad (13)$$

which is related to the Laplace transform by means of the identity:

$$\mathcal{L}_4\{f(x); y\} = \frac{1}{4} \mathcal{L}\{f(x^{1/4}); y^4\}, \quad (14)$$

and to the $\mathcal{L}_2$ -transform by means of the identity:

$$\mathcal{L}_4\{f(x); y\} = \frac{1}{2} \mathcal{L}_2\{f(x^{1/2}); y^2\}. \quad (15)$$

The $\mathcal{P}_4$ -transform is defined here as follows:

$$\mathcal{P}_4\{f(x); y\} = \int_0^\infty \frac{x^3 f(x)}{x^4 + y^4} dx, \quad (16)$$

which is related to the Stieltjes transform (4) by means of the identity:

$$\mathcal{P}_4\{f(x); y\} = \frac{1}{4} \mathcal{S}\{f(x^{1/4}); y^4\} \quad (17)$$

and to the Widder potential transform (1) by means of the identity:

$$\mathcal{P}_4\{f(x); y\} = \frac{1}{2} \mathcal{P}\{f(x^{1/2}); y^2\}. \quad (18)$$

The $\mathcal{F}_{S,2}$ -transform is defined here as follows:

$$\mathcal{F}_{S,2}\{f(x); y\} = \int_0^\infty x \sin(x^2 y^2) f(x) dx, \quad (19)$$

which is related to the Fourier sine transform (8) by means of the identity:

$$\mathcal{F}_{S,2}\{f(x); y\} = \frac{1}{2} \mathcal{F}_S\{f(x^{1/2}); y^2\}. \quad (20)$$

Similarly, the $\mathcal{F}_{C,2}$ -transform is defined here as follows:

$$\mathcal{F}_{C,2}\{f(x); y\} = \int_0^\infty x \cos(x^2 y^2) f(x) dx, \quad (21)$$

which is related to the Fourier cosine transform (9) by means of the identity:

$$\mathcal{F}_{C,2}\{f(x); y\} = \frac{1}{2} \mathcal{F}_C\{f(x^{1/2}); y^2\}. \quad (22)$$

The $\mathcal{H}_{v,2}$ -transform is defined here as follows:

$$\mathcal{H}_{v,2}\{f(x); y\} = \int_0^\infty (x^2 y) J_v(x^2 y^2) f(x) dx, \quad (23)$$

which is related to the $\mathcal{H}_v$ -transform in (11) by means of the identity:

$$\mathcal{H}_{v,2}\{f(x); y\} = \frac{1}{2} \mathcal{H}_v\{f(x^{1/2}); y^2\}. \quad (24)$$

Finally, the $\mathcal{K}_{v,2}$ -transform is defined here as follows:

$$\mathcal{K}_{v,2}\{f(x); y\} = \int_0^\infty (x^2 y) K_v(x^2 y^2) f(x) dx, \quad (25)$$

which is related to the $\mathcal{K}_v$ -transform in (12) by means of the identity:

$$\mathcal{K}_{v,2}\{f(x); y\} = \frac{1}{2} \mathcal{K}_v\{f(x^{1/2}); y^2\}. \quad (26)$$

In this article, we first show that the second iterate of the $\mathcal{L}_4$ -transform in (13) is a constant multiple of the $\mathcal{P}_4$ -transform defined by (16). By using this iteration identity, we establish a Parseval–Goldstein type theorem relating the $\mathcal{L}_2$ -transform and the $\mathcal{P}_4$ -transform. The Parseval–Goldstein type theorem established here yields new identities for the various integral transforms introduced above. As applications of the resulting identities and theorems, some illustrative examples are also given.

2. The main Parseval–Goldstein type theorem

In the following lemma, we show that the second iterate of the $\mathcal{L}_4$ -transform in (13) is essentially the $\mathcal{P}_4$ -transform defined by (16).

LEMMA *The following iteration identity holds true:*

$$\mathcal{L}_4^2\{f(x); y\} = \mathcal{L}_4\{\mathcal{L}_4\{f(x); u\}; y\} = \frac{1}{4} \mathcal{P}_4\{f(x); y\}, \quad (27)$$

provided that the integrals involved converge absolutely.

Proof Using the definition (13) of the $\mathcal{L}_4$ -transform, we have

$$\mathcal{L}_4^2\{f(x); y\} = \int_0^\infty u^3 \exp(-u^4 y^4) \cdot \left(\int_0^\infty x^3 \exp(-x^4 u^4) f(x) dx \right) du. \quad (28)$$

Upon changing the order of integration, which is permissible by absolute convergence of the integrals involved, it follows from (28) that

$$\mathcal{L}_4^2\{f(x); y\} = \int_0^\infty x^3 f(x) \left(\int_0^\infty u^3 \exp[(-x^4 + y^4) u^4] du \right) dx. \quad (29)$$

By evaluating the inner integral on the right-hand side of (29), we obtain

$$\mathcal{L}_4^2\{f(x); y\} = \frac{1}{4} \int_0^\infty \frac{x^3 f(x)}{x^4 + y^4} dx. \quad (30)$$

Now, by applying the definition (16) on the right-hand side of (30), we deduce the iteration identity (27) asserted by the Lemma. ■

Example 1: We illustrate the above Lemma by showing that

$$\mathcal{P}_4\{\sin(a^2 x^2); y\} = \frac{\pi}{4} \exp(-a^2 y^2) \quad (31)$$

and

$$\mathcal{P}_4\{x^{-2} \cos(a^2 x^2); y\} = \frac{\pi}{4} y^{-2} \exp(-a^2 y^2). \quad (32)$$

Demonstration. If we set

$$f(x) = \sin(a^2 x^2) \quad (33)$$

in the assertion (27) of the Lemma and then use the identity (15) as well as a known formula [10, p. 497, Entry 3.952-1], we find that

$$\mathcal{L}_4\{\sin(a^2 x^2); u\} = \frac{1}{2} \mathcal{L}_2\{\sin(a^2 x); u^2\} = \frac{a^2 \sqrt{\pi}}{8 u^6} \exp\left(-\frac{a^4}{4u^4}\right). \quad (34)$$

Now we apply the $\mathcal{L}_4$ -transform to (34) and use the relation (14). We thus obtain

$$\begin{aligned} \mathcal{L}_4^2\{\sin(a^2 x^2); y\} &= \frac{a^2 \sqrt{\pi}}{8} \mathcal{L}_4\left\{u^{-6} \exp\left(-\frac{a^4}{4u^4}\right); y\right\} \\ &= \frac{a^2 \sqrt{\pi}}{32} \mathcal{L}\left\{u^{-3/2} \exp\left(-\frac{a^4}{4u}\right); y^4\right\}. \end{aligned} \quad (35)$$

Making use of another known formula [11, Vol. 1, p. 146, Entry (28)], the Laplace transform on the right-hand side of (34) is given by

$$\mathcal{L}\left\{u^{-3/2} \exp\left(-\frac{a^4}{4u}\right); y^4\right\} = \frac{2 \sqrt{\pi}}{a^2} \exp(-a^2 y^2). \quad (36)$$

The assertion (31) follows upon substituting the result (36) into (35).

Similarly, in order to demonstrate (32), we set

$$f(x) = \frac{\cos(a^2 x^2)}{x^2} \quad (37)$$

and use the formula [11, Vol. 1, p. 158, Entry (67)] to obtain

$$\mathcal{L}_4\{f(x); u\} = \frac{\sqrt{\pi}}{4 u^2} \exp\left(-\frac{a^4}{4u^4}\right). \quad (38)$$

Applying the $\mathcal{L}_4$ -transform to (38), using the known formula [11, Vol. 1, p. 146, Entry (27)], and the assertion (27) of the Lemma, we arrive at (32).

Remark 1 Alternatively, we can compute the $\mathcal{P}_4$ -transforms in (31) and (32) by using the relationship (17) and the known formulas [11, Vol. 2, p. 219, Entry (36)] and [11, Vol. 2, p. 221, Entry (55)]. Here we have chosen to give the longer way because we will need the formulas (34) and (35) later in our investigation.

Example 2: We show that

$$\mathcal{P}_4\{x^{2\nu} J_\nu(a^2 x^2); y\} = \frac{1}{2} y^{2\nu} K_\nu(a^2 y^2) \quad (\Re(\nu) > -1). \quad (39)$$

Demonstration. If we set

$$f(x) = x^{2\nu} J_\nu(a^2 x^2) \quad (40)$$

in the Lemma and then use the identity (14) and the known formula [11, Vol. 1, p. 185, Entry (30)], we find that

$$\mathcal{L}_4\{f(x); u\} = \frac{1}{4} \mathcal{L}\{x^{\nu/2} J_\nu(a^2 x^{1/2}); u^4\} = \frac{a^{2\nu}}{2^{v+2} u^{4v+4}} \exp\left(-\frac{a^4}{4u^4}\right). \quad (41)$$

Now we apply the $\mathcal{L}_4$ -transform to (41) and use the relationship (14). We thus obtain

$$\mathcal{L}_4^2\{f(x); y\} = \frac{a^{2\nu}}{2^{v+4}} \mathcal{L}_4\left\{u^{-4\nu-4} \exp\left(-\frac{a^4}{4u^4}\right); y\right\} = \frac{a^{2\nu}}{2^{v+2}} \mathcal{L}\left\{u^{-\nu-1} \exp\left(-\frac{a^4}{4u}\right); y^4\right\}. \quad (42)$$

Using the known formula [11, Vol. 1, p. 146, Entry (29)], the Laplace transform on the right-hand side of (42) is given by

$$\mathcal{L}\left\{u^{-\nu-1} \exp\left(-\frac{a^4}{4u}\right); y^4\right\} = \frac{2^{\nu+1}}{a^{2\nu}} y^{2\nu} K_\nu(-a^2 y^2). \quad (43)$$

The assertion (39) follows upon substituting the result (43) into (42) and using the iteration identity (27) of the Lemma.

Remark 2 Alternatively, we can calculate the $\mathcal{P}_4$ -transform in (39) by using the relationship (17) and the known formula [11, Vol. 2, p. 225, Entry (10)]. Here we have chosen to give the longer way because we will need the formulas in (41) later in our investigation.

Remark 3 It is well known that

$$K_{(1/2)}(x) = K_{-(1/2)}(x) = \sqrt{\frac{\pi}{2x}} e^{-x} \quad (44)$$

and

$$J_{(1/2)}(x) = \sqrt{\frac{\pi}{2x}} \sin(x) \quad \text{and} \quad J_{-(1/2)}(x) = \sqrt{\frac{\pi}{2x}} \cos(x). \quad (45)$$

Thus, by setting $\nu = 1/2$ in the result (39) and using the formulas (44) and (45), we obtain the results of Example 1.

THEOREM *The following Parseval–Goldstein type identities hold true:*

$$\int_0^\infty y^3 \mathcal{L}_4\{f(x); y\} \mathcal{L}_4\{g(u); y\} dy = \frac{1}{4} \int_0^\infty x^3 f(x) \mathcal{P}_4\{g(u); x\} dx, \quad (46)$$

$$\int_0^\infty y^3 \mathcal{L}_4\{f(x); y\} \mathcal{L}_4\{g(u); y\} dy = \frac{1}{4} \int_0^\infty u^3 g(u) \mathcal{P}_4\{f(x); u\} du, \quad (47)$$

and

$$\int_0^\infty x^3 f(x) \mathcal{P}_4\{g(u); x\} dx = \int_0^\infty u^3 g(u) \mathcal{P}_4\{f(x); u\} du, \quad (48)$$

provided that the integrals involved converge absolutely.

Proof We only give here the proof of the Parseval–Goldstein identity (46) because the proof of (47) is similar and the identity (48) follows easily from the assertions (46) and (47). Indeed, by the definition (13) of the $\mathcal{L}_4$ -transform, we have

$$\begin{aligned} & \int_0^\infty y^3 \mathcal{L}_4\{f(x); y\} \mathcal{L}_4\{g(u); y\} dy \\ &= \int_0^\infty y^3 \mathcal{L}_4\{g(u); y\} \left(\int_0^\infty x^3 \exp(-x^4 y^4) f(x) dx \right) dy. \end{aligned} \quad (49)$$

Changing the order of integration (which is permissible by absolute convergence of the integrals involved) and using the definition (13) of the $\mathcal{L}_4$ -transform once again, we find from (49) that

$$\begin{aligned} \int_0^\infty y^3 \mathcal{L}_4\{f(x); y\} \mathcal{L}_4\{g(u); y\} dy &= \int_0^\infty x^3 f(x) \left(\int_0^\infty y^3 \exp(-x^4 y^4) \mathcal{L}_4\{g(u); y\} dy \right) dx \\ &= \int_0^\infty x^3 f(x) \mathcal{L}_4\left\{ \mathcal{L}_4\{g(u); y\}; x \right\} dx. \end{aligned} \quad (50)$$

Now the assertion (46) follows from (50) and (27) of the Lemma. ■

3. A set of useful corollaries

Several interesting consequences of our main theorem will be presented in this section. For this purpose, we first introduce the $\mathcal{E}_{4,1}$ -transform as follows:

$$\mathcal{E}_{4,1}\{f(x); y\} = \int_0^\infty x^3 \exp(x^4 y^4) E_1(x^4 y^4) f(x) dx, \quad (51)$$

where $E_1(x)$ is the exponential integral function defined by

$$E_1(x) = -\text{Ei}(-x) = \int_x^\infty \frac{e^{-u}}{u} du = \int_1^\infty \frac{e^{-xt}}{t} dt. \quad (52)$$

The $\mathcal{E}_{4,1}$ -transform is related to the $\mathcal{E}_{2,1}$ -transform (see [12]):

$$\mathcal{E}_{2,1}\{f(x); y\} = \int_0^\infty x \exp(x^2 y^2) E_1(x^2 y^2) f(x) dx, \quad (53)$$

by means of the following identity:

$$\mathcal{E}_{4,1}\{f(x); y\} = \frac{1}{2} \mathcal{E}_{2,1}\{f(x^{1/2}); y^2\}. \quad (54)$$

COROLLARY 1 *The following identities hold true for the $\mathcal{L}_4$ -transform, the $\mathcal{P}_4$ -transform and the $\mathcal{E}_{4,1}$ -transform:*

$$\mathcal{P}_4\left\{ \mathcal{L}_4\{f(x); y\}; z \right\} = \mathcal{L}_4\left\{ \mathcal{P}_4\{f(x); u\}; z \right\} \quad (55)$$

and

$$\mathcal{E}_{4,1}\{f(x); y\} = 4 \mathcal{P}_4\{\mathcal{L}_4\{f(x); u\}; z\}, \quad (56)$$

provided that each member of the assertions (55) and (56) exists.

Proof Upon setting

$$g(u) = \exp(-u^4 z^4) \quad (57)$$

in the assertion (47) of our Theorem, if we use the definition (13) of the $\mathcal{L}_4$ -transform, we find that

$$\mathcal{L}_4\{g(u); y\} = \int_0^\infty u^3 \exp[-(y^4 + z^4)u^4] du = \frac{1}{4(y^4 + z^4)}. \quad (58)$$

Now the assertion (55) follows upon substituting the equations (57) and (58) into the identity (47) asserted by our Theorem.

In order to prove (56), we substitute (57) and (58) into (46). We thus obtain

$$\int_0^\infty \frac{y^3}{y^4 + z^4} \mathcal{L}_4\{f(x); y\} dy = \int_0^\infty x^3 f(x) \mathcal{P}_4\{\exp(-u^4 z^4); x\} dx. \quad (59)$$

If we evaluate the $\mathcal{P}_4$ -transform on the right-hand side of (59) by using the identity (17) and the known formula [11, Vol. 2, p. 217, Entry (11)], we get

$$\mathcal{P}_4\{\exp(-u^4 z^4); x\} = \frac{1}{4} \mathcal{S}\{\exp(-u z^4); x^4\} = \frac{1}{4} \exp(x^4 z^4) E_1(x^4 z^4). \quad (60)$$

The assertion (56) is obtained upon inserting (60) into (59) and using the definition (51) of the $\mathcal{E}_{4,1}$ -transform. ■

COROLLARY 2 *The following identities hold true:*

$$\mathcal{L}_4 \left\{ \frac{1}{v^2} \mathcal{L}_4 \left\{ f(x); \frac{1}{\sqrt{2}v} \right\}; z \right\} = \frac{\sqrt{\pi}}{4z^2} \mathcal{L}_2\{x^2 f(x); z\} \quad (61)$$

$$\mathcal{L}_4 \left\{ \frac{1}{v^2} \mathcal{L}_4 \left\{ f(x); \frac{1}{\sqrt{2}v} \right\}; z \right\} = \frac{1}{\sqrt{\pi} z^2} \mathcal{F}_{S,2} \left\{ u^2 \mathcal{P}_4\{f(x); u\}; z \right\}, \quad (62)$$

and

$$\mathcal{L}_2\{x^2 f(x); z\} = \frac{4}{\pi} \mathcal{F}_{S,2} \left\{ u^2 \mathcal{P}_4\{f(x); u\}; z \right\}, \quad (63)$$

provided that each member of the assertions (61), (62), and (63) exists.

Proof By setting

$$g(u) = \sin(z^2 u^2) \quad (64)$$

in the assertion (47) of the Theorem and applying the results (31) and (34) in each of the identities (46), (47), and (48) asserted by our Theorem, we arrive at the assertions (61), (62), and (63), respectively. ■

COROLLARY 3 *The following identities hold true:*

$$\mathcal{L}_4 \left\{ \frac{1}{v^6} \mathcal{L}_4 \left\{ f(x); \frac{1}{\sqrt{2}v} \right\}; z \right\} = \frac{\sqrt{\pi}}{2} \mathcal{L}_2\{f(x); z\}, \quad (65)$$

$$\mathcal{L}_4 \left\{ \frac{1}{v^6} \mathcal{L}_4 \left\{ f(x); \frac{1}{\sqrt{2}v} \right\}; z \right\} = \frac{2}{\sqrt{\pi}} \mathcal{F}_{C,2} \left\{ \mathcal{P}_4\{f(x); u\}; z \right\}, \quad (66)$$

and

$$\mathcal{L}_2\{f(x); z\} = \frac{4}{\pi} \mathcal{F}_{C,2} \left\{ \mathcal{P}_4\{f(x); u\}; z \right\}, \quad (67)$$

provided that each member of the assertions (65), (66), and (67) exists.

Proof Upon setting

$$g(u) = \cos(z^2 u^2) \quad (68)$$

in the assertion (47) of our Theorem, if we make use of the results (31) and (34) in each of the identities (46), (47), and (48) asserted by the Theorem, we obtain the assertions (65), (66), and (67), respectively. ■

4. Illustrative examples

An interesting illustration for the identity (55) asserted by Corollary 1 is contained in the following example.

Example 3: We show that

$$\mathcal{P}_4 \left\{ z^{-4\nu-4} \exp \left(-\frac{a^4}{4z^4} \right); y \right\} = \frac{\Gamma(\nu+1)}{y^{4\nu+4}} \exp \left(\frac{a^4}{4z^4} \right) \Gamma \left(-\nu; \frac{a^4}{4y^4} \right) \quad (\Re(\nu) > -1). \quad (69)$$

Demonstration. If we set

$$f(x) = x^{2\nu} J_\nu(a^2 x^2) \quad (70)$$

in the assertion (55) of Corollary 1 and apply the formula (41), we get

$$\mathcal{P}_4 \{ \mathcal{L}_4 \{ f(x); z \}; y \} = \frac{a^{2\nu}}{2^{\nu+2}} \mathcal{P}_4 \left\{ z^{-4\nu-4} \exp \left(-\frac{a^4}{4z^4} \right); y \right\}. \quad (71)$$

Now, by using the identity (54) and the result [12, Eq. (49)], we obtain

$$\begin{aligned} \mathcal{E}_{4,1} \{ x^{2\nu} J_\nu(a^2 x^2); y \} &= \frac{1}{2} \mathcal{E}_{2,1} \{ x^\nu J_\nu(a^2 x); y^2 \} \\ &= \frac{a^{2\nu}}{2^{\nu+2}} \Gamma(\nu+1) y^{-4\nu-4} \exp \left(-\frac{a^4}{4y^4} \right) \Gamma \left(-\nu; \frac{a^4}{4y^4} \right) \quad (\Re(\nu) > -1). \end{aligned} \quad (72)$$

Making use of the equations (71) and (72) in conjunction with the identity (55), we deduce the assertion (69) in Example 3.

Remark 4 Using the identity (18) in the assertion (69) of Example 3, we have

$$\mathcal{P} \left\{ z^{-2\nu-2} \exp \left(-\frac{\alpha^2}{4z^2} \right); u \right\} = \frac{2 \Gamma(\nu+1)}{u^{2\nu+2}} \exp \left(\frac{\alpha^2}{4u^2} \right) \Gamma \left(-\nu; \frac{\alpha^2}{4u^2} \right) \quad (\Re(\nu) > -1). \quad (73)$$

Similarly, by using the identity (17) in the assertion (69) of Example 3, we obtain

$$\mathcal{S} \left\{ z^{-\nu-1} \exp \left(-\frac{\alpha}{4z} \right); u \right\} = \frac{4 \Gamma(\nu+1)}{u^{\nu+1}} \exp \left(\frac{\alpha}{4u} \right) \Gamma \left(-\nu; \frac{\alpha}{4u} \right) \quad (\Re(\nu) > -1). \quad (74)$$

Next, an illustration for Corollary 2 is provided by Example 4 below.

Example 4: We show that

$$\mathcal{L}_2\{x^{2\nu+2} J_\nu(a^2 x^2); u\} = \frac{2^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{3}{2}\right) \frac{a^{2\nu} z^2}{(z^4 + a^4)^{\nu+(3/2)}} \quad (\Re(\nu) > -1) \quad (75)$$

and

$$\mathcal{F}_{S,2}\{x^{2\nu+2} K_\nu(a^2 x^2); u\} = 2^{\nu-1} \sqrt{\pi} \Gamma\left(\nu + \frac{3}{2}\right) \frac{a^{2\nu} z^2}{(z^4 + a^4)^{\nu+(3/2)}} \quad (\Re(\nu) > -1). \quad (76)$$

Demonstration. If we set

$$f(x) = x^{2\nu} J_\nu(a^2 x^2) \quad (77)$$

in Corollary 2 and use the formula (41), we obtain

$$\mathcal{L}_4\left\{x^{2\nu} J_\nu(a^2 x^2); \frac{1}{\sqrt{2}u}\right\} = 2^\nu a^{2\nu} v^{4\nu+4} \exp(-a^4 v^4). \quad (78)$$

We now apply the $\mathcal{L}_4$ -transform to (78) and use the relation (14). We thus find that

$$\begin{aligned} \mathcal{L}_4\left\{\frac{1}{v^2} \mathcal{L}_4\left\{x^{2\nu} J_\nu(a^2 x^2); \frac{1}{\sqrt{2}u}\right\}; z\right\} &= 2^\nu a^{2\nu} \mathcal{L}_4\{v^{4\nu+2} \exp(-a^4 v^4); z\} \\ &= a^{2\nu} 2^{\nu-2} \mathcal{L}\{v^{\nu+1/2} \exp(-a^4 v); z^4\}. \end{aligned} \quad (79)$$

Using the known formula [11, Vol. 1, p. 144, Entry (3)], the Laplace transform on the right-hand side of (79) is given by

$$\mathcal{L}\{v^{\nu+1/2} \exp(-a^4 v); z^4\} = \Gamma\left(\nu + \frac{3}{2}\right) (z^4 + a^4)^{-\nu-(3/2)}. \quad (80)$$

The assertion (75) of Example 4 follows upon substituting the result (80) into (79) and using the identity (61) of Corollary 2. The assertion (76) of Example 4 follows upon using the results (79) and (80) in conjunction with (62) of Corollary 2 and (39) of Example 2.

Remark 5 Using the identity (10) in the formula (75) of Example 4, we obtain the following consequence (cf. [11, p. 182, Entry (8)]; see also [14] and [15]):

$$\mathcal{L}\{x^{\nu+1} J_\nu(ax); y\} = \frac{2^{\nu+1} a^\nu}{\sqrt{\pi}} \Gamma\left(\nu + \frac{3}{2}\right) \frac{y}{(y^2 + a^2)^{\nu+(3/2)}} \quad (\Re(\nu) > -1). \quad (81)$$

Similary, by using the identity (20) in the formula (4) of Example 4, we obtain the following result (cf. [11, p. 105, Entry (49)]):

$$\mathcal{F}_S\{x^{\nu+1} K_\nu(ax); y\} = \sqrt{\pi} (2a)^\nu \Gamma\left(\nu + \frac{3}{2}\right) \frac{y}{(y^2 + a^2)^{\nu+(3/2)}} \quad \left(\Re(\nu) > -\frac{3}{2}\right). \quad (82)$$

Finally, as an illustration of Corollary 3, we compute the $\mathcal{F}_{C,2}$ -transform of the Thompson function of order zero which may be defined as follows (cf. [13, p. 545, Entry (55:3:7)]):

$$\ker(x) = \int_0^\infty \frac{t^3 J_0(xt)}{1+t^4} dt. \quad (83)$$

Example 5: We show that

$$\mathcal{F}_{C,2}\{\ker(bu); z\} = \frac{\pi}{8z^2} \exp\left(-\frac{b^2}{4z^2}\right). \quad (84)$$

Demonstration. Upon setting

$$f(x) = J_0(bx) \quad (85)$$

in Corollary 3, if we make use of the the known formula [10, p. 664, Entry (6.536)], we obtain

$$\mathcal{P}_4\{J_0(bx); u\} = \ker(bu). \quad (86)$$

Using the relation (10) and another known formula [11, p. 185, Entry (25)], we find that

$$\mathcal{L}_2\{J_0(bx); z\} = \frac{1}{2} \mathcal{L}\{J_0(bx^{1/2}); z^2\} = \frac{1}{2z^2} \exp\left(-\frac{b^2}{4z^2}\right). \quad (87)$$

The assertion (84) of Example 5 follows upon substituting the results (86) and (87) into (67) of Corollary 3.

We conclude by remarking that many other infinite integrals can be evaluated in this manner by applying the Lemma as well as the Theorem and its corollaries considered here (see also [16], [17] and [18]).

Acknowledgements

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A new Kapteyn series

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We study the Kapteyn series $\sum_{n=1}^{\infty} t^n J_n(nz)$. We find a series representation in powers of z and analyze its radius of convergence.

Keywords: Kapteyn series; Power series; Bessel functions

MSC Classification: Primary: 42C10; Secondary: 30B50, 33C10

1. Introduction

Series of the form

$$\sum_{n=0}^{\infty} \alpha_n J_{n+\nu}[(n+\nu)z], \quad \nu \in \mathbb{C}, \quad (1)$$

where $J_n(\cdot)$ is the Bessel function of the first kind, are called Kapteyn series. The series (1) is convergent and represents an analytic function throughout the domain in which [1, 17.3]

$$\Omega(z) < \liminf_{n \rightarrow \infty} |\alpha_n|^{-(1/\nu+n)}, \quad (2)$$

where

$$\Omega(z) = \left| \frac{z \exp[\sqrt{1-z^2}]}{1 + \sqrt{1-z^2}} \right|. \quad (3)$$

Kapteyn series first appeared in Lagrange's solution [2]

$$E = M + 2 \sum_{n=1}^{\infty} \frac{1}{n} J_n(\varepsilon n) \sin(Mn) \quad (4)$$

of Kepler's equation [3]

$$M = E - \varepsilon \sin(M).$$

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The solution (4) was independently discovered by Friedrich Bessel in ref. [4], where he introduced the functions which now bear his name.

Kapteyn series were systematically studied by Willem Kapteyn in his article [5], where he proved the following expansion theorem.

THEOREM 1 *Let $f(z)$ be a function which is analytic throughout the region*

$$D_a = \{z \in \mathbb{C} \mid \Omega(z) \leq a\},$$

with $a \leq 1$. Then,

$$f(z) = \alpha_0 + 2 \sum_{n=1}^{\infty} \alpha_n J_n(nz), \quad z \in D_a$$

where

$$\alpha_n = \frac{1}{2\pi i} \oint \Theta_n(z) f(z) dz$$

and the path of integration is the curve on which $\Omega(z) = a$. The function $\Theta_n(z)$ is the Kapteyn polynomial defined by

$$\begin{aligned} \Theta_0(z) &= \frac{1}{z}, \\ \Theta_n(z) &= \frac{1}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-2k)^2(n-k-1)!}{k!} \left(\frac{nz}{2}\right)^{2k-n-1}, \quad n \geq 1, \end{aligned}$$

where $\lfloor \cdot \rfloor$ denotes the integer-part function.

Proof See ref. [1, 17.4]. ■

As a corollary [1], one finds that if the Taylor series for $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$\begin{aligned} \alpha_0 &= a_0, \\ \alpha_n &= \frac{1}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-2k)^2(n-k-1)!}{k!(n/2)^{n-2k+1}} a_{n-2k}, \quad n \geq 1. \end{aligned} \tag{5}$$

In ref. [6], Kapteyn established the fundamental formula

$$\frac{1}{1-z} = 1 + 2 \sum_{n=1}^{\infty} J_n(nz), \quad \Omega(z) < 1, \tag{6}$$

using the integral representation [1]

$$J_n(nz) = \frac{1}{2\pi i} \int_{\mathcal{H}} \left\{ \frac{\exp[(z/2)(s - (1/s))]}{s} \right\}^n \frac{ds}{s},$$

where the Hankel contour $\mathcal{H}$ encircles the origin once counterclockwise.

The purpose of this article is to generalize equation (6) by means of the function

$$F(z, t) = \sum_{n=1}^{\infty} t^n J_n(nz), \quad z \in \mathbb{C}, \quad t \in \mathbb{R}, \quad (7)$$

so that equation (6) corresponds to the particular case $t = 1$. In section 2, we write $F(z, t)$ as a series in powers of z ,

$$F(z, t) = \sum_{n=1}^{\infty} A_n(t) z^n, \quad (8)$$

and find the coefficients $A_n(t)$. Although one could replace $\alpha_n = t^n$ in equation (5) and solve the problem

$$t^n = \frac{1}{4} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(n-2k)^2 (n-k-1)!}{k! (n/2)^{n-2k+1}} A_{n-2k}(t),$$

we take a different approach that only uses the differential equation satisfied by the Bessel functions. In section 3, we analyze the radius of convergence of equation (8) for different ranges of t . Finally, in section 4, we solve the inversion problem [equation (5)] for arbitrary α_n .

2. Power series

Since the Bessel function $J_n(z)$ is a solution of ref. [1]

$$z^2 J_n'' + z J_n' + (z^2 - n^2) J_n = 0,$$

the function $y_n(z) = J_n(nz)$ satisfies the ordinary differential equation (ODE)

$$z^2 y_n'' + z y_n' + n^2 (z^2 - 1) y_n = 0. \quad (9)$$

We also observe that

$$t^2 (t^n)'' + t (t^n)' = n^2 t^n. \quad (10)$$

Using equations (9) and (10) in equation (7), we have

$$z^2 F_{zz} + z F_z = (1 - z^2) (t^2 F_{tt} + t F_t), \quad (11)$$

where the subscripts denote partial derivatives.

Replacing equation (8) in (11), we get the equation

$$t^2 A_n'' + t A_n' - n^2 A_n = t^2 A_{n-2}'' + t A_{n-2}', \quad n \geq 1, \quad (12)$$

where we define $A_n \equiv 0$ for $n = -1, 0$.

From equations (7) and (8), we have

$$0 = F(z, 0) = \sum_{n=1}^{\infty} A_n(0)z^n,$$

and from equation (6), we see that

$$\sum_{n=1}^{\infty} \frac{1}{2} z^n = \frac{z}{2(1-z)} = F(z, 1) = \sum_{n=1}^{\infty} A_n(1)z^n,$$

which together imply that

$$A_n(0) = 0, \quad A_n(1) = \frac{1}{2}, \quad n \geq 1. \quad (13)$$

Solving equation (12) with the boundary conditions [equations (13)], we obtain

$$\begin{aligned} A_1(t) &= \frac{t}{2}, \quad A_2(t) = \frac{t^2}{2}, \quad A_3(t) = \frac{9}{16}t^3 - \frac{1}{16}t \\ A_4(t) &= \frac{2}{3}t^4 - \frac{1}{6}t^2, \quad A_5(t) = \frac{625}{768}t^5 - \frac{81}{256}t^3 + \frac{1}{384}t, \dots \end{aligned}$$

Thus, the function $A_n(t)$ is a polynomial in t of degree n . Writing

$$A_n(t) = \sum_{k=0}^n C_k^n t^k \quad (14)$$

and using equation (12), we get the recurrence

$$\begin{aligned} (n^2 - k^2)C_k^n &= -k^2 C_k^{n-2}, \quad \text{for } k = 0, 1, \dots, n-2, \\ (n^2 - k^2)C_k^n &= 0, \quad \text{for } k = n-1, n. \end{aligned} \quad (15)$$

To solve equation (15), we recall that the double factorial function satisfies

$$(n-k)!! = (n-k)(n-k-2)!!, \quad (n+k)!! = (n+k)(n+k-2)!!,$$

which gives

$$(n^2 - k^2) \frac{1}{(n-k)!! \times (n+k)!!} = \frac{1}{(n-k-2)!! \times (n+k-2)!!}.$$

This suggests the solution

$$C_k^n = \frac{\cos(((n-k)/2)\pi)k^n}{(n-k)!! \times (n+k)!!}, \quad 0 \leq k \leq n. \quad (16)$$

Therefore,

$$A_n(t) = \sum_{k=0}^n \frac{\cos(((n-k)/2)\pi)k^n}{(n-k)!! \times (n+k)!!} t^k \quad (17)$$

or, after rearranging the terms,

$$A_n(t) = \frac{(-1)^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n t^{n-2k}, \quad (18)$$

It is clear from equation (17) that $A_n(0) = 0$. To show that $A_n(1) = 1/2$, we use the identity [7, 0.154, #5]

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (k + \alpha)^n = (-1)^n n!, \quad (19)$$

where α is arbitrary. From equation (18), we have

$$A_n(t) + A_n\left(\frac{1}{t}\right) = \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n t^{n-2k}. \quad (20)$$

Setting $t = 1$ in equation (20), we have

$$2A_n(1) = \frac{(-1)^n}{n!} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n,$$

and from equation (19), we conclude that $2A_n(1) = 1$.

3. Radius of convergence

We shall now find the radius of convergence $R(t)$ of the power series [equation (8)]. From equation (7), we observe that $R(t) \rightarrow 0$ as $t \rightarrow \infty$ and $R(t) \rightarrow \infty$ as $t \rightarrow 0$, while equation (6) gives $R(1) = 1$ (figure 1). Since from equation (18), we have $A_n(-t) = (-1)^n A_n(t)$, we shall limit our analysis to $t > 0$.

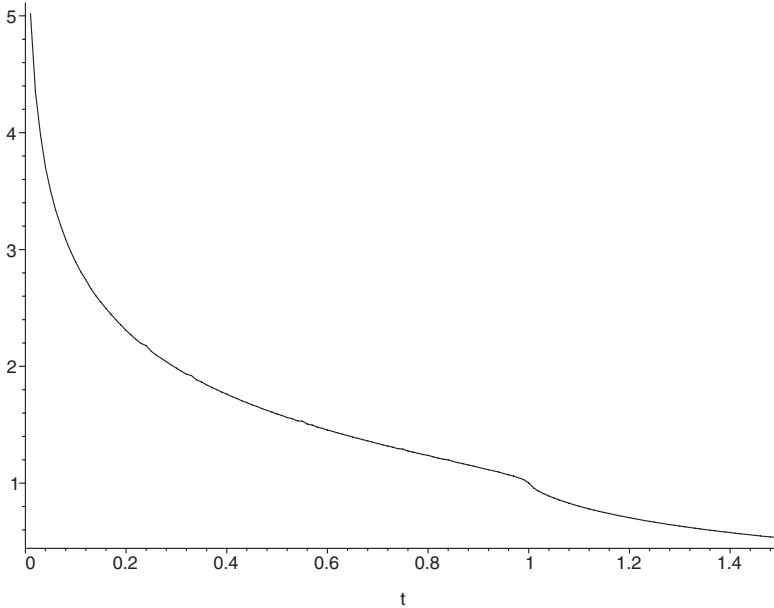


Figure 1. A sketch of $|A_{500}(t)|^{-(1/500)} \simeq R(t)$.

When $t \gg 1$, the largest term in the sum [equation (18)] corresponds to $k = 0$, and therefore,

$$A_n(t) \sim \frac{1}{n!} \left(\frac{nt}{2} \right)^n, \quad t \longrightarrow \infty. \quad (21)$$

Using Stirling's formula [8]

$$n! \sim \sqrt{2\pi n} n^n e^{-n}, \quad n \longrightarrow \infty,$$

we obtain

$$A_n(t) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{et}{2} \right)^n, \quad t \longrightarrow \infty. \quad (22)$$

Thus,

$$R(t) = \lim_{n \rightarrow \infty} |A_n(t)|^{-(1/n)} \sim \frac{2e^{-1}}{t}, \quad t \longrightarrow \infty. \quad (23)$$

When $t < 1$, equation (18) is highly oscillatory and there is no approximation like equation (21) valid in this range.

To find a formula for $R(t)$ valid for all t , we observe that, for a fixed value $t > 0$, we have

$$\ln[|A_n(t)|] \simeq \beta(t)n, \quad (24)$$

for some function $\beta(t)$ (figure 2).

Thus, we consider an asymptotic expansion of the form

$$A_n(t) \sim \frac{1}{2} \theta_n(t) [\psi(t)]^n, \quad n \longrightarrow \infty, \quad (25)$$

where $\theta_n(t)$ denotes the sign of $A_n(t)$ and $\psi(1) = 1$. Replacing equation (25) in equation (12) and setting

$$\theta'_n(t) = \theta''_n(t) = 0, \quad \theta'_{n-2}(t) = \theta''_{n-2}(t) = 0,$$

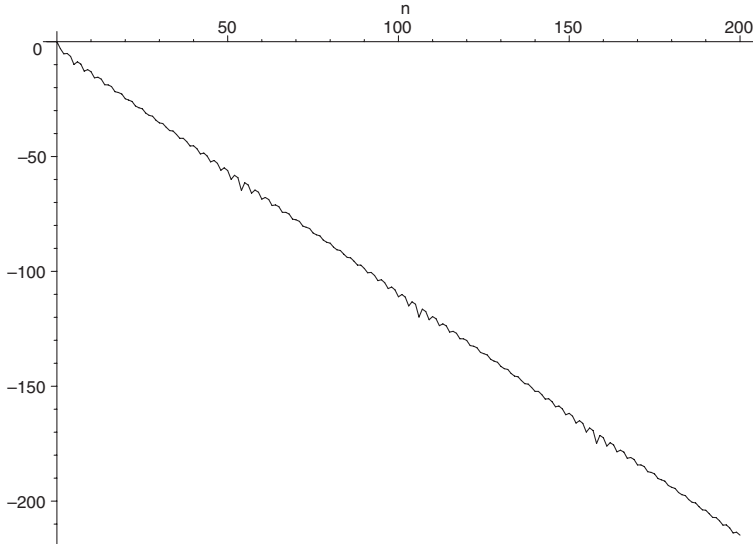


Figure 2. A sketch of $\ln[|A_n(0.1)|]$.

we obtain, to leading order in n ,

$$(t\psi')^2(\theta_n\psi^2 - \theta_{n-2}) - \theta_n\psi^4 = 0. \quad (26)$$

Since

$$\theta_n = \pm 1, \quad \theta_{n-2} = \pm 1,$$

and $\psi(t)$ is increasing for $t > 0$, we get

$$\psi' = \frac{\psi^2}{t\sqrt{\psi^2 \pm 1}}. \quad (27)$$

Solving equation (27) subject to $\psi(1) = 1$, we obtain the implicit solutions

$$\ln(t) + \frac{[\psi^2(t) + 1]^{3/2}}{\psi(t)} - \psi(t)\sqrt{\psi^2(t) + 1} - \operatorname{arcsinh}[\psi(t)] - \sqrt{2} + \ln(\sqrt{2} + 1) = 0, \quad (28)$$

for $0 < t \leq 1$ and

$$\ln(t) - \frac{[\psi^2(t) - 1]^{3/2}}{\psi(t)} + \psi(t)\sqrt{\psi^2(t) - 1} - \ln[\psi(t) + \sqrt{\psi^2(t) - 1}] = 0 \quad (29)$$

for $t \geq 1$.

Although we cannot solve equations (28) and (29) exactly, we can consider the limiting cases as $t \rightarrow 0$ and $t \rightarrow \infty$. Since $\psi(t) \rightarrow 0$ as $t \rightarrow 0$, we obtain from equation (28)

$$\psi(t) \sim \frac{1}{-\ln(t) + \sqrt{2} + \ln(\sqrt{2} - 1)}, \quad t \rightarrow 0. \quad (30)$$

When $t \rightarrow \infty$, we have $\psi(t) \rightarrow \infty$, and we get from equation (29)

$$\psi(t) \sim \frac{e}{2}t, \quad t \rightarrow \infty, \quad (31)$$

which agrees with equation (22).

Since $R(t) = 1/\psi(t)$, we have from equations (28) and (29), after exponentiating

$$e^{-\sqrt{2}}(1 + \sqrt{2}) \frac{R(t) \exp\left[\sqrt{1 + R^2(t)}\right]}{1 + \sqrt{1 + R^2(t)}} t = 1, \quad 0 < t \leq 1, \quad (32)$$

and

$$\frac{R(t) \exp\left[\sqrt{1 - R^2(t)}\right]}{1 + \sqrt{1 - R^2(t)}} t = 1, \quad t \geq 1. \quad (33)$$

In figure 3, we graph the solutions of equations (32) and (33) and the approximate value of $R(t)$ given by $|A_{500}(t)|^{-(1/500)}$.

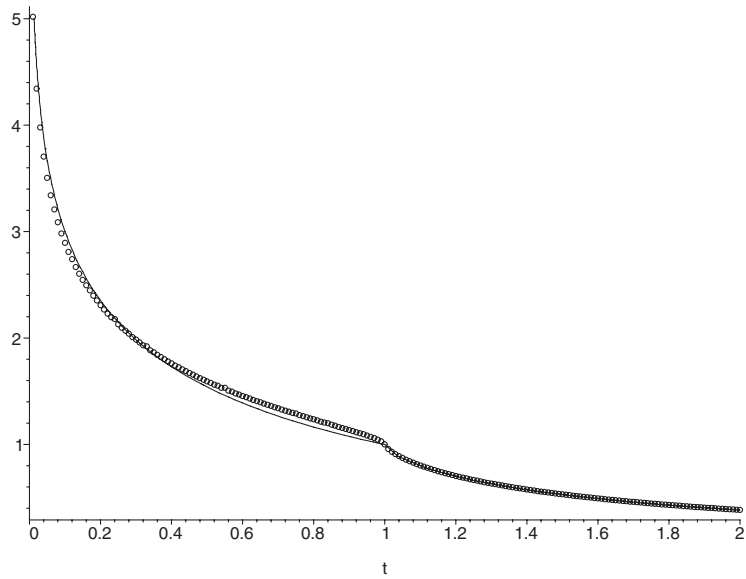


Figure 3. A comparison of $R(t)$ (solid curve) and $|A_{500}(t)|^{-(1/500)}$ (ooo).

Using equation (2) with $\alpha_n = t^n$, we conclude that the Kapteyn series (7) converges for those $t \in \mathbb{R}$ and $z \in \mathbb{C}$ such that

$$\left| \frac{z \exp[\sqrt{1-z^2}]}{1 + \sqrt{1-z^2}} t \right| < 1. \tag{34}$$

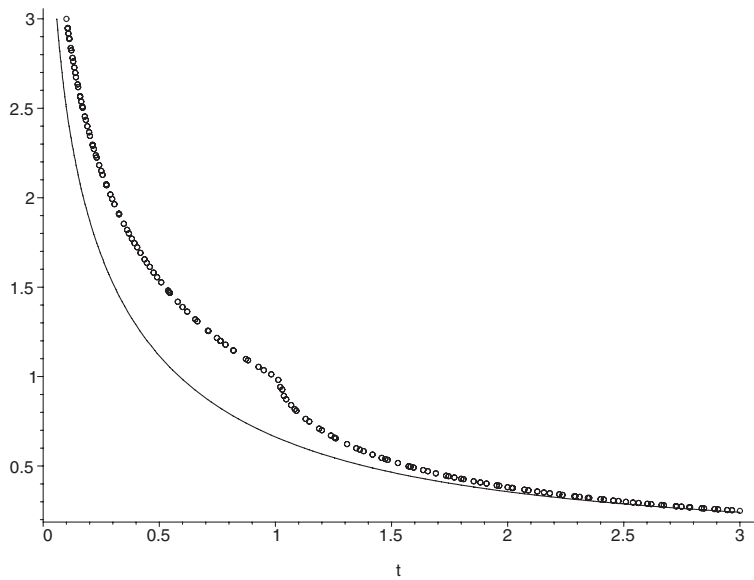


Figure 4. A comparison of $R(t)$ (solid curve) and $|A_{500}(t)|^{-(1/500)}$ (ooo).

Replacing $z = re^{i\omega}$ in (34), we find that the minimum value of r corresponds to $\omega = \pm(\pi/2)$. Thus, for $t > 0$, the Kapteyn series (7) will converge inside the circle $|z| < r(t)$, with

$$\frac{r(t) \exp[\sqrt{1 + r(t)^2}]}{1 + \sqrt{1 + r(t)^2}} t = 1. \quad (35)$$

As it was observed in ref. [1], the circles of convergence of the Kapteyn series (7) and the power series (8) are not equal, the former being slightly smaller than the latter. However, the radius $r(t)$ given by equation (35) and $R(t)$ are asymptotically equal as $t \rightarrow 0$ and $t \rightarrow \infty$ (figure 4).

We summarize our results in the following theorem.

THEOREM 2 *The Kapteyn series*

$$F(z, t) = \sum_{n=1}^{\infty} t^n J_n(nz)$$

converges for those $t \in \mathbb{R}$ and $z \in \mathbb{C}$ such that

$$\left| \frac{z \exp[\sqrt{1 - z^2}]}{1 + \sqrt{1 - z^2}} t \right| < 1.$$

The function $F(z, t)$ admits the power series representation

$$F(z, t) = \sum_{n=1}^{\infty} A_n(t) z^n,$$

with coefficients

$$A_n(t) = \frac{(-1)^n}{n!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \left(k - \frac{n}{2}\right)^n t^{n-2k}$$

and radius of convergence $R(t)$ defined by the implicit equations

$$e^{-\sqrt{2}}(1 + \sqrt{2}) \frac{R(t) \exp[\sqrt{1 + R^2(t)}]}{1 + \sqrt{1 + R^2(t)}} |t| = 1, \quad 0 < |t| \leq 1,$$

$$\frac{R(t) \exp[\sqrt{1 - R^2(t)}]}{1 + \sqrt{1 - R^2(t)}} |t| = 1, \quad |t| \geq 1.$$

4. The general problem

We shall now consider the problem of finding a_n in terms of α_n , where

$$\sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} \alpha_n J_n(nz), \quad (36)$$

for arbitrary a_n and α_n . Rewriting the power series of $J_n(nz)$ [1]

$$J_n(nz) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{nz}{2}\right)^{n+2j}$$

in the form

$$J_n(nz) = \left(\frac{n}{2}\right)^n \sum_{j=0}^{\infty} \frac{\cos((\pi/2)j)(n/2)^j}{(j/2)!(n + (j/2))!} z^{n+j},$$

we have,

$$\begin{aligned} \sum_{n=1}^{\infty} \alpha_n J_n(nz) &= \sum_{n=1}^{\infty} \alpha_n \left(\frac{n}{2}\right)^n \sum_{j=0}^{\infty} \frac{\cos((\pi/2)j)(n/2)^j}{(j/2)!(n + (j/2))!} z^{n+j} \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^k \alpha_n \left(\frac{n}{2}\right)^n \frac{\cos[(\pi/2)(k-n)](n/2)^{k-n}}{((k-n)/2)!((k+n)/2)!} z^k. \end{aligned}$$

Therefore, we obtain from equation (36) that

$$a_n = \sum_{n=1}^k \alpha_n \frac{\cos[\pi/2(k-n)](n/2)^k}{((k-n)/2)!((k+n)/2)!}$$

or

$$a_k = \sum_{n=1}^k \alpha_n \frac{\cos[\pi/2(k-n)]}{(k-n)!!(k+n)!!} n^k. \quad (37)$$

In particular, setting $\alpha_n = t^n$ in equation (37), we recover equation (17).

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Starlike and convex functions of complex order involving the Dziok-Srivastava operator

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Making use of Dziok-Srivastava operator we investigate starlike and convex functions of complex order defined by means of Hadamard product.

Keywords: Hadamard product; Starlike and convex functions of complex order; Dziok-Srivastava operator

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic and univalent in the unit disk $U = \{z : |z| < 1\}$.

A function $f \in \mathcal{A}$ is starlike of complex order γ ($\gamma \in \mathbb{C} - \{0\}$), that is $S_o^*(\gamma)$, if it also satisfies the following inequality

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 (z \in U; \gamma \in \mathbb{C} - \{0\}). \quad (2)$$

Let $S_1^*(\gamma)$ denote the class of functions $f(z) \in \mathcal{A}$ satisfying

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < |\gamma| (\gamma \in \mathbb{C} - \{0\}). \quad (3)$$

We note that $S_1^*(\gamma)$ is a subclass of $S_o^*(\gamma)$ [1].

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Furthermore, a function $f \in \mathcal{A}$ is convex of complex order γ ($\gamma \in \mathbb{C} - \{0\}$), that is $K_o(\gamma)$, if it also satisfies the following inequality

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \right\} > 0 (z \in U; \gamma \in \mathbb{C} - \{0\}). \quad (4)$$

Let $K_1(\gamma)$ denote the class of functions $f(z) \in \mathcal{A}$, satisfying

$$\left| \frac{zf''(z)}{f'(z)} \right| < |\gamma| (\gamma \in \mathbb{C} - \{0\}). \quad (5)$$

We note that

$$f(z) \in K_o(\gamma) \text{ if, and only if, } zf'(z) \in S_o^*(\gamma) \text{ for } \gamma \in \mathbb{C} - \{0\} \quad (6)$$

and

$$f(z) \in K_1(\gamma) \text{ if, and only if, } zf'(z) \in S_1^*(\gamma) \text{ for } \gamma \in \mathbb{C} - \{0\}. \quad (7)$$

We also have $K_1(\gamma) \subset K_o(\gamma)$.

The classes $S_o^*(\gamma)$ and $K_o(\gamma)$ denote, respectively, the familiar classes of starlike and convex functions of complex order $\gamma \neq 0$ in U , which were considered earlier by Nasr and Aouf [2] and Wiatrowski [3], respectively (see also [4] and [5]). Moreover, by taking $\gamma = 1 - \alpha$ ($0 \leq \alpha < 1$), we observe that $S_o^*(1 - \alpha) = S^*(\alpha)$ and $K_o(1 - \alpha) = K(\alpha)$, where $S^*(\alpha)$ denotes the class of starlike functions of order α in U and $K(\alpha)$ denotes the class of convex functions of order α in U , which were introduced by Robertson [6].

For two functions $f(z) \in \mathcal{A}$, of the form (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g * f)(z). \quad (8)$$

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \dots, l$) and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$), the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \frac{z^k}{k!} \quad (l \leq m + 1; l, m \in N_0 := N \cup \{0\}; z \in U). \quad (9)$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & k = 0 \\ x(x+1)(x+2) \cdots (x+k-1), & k \in N. \end{cases} \quad (10)$$

Corresponding to the function $h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) := {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)$, Dizok–Srivastava [7] defined a linear operator

$$\begin{aligned} H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) f(z) &= h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{(\alpha_1)_{k-1} \cdots (\alpha_l)_{k-1}}{(\beta_1)_{k-1} \cdots (\beta_m)_{k-1}} \frac{a_k z^k}{(k-1)!}. \end{aligned} \quad (11)$$

To make the notation simple, we write

$$H_m^l[\alpha_1]f(z) = H_m^l(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)f(z).$$

We note that many subclasses of analytic functions, associated with the Dziok–Srivastava operator $H_m^l[\alpha_1]f(z)$ and many special cases, were investigated recently by Dziok–Srivastava [7] and Liu [8]. Also we note that the following special cases are of the Dziok–Srivastava linear operator.

Remark 1 For $f(z) \in A$,

$$H_1^2(\alpha, 1; \beta)f(z) = L(\alpha, \beta)f(z),$$

which was considered by Carlson and Shaffer [9].

Remark 2 For $f(z) \in A$,

$$H_1^2(n+1, 1; 1)f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = D^n f(z),$$

the symbol $D^n f(z)$ was introduced by Ruscheweyh [10] and is called Ruscheweyh derivative of $f(z)$ of n th order.

Remark 3 For $f(z) \in A$,

$$H_1^2(c+1, 1; c+2)f(z) = \frac{c+1}{(z^c)} \int_0^z t^{c-1} f(t) dt = J_c f(z),$$

where $c > -1$. The operator J_c was introduced by Bernardi [11]. In particular, the operator J_1 was studied earlier by Libera [12] and Livingston [13].

Remark 4 For $f(z) \in A$,

$$H_1^2(2, 1; 2-\lambda)f(z) = \Gamma(2-\lambda)z^\lambda D_z^\lambda f(z) = \Omega^\lambda f(z), \lambda \notin N - \{1\}.$$

The operator Ω^λ was introduced by Srivastava–Owa [14] (cf, e.g. [15–17]) and Ω^λ is also called Srivastava–Owa fractional derivative operator, where $D_z^\lambda f(z)$ denotes the fractional derivative of $f(z)$ of order λ , studied by Owa [18] and others [15, 16]. The main object of the present paper is to investigate the starlike and convex functions of complex order involving Dziok–Srivastava operator defined by means of Hadamard product.

2. Main results

In order to prove our main results, we shall require the following lemma due to Ruscheweyh and Sheil-Small [19].

LEMMA 1 Let $\phi(z)$ and $g(z)$ be analytic in U and satisfy $\phi(0) = g(0) = 0$, $\phi'(0) \neq 0$ and $g'(0) \neq 0$. Suppose that for each $\sigma(|\sigma| = 1)$ and $\rho(|\rho| = 1)$

$$\phi(z) * \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) g(z) \neq 0 (z \in U - \{0\}).$$

Then for each function $F(z)$ analytic in the unit disc U and satisfying the inequality $\operatorname{Re}\{F(z)\} > 0$, $z \in U$, we have

$$\operatorname{Re} \left(\frac{\phi * G(z)}{\phi * g(z)} \right) \neq 0 (z \in U),$$

where $G(z) = F(z)g(z)$.

Applying the above lemma we derive the following theorems.

THEOREM 1 Let $f(z) \in S_o^*(\gamma)$ ($\gamma \in C - \{0\}$) and let

$$h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 (z \in U - \{0\}), \quad (12)$$

for each $\sigma(|\sigma| = 1)$, $\rho(|\rho| = 1)$ and for $\alpha_j \in C$ ($j = 1, 2, \dots, l$) and $\beta_j \in C - \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$). Then we have $H_m^l[\alpha_1]f(z) \in S_o^*(\gamma)$.

Proof It is sufficient to show that

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) \right\} > 0, \quad (13)$$

for $z \in U$. We notice from (9), which readily yields

$$\begin{aligned} 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) &= 1 + \frac{1}{\gamma} \left(\frac{H_m^l[\alpha_1](zf'(z))}{H_m^l[\alpha_1]f(z)} - 1 \right) \\ &= 1 + \frac{1}{\gamma} \left(\frac{h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * (zf'(z))}{h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z)} - 1 \right) \\ &= 1 + \frac{1}{\gamma} \left[\frac{h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * [(\gamma - 1)f(z) + zf'(z)]}{h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * \gamma f(z)} \right]. \end{aligned}$$

Therefore, setting $\phi(z) = h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$, $g(z) = \gamma f(z)$ and $F(z) = 1 + (1/\gamma)((zf'(z)/f(z)) - 1)$ in Lemma 1, we find that

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{z(H_m^l[\alpha_1]f(z))'}{H_m^l[\alpha_1]f(z)} - 1 \right) \right\} > 0,$$

which implies that $H_m^l[\alpha_1]f(z) \in S_o^*(\gamma)$. ■

The following corollaries are special cases of Theorem 1.

COROLLARY 1 [20] Let the function $f(z)$ defined by (1) be in the class $S_o^*(\gamma)$ and let

$$H_1^2(\alpha, 1; \beta) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$ and for $\beta \neq 0, -1, -2, \dots$. Then we have $L(\alpha, \beta)f(z) \in S_o^*(\gamma)$.

COROLLARY 2 Let the function $f(z)$ defined by (1) be in the class $S_o^*(\gamma)$ and let

$$H_1^2(n+1, 1; 1) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $D^n f(z) \in S_o^*(\gamma)$.

COROLLARY 3 Let the function $f(z)$ defined by (1) be in the class $S_o^*(\gamma)$ and let

$$H_1^2(c+1, 1; c+2) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $J_c f(z) \in S_o^*(\gamma)$.

COROLLARY 4 Let the function $f(z)$ defined by (1) be in the class $S_o^*(\gamma)$ and let

$$H_1^2(2, 1; 2 - \lambda) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) f(z) \neq 0 \quad (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $\Omega^\lambda f(z) \in S_o^*(\gamma)$.

THEOREM 2 Let $f(z) \in K_o(\gamma)$ ($\gamma \in C - \{0\}$) and let

$$H_m^l[\alpha_1] \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) z f'(z) \neq 0 \quad (z \in U - \{0\}), \quad (14)$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$ and for $\alpha_j \in C$ ($j = 1, 2, \dots, l$) and $\beta_j \in C - \{0, -1, -2, \dots\}$ ($j = 1, 2, \dots, m$). Then we have $H_m^l[\alpha_1]f(z) \in K_o(\gamma)$.

Proof Notice that $f(z) \in K_o(\gamma)$ if, and only if, $zf'(z) \in S_o^*(\gamma)$. With the aid of Theorem 1, we conclude that

$$\begin{aligned} f(z) \in K_o(\gamma) &\iff zf'(z) \in S_o^*(\gamma) \\ &\implies H_m^l[\alpha_1]zf'(z) \in S_o^*(\gamma) \\ &\iff z(H_m^l[\alpha_1]f(z))' \in S_o^*(\gamma) \\ &\iff H_m^l[\alpha_1]f(z) \in K_o(\gamma), \end{aligned}$$

which completes the proof of Theorem 2. ■

The following corollaries are special cases of Theorem 2.

COROLLARY 5 [20] *Let the function $f(z)$ defined by (1) be in the class $K_o(\gamma)$ and let*

$$H_1^2(\alpha, 1; \beta) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) z f'(z) \neq 0 (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$ and for $\beta \neq 0, -1, -2, \dots$. Then we have $L(\alpha, \beta)f(z) \in K_o(\gamma)$.

COROLLARY 6 *Let the function $f(z)$ defined by (1) be in the class $K_o(\gamma)$ and let*

$$H_1^2(n+1, 1; 1) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) z f'(z) \neq 0 (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $D^n f(z) \in K_o(\gamma)$.

COROLLARY 7 *Let the function $f(z)$ defined by (1) be in the class $K_o(\gamma)$ and let*

$$H_1^2(c+1, 1; c+2) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) z f'(z) \neq 0 (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $J_c f(z) \in K_o(\gamma)$.

COROLLARY 8 *Let the function $f(z)$ defined by (1) be in the class $K_o(\gamma)$ and let*

$$H_1^2(2, 1; 2 - \lambda) \left(\frac{1 + \rho\sigma z}{1 - \sigma z} \right) z f'(z) \neq 0 (z \in U - \{0\}),$$

for each $\sigma (|\sigma| = 1)$, $\rho (|\rho| = 1)$. Then $\Omega^\lambda f(z) \in S_o^*(\gamma)$.

In order to establish our next two results, we need to recall, here, the following lemma by Ruscheweyh and Sheil-Small [19].

LEMMA 2 *Let $\phi(z)$ be convex and $g(z)$ be starlike in U . Then for each function $F(z)$ analytic in U and satisfying $\operatorname{Re}\{F(z)\} > 0$ ($z \in U$), we have*

$$\operatorname{Re} \left\{ \frac{\phi * Fg(z)}{\phi * g(z)} \right\} > 0 (z \in U).$$

Using the above lemma, we get the following theorems.

THEOREM 3 *Let $h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ be convex and let $f(z) \in S_1^*(\gamma) (|\gamma| < 1)$. Then $H_m^l[\alpha_1]f(z) \in S_o^*(\gamma)$.*

THEOREM 4 *Let $h(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$ be convex and let $f(z) \in K_1(\gamma) (|\gamma| < 1)$. Then $H_m^l[\alpha_1]f(z) \in K_o(\gamma)$.*

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On Marcel Riesz kernel $H_\alpha(P \pm i0, n)$

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In this article, the author generalizes the kernel $H_\alpha = H_\alpha(P \pm i0, n)$ given by Trione [Trione, S.E., 1980, *Distributional Products*, Cursos de Matemática, Vol. 3 (Buenos Aires, Argentina, IAM-CONICET).]. The author proves several of their properties, for example that H_{2k} is a fundamental solution from operator given by formula (22). Moreover, the author uses this kernel to evaluate some distributional products.

Keywords: Multiplicative products of distributions; Convolution products of distributions; Ultra-hyperbolic operator

1991 Mathematics Subject Classification: 46F-35L

1. Introduction

In ref. [1], Trione generalized the kernel of Marcel Riesz R_α (cf. [2]) for the quadratic form $P = P(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, where $n = p + q$, and proved that it has analogous properties. In this article, we generalize the kernel $H_\alpha(P \pm i0, n)$ given in ref. [1] for any non-degenerate quadratic form P . Moreover, we generalize several products that appear in ref. [1]. The plan of exposition is as follows: in section 2, we define the distribution $(P \pm i0)^\lambda$ and give some of their properties. In section 3, we define the distribution $(m^2 + P \pm i0)^\lambda$. In section 4, we evaluate the products $(m^2 + P \pm i0)^\lambda \cdot (m^2 + P \pm i0)^\mu$, $(P \pm i0)^\lambda \cdot (P \pm i0)^\mu$ and $(P \pm i0)^{-m} \cdot L^k \delta$, formulas (18), (20) and (21), respectively. In section 5, we define the distributions $H_\alpha(P \pm i0, n)$, formula (29); also we prove some of their properties, for example that H_{2k} is a fundamental solution from the operator $L^k = (\sum_{i,j=1}^n g^{j,k} \partial^2 / \partial x_i \partial x_j)^k$. Finally, we evaluate the products of the form $L^h(P \pm i0)^{-m} \cdot L^k \delta$, formulas (37), (43) and (49).

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2. The distributions $(P \pm i0)^\lambda$

Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space $\mathbb{R}^n$. Consider a non-degenerate quadratic form in n variables

$$P = P(x) = \sum_{i,j=1}^n g_{i,j} x_i x_j, \quad (1)$$

where $g_{i,j} \in \mathbb{R}$.

The distribution $(P \pm i0)^\lambda$ is defined by

$$(P \pm i0)^\lambda = \lim_{\epsilon \rightarrow 0} \{P \pm i\epsilon |x|^2\}^\lambda, \quad (2)$$

where $\epsilon > 0$, $|x|^2 = x_1^2 + \dots + x_n^2$, $\lambda \in \mathbb{C}$.

The distributions $(P \pm i0)^\lambda$ are an important contribution of Gelfand ([3], p. 274). The distributions $(P \pm i0)^\lambda$ are analytic in λ everywhere except at $\lambda = (n/2) - k$, $k = 0, 1, \dots$, where they have simple poles ([3], p. 275).

The distribution $(P \pm i0)^\lambda$ can be expressed as follows ([3], p. 576):

$$(P + i0)^\lambda = P_+^\lambda + e^{+i\pi\lambda} P_-^\lambda \quad (3)$$

and

$$(P - i0)^\lambda = P_+^\lambda + e^{-i\pi\lambda} P_-^\lambda. \quad (4)$$

For $\text{Re } \lambda > 0$, we have

$$P_+^\lambda = \begin{cases} P^\lambda & \text{if } P \geq 0, \\ 0 & \text{if } P < 0. \end{cases} \quad (5)$$

and

$$P_-^\lambda = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^\lambda & \text{if } P \leq 0. \end{cases} \quad (6)$$

Furthermore, the Fourier transform of distributions $(P \pm i0)$ is ($m = 0$, formula (9), p. 289, ref. [3])

$$\begin{aligned} \mathcal{F}[(P \pm i0)] &= [(P \pm i0)^\lambda]^\wedge = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x, y \rangle} (P \pm i0)^\lambda dx \\ &= \frac{e^{\mp i(\pi/2)q} 2^{2\lambda+n} \pi^{n/2} \Gamma(\lambda + (n/2))}{(2\pi)^{n/2} \Gamma(-\lambda) \sqrt{|\Delta|}} (Q \mp i0)^{-\lambda - (n/2)}, \end{aligned} \quad (7)$$

where

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i, \quad (8)$$

$$Q = Q(y) = \sum_{i,j=1}^n g^{i,j} y_i y_j, \quad (9)$$

$$\sum_{j=1}^n g_{i,j} g^{j,k} = \delta_i^k = 1 \text{ for } i = k \text{ and } \delta_i^k = 0 \text{ for } i \neq k, \quad (10)$$

Δ is the determinant of the coefficient of P and q the number of negative terms of P . The residue of $(P \pm i0)^\lambda$ in $\lambda = -(n/2) - k, k = 0, 1, \dots$ is given by (see p. 276 of [3]):

$$\text{Res}_{\lambda=-(n/2)-k}(P \pm i0)^\lambda = \frac{e^{\mp i(\pi/2)q} \pi^{n/2}}{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}} \left(\sum_{i,j=1}^n g^{j,k} \frac{\partial^2}{\partial x_i \partial x_j} \right)^k \delta. \quad (11)$$

3. The distributions $(m^2 + P \pm i0)^\lambda$

The distributions $(m^2 + P \pm i0)^\lambda$ are defined in an analogue manner as the distributions $(P \pm i0)^\lambda$. Let us put [3], p. 289)

$$(m^2 + P \pm i0)^\lambda = \lim_{\epsilon \rightarrow 0} \{m^2 + P \pm i\epsilon|x|^2\}^\lambda, \quad (12)$$

where m is a positive real number.

We can prove, without difficulty, that the following formula is valid [3], p. 566)

$$(m^2 + P \pm i0)^\lambda = (m^2 + P)_+^\lambda + e^{\pm i\pi\lambda} (m^2 + P)_-^\lambda, \quad (13)$$

here

$$(m^2 + P)_+^\lambda = \begin{cases} (m^2 + P)^\lambda & \text{if } m^2 + P \geq 0, \\ 0 & \text{if } m^2 + P < 0, \end{cases} \quad (14)$$

$$(m^2 + P_-)^\lambda = \begin{cases} 0 & \text{if } m^2 + P > 0, \\ (-m^2 - P)^\lambda & \text{if } m^2 + P \leq 0. \end{cases} \quad (15)$$

From equation (13), we conclude immediately that

$$(m^2 + P + i0)^\lambda = (m^2 + P - i0)^\lambda = (m^2 + P)^\lambda, \quad (16)$$

when $\lambda = k = \text{positive integer}$.

We observe that $(m^2 + P \pm i0)^\lambda$ are entire distributional functions of λ .

This is the principal difference between the distributions, formally analogue, $(P \pm i0)^\lambda$, which have poles at the points $\lambda = -(n/2) - k, k = 0, 1, 2, \dots$

It can be proved ([4]; p. 573, formula (2.14) and p. 275, formula (3.5)) that

$$(m^2 + P \pm i0)^{-k} = F_p(m^2 + P)^{-k} \mp \frac{i\pi(-1)^{k-1}}{(k-1)!} \delta^{k-1}(m^2 + P), \quad (17)$$

$k = 0, 1, \dots, F_p = \text{finite part}$.

4. The products $(m^2 + P \pm i0)^\lambda \cdot (m^2 + P \pm i0)^\mu, (P \pm i0)^\lambda \cdot (P \pm i0)^\mu$ and $(P \pm i0)^{-m} \cdot L^k \delta$

In this section, we generalize the product that appears in p. 23 of ref. [1]; they are proved to be in analogous form. Moreover, we obtain a generalization of the Theorem 12 of ref. [5].

THEOREM 1 *Hypothesis: Let $\lambda, \mu \in \mathbb{C}$ and $m^2 \neq 0$.*

$$(m^2 + P \pm i0)^\lambda \cdot (m^2 + P \pm i0)^\mu = (m^2 + P \pm i0)^{\lambda+\mu}. \quad (18)$$

Proof Let $\operatorname{Re} \lambda, \operatorname{Re} \mu > 0$, we approximate both terms for smooth function of identical structure and obtain

$$\lim_{\epsilon \rightarrow 0} (m^2 + P \pm i\epsilon)^\lambda \cdot (m^2 + P \pm i\epsilon)^\mu = \lim_{\epsilon \rightarrow 0} (m^2 + P \pm i\epsilon)^{\lambda+\mu} = (m^2 + P \pm i0)^{\lambda+\mu}. \quad (19)$$

Then by analytical continuation, the formula (18) is valid for every $\lambda, \mu \in \mathbb{C}$. ■

THEOREM 2 *Hypothesis:*

- (1) $\lambda, \mu \in \mathbb{C}$,
- (2) λ, μ and $\lambda + \mu$ different from $-(n/2) - k, k = 0, 1, \dots$

Thesis:

$$(P \pm i0)^\lambda \cdot (P \pm i0)^\mu = (P \pm i0)^{\lambda+\mu}. \quad (20)$$

Proof It is identical to Theorem 1, putting $m = 0$. ■

Now, using Theorem 2, we extend theorem 12 of ref. [5].

THEOREM 3 *Hypothesis: Let m be a positive integer and k a non-negative integer.*

Thesis: The following formula is valid

$$(P \pm i0)^{-m} \cdot L^k \delta = CL^{k+m} \delta, \quad (21)$$

where

$$L^k = \left(\sum_{i,j=1}^n g^{j,k} \frac{\partial^2}{\partial x_i \partial x_j} \right)^k, \quad (22)$$

and

$$C = \left\{ 4^m (k+1) \cdots (k+m) \left(\frac{n}{2} + k \right) \cdots \left(\frac{n}{2} + k + m - 1 \right) \right\}^{-1}. \quad (23)$$

Proof From formula (11), we obtain

$$\lim_{\epsilon \rightarrow 0} \frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2}} \epsilon (P \pm i0)^{\epsilon - (n/2) - k} = L^k \delta. \quad (24)$$

We put, by definition,

$$(P \pm i0)^{-m} \cdot L^k \delta := \lim_{\epsilon \rightarrow 0} (P \pm i0)^{-m} \frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2}} \epsilon (P \pm i0)^{\epsilon - (n/2) - k}. \quad (25)$$

Taking into account Theorem 2 and formula (24), we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2}} \epsilon (P \pm i0)^{\epsilon - (n/2) - k - m} \\ &= \lim_{\epsilon \rightarrow 0} \frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2}} \frac{C_\pm}{C_\pm} \epsilon (P \pm i0)^{\epsilon - (n/2) - k - m} \\ &= \frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2} C_\pm} L^{k+m} \delta, \end{aligned} \quad (26)$$

where

$$C_\pm = \frac{e^{\mp i(\pi/2)} q \pi^{n/2}}{4^{k+m} k! \Gamma((n/2) + k + m) \sqrt{|\Delta|}}. \quad (27)$$

We obtain formula (21) of

$$\frac{4^k k! \Gamma((n/2) + k) \sqrt{|\Delta|}}{e^{\mp i(\pi/2)q} \pi^{n/2} C_\pm} = \left\{ 4^m (k+1) \cdots (k+m) \left(\frac{n}{2} + k\right) \cdots \left(\frac{n}{2} + k + m - 1\right) \right\}^{-1}. \quad (28)$$

■

5. The kernel of Marcel Riesz $H_\alpha(P \pm i0, n)$

In this section, we generalize, for any non-degenerate quadratic form, the kernel of Marcel Riesz $H_\alpha(P \pm i0, n)$, given by Trione [1]. Moreover, we prove several properties, for example that H_{2k} is a fundamental solution of the operator given by formula (22).

Let

$$H_\alpha = H_\alpha(P \pm i0) = \frac{e^{i\alpha(\pi/2)} e^{\pm iq(\pi/2)} \Gamma((n-\alpha)/2) \sqrt{|\Delta|}}{2^\alpha \pi^{n/2} \Gamma(\alpha/2)} (P \pm i0)^{(\alpha-n)/2}, \quad (29)$$

here $\alpha \in \mathbb{C}$, $\alpha \neq n + 2h$ with $h = 0, 1, \dots$ and $(P \pm i0)^{(\alpha-n)/2}$ is given by formula (2).

The distribution $H_\alpha(P \pm i0, n)$ have analogous properties to the kernel of Marcel Riesz ([4], pp. 1–22).

We begin by proving the

THEOREM 4 *The following formula is valid*

$$(H_\alpha \cdot H_{-2k})^\wedge = (2\pi)^{n/2} (H_\alpha)^\wedge \cdot (H_{-2k})^\wedge, \quad (30)$$

where $k = 0, 1, \dots$ and $\alpha \neq n + 2h$ with $h = 0, 1, \dots$

Proof We observe that, taking into account formulas (7) and (29), we have

$$\{H_\alpha(P \pm i0)\}^\wedge = e^{i\alpha(\pi/2)} \frac{(Q \mp i0)^{-(\alpha/2)}}{(2\pi)^{n/2}}, \quad (31)$$

then putting $\alpha = -2k$, we obtain

$$\{H_{-2k}(P \pm i0)\}^\wedge = (-1)^k \frac{(Q \mp i0)^k}{(2\pi)^{n/2}}. \quad (32)$$

From the preceding formulas, we have

$$H_{-2k} = L^k(\delta), \quad (33)$$

where L^k is given by the formula (22). Since H_{-2k} is a convolutor of D' (see formula (VII,8;5), p. 268 of [6]), we obtain formula (30). ■

THEOREM 5 *The following formula is valid*

$$H_\alpha * H_{-2k} = H_{\alpha-2k}, \quad (34)$$

where $k = 0, 1, \dots$ and $\alpha, \alpha - 2k \neq n + 2h$ with $h = 0, 1, \dots$

Proof Taking into account formula (31), we have

$$(H_\alpha)^\wedge \cdot (H_{-2k})^\wedge = (2\pi)^{n/2} (H_{\alpha-2k})^\wedge, \quad (35)$$

and from Theorem 4, we arrive at the formula (34). ■

Finally, we arrive at our main result: “ $H_{2k}(P \pm i0, n)$ is a fundamental solution from operator L^k given by formula (22)”.

THEOREM 6 *The following formula is true*

$$L^k(H_{2k}) = \delta, \quad (36)$$

where $2k \neq n + 2h$, $h, k = 0, 1, \dots$

Proof We obtain formula (36) from formulas (33) and (34). ■

6. Products of the form $L^h(P \pm i0)^{-m} \cdot L^k \delta$

In this section, we evaluate the products of the form $L^h(P \pm i0)^{-m+h} \cdot L^k \delta$. The method is to approximate $L^k \delta$ by the distribution $H_\alpha(P \pm i0, n)$.

THEOREM 7 *The following formula is valid*

$$(P \pm i0)^{-m} \cdot \delta = \frac{\Gamma(n/2)2^{-2m}}{\Gamma((n+2m)/2)\Gamma(m+1)} L^m \delta, \quad (37)$$

where $2m \neq n + 2h$, $h = 0, 1, \dots$

Proof Taking by definition

$$\begin{aligned} (P \pm i0)^{-m} \cdot \delta &:= \lim_{\epsilon \rightarrow 0} (P \pm i0)^{-m} H_\epsilon \\ &= \lim_{\epsilon \rightarrow 0} (P \pm i0)^{-m} \frac{e^{\pm i q(\pi/2)} e^{i \epsilon(\pi/2)} \Gamma((n-\epsilon)/2) \sqrt{|\Delta|}}{2^\epsilon \pi^{n/2} \Gamma(\epsilon/2)} (P \pm i0)^{(\epsilon-n)/2} \\ &= \lim_{\epsilon \rightarrow 0} \frac{e^{\pm i q(\pi/2)} e^{i \epsilon \pi/2} \Gamma((n-\epsilon)/2) \sqrt{|\Delta|}}{2^\epsilon \pi^{n/2} \Gamma(\epsilon/2)} (P \pm i0)^{(\epsilon-n-2m)/2}. \end{aligned} \quad (38)$$

By multiplying and dividing the last formula by the factor

$$\frac{e^{i(-2m)(\pi/2)} \Gamma((n-\epsilon+2m)/2)}{2^{-2m} \Gamma((\epsilon-2m)/2)}, \quad (39)$$

we have

$$(P \pm i0)^{-m} \cdot \delta := \lim_{\epsilon \rightarrow 0} \frac{\Gamma((n-\epsilon)/2)}{\Gamma(\epsilon/2)} \frac{\Gamma((\epsilon-2m)/2) 2^{-2m}}{\Gamma((n-\epsilon+2m)/2) e^{i(-2m)(\pi/2)}} H_{\epsilon-2m}.$$

Taking into account (cf. p. 3, formula (4) of ref. [7])

$$\frac{\Gamma(Z)}{\Gamma(Z-m)} = (-1)^m \frac{\Gamma(-Z+m+1)}{\Gamma(-Z+1)}, \quad (40)$$

then, we obtain

$$\lim_{\epsilon \rightarrow 0} \frac{\Gamma((\epsilon - 2m)/2)}{\Gamma(\epsilon/2)} = \lim_{\epsilon \rightarrow 0} (-1)^m \frac{\Gamma((-\epsilon/2) + 1)}{\Gamma((-\epsilon + 2m + 2)/2)} = (-1)^m \frac{1}{\Gamma(m + 1)}. \quad (41)$$

Finally, from formula (41), we can conclude that

$$(P \pm i0)^{-m} \cdot \delta = \frac{(-1)^m \Gamma(n/2) 2^{-2m}}{\Gamma((n + 2m)/2) e^{i(-2m)(\pi/2)} \Gamma(m + 1)} L^m \delta. \quad (42)$$

■

THEOREM 8 *The following formula is valid*

$$(P \pm i0)^{-m} \cdot L^k \delta = \frac{(-1)^k \Gamma((n + 2k)/2) 2^{-2m}}{\Gamma((n + 2m + 2k)/2) \Gamma(k + m + 1)} L^{m+k} \delta, \quad (43)$$

where $2m + 2k \neq n + 2h$, $k, h = 0, 1, \dots$

Proof

$$\begin{aligned} & (P \pm i0)^{-m} \cdot L^k \delta \\ &:= \lim_{\epsilon \rightarrow 0} (P \pm i0)^{-m} H_{-2k+\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} (P \pm i0)^{-m} \frac{e^{\pm i q(\pi/2)} e^{i(\epsilon - 2k)(\pi/2)} \Gamma((n - \epsilon + 2k)/2) \sqrt{|\Delta|}}{2^{\epsilon - 2k} \pi^{n/2} \Gamma((\epsilon - 2k)/2)} (P \pm i0)^{(-2k + \epsilon - n)/2} \\ &= \lim_{\epsilon \rightarrow 0} \frac{e^{\pm i q(\pi/2)} e^{i(\epsilon - 2k)(\pi/2)} \Gamma((n - \epsilon + 2k)/2) \sqrt{|\Delta|}}{2^{\epsilon - 2k} \pi^{n/2} \Gamma((\epsilon - 2k)/2)} (P \pm i0)^{(-2k + \epsilon - n - 2m)/2}. \end{aligned} \quad (44)$$

By multiplying and dividing the last formula by the factor

$$\frac{e^{i(-2m)(\pi/2)} \Gamma((n - \epsilon + 2k + 2m)/2)}{2^{-2m} \Gamma((\epsilon - 2k - 2m)/2)}, \quad (45)$$

it results

$$(P \pm i0)^{-m} \cdot \delta := \lim_{\epsilon \rightarrow 0} \frac{\Gamma((n + 2k - \epsilon)/2)}{\Gamma((-2k + \epsilon)/2)} \frac{\Gamma((\epsilon - 2k - 2m)/2) 2^{-2m}}{\Gamma((n - \epsilon + 2k + 2m)/2) e^{i(-2m)(\pi/2)}} H_{\epsilon - 2k - 2m}.$$

Taking into account (cf. formula (4), p. 3 of ref. [7])

$$\frac{\Gamma(Z)}{\Gamma(Z - m)} = (-1)^m \frac{\Gamma(-Z + m + 1)}{\Gamma(-Z + 1)}, \quad (46)$$

we obtain that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\Gamma((\epsilon - 2k - 2m)/2)}{\Gamma((\epsilon - 2k)/2)} &= \lim_{\epsilon \rightarrow 0} (-1)^{k+m} \frac{\Gamma((-\epsilon/2) + 1)}{\Gamma((-\epsilon + 2m + 2k + 2)/2)} \\ &= (-1)^{m+k} \frac{1}{\Gamma(k + m + 1)}. \end{aligned} \quad (47)$$

Finally, from formula (47), we arrive at

$$(P \pm i0)^{-m} \cdot L^k \delta = \frac{(-1)^{m+k} \Gamma((n + 2k)/2) 2^{-2m}}{\Gamma((n + 2m + 2k)/2) e^{i(-2m)(\pi/2)} \Gamma(k + m + 1)} L^{m+k} \delta, \quad (48)$$

which proves Theorem 8. ■

THEOREM 9 *The following formula is valid*

$$L^h(P \pm i0)^{-m+h} \cdot L^k \delta = \frac{(4^h(-m+1) \cdots (-m+h)(-m+(n/2)) \cdots (-m+(n/2) + h - 1))(-1)^k \Gamma(n/2) 2^{-2m}}{\Gamma((n+2m)/2) \Gamma(k+m+1)} - L^{m+k} \delta, \quad (49)$$

where $2m + 2k \neq n + 2h$, $k, h, = 0, 1, \dots$

Proof Taking into account p. 272 of [3], we have

$$LP^{-m+h} = 4^h(-m+1) \cdots (-m+h) \left(-m + \frac{n}{2}\right) \cdots \left(-m + \frac{n}{2} + h - 1\right) P^\lambda \quad (50)$$

and from definition (2) is

$$L(P \pm i0)^{-m+h} = 4^h(-m+1) \cdots (-m+h) \left(-m + \frac{n}{2}\right) \cdots \left(-m + \frac{n}{2} + h - 1\right) (P \pm i0)^{-m}. \quad (51)$$

Then

$$L^h(P \pm i0)^{-m+h} \cdot L^k \delta := 4^h(-m+1) \cdots (-m+h) \left(-m + \frac{n}{2}\right) \cdots \left(-m + \frac{n}{2} + h - 1\right) (P \pm i0)^{-m} \cdot L^k \delta. \quad (52)$$

Finally, from the previous Theorem, we arrive at the formula (49). ■

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Logarithmically completely monotonic functions concerning gamma and digamma functions

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For given real numbers $a \geq 0$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$, let $F_{a,b,c}(x) = [\Gamma(x+1)]^{1/x} (1+a/x)^{x+b}/x^c$ and $\phi_{a,b,c}(x) = \psi''(x) + [2 + (b+c)x - 2x^2]/x^3 + [3a(2a-b) + (6a-b)x + 2x^2]/(x+a)^3$ with $x \in (0, \infty)$, where $\Gamma(x)$ and $\psi(x)$ are the well-known Euler gamma function and the psi or digamma function, respectively. In this article, it is revealed that the function $F_{a,b,c}(x)$ for $2a \leq 3b \leq -3c$ and its reciprocal $1/F_{a,b,c}(x)$ for $2a \leq 3b$ and $1+2b+c \geq 0$ are logarithmically completely monotonic in $(0, \infty)$, while the function $\phi_{a,b,c}(x)$ for $0 \leq 2a \leq 3b$ and $1+2b+c \geq 0$ and its negative $-\phi_{a,b,c}(x)$ for $0 \leq 2a \leq 3b$ and $b+c \leq 0$ are completely monotonic in $(0, \infty)$.

Keywords: Completely monotonic function; Logarithmically completely monotonic function; Gamma function; Polygamma function

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1. Introduction

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1)$$

for $x \in I$ and $n \geq 0$. Let $\mathcal{C}[I]$ denote the set of completely monotonic functions on I .

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k [\ln f(x)]^{(k)} \geq 0 \quad (2)$$

for $k \in \mathbb{N}$ on I . Let $\mathcal{L}[I]$ stand for the set of logarithmically completely monotonic functions on I .

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A function f is called a Stieltjes transform if it can be expressed in the form

$$f(x) = a + \int_0^{\infty} \frac{d\mu(s)}{s+x}, \quad (3)$$

where a is a nonnegative number and μ a non-negative measure on $[0, \infty)$ satisfying

$$\int_0^{\infty} \frac{1}{1+s} d\mu(s) < \infty.$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

The notion ‘logarithmically completely monotonic function’ was posed explicitly and a useful and significant relation $\mathcal{L}[I] \subset \mathcal{C}[I]$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in refs. [1–3]. Stimulated by the papers refs. [2, 3], among other things, it is proved in ref. [4] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$. For more information, please refer to refs. [4–7].

One motivation for studying completely monotonic functions is the connection between infinitely divisible probability distributions (measure) on $\mathbb{R}$ and $[0, \infty)$ and completely monotonic functions related to their Fourier and Laplace transforms (see ref. [8, p. 161]). The class of logarithmically completely monotonic functions can be characterized as the infinitely divisible completely monotonic functions, which are established by Horn in ref. [9, Theorem 4.4] and restated in ref. [4, Theorem 1.1].

The logarithmically completely monotonic functions have applications in many branches. For example, they play an important role in potential theory, probability theory, physics, theory of integral transforms, numerical and asymptotic analysis, and combinatorics. Some related references are listed in ref. [10].

It is well known that the classical Euler gamma function $\Gamma(x)$ is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \quad (4)$$

for $\operatorname{Re} z > 0$, which is one of the most important special functions [11–13] and has extensive applications in many disciplines including statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x) = [\ln \Gamma(x)]'$, is called psi or digamma function.

In ref. [2], it is proved that the function $[\Gamma(x+1)]^{1/x}/x^\alpha$ for $\alpha \geq 1$ is logarithmically completely monotonic in $(0, \infty)$. In ref. [7], the authors found that the function

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^x \quad (5)$$

is logarithmically completely monotonic in $(0, \infty)$. In fact, this conclusion can be verified simply by writing

$$\Phi(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1} \left(1 + \frac{1}{x}\right)^{x+1}, \quad (6)$$

and using the fact in ref. [14] that the function $(1+a/x)^{x+b}$ for $a > 0$ and $b \in \mathbb{R}$ is logarithmically completely monotonic in $(0, \infty)$ if and only if $2b \geq a$, and by a result in ref. [2] that the function $[\Gamma(x+1)]^{1/x}/(x+1)^\alpha$ for $\alpha \geq 1$ is logarithmically completely monotonic in $(-1, \infty)$. Motivated by refs. [2, 7], among other things, the paper [4] presented a better result that says that the functions $\Phi(x)$ and $\ln \Phi(x)$ are both Stieltjes transforms.

Define for $x \in (0, \infty)$

$$\Phi_c(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c} \left(1 + \frac{1}{x}\right)^x. \quad (7)$$

It is clear that $\Phi_1(x) = \Phi(x)$. Recently, among other things, the following necessary and sufficient conditions are established in ref. [15]: The function $\Phi_c(x)$ is logarithmically completely monotonic in $(0, \infty)$ if and only if $c \geq 1$, so is the function $1/\Phi_c(x)$ if and only if $c \leq 0$.

Define for $x \in (0, \infty)$

$$\Psi_b(x) = \frac{[\Gamma(x+1)]^{1/x}}{x} \left(1 + \frac{1}{x}\right)^{x+b}. \quad (8)$$

It is obvious that $\Psi_0(x) = \Phi_0(x) = \Phi(x)$. One of the main results in ref. [14] states that there exists a constant $b^* \geq -1$ such that the function $\Psi_b(x)$ is logarithmically completely monotonic in $(0, \infty)$ if $b \geq b^*$.

For given real numbers $a \geq 0$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$, define, as in ref. [7],

$$F_{a,b,c}(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c} \left(1 + \frac{a}{x}\right)^{x+b} \quad (9)$$

for $x \in (0, \infty)$. It is apparent that $F_{1,0,1}(x) = \Phi(x)$, $F_{1,0,c}(x) = \Phi_c(x)$, and $F_{1,b,1}(x) = \Psi_b(x)$. At the end of ref. [10], the following open problem was posed: Under what conditions on a , b , and c is the function $F_{a,b,c}(x)$ defined by equation (9) completely monotonic or a Stieltjes transform in $(0, \infty)$?

The purpose of this article is to give sufficient conditions such that this open problem is partially affirmative. Our main results are as follows.

THEOREM 1 *The functions*

$$\begin{cases} F_{a,b,c}(x) & \text{for } 2a \leq 3b \leq -3c \\ \frac{1}{F_{a,b,c}(x)} & \text{for } 2a \leq 3b \text{ and } 1 + 2b + c \geq 0 \end{cases}$$

are logarithmically completely monotonic in $(0, \infty)$.

As a direct consequence of the proof of Theorem 1, we have the following

COROLLARY 1 *The function*

$$\phi_{a,b,c}(x) = \psi''(x) + \frac{2 + (b+c)x - 2x^2}{x^3} + \frac{3a(2a-b) + (6a-b)x + 2x^2}{(x+a)^3} \quad (10)$$

is completely monotonic in $(0, \infty)$ if $0 \leq 2a \leq 3b$ and $1 + 2b + c \geq 0$, and its negative $-\phi_{a,b,c}(x)$ is also completely monotonic in $(0, \infty)$ if $2a \leq 3b \leq -3c$.

2. Lemmas

In order to prove our main result, the following lemmas are necessary.

LEMMA 1 ([11–13]) For $x > 0$ and $r > 0$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt. \quad (11)$$

LEMMA 2 ([11–13]) The polygamma functions $\psi^{(k)}(x)$ can be expressed for $x > 0$ and $k \in \mathbb{N}$ as

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} dt, \quad (12)$$

where $\gamma = 0.57721566\dots$ is the Euler–Mascheroni constant.

For $i \in \mathbb{N}$,

$$\psi^{(i-1)}(x+1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i}. \quad (13)$$

LEMMA 3 ([11–13]) For all large x ,

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{\ln(2\pi)}{2} + \frac{1}{12x} + O\left(\frac{1}{x}\right), \quad (14)$$

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O\left(\frac{1}{x^2}\right), \quad (15)$$

$$(-1)^{n+1} \psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + \frac{(n+1)!}{12x^{n+2}} + O\left(\frac{1}{x^{n+2}}\right). \quad (16)$$

3. Proof of Theorem 1

Without loss of generality, in what follows let us set $a > 0$.

Taking the logarithm of the function $F_{a,b,c}(x)$ defined in equation (9) and differentiating yields

$$\ln F_{a,b,c}(x) = (x+b) \ln \left(1 + \frac{a}{x}\right) + \frac{\ln \Gamma(x+1)}{x} - c \ln x, \quad (17)$$

$$[\ln F_{a,b,c}(x)]' = \ln \left(1 + \frac{a}{x}\right) - \frac{a(x+b)}{x(x+a)} + \frac{x\psi(x+1) - \ln \Gamma(x+1)}{x^2} - \frac{c}{x}, \quad (18)$$

and

$$\begin{aligned} [\ln F_{a,b,c}(x)]^{(n)} &= (-1)^{n-1}(n-1)!(x+b) \left[\frac{1}{(x+a)^n} - \frac{1}{x^n} \right] \\ &\quad + (-1)^n(n-2)! \ln \left[\frac{1}{(x+a)^{n-1}} - \frac{1}{x^{n-1}} \right] + \frac{h_n(x)}{x^{n+1}} + (-1)^n(n-1)! \frac{c}{x^n} \\ &= (-1)^n(n-2)! \left[\frac{(n-1)(b+c) - x}{x^n} + \frac{x+na - (n-1)b}{(x+a)^n} \right] + \frac{h_n(x)}{x^{n+1}}, \end{aligned} \quad (19)$$

where $n \geq 2$, $\psi^{(-1)}(x+1) = \ln \Gamma(x+1)$, $\psi^{(0)}(x+1) = \psi(x+1)$, and

$$h_n(x) = \sum_{k=0}^n \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!}, \quad (20)$$

$$h'_n(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0, & \text{if } n \text{ is odd,} \\ < 0, & \text{if } n \text{ is even.} \end{cases} \quad (21)$$

Therefore, we have

$$\begin{aligned} & (-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)} + (-1)^{n+1} h_n(x) \\ &= (n-2)! \left\{ (n-1)(b+c) - x + \frac{x^n [x + na - (n-1)b]}{(x+a)^n} \right\} x \end{aligned} \quad (22)$$

and

$$\begin{aligned} & \frac{d\{(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)}\}}{dx} \\ &= (-1)^n x^n \psi^{(n)}(x+1) + (n-2)! \left\{ (n-1)(b+c) - 2x \right. \\ & \quad \left. + \frac{x^n [a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2]}{(x+a)^{n+1}} \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x+1) + (n-2)! \left[\frac{(n-1)(b+c) - 2x}{x^n} \right. \right. \\ & \quad \left. \left. + \frac{a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2}{(x+a)^{n+1}} \right] \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x) + \frac{n!}{x^{n+1}} + (n-2)! \left[\frac{(n-1)(b+c) - 2x}{x^n} \right. \right. \\ & \quad \left. \left. + \frac{a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2}{(x+a)^{n+1}} \right] \right\} \\ &= x^n \left\{ (-1)^n \psi^{(n)}(x) + (n-2)! \left[\frac{n(n-1) + (n-1)(b+c)x - 2x^2}{x^{n+1}} \right. \right. \\ & \quad \left. \left. + \frac{a(b+an+an^2-bn^2) + (2a+b+2an-bn)x + 2x^2}{(x+a)^{n+1}} \right] \right\} \\ &\triangleq x^n \{(-1)^n \psi^{(n)}(x) + (n-2)! [g_n(x) + h_n(x)]\}. \end{aligned}$$

By induction, it follows that

$$g'_n(x) = -(n-1)g_{n+1}(x) \quad \text{and} \quad h'_n(x) = -(n-1)h_{n+1}(x). \quad (23)$$

This implies

$$g_2^{(n-2)}(x) = (-1)^n (n-2)! g_n(x) \quad (24)$$

and

$$h_2^{(n-2)}(x) = (-1)^n (n-2)! h_n(x), \quad (25)$$

therefore,

$$\frac{d\{(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)}\}}{dx} = (-1)^n x^n [\psi''(x) + g_2(x) + h_2(x)]^{(n-2)}. \quad (26)$$

Utilizing equations (11) and (12) and straightforward calculating yields

$$\begin{aligned} \phi_{a,b,c}(x) &\triangleq \psi''(x) + g_2(x) + h_2(x) \\ &= \psi''(x) + \frac{2 + (b+c)x - 2x^2}{x^3} + \frac{3a(2a-b) + (6a-b)x + 2x^2}{(x+a)^3} \\ &\quad [2a^3 + (6a^2 + a^3b + a^3c)x + (6a - 2a^3 + 3a^2b + 3a^2c)x^2 \\ &\quad + (2 + 3ac)x^3 + cx^4] \\ &= \psi''(x) + \frac{2a^3 + (6a^2 + a^3b + a^3c)x + (6a - 2a^3 + 3a^2b + 3a^2c)x^2 \\ &\quad + (2 + 3ac)x^3 + cx^4}{x^3(x+a)^3} \\ &= \psi''(x) + \frac{2}{x^3} + \frac{b+c}{x^2} - \frac{2}{x} + \frac{2a(a-b)}{(x+a)^3} + \frac{2a-b}{(x+a)^2} + \frac{2}{x+a} \\ &= -\int_0^\infty \frac{t^2 e^{-xt}}{1-e^{-t}} dt + \int_0^\infty t^2 e^{-xt} dt + (b+c) \int_0^\infty t e^{-xt} dt \\ &\quad - 2 \int_0^\infty e^{-xt} dt + a(a-b) \int_0^\infty t^2 e^{-(x+a)t} dt \\ &\quad + (2a-b) \int_0^\infty t e^{-(x+a)t} dt + 2 \int_0^\infty e^{-(x+a)t} dt \\ &= \int_0^\infty \{[a(b-a)t^2 + (b-2a)t - 2] + [(b+c)t - 2]e^{(a+1)t} \\ &\quad + [2 - (b+c)t - t^2]e^{at} + [2 + (2a-b)t + a(a-b)t^2]e^t\} \frac{e^{-(x+a)t}}{e^t - 1} dt \\ &\triangleq \int_0^\infty \{\mathcal{F}(t) + \mathcal{G}(t)e^{(a+1)t} - [\mathcal{G}(t) + t^2]e^{at} - \mathcal{F}(t)e^t\} \frac{e^{-(x+a)t}}{e^t - 1} dt \\ &= \int_0^\infty \{e^{at}(e^t - 1)\mathcal{G}(t) + (1 - e^t)\mathcal{F}(t) - t^2 e^{at}\} \frac{e^{-(x+a)t}}{e^t - 1} dt \\ &= \int_0^\infty \left[\mathcal{G}(t) - \frac{\mathcal{F}(t)}{e^{at}} - \frac{t^2}{e^t - 1} \right] e^{-xt} dt \\ &= \int_0^\infty \left[b + c - \frac{\mathcal{F}(t)}{te^{at}} - \frac{t}{e^t - 1} - \frac{2}{t} \right] te^{-xt} dt \\ &= \int_0^\infty \left[b + c - \frac{\mathcal{F}(t) + 2e^{at}}{te^{at}} - \frac{t}{e^t - 1} \right] te^{-xt} dt \\ &= \int_0^\infty \left[b + c - \frac{a(b-a)t^2 + (b-2a)t + 2(e^{at} - 1)}{te^{at}} - \frac{t}{e^t - 1} \right] te^{-xt} dt \\ &\triangleq \int_0^\infty \left[b + c - \mathcal{H}_{a,b}(t) - \frac{t}{e^t - 1} \right] te^{-xt} dt \\ &\triangleq \int_0^\infty \mathcal{T}_{a,b,c}(t) te^{-xt} dt. \end{aligned}$$

It is easy to see that the function $t/(e^t - 1)$ is strictly decreasing in $(0, \infty)$ and, by using L'Hospital's rule,

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{t}{e^t - 1} &= 1, & \lim_{t \rightarrow \infty} \frac{t}{e^t - 1} &= 0, \\ \lim_{t \rightarrow 0} \mathcal{H}_{a,b}(t) &= b, & \lim_{t \rightarrow \infty} \mathcal{H}_{a,b}(t) &= 0.\end{aligned}$$

Direct computation gives

$$\begin{aligned}\frac{d\mathcal{H}_{a,b}(t)}{dt} &= \frac{2 + 2at + a^2t^2 + a^2(a-b)t^3 - 2e^{at}}{t^2e^{at}} \\ &= \frac{a[a(2 + 2u + u^2 + u^3 - 2e^u) - bu^3]}{u^2e^u} \\ &= \frac{a^2u}{e^u} \left[\frac{2 + 2u + u^2 - 2e^u}{u^3} + 1 - \frac{b}{a} \right] \\ &\triangleq \frac{a^2u}{e^u} [\mathcal{P}(u) + 1 - \frac{b}{a}],\end{aligned}$$

where $u = at \in (0, \infty)$. It is not difficult to prove that the function $\mathcal{P}(u)$ is strictly decreasing in $u \in (0, \infty)$ and

$$\lim_{u \rightarrow 0} \mathcal{P}(u) = -\frac{1}{3}, \quad \lim_{u \rightarrow \infty} \mathcal{P}(u) = -\infty.$$

When $1 - b/a \leq 1/3$, it is deduced that $(d/dt)\mathcal{H}_{a,b}(t) \leq 0$ and $\mathcal{H}_{a,b}(t)$ is strictly decreasing in $t \in (0, \infty)$. In conclusion, if $1 - b/a \leq 1/3$, then the function $\mathcal{T}_{a,b,c}(t)$ is strictly increasing in $t \in (0, \infty)$ and $1 + 2b + c < \mathcal{T}_{a,b,c}(t) < b + c$. Hence, when $2a \leq 3b$ and $1 + 2b + c \geq 0$, the function $\phi_{a,b,c}(x)$ is completely monotonic in $(0, \infty)$; when $2a \leq 3b$ and $b + c \leq 0$, the function $-\phi_{a,b,c}(x)$ is completely monotonic in $(0, \infty)$, since

$$\phi_{a,b,c}^{(i)} = (-1)^i \int_0^\infty \mathcal{T}_{a,b,c}(t) t^{i+1} e^{-xt} dt, \quad i \in \mathbb{N}. \quad (27)$$

Combining equations (26) and (27) shows that for $n \geq 2$

$$\frac{d\{(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)}\}}{dx} = x^n \int_0^\infty \mathcal{T}_{a,b,c}(t) t^{n-1} e^{-xt} dt,$$

thus, the function $(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)}$ is

$$\begin{cases} \text{decreasing} & \text{for } 2a \leq 3b \text{ and } b + c \leq 0 \\ \text{increasing} & \text{for } 2a \leq 3b \text{ and } 1 + 2b + c \geq 0 \end{cases}$$

in $(0, \infty)$. From

$$\lim_{x \rightarrow 0} \{(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)}\} = 0$$

by direct calculation, it follows for $n \geq 2$ that

$$(-1)^n x^{n+1} [\ln F_{a,b,c}(x)]^{(n)} \begin{cases} \geq 0 & \text{for } 2a \leq 3b \text{ and } b + c \leq 0 \\ \leq 0 & \text{for } 2a \leq 3b \text{ and } 1 + 2b + c \geq 0 \end{cases}$$

and

$$(-1)^n [\ln F_{a,b,c}(x)]^{(n)} \begin{cases} \geq 0 & \text{for } 2a \leq 3b \text{ and } b+c \leq 0 \\ \leq 0 & \text{for } 2a \leq 3b \text{ and } 1+2b+c \geq 0 \end{cases} \quad (28)$$

in $(0, \infty)$. This implies the function $[\ln F_{a,b,c}(x)]'$ is

$$\begin{cases} \text{increasing} & \text{for } 2a \leq 3b \text{ and } b+c \leq 0 \\ \text{decreasing} & \text{for } 2a \leq 3b \text{ and } 1+2b+c \geq 0 \end{cases}$$

in $(0, \infty)$. It is easy to obtain $\lim_{x \rightarrow \infty} [\ln F_{a,b,c}(x)]' = 0$ by using Lemma 3, so

$$[\ln F_{a,b,c}(x)]' \begin{cases} \leq 0 & \text{for } 2a \leq 3b \text{ and } b+c \leq 0 \\ \geq 0 & \text{for } 2a \leq 3b \text{ and } 1+2b+c \geq 0 \end{cases}$$

in $(0, \infty)$. This means that equation (28) holds in $(0, \infty)$ for all natural numbers $n \in \mathbb{N}$, that is, the functions

$$\begin{cases} F_{a,b,c}(x) & \text{for } 2a \leq 3b \leq -3c \\ \frac{1}{F_{a,b,c}(x)} & \text{for } 2a \leq 3b \text{ and } 1+2b+c \geq 0 \end{cases}$$

are logarithmically completely monotonic in $(0, \infty)$. The proof is complete.

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Corrigendum to Fourier–Feynman Transforms and Wiener Integrals in the Fresnel Class on Abstract Wiener Spaces, Vol. 16, No. 3, April 2005, 255–261

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In the paper listed above, the author established some relationships between Fourier–Feynman transforms and Wiener integrals for functions in the Fresnel class of the form $F(x) = \int_H \exp\{i(h, x)^\sim\} d\mu(h)$, $\mu \in \mathcal{M}(H)$ on the abstract Wiener space. But the author has found some mistakes in Theorems 1–3. For example, the statement that the function, $\exp\{(1-z)/2|h|^2[(h, x)^\sim]^2\} \cdot F(x+y)$ is a Wiener integrable function of $x \in B$ in Theorem 1 is incorrect, as the function $\exp\{(1-z)/2|h|^2[(h, x)^\sim]^2\}$ is not independent from the function $F(x) = \int_H \exp\{i(h, x)^\sim\} d\mu(h)$. So, those theorems in the paper should be corrected as follows:

THEOREM 1 *Let (B, H, m) be an abstract Wiener space and let F be functions in the Fresnel class $\mathcal{F}(B)$. Then for every $z \in \mathbf{C}_+$ and for $s - a.e. y \in B$, $\int_H \exp\{(1-z)/2|h|^2[(h, x)^\sim]^2\} \cdot \exp\{i(h, x+y)^\sim\} d\mu(h)$ is a Wiener integrable function of $x \in B$.*

Proof By Wiener integration formula, we have that for every $z \in \mathbf{C}_+$ and for $s - a.e. y \in B$,

$$\begin{aligned} & \int_B \left[\int_H \exp \left\{ \frac{(1-z)}{2|h|^2} [(h, x)^\sim]^2 \right\} \cdot \exp\{i(h, x+y)^\sim\} d\mu(h) \right] dm(x) \\ &= \int_H \int_B \exp \left\{ \frac{(1-z)}{2|h|^2} [(h, x)^\sim]^2 \right\} \cdot \exp\{i(h, x)^\sim + i(h, y)^\sim\} dm(x) d\mu(h) \\ &= \int_H \exp\{i(h, y)^\sim\} \left[\int_B \exp \left\{ \frac{(1-z)}{2|h|^2} [(h, x)^\sim]^2 + i(h, x)^\sim \right\} dm(x) \right] d\mu(h) \end{aligned}$$

$$\begin{aligned}
&= \int_H \exp\{i(h, y)^\sim\} \cdot \left[(2\pi|h|^2)^{-(1/2)} \right. \\
&\quad \cdot \left. \int_{-\infty}^{\infty} \exp\left\{\frac{(1-z)}{2|h|^2}u^2 + iu\right\} \cdot \exp\left\{-\frac{1}{2|h|^2}u^2\right\} du \right] d\mu(h) \\
&= \int_H \exp\{i(h, y)^\sim\} \cdot \left[(2\pi|h|^2)^{-(1/2)} \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{z}{2|h|^2}u^2 + iu\right\} du \right] d\mu(h) \\
&= \int_H \exp\{i(h, y)^\sim\} \cdot [(2\pi|h|^2)^{-(1/2)} \cdot \sqrt{\frac{2\pi|h|^2}{z}} \cdot \exp\left\{-\frac{|h|^2}{2z}\right\}] d\mu(h) \\
&= \frac{1}{\sqrt{z}} \int_H \exp\left\{-\frac{|h|^2}{2z} + i(h, y)^\sim\right\} d\mu(h). \tag{1}
\end{aligned}$$

We can easily show that the last integral has a finite value for every $z \in \mathbf{C}_+$ and for $s - a.e.$ $y \in B$. Thus the proof of this theorem is complete. ■

THEOREM 2 *Let (B, H, m) be an abstract Wiener space and let F be functions in the Fresnel class $\mathcal{F}(B)$. Then for every $z \in \mathbf{C}_+$ and for $s - a.e.$ $y \in B$,*

$$\begin{aligned}
&I^{\text{aw}}(F(\cdot + y) : z) \\
&= \sqrt{z} \int_B \left[\int_H \exp\left\{\frac{(1-z)}{2|h|^2}[(h, x)^\sim]^2\right\} \cdot \exp\{i(h, x + y)^\sim\} d\mu(h) \right] dm(x). \tag{2}
\end{aligned}$$

Proof First we know that the analytic Wiener integral $I^{\text{aw}}(F(\cdot + y)) : z$ of functions F in the Fresnel class $\mathcal{F}(B)$ is given by $I^{\text{aw}}(F(\cdot + y)) : z = \int_B F(z^{-(1/2)}x + y) dm(x) = \int_H \exp\{-(|h|^2/2z) + i(h, y)^\sim\} d\mu(h)$, for every $z \in \mathbf{C}_+$ and for $s - a.e.$ $y \in B$. Using the first and last equation in the proof of Theorem 1, we have that for every $z \in \mathbf{C}_+$ and for $s - a.e.$ $y \in B$,

$$\begin{aligned}
&\int_B \left[\int_H \exp\left\{\frac{(1-z)}{2|h|^2}[(h, x)^\sim]^2\right\} \cdot \exp\{i(h, x + y)^\sim\} d\mu(h) \right] dm(x) \\
&= \frac{1}{\sqrt{z}} \int_H \exp\left\{-\frac{|h|^2}{2z} + i(h, y)^\sim\right\} d\mu(h) \\
&= \frac{1}{\sqrt{z}} I^{\text{aw}}(F(\cdot + y) : z). \tag{3}
\end{aligned}$$

Therefore, we have the desired result. ■

THEOREM 3 *Let (B, H, m) be an abstract Wiener space and let $\{z_k\}_{k=1}^{\infty}$ be a sequence of complex numbers from $\mathbf{C}_+$ such that z_k approaches to $-iq$ through $\mathbf{C}_+$, where q is a non-zero real number and $i^2 = -1$. Then an L_1 -analytic Fourier-Feynman transform of functions F in the Fresnel class $\mathcal{F}(B)$ is expressed as the limit of Wiener integrals:*

$$\begin{aligned}
&(T_q^{(1)}(F))(y) \\
&= \lim_{k \rightarrow \infty} \sqrt{z_k} \int_B \left[\int_H \exp\left\{\frac{(1-z_k)}{2|h|^2}[(h, x)^\sim]^2\right\} \cdot \exp\{i(h, x + y)^\sim\} d\mu(h) \right] dm(x), \tag{4}
\end{aligned}$$

for $s - a.e.$ $y \in B$, whenever $z_k \rightarrow -iq$ through $\mathbf{C}_+$.

Proof We know that for $s - a.e. y \in B$, an L_1 -analytic Fourier–Feynman transform of functions F in the Fresnel class $\mathcal{F}(B)$ is given by $(T_q^{(1)}(F))(y) = \lim_{k \rightarrow \infty} I^{\text{aw}}(F(\cdot + y) : z_k) = \lim_{k \rightarrow \infty} \int_H \exp\{-(|h|^2/2z_k) + i(h, y)^\sim\} d\mu(h) = \int_H \exp\{-(i/2q)|h|^2 + i(h, y)^\sim\} d\mu(h)$, whenever $z_k \rightarrow -iq$ through $\mathbf{C}_+$. Using the first and last equations in the proof of Theorem 1, we have that for $s - a.e. y \in B$,

$$\begin{aligned} & \lim_{k \rightarrow \infty} \sqrt{z_k} \cdot \int_B \left[\int_H \exp \left\{ \frac{(1-z)}{2|h|^2} [(h, x)^\sim]^2 \right\} \cdot \exp\{i(h, x+y)^\sim\} d\mu(h) \right] dm(x) \\ &= \lim_{k \rightarrow \infty} \sqrt{z_k} \cdot \left[\frac{1}{\sqrt{z_k}} \int_H \exp \left\{ -\frac{|h|^2}{2z_k} + i(h, y)^\sim \right\} d\mu(h) \right] \\ &= \lim_{k \rightarrow \infty} \int_H \exp \left\{ -\frac{|h|^2}{2z_k} + i(h, y)^\sim \right\} d\mu(h) \\ &= \int_H \exp \left\{ -\frac{i}{2q} |h|^2 + i(h, y)^\sim \right\} d\mu(h) \\ &= (T_q^{(1)}(F))(y), \end{aligned} \tag{5}$$

whenever $z_k \rightarrow -iq$ through $\mathbf{C}_+$. ■

Remark In Corollary 5, 7 and Corollary 11 in the paper, 2007, ‘Far East Journal on Mathematics and Science, **25**(1), 93–110’, the author finds rather nice relationships without using the double integral.

Monomiality and multi-index multi-variable special polynomials

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In this paper, we will use the operational recipes from monomiality and umbral calculus to deal with multi-dimensional polynomials.

Keywords: Monomiality principle; Hermite polynomials; Multi-variable; Multi-index polynomials

MSC (2000): 33C45; 33C47; 33C50

1. Introduction

The operational rule

$$\exp(\tau \partial_{x,y}) \{x^m y^n\} = h_{m,n}(x, y|\tau). \quad (1)$$

defines the incomplete 2D Hermite polynomials (I2DHP) $h_{m,n}(x, y|\tau)$ [1] (see also [2, 3]):

$$h_{m,n}(x, y|\tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s x^{m-s} y^{n-s}}{s!(m-s)!(n-s)!}. \quad (2)$$

In equation (1), we have denoted by $\partial_{x,y}$ the derivative $(\partial^2/\partial x \partial y)$ and from now on we will assume that $\partial_{x,y} = \partial_{y,x}$.

The recurrence relations of this polynomial family can easily be derived using the formalism underlying the monomiality principle [4] by defining a set of operators, $\hat{M}_x, \hat{M}_y, \hat{P}_x, \hat{P}_y$ whose

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action on $h_{m,n}(x, y|\tau)$ is specified by

$$\begin{aligned}\hat{M}_x h_{m,n}(x, y|\tau) &= h_{m+1,n}(x, y|\tau), & \hat{M}_y h_{m,n}(x, y|\tau) &= h_{m,n+1}(x, y|\tau), \\ \hat{P}_x h_{m,n}(x, y|\tau) &= m h_{m-1,n}(x, y|\tau), & \hat{P}_y h_{m,n}(x, y|\tau) &= n h_{m,n-1}(x, y|\tau).\end{aligned}\quad (3)$$

According to the language of monomiality these operators act as multiplicative and derivative operators respectively, with the further conditions

$$[\hat{P}_\alpha, \hat{M}_\beta] = \delta_{\alpha,\beta}, \quad [\hat{P}_\alpha, \hat{P}_\beta] = [\hat{M}_\alpha, \hat{M}_\beta] = 0, \quad ([\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}). \quad (4)$$

We can derive the explicit form of the above operators by noting that

$$\begin{aligned}\exp(\tau \partial_{x,y})[x\{x^m y^n\}] &= h_{m+1,n}(x, y|\tau), \\ \exp(\tau \partial_{x,y})[\{x^m y^n\}y] &= h_{m,n+1}(x, y|\tau).\end{aligned}\quad (5)$$

The multiplicative operators $\hat{M}$ are therefore defined as

$$\begin{aligned}\hat{M}_x &= \exp(\tau \partial_{x,y})x \exp(-\tau \partial_{x,y}), \\ \hat{M}_y &= \exp(\tau \partial_{x,y})y \exp(-\tau \partial_{x,y}).\end{aligned}\quad (6)$$

The use of the Hausdorff identity [5]

$$\exp(\xi \hat{A})\hat{B} \exp(-\xi \hat{A}) = \hat{B} + \xi[\hat{A}, \hat{B}] + \frac{\xi^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{\xi^3}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \quad (7)$$

and the fact that

$$[\partial_{x,y}, x] = \partial_y, \quad [\partial_{x,y}, y] = \partial_x \quad (8)$$

leads to the following identification

$$\hat{M}_x = x + \tau \partial_y, \quad \hat{M}_y = y + \tau \partial_x, \quad (9)$$

which are easily shown to be commuting quantities. As to the derivative operator we can use an analogous argument thus getting

$$\exp(\tau \partial_{x,y})[\partial_x \{x^m y^n\}] = m h_{m-1,n}(x, y|\tau), \quad (10)$$

which yields

$$\hat{P}_x = \exp(\tau \partial_{x,y})\partial_x \exp(-\tau \partial_{x,y}) = \partial_x, \quad (11)$$

as a consequence of the fact that

$$[\partial_{x,y}, \partial_x] = 0. \quad (12)$$

The derivative operators $\hat{P}_x$ as well as $\hat{P}_y$ can therefore be identified with the ordinary derivative as it happens for the classical polynomials belonging to the Appell family [6].

According to the previous example, we have a simple mean to ‘generate’ families of special polynomials and to study their properties.

The I2DHP $h_{m,n}(x, y|\tau)$ can *e.g.* be defined according to the further operational rule

$$h_{m,n}(x, y|\tau) = (x + \tau \partial_y)^m (y + \tau \partial_x)^n \{1\}, \quad (13)$$

in which the multiplicative operators are assumed to act on the ‘vacuum’, we will neglect ‘1’ in the following for the sake of conciseness. More in general the use of the operational identities can also be exploited to state the following generalized Burchnell identity:

$$\hat{M}_x^m \hat{M}_y^n = m!n! \sum_{p,q=0}^{\min(m,n)} \frac{\tau^{p+q}}{p!q!(m-q)!(n-p)!} h_{m-q,n-p}(x, y|\tau) \partial_{px, qy} \quad (14)$$

and the relevant generating function

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} h_{m+n,p}(x, y|\tau) = \exp(xu + yv + \tau uv) h_{q,p}(x + \tau v, y + \tau u|\tau). \quad (15)$$

In this paper, we will enter more deeply into the theory of multi-dimensional polynomials and we will see that the use of operational ‘recipes’ from monomiality and umbral calculus can be a fairly powerful tool to deal with these families of polynomials.

2. Appell type mixed polynomials

Before getting into the specific body of the problem, let us consider as a further example, the polynomials

$$h_{m,n}^{(p,q)}(x, y|\tau) = \exp(\tau \partial_{px, qy}) \{x^m y^n\} = m!n! \sum_{r=0}^{\min([m/p], [n/q])} \frac{\tau^r x^{m-pr} y^{n-qr}}{r!(m-pr)!(n-qr)!}, \quad (16)$$

whose multiplicative operators are easily shown to be

$$\hat{M}'_x = x + p\tau \partial_{(p-1)x, qy}, \quad \hat{M}'_y = y + q\tau \partial_{px, (q-1)y}, \quad (17)$$

while the derivative operators are the same as in the ordinary case. The commutativity between the operators (x and y counterparts) is easily shown to hold. The other properties which the I2DHP $h_{m,n}(x, y|\tau)$ satisfy also hold for $h_{m,n}^{(p,q)}(x, y|\tau)$.

Let us now consider the problem in more general terms and ask what kind of polynomials are defined through the multiplicative and derivative operators

$$\begin{aligned} \hat{M} &= x + f(\partial_x) \\ \hat{P} &= \partial_x. \end{aligned} \quad (18)$$

It is easily shown that the associated family of polynomials can be defined through the operational identity

$$a_n(x) = A(\partial_x) \{x^n\}, \quad A(\eta) = \exp \left(\int_0^\eta f(\xi) d\xi \right). \quad (19)$$

These polynomials belong to the Appell family. To provide an example, we recall that the Bernoulli polynomials $B_n(x)$ [7] are defined by the generating function

$$\frac{t \exp(xt)}{\exp(t) - 1} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(x). \quad (20)$$

It is therefore, clear that for Bernoulli polynomials $B_n(x)$, we have

$$A(\eta) = \frac{\eta}{\exp(\eta) - 1}. \quad (21)$$

Also, we note that for the truncated exponential polynomials $e_n(x)$ [8], defined by the generating function

$$\frac{\exp(xt)}{1-t} = \sum_{n=0}^{\infty} t^n e_n(x), \quad (22)$$

we have

$$A(\eta) = \frac{1}{1-\eta}. \quad (23)$$

In the following, we will discuss families of multi-dimensional Appell polynomials.

An example is provided by

$$\begin{aligned} a_{m,n}(x, y|\tau) &= A(\partial_x)B(\partial_y)C(\tau\partial_{x,y})\{x^m y^n\} = C(\tau\partial_{x,y})\{a_m(x)b_n(y)\} \\ &= m!n! \sum_{s=0}^{\min(m,n)} \frac{C_s \tau^s}{s!(m-s)!(n-s)!} a_{m-s}(x)b_{n-s}(y), \end{aligned} \quad (24)$$

which in umbral form [9] can be written as

$$h_{m,n}([a], [b]|\tau_C). \quad (25)$$

We can easily prove that the associated multiplicative operators write

$$\hat{M}_x = x + \frac{A'}{A} + \frac{C'}{C}\partial_y, \quad \hat{M}_y = y + \frac{B'}{B} + \frac{C'}{C}\partial_x, \quad (26)$$

while the derivative operators are defined in terms of the ordinary derivatives. The commutativity of the multiplicative operators is ensured by the fact that

$$\left[x, \frac{C'}{C}\partial_x \right] = \left[y, \frac{C'}{C}\partial_y \right]. \quad (27)$$

Just to give an example we note that for

$$A(t) = \exp(wt^2), \quad B(t) = \exp(zt^2), \quad C(t) = \exp(\tau t), \quad (28)$$

we obtain

$$\begin{aligned} \exp(w\partial_x^2)\exp(z\partial_y^2)\exp(\tau\partial_{x,y})\{x^m y^n\} &= H_{m,n}(x, w; y, z|\tau) \\ &= m!n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r}{r!(m-r)!(n-r)!} H_{m-r}(x, w)H_{n-r}(y, z), \end{aligned} \quad (29)$$

where $H_{m,n}(x, w; y, z|\tau)$ are the 4-variable 2-index Hermite polynomials (4V2IHP) [4], which are defined in terms of the Hermite–Kampé de Fériet polynomials $H_n(x, y)$ [6].

The relevant multiplicative operators can be written as:

$$\hat{M}_x'' = x + 2w\partial_x + \tau\partial_y, \quad \hat{M}_y'' = y + 2z\partial_y + \tau\partial_x. \quad (30)$$

As a further example, we consider the mixed truncated exponential polynomials for which we have

$$\begin{aligned} e_{m,n}(x, y|\tau) &= \frac{1}{1 - \partial_x} \frac{1}{1 - \partial_y} \frac{1}{1 - \tau\partial_{x,y}} \{x^m y^n\} = \frac{1}{1 - \tau\partial_{x,y}} e_m(x) e_n(y) \\ &= \sum_{r=0}^{\min(m,n)} \tau^r \partial_{rx, ry} e_m(x) e_n(y) = \sum_{r=0}^{\min(m,n)} \tau^r e_{m-r}(x) e_{n-r}(y) \end{aligned} \quad (31)$$

and the relevant multiplicative operators write

$$\hat{M}_x''' = x + \frac{1}{1 - \partial_x} - \tau \frac{\partial_y}{1 - \tau\partial_{x,y}}, \quad \hat{M}_y''' = y + \frac{1}{1 - \partial_y} - \tau \frac{\partial_x}{1 - \tau\partial_{x,y}}. \quad (32)$$

Before concluding this section we want to discuss a further example, which is rather artificial but effective for our purposes and consider the following family of polynomials

$${}_B e_{m,n}(x, y|\tau) = \frac{1}{1 - \partial_x} \frac{\partial_y}{\exp(\partial_y) - 1} \exp(\tau\partial_{x,y}) \{x^m y^n\}, \quad (33)$$

which are Hermite–Bernoulli truncated polynomials (HBTP) and write

$${}_B e_{m,n}(x, y|\tau) = n! \sum_{r=0}^{\min(m,n)} \frac{\tau^r}{r!(n-r)!} e_{m-r}(x) B_{n-r}(y). \quad (34)$$

The relevant multiplicative operators are

$$\hat{M}_x^{iv} = x + \frac{1}{1 - \partial_x} + \tau\partial_y, \quad \hat{M}_y^{iv} = y + \frac{\exp(\partial_y)(1 - \partial_y) - 1}{\partial_y(\exp(\partial_y) - 1)} + \tau\partial_x. \quad (35)$$

It is also evident that the HBTP ${}_B e_{m,n}(x, y|\tau)$ can be written in different forms, indeed, as a consequence of the fact that the various operators, appearing in the r.h.s. of equation (33) commute, we can write the HBTP ${}_B e_{m,n}(x, y|\tau)$ as:

$${}_B e_{m,n}(x, y|\tau) = \sum_{r=0}^m \sum_{p=0}^n \frac{m!n!B_p}{(m-r)!(n-p)!p!} h_{m-r, n-p}(x, y|\tau), \quad (36)$$

where B_p denotes the p -th Bernoulli number [7].

3. Monomiality, Sheffer polynomials and multi-index families

The Appell polynomials are characterized by the fact that the relevant derivative operator is just the ordinary derivative. Among the polynomials encountered in Quantum Mechanics, Hermite and Laguerre polynomials are of Sheffer-type [10]. The Laguerre polynomials unlike

the Hermite family can not be classified as Appell. We note indeed that for the 2-variable Laguerre polynomials (2VLP) $L_n(x, y)$ [4], the multiplicative and derivative operators write

$$\hat{M} = y - \hat{D}_x^{-1}, \quad \hat{P} = -\partial_x x \partial_x = -\frac{\partial}{\partial \hat{D}_x^{-1}} = {}_L \hat{D}_x. \quad (37)$$

In particular, the operator $\hat{D}_x^{-1}$ is the inverse of the ordinary derivative and is defined in such a way that

$$\hat{D}_x^{-n} \{1\} = \frac{x^n}{n!} \quad (n \in \mathbf{N}_0 := \mathbf{N} \cup \{0\}). \quad (38)$$

Albeit the Laguerre polynomials are not Appell we can just use the previous formalism to get a complete analogy with the Appell family.

We can use indeed the operators (37), which satisfy a Weyl algebra to write the 2VLP in the form

$$L_n(x, y) = (y - \hat{D}_x^{-1})^n = n! \sum_{r=0}^n \frac{(-1)^r x^r y^{n-r}}{(n-r)!(r!)^2}, \quad (39)$$

so that within this framework the 2VLP are essentially the ordinary monomials generated by

$$G(t, x, y) = \exp(yt) \exp(-t \hat{D}_x^{-1}). \quad (40)$$

Along with 2VLP one can introduce the 2-variable associated Laguerre polynomials (2VALP) $L_n^{(\alpha)}(x, y)$ [11], as follows:

$$L_n^{(\alpha)}(x, y) = n! \sum_{r=0}^n \frac{(-1)^r x^r y^{n-r}}{r!(n-r)!\Gamma(r+\alpha+1)}, \quad (41)$$

for which the relevant derivative operator writes:

$${}_L \hat{D}_x^{(\alpha)} = -\partial_x x \partial_x - \alpha \partial_x. \quad (42)$$

The 2VALP are specified by the generating function

$$(1-yt)^{-\alpha-1} \exp\left(-\frac{xt}{1-yt}\right) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} t^n L_n^{(\alpha)}(x, y) \quad (|yt| < 1). \quad (43)$$

We can now define mixed polynomials in a fairly straightforward way. For example

$$\begin{aligned} \exp(\tau_L \hat{D}_x {}_L \hat{D}_z) \{(y - \hat{D}_x^{-1})^m (w - \hat{D}_z^{-1})^n\} &= L_{m,n}(x, y; z, w | \tau) \\ &= m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s}{s!(m-s)!(n-s)!} L_{m-s}(x, y) L_{n-s}(z, w) \end{aligned} \quad (44)$$

and we call $L_{m,n}(x, y; z, w | \tau)$ as 4-variable 2-index Laguerre polynomials (4V2ILP). For $y = w = 0$ these polynomials reduce to

$$L_{m,n}(x, z | \tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{(-1)^{m+n} \tau^s x^{m-s} z^{n-s}}{s!((m-s)!)^2((n-s)!)^2}, \quad (45)$$

this can be called 2-variable 2-index Laguerre polynomials (2V2ILP). The relevant multiplicative operators are just provided by

$$\hat{M}_x^v = -\hat{D}_x^{-1} + \tau_L \hat{D}_z, \quad \hat{M}_z^v = -\hat{D}_z^{-1} + \tau_L \hat{D}_x \quad (46)$$

and these polynomials satisfy the following partial differential equation

$$\partial_\tau L_{m,n}(x, z|\tau) = (\partial_x x \partial_x)(\partial_z z \partial_z) L_{m,n}(x, z|\tau). \quad (47)$$

Regarding the Laguerre polynomials we have a further degree of freedom and we can indeed introduce associated forms like

$$L_{m,n}^{(\alpha,\beta)}(x, z|\tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{(-1)^{m+n} \tau^s x^{m-s} z^{n-s}}{s!(m-s)!(n-s)!\Gamma(m-s+\alpha+1)\Gamma(n-s+\beta+1)}. \quad (48)$$

Before concluding this section, we want to introduce a family of mixed Sheffer-type polynomials. First, we recall that the Bessel polynomials $b_m(x)$ [12] (see also [13]), are quasi-monomials with respect to the operators:

$$\hat{M}' = \frac{x}{1 - \partial_x}, \quad \hat{P}' = -\frac{1}{2}\partial_x^2 + \partial_x. \quad (49)$$

These polynomials are of Sheffer-type and are generated by

$$G(t, x) = \exp(x(1 - \sqrt{1 - 2t})). \quad (50)$$

Next, we recall other Sheffer-type polynomials, that are lower factorial polynomials $F_n(x)$ [14], which are defined as [13]:

$$F_n(x) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}. \quad (51)$$

These polynomials are generated by

$$G^*(t, x) = \exp(x\ell_n(1+t)) = (1+t)^x \quad (52)$$

and their associated raising and lowering operators are given by

$$\hat{M}'' = x \exp(-\partial_x), \quad \hat{P}'' = \exp(\partial_x) - 1. \quad (53)$$

Now, let us define what we call as Bessel factorial polynomials (BFP), as follows:

$${}_F b_{m,n}(x, y|\tau) = m!n! \sum_{s=0}^{\min(m,n)} \frac{\tau^s}{s!(m-s)!(n-s)!} b_{m-s}(x) F_{n-s}(y). \quad (54)$$

The relevant multiplicative operators are provided by

$$\hat{M}_x^{vi} = \frac{x}{1 - \partial_x} + \tau(\exp(\partial_y) - 1), \quad \hat{M}_y^{vi} = y \exp(-\partial_y) + \tau \left(-\frac{1}{2}\partial_x^2 + \partial_x \right). \quad (55)$$

Also, we note that the BFP ${}_F b_{m,n}(x, y|\tau)$ satisfy the following differential relation

$$\partial_\tau {}_F b_{m,n}(x, y|\tau) = \left(-\frac{1}{2}\partial_x^2 + \partial_x \right) ({}_F b_{m,n}(x, y+1|\tau) - {}_F b_{m,n}(x, y|\tau)). \quad (56)$$

Further comments on the various aspects touched in this section, will be presented in the concluding remarks.

4. Concluding remarks

Very recently Dattoli *et al.* [15] have established generalized forms of Gleisher operational rules and used them to derive new families of generating relations involving multi-dimensional Hermite polynomials. However, the operational calculus approach to special polynomials is of much wider applicability. To provide a further application, we consider the following identity:

$$\exp(\tau \partial_y \partial_{2x}) \{\exp(-x^2 - y^2)\} = \frac{1}{\sqrt{1 + 4\tau \partial_y}} \exp\left(-\frac{x^2}{1 + 4\tau \partial_y}\right) \exp(-y^2), \quad (57)$$

this can be viewed as a kind of generalization of the well-known Gleisher identity [7]:

$$\exp(y \partial_{2x}) \{\exp(-x^2)\} = \frac{1}{\sqrt{1 + 4y}} \exp\left(-\frac{x^2}{1 + 4y}\right). \quad (58)$$

Now, we use the generating function (43) of 2VALP $L_n^{(\alpha)}(x, y)$ in the r.h.s. of equation (57) to get

$$\exp(\tau \partial_y \partial_{2x}) \{\exp(-x^2 - y^2)\} = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + \frac{1}{2})} L_n^{(-1/2)}(x^2, -4\tau \partial_y) \exp(-y^2), \quad (59)$$

which on using equation (41) and the Rodrigues formula for Hermite polynomials $H_n(x)$ yields

$$\exp(\tau \partial_y \partial_{2x}) \{\exp(-x^2 - y^2)\} = \sum_{n=0}^{\infty} \sum_{k=0}^n \binom{n}{k} \frac{(-1)^k n! x^{2k} (4\tau)^{n-k}}{\Gamma(n + (1/2)) \Gamma(k + (1/2))} H_{n-k}(y) \exp(-y^2). \quad (60)$$

In order to view another possibility of application of operational approach, we consider the solution of a kind of generalized heat equation, as *e.g.*

$$\begin{aligned} \frac{\partial}{\partial \tau} \phi(x, y, \tau) &= \partial_{2x, y} \phi(x, y, \tau), \\ \phi(x, y, 0) &= F(x, y). \end{aligned} \quad (61)$$

The solution of the above equation reduces to the following operational problem

$$\phi(x, y, \tau) = \exp(\tau \partial_{2x, y}) \{F(x, y)\}. \quad (62)$$

Further, if the function $F(x, y)$ can be expanded as:

$$F(x, y) = \sum_{m, n=0}^{\infty} a_{m, n} x^m y^n. \quad (63)$$

Then, in view of definition (16), we can write the solution of equation (61) as follows:

$$\exp(\tau \partial_{2x, y}) \{F(x, y)\} = \sum_{m, n=0}^{\infty} a_{m, n} h_{m, n}^{(2, 1)}(x, y | \tau), \quad (64)$$

provided that the series converges.

Furthermore, we consider the following equation:

$$\begin{aligned}\frac{\partial}{\partial \tau} \psi(x, y, \tau) &= \partial_{2x, 2y} \psi(x, y, \tau), \\ \psi(x, y, 0) &= g(x, y)\end{aligned}\quad (65)$$

The solution of the above equation is given by

$$\psi(x, y, \tau) = \exp(\tau \partial_{2x, 2y}) \{g(x, y)\}. \quad (66)$$

If the function $g(x, y)$ has an expansion of the form similar to that of function $F(x, y)$ given in equation (63), then the solution of equation (65) can be written as

$$\psi(x, y, \tau) = \sum_{m, n=0}^{\infty} a_{m, n} h_{m, n}^{(2, 2)}(x, y | \tau). \quad (67)$$

We can also use the integral formula

$$\exp(\lambda \hat{c}^2) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\xi^2 + 2\xi \sqrt{\lambda} \hat{c}) d\xi \quad (68)$$

in the r.h.s. of equation (66), to get

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\xi^2 + 2\xi \sqrt{\tau} \partial_{x, y}) g(x, y) d\xi, \quad (69)$$

which by means of Crofton-like identity [5], further simplifies to

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \exp(-\xi^2) g(x + 2\xi \sqrt{\tau} \partial_y, y + 2\xi \sqrt{\tau} \partial_x) d\xi. \quad (70)$$

Again, expanding $g(x, y)$ using equation (63), we can simplify the solution (69) as follows:

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \sum_{m, n=0}^{\infty} a_{m, n} \int_{-\infty}^{+\infty} \exp(-\xi^2 + 2\xi \sqrt{\tau} \partial_{x, y}) \{x^m y^n\} d\xi, \quad (71)$$

which by using definition (1), can be written as

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \sum_{m, n=0}^{\infty} a_{m, n} \int_{-\infty}^{+\infty} \exp(-\xi^2) h_{m, n}(x, y | 2\xi \sqrt{\tau}) d\xi. \quad (72)$$

Now we simplify the above equation by making use of equation (2), to get

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \sum_{m, n=0}^{\infty} a_{m, n} m! n! \sum_{s=0}^{\min(m, n)} \frac{(2\sqrt{\tau})^s x^{m-s} y^{n-s}}{s!(m-s)!(n-s)!} \int_{-\infty}^{+\infty} \exp(-\xi^2) \xi^s d\xi. \quad (73)$$

Finally, by using umbral concepts [9], we express the solution (73) in the following form:

$$\psi(x, y, \tau) = \frac{1}{\sqrt{\pi}} \sum_{m, n=0}^{\infty} a_{m, n} h_{m, n}(x, y | (2\sqrt{\tau})_f), \quad (74)$$

where

$$f_s = \int_{-\infty}^{+\infty} \exp(-\xi^2) \xi^s d\xi = \begin{cases} \Gamma\left(\frac{s+1}{2}\right) & : \text{if } s \text{ is even,} \\ 0 & : \text{if } s \text{ is odd.} \end{cases} \quad (75)$$

In view of equations (67) and (74), we also find that

$$h_{m,n}^{(2,2)}(x, y|\tau) = \frac{1}{\sqrt{\pi}} h_{m,n}(x, y|(2\sqrt{\tau})_f). \quad (76)$$

In this paper, we have seen the possibilities of defining families of entangled polynomials; we have defined the underlying operational calculus and shown that these may have relevance to the solutions of partial differential equations. In a forthcoming investigation we will study multi-variable orthogonal polynomials and discuss applications to physical problems in Quantum Mechanics.

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Some theorems on integrability of Laplace transforms and their applications

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In this article, we study the Laplace transforms of functions, integrable on $\mathbb{R}_+$ with some weight. We obtain two theorems of the Paley–Wiener type and apply them to the problem of completeness of systems of exponentials in weighted L^p spaces on $\mathbb{R}_+$.

Keywords: Laplace transform; Bergman spaces; Systems of exponentials

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1. Introduction

Let A be a Banach space of functions on $\mathbb{R}_+$ and let B be a Banach space of analytic functions in the half-plane $\Re w > 0$. The pairs of spaces A and B are studied, for which the operator

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt \quad (1)$$

is a bounded operator from A to B or the inverse operator is bounded. This problem has been studied, for example, by Benedetto *et al.* [1]. They considered the weighted spaces L^p for A and the weighted Hardy spaces for B . Sedletsii [2] studied the pairs of the weighted L^p spaces, with t^α for a weight, and the Bergman spaces. We are going to generalize his result.

DEFINITION 1 A positive function $l(x)$ is called slowly varying in the infinity if it is measurable on $[A, \infty)$, $A > 0$ and for any fixed $\lambda > 0$

$$\lim_{x \rightarrow \infty} \frac{l(\lambda x)}{l(x)} = 1. \quad (2)$$

A function $l(x)$ is called slowly varying in zero if $l(1/x)$ is slowly varying in the infinity.

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For example, $\log x$ is slowly varying in the infinity. We usually write ‘slowly varying’ meaning ‘slowly varying in the infinity’. Throughout this article, let $w = u + iv$, $z = x + iy = re^{i\theta}$, $l(t)$ be a slowly varying function, positive on the half-line $t \geq 0$ and $1/q + 1/p = 1$.

DEFINITION 2 A function $f(z)$, analytic in the unit disc $|z| < 1$, is said to belong to the space $A_{\alpha, l(t)}^p \{|z| < 1\}$, $0 < p < \infty$, $\alpha \geq -1$, if

$$\|f\|_{p, \alpha, l}^p = \int_{|z| < 1} |f(z)|^p (1 - |z|^2)^{\alpha l} \left(\frac{1}{1 - |z|} \right) dx dy < \infty.$$

A function $f(w)$, analytic in the half-plane $\Re w > 0$, is said to belong to the space $A_{\alpha, l(t)}^p \{\Re w > 0\}$ if

$$\|f\|_{p, \alpha, l}^p = \int_{u > 0} |f(w)|^p u^{\alpha l} \left(\frac{1}{u} \right) du dv < \infty.$$

A function $f(x)$ is said to belong to the space $L_{\alpha, l(t)}^p$ on $\mathbb{R}_+$ if

$$\|f\|_{p, \alpha, l}^p = \int_0^\infty |f(t)|^p t^{\alpha l} \left(\frac{1}{t} \right) dt < \infty.$$

Our main result in this article is the following. We show that if $g(t) \in L_{\beta, l(t)}^q$ ($2 \leq q < \infty$, $\beta < q - 2$), then $G(w) \in A_{q-3-\beta, l(1/t)}^q$. On the contrary, if $1 < q \leq 2$, $\beta < q - 2$ and $G(w) \in A_{q-3-\beta, l(1/t)}^q$, then $g(t) \in L_{\beta, l(t)}^q$. The result holds for all slowly varying functions, satisfying just some natural conditions. If we put $l \equiv 1$, we obtain the result of Sedletskii as a corollary. Besides, similar theorems will be proved for the boundary case $\beta = q - 2$ and they will be applied to the problem of completeness of systems of exponentials in $L_{p-2, l(t)}^p$ spaces.

2. Some auxiliary propositions

THEOREM 1 (Hardy–Littlewood [3])

1. Let $2 \leq p < \infty$, $f \in L_{p-2}^p(\mathbb{R})$. Then $\hat{f} \in L_p(\mathbb{R})$ and $\|\hat{f}\|_p \leq C_p \|f\|_{p, p-2}$.
2. Let $1 < p \leq 2$, $f \in L^p(\mathbb{R})$. Then $\hat{f} \in L_{p-2}^p(\mathbb{R})$ and $\|\hat{f}\|_{p, p-2} \leq C_p \|f\|_p$.

LEMMA 1 Let $1 < p < \infty$, $\alpha > -1$. The incompleteness of the system $\{e^{-\lambda_n t}\}_{n=1}^\infty$, $\Re \lambda_n > 0$ in $L_{\alpha, l(t)}^p$ is equivalent to the existence of a nontrivial analytic function of the kind

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt, \quad u = \Re w > 0, \quad g \in L_{-\alpha q/p, l(t)^{-q/p}}^q, \quad (3)$$

such that $G(\lambda_n) = 0$.

Proof The system $\{e^{-\lambda_n t}\}$ is incomplete in $L_{\alpha, l(t)}^p$ if and only if there exists a nontrivial functional g in $(L_{\alpha, l(t)}^p)^*$ such that

$$\int_{\mathbb{R}_+} e^{-\lambda_n t} g(t) dt = 0, \quad n = 1, \dots, \infty.$$

As $(L_{\alpha, l(t)}^p)^* = L_{-\alpha q/p, l(t)^{-q/p}}^q$, this is equivalent to the existence of a nontrivial function $G(w)$ of the form (3) with $g \in L_{-\alpha q/p, l(t)^{-q/p}}^q$ such that $G(\lambda_n) = 0$. ■

Denote

$$\mathcal{L}(x) = \int_0^{1/x} \frac{l(1/t)}{t} dt.$$

LEMMA 2 [4]

Let $l(t)$ be a bounded function such that

$$\mathcal{L}(1) = \int_0^1 \frac{l(1/t)}{t} dt < \infty. \quad (4)$$

If $l(t)$ is slowly varying in the infinity (in zero) then the function $\mathcal{L}(x)$ is also slowly varying in the infinity (in zero).

Denote

$$F_\alpha(x) = \int_{\mathbb{R}_+} e^{-xt} t^\alpha l\left(\frac{1}{t}\right) dt.$$

THEOREM 2 [2]

1. Let $\alpha > -1$ and let function $l(t)$ be slowly varying in the infinity (in zero). If $F_\alpha(x) < \infty$, $x > 0$, then $F_\alpha(x) \sim cl(x)x^{-1-\alpha}$, $x \rightarrow \infty$ ($x \rightarrow 0$).
2. If function $l(t)$ is slowly varying in the infinity and $F_{-1}(x) < \infty$, $x > 0$, then $F_{-1}(x) \sim \mathcal{L}(x)$, $x \rightarrow \infty$.
3. If function $l(t)$ is slowly varying in zero and $F_{-1}(x) < \infty$, $x > 0$, $F_{-1}(0) = \infty$, then $F_{-1}(x) \sim \mathcal{L}(x)$, $x \rightarrow 0$.

LEMMA 3 Let l be a slowly varying function, bounded and separated from zero in all intervals (a, b) , $0 \leq a < b < \infty$. If $f \in A_{-1, l(t)}^p \{\Re w > 0\}$, then for any $\varepsilon > 0$

$$g_\varepsilon(z) = f\left(\frac{1+z}{1-z}\right) \left(\frac{1}{1-z}\right)^{2/p-\varepsilon}$$

belongs to $A_{-1, l(t)}^p \{|z| < 1\}$. On the contrary, if for any $\varepsilon > 0$ function $g_{-\varepsilon}(z)$ is from $A_{-1, l(t)}^p \{|z| < 1\}$, then $f(z) \in A_{-1, l(t)}^p \{\Re w > 0\}$.

Proof Let $z = (w-1)/(w+1)$ be a conformal mapping of a half-plane $\Re w > 0$ to the unit circle $|z| < 1$. As the Jacobian $J = |w_z'|^2 = 4/|1-z|^4$, and $u = (1-|z|^2)/(|1-z|^2)$, we have

$$\begin{aligned} \int_{u>0} |f(w)|^p \frac{l(1/u)}{u} du dv &= 4 \int_{|z|<1} \left| f\left(\frac{1+z}{1-z}\right) \right|^p \frac{|1-z|^2}{1-|z|^2} l\left(\frac{|1-z|^2}{1-|z|^2}\right) \frac{1}{|1-z|^4} dx dy \\ &= 4 \int_{|z|<1} \left| f\left(\frac{1+z}{1-z}\right) \frac{1}{(1-z)^{2/p}} \right|^p \frac{1}{1-|z|^2} \\ &\quad \times l\left(\frac{|1-z|^2}{1-|z|^2}\right) dx dy. \end{aligned}$$

The lemma is proved if we can show that for any $\varepsilon > 0$, there exist such constants $c_1 = c_1(\varepsilon)$, $c_2 = c_2(\varepsilon)$, which for all z , $|z| < 1$

$$\begin{aligned} |1-z|^\varepsilon l\left(\frac{|1-z|^2}{1-|z|^2}\right) &\leq c_1 l\left(\frac{1}{1-|z|^2}\right), \\ |1-z|^{-\varepsilon} l\left(\frac{|1-z|^2}{1-|z|^2}\right) &\geq c_2 l\left(\frac{1}{1-|z|^2}\right). \end{aligned}$$

It is enough to show, in turn, that for all $0 < \lambda \leq 2$ and for all $x \geq 0$

$$\lambda^\varepsilon l(\lambda x) \leq c l(x), \quad (5)$$

$$\lambda^{-\varepsilon} l(\lambda x) \geq c l(x). \quad (6)$$

We will prove equation (5), as equation (6) follows immediately from equation (5) if we denote $l_1(x) = 1/l(x)$.

Fix $\varepsilon > 0$ and let $\lambda \in [1, 2]$. As l is slowly varying, equation (2) holds uniformly on any interval $[a, b]$, $0 < a < b < \infty$ [4]. Hence, for some $C > 0$

$$l(\lambda x) \leq 2^\varepsilon l(x), \quad \text{for all } x \geq C, \quad \lambda \in [1, 2]. \quad (7)$$

Denote

$$M := \frac{\sup_{0 \leq x \leq C} l(x)}{\inf_{0 \leq x \leq C} l(x)}.$$

The function l is bounded and separated from zero in all finite intervals, hence $0 < M < \infty$. We assume $M \geq 2^\varepsilon$ and then

$$l(\lambda x) \leq M l(x), \quad x \geq 0, \quad \lambda \in [1, 2]. \quad (8)$$

Thus

$$\frac{l((\lambda/2)x)}{l(x)} = \frac{l(\lambda(x/2))}{l(x/2)} \frac{l(x/2)}{l(x)} \leq M 2^\varepsilon, \quad \lambda \in [1, 2]. \quad (9)$$

If $x \leq C$, equation (9) holds because of the choice of M . If $x \geq C$, then the second multiplier in equation (9) is less than 2^ε , which follows from equation (7). The first multiplier is less than M , it follows from equation (8).

Using the induction, we can obtain that for any $n \in \mathbb{N} \cup \{0\}$ and for any $x \geq 0$

$$(2^{-n}\lambda)^\varepsilon \frac{l((2^{-n}\lambda)x)}{l(x)} \leq (2^{-n})^\varepsilon \frac{l((2^{-n}\lambda)x)}{l(x)} \leq M.$$

Since λ is arbitrary from $[1, 2]$, inequality (5) holds and hence, the lemma itself is proved. ■

3. Laplace transform: case $\alpha > -1$

THEOREM 3 *Let $2 \leq q < \infty$, $\beta < q - 2$, and let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero in all intervals of the kind (a, b) , $0 < a < b < \infty$. If $g \in L_{\beta, l(1/t)}^q$, then function (1) belongs to the space $A_{q-3-\beta, l(t)}^q \{\Re w > 0\}$ and*

$$\|G(w)\|_{q, q-3-\beta, l(t)} \leq c \|g\|_{q, \beta, l(1/t)},$$

where c does not depend on g .

Proof Fix $u > 0$. Function $G(u + iv)$ is the Fourier transform of the function $e^{-ut}g(t)$, $t > 0$. We apply the Hardy–Littlewood theorem

$$\int_{\mathbb{R}} |G(u + iv)|^q dv \leq c \int_{\mathbb{R}_+} e^{-qut} |g(t)|^q t^{q-2} dt. \quad (10)$$

Integrating this inequality with respect to $u^{q-3-\beta} \cdot l(1/u) du$ over the $\mathbb{R}_+$, we obtain

$$\begin{aligned} \|G(w)\|_{q,q-3-\beta,l(t)}^q &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} |G(u + iv)|^q u^{q-3-\beta} l\left(\frac{1}{u}\right) du dv \\ &\leq c \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} e^{-qut} |g(t)|^q t^{q-2} dt \right) u^{q-3-\beta} l\left(\frac{1}{u}\right) du \\ &= c \int_{\mathbb{R}_+} |g(t)|^q t^{q-2} \int_{\mathbb{R}_+} e^{-qut} u^{q-3-\beta} l(1/u) du dt. \end{aligned} \quad (11)$$

As $\beta < q - 2$ and $l(u)$ are bounded, we have

$$F_{q-3-\beta}(qt) = \int_{\mathbb{R}_+} e^{-qut} u^{q-3-\beta} l\left(\frac{1}{u}\right) du < \infty, \quad t > 0.$$

Besides, the function $l(t)$ is slowly varying in zero and in the infinity, and so we can apply Theorem 2, statement 1, to obtain

$$F_{q-3-\beta}(qt) \sim cl(qt)(qt)^{-(q-2-\beta)} \asymp l(t)t^{-(q-2-\beta)}, \quad t \rightarrow \infty, \quad t \rightarrow 0.$$

Hence, for some $0 < T_0 < T < \infty$, we have $F_{q-3-\beta}(qt) \leq cl(t)(t)^{-(q-2-\beta)}$ for all $t \geq T$ or $0 < t \leq T_0$. As $l(t)$ is bounded and separated from zero on finite intervals, the last inequality holds also for $T_0 \leq t \leq T$. Together with equation (11), this yields

$$\|G(w)\|_{q,q-3-\beta,l(t)}^q \leq c \int_{\mathbb{R}_+} |g(t)|^q t^\beta l(t) dt = c \|g(w)\|_{q,\beta,l(1/t)}^q.$$

■

THEOREM 4 Let $1 < q \leq 2$, $\beta < q - 2$ and let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. If $G(w) \in A_{q-3-\beta,l(t)}^q (\Re w > 0)$, then $G(w)$ can be represented in form (1) with $g \in L_{\beta,l(1/t)}^q$ and

$$\|g\|_{q,\beta,l(1/t)} \leq c \|G(w)\|_{q,q-3-\beta,l(t)}, \quad (12)$$

with c not depending on G .

Proof Let $u_0 = \Re w_0 > 0$, $K := \{w : |w - w_0| < u_0/2\}$. As $|G(w_0)|^q$ is subharmonic

$$u_0^{q-1-\beta} l\left(\frac{1}{u_0}\right) |G(w_0)|^q \leq \frac{u_0^{q-1-\beta} l(1/u_0)}{\pi (u_0/2)^2} \iint_K |G(w)|^q du dv.$$

If $w \in K$, then $u_0/2 < u < 3u_0/2$. As the functions $l(x), l(1/x)$ are slowly varying, equation (2) holds uniformly on λ on all intervals of the kind $[a, b]$, $0 < a < b < \infty$. Hence, for some constants $U, V > 0$, we have $l(u)/l(u_0) \in [c, C]$, $0 < c < 1 < C < \infty$

while $u_0 > U$, $u_0 < V$. As l is bounded and separated from zero, the expression $l(u)/l(u_0)$ is bounded while $V < u_0 < U$. Thus

$$u_0^{q-1-\beta} l\left(\frac{1}{u_0}\right) |G(w_0)|^q \leq c \iint_K |G(w)|^q u^{q-3-\beta} l\left(\frac{1}{u}\right) du dv \leq c \|G(w)\|_{q,q-3-\beta,l}^q.$$

Hence, $|G(w)|^q \leq c (u^{q-1-\beta} l(1/u))^{-1}$, $u = \Re w > 0$. The function $l(1/u)$ is separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. Besides, $q - 1 - \beta > 1$; therefore, $u^{q-1-\beta} l(1/u) \rightarrow \infty$, $u \rightarrow \infty$ [4]. It follows that the expression $u^{q-1-\beta} l(1/u)$ is separated from zero while $u \geq \delta$, $\delta > 0$. Hence, $|G(w)|$ is bounded in any half-plane of the form $\Re w \geq \delta > 0$. But, in this case, for a fixed $\varepsilon > 0$, the function

$$G_\varepsilon(w) = G(w) \left(\frac{1 - e^{-\varepsilon w}}{\varepsilon w} \right)^2, \quad \Re w > 0 \quad (13)$$

is a Laplace transform of some original $g_\varepsilon(t)$, *i.e.*

$$G_\varepsilon(w) = \int_{\mathbb{R}_+} e^{-wt} g_\varepsilon(t) dt, \quad \Re w > 0. \quad (14)$$

For fixed $u > 0$, the function $e^{-ut} g_\varepsilon(t)$ is the inverse Fourier transform of the function $G_\varepsilon(u + iv)$. As $G_\varepsilon(u + iv) \in L^q(\mathbb{R})$, by the Hardy–Littlewood theorem

$$\int_{\mathbb{R}_+} e^{-qu} |g_\varepsilon(t)|^q t^{q-2} dt \leq c \int_{\mathbb{R}} |G_\varepsilon(u + iv)|^q dv. \quad (15)$$

We integrate this inequality with respect to the measure $u^{q-3-\beta} l(1/u) du$ over the half-line $u > 0$

$$\begin{aligned} & \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} e^{-qu} |g_\varepsilon(t)|^q t^{q-2} dt \right) u^{q-3-\beta} l\left(\frac{1}{u}\right) du \\ & \leq c \int_{\mathbb{R}_+} \int_{\mathbb{R}} |G_\varepsilon(u + iv)|^q u^{q-3-\beta} l\left(\frac{1}{u}\right) du dv = \|G_\varepsilon(w)\|_{q,q-3-\beta,l(t)}^q. \end{aligned} \quad (16)$$

We use the same arguments as in Theorem 3 to estimate the left side of equation (16) and apply Theorem 2, statement 1, to obtain that

$$F_{q-3-\beta}(qt) \sim cl(qt)(qt)^{-(q-2-\beta)} \asymp l(t)t^{-(q-2-\beta)}, \quad t \rightarrow \infty, \quad t \rightarrow 0.$$

Hence, for some $T, T_0 \geq 0$, we have $F_{q-3-\beta}(qt) \geq cl(t)t^{-(q-2-\beta)}$ for all $t \geq T$, $0 < t \leq T_0$. Moreover, the same inequality holds for $T_0 \leq t \leq T$ because l is bounded on finite intervals. The last estimate together with equation (16) yields

$$\|g_\varepsilon(w)\|_{q,\beta,l(1/t)}^q = c \int_{\mathbb{R}_+} |g_\varepsilon(t)|^q t^\beta l(t) dt \leq c \|G_\varepsilon(w)\|_{q,q-3-\beta,l(t)}^q \leq c \|G(w)\|_{q,q-3-\beta,l(t)}^q. \quad (17)$$

Thus, for some constant c , independent of ε , we have $\|g_\varepsilon(t)\|_{q,\beta,l(1/t)} \leq c$, $0 < \varepsilon < 1$, and we can choose a weakly convergent sequence $g_{\varepsilon_i}(t)$, $\varepsilon_i \rightarrow 0$ with $g(t)$ for a weak limit, $g(t) \in L_{\beta,l(1/t)}^q$. As $G_\varepsilon(w) \rightarrow G(w)$, $\varepsilon \rightarrow 0$ for any w , $\Re w > 0$, equation (14) implies equation (1), and equation (17) implies equation (12). ■

4. Laplace transform: case $\alpha = -1$

THEOREM 5 Let $2 \leq q < \infty$ and let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. Suppose $\mathcal{L}(1) < \infty$, $\lim_{x \rightarrow 0} \mathcal{L}(x) = \infty$. If $g \in L_{q-2, \mathcal{L}(1/t)}^q$, then function (1) is from $A_{-1, l(t)}^q \{\Re w > 0\}$ space and

$$\|G(w)\|_{q, -1, l(t)} \leq c \|g\|_{q, q-2, \mathcal{L}(1/t)},$$

with c not depending on g .

Proof For any fixed $u > 0$, the function $G(u + iv)$ is the Fourier transform of the function $e^{-ut}g(t)$, $t > 0$. As $\mathcal{L}(t) \geq c$, $t \in U(0)$, we have $e^{-ut}g(t) \in L_{q-2}^q$. Applying the Hardy–Littlewood theorem, we can see that equation (10) holds. Integrating it with respect to $(1/u)l(1/u) du$ over the $\mathbb{R}_+$, we obtain

$$\begin{aligned} \|G(w)\|_{q, -1, l(t)}^q &= \int_{\mathbb{R}_+} \int_{\mathbb{R}} |G(u + iv)|^q \frac{l(1/u)}{u} du dv \\ &\leq c \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} e^{-qu} |g(t)|^q t^{q-2} dt \right) \frac{l(1/u)}{u} du \\ &= c \int_{\mathbb{R}_+} |g(t)|^q t^{q-2} \int_{\mathbb{R}_+} e^{-qu} \frac{l(1/u)}{u} du dt. \end{aligned}$$

We apply Theorem 2, proposition 2, 3 and Lemma 2 to obtain $F_{-1}(qt) \sim \mathcal{L}(qt) \sim \mathcal{L}(t)$ while $t \rightarrow \infty$, $t \rightarrow 0$. Hence, for some T, T_0 , we have $F_{-1}(qt) \leq c\mathcal{L}(t)$ while $t \geq T$ or $0 < t \leq T_0$. As $\mathcal{L}(x)$ is separated from zero, the same estimate holds for $T_0 \leq t \leq T$. Thus,

$$\|G(w)\|_{q, -1, l(t)}^q \leq c \int_{\mathbb{R}_+} |g(t)|^q t^{q-2} \mathcal{L}(t) dt = c \|g(w)\|_{q, q-2, \mathcal{L}(1/t)}^q.$$

The theorem is proved. ■

THEOREM 6 Let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. Let $\mathcal{L}(1) < \infty$, $\lim_{x \rightarrow 0} \mathcal{L}(x) = \infty$. If $G(w) \in A_{-1, l(t)}^q \{\Re w > 0\}$, $1 < q \leq 2$, then $G(w)$ can be represented in form (1), with $g(t) \in L_{q-2, \mathcal{L}(1/t)}^q$ and

$$\|g\|_{q, q-2, \mathcal{L}(1/t)} \leq c \|G(w)\|_{q, -1, l(t)}, \quad (18)$$

with c independent of $G(w)$.

Proof As in Theorem 4, function (13) is the Laplace transform of some original $g_\varepsilon(t)$, i.e. equation (14) holds. As for fixed $u > 0$ the function $G_\varepsilon(u + iv) \in L^q(\mathbb{R})$, by the Hardy–Littlewood theorem, the inequality (15) holds. Integrate it with respect to the measure $1/ul(1/u)du$ over the half-line $u > 0$:

$$\begin{aligned} \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} e^{-qu} |g_\varepsilon(t)|^q t^{q-2} dt \right) \frac{l(1/u)}{u} du &\leq c \\ \int_{\mathbb{R}_+} \int_{\mathbb{R}} |G_\varepsilon(u + iv)|^q \frac{l(1/u)}{u} du dv &= \|G_\varepsilon(w)\|_{q, -1, l(t)}^q. \end{aligned} \quad (19)$$

We use the same arguments as in Theorem 5 to estimate the left part of equation (19). Apply Theorem 2 and obtain that $F_{-1}(qt) \sim \mathcal{L}(qt) \sim \mathcal{L}(t)$ while $t \rightarrow \infty$, $t \rightarrow 0$. Hence, for

some $T, T_0 \geq 0$, we have $F_{-1}(qt) \geq c\mathcal{L}(t)$ for all $t \geq T, t \leq T_0$ and the same inequality holds for $T_0 \leq t \leq T$ because $\mathcal{L}(x)$ is bounded on finite intervals. The last estimate together with equation (19) yields

$$\|g_\varepsilon(w)\|_{q,q-2,\mathcal{L}(1/t)}^q = c \int_{\mathbb{R}_+} |g_\varepsilon(t)|^q t^{q-2} \mathcal{L}(t) dt \leq c \|G_\varepsilon(w)\|_{q,-1,l(t)}^q \leq c \|G(w)\|_{q,-1,l(t)}^q. \quad (20)$$

Thus, for some constant c , independent of ε , $\|g_\varepsilon(t)\|_{q,q-2,\mathcal{L}(1/t)} \leq c$ and we can choose a weakly convergent sequence $g_{\varepsilon_i}(t), \varepsilon_i \rightarrow 0$ with $g(t)$ for a weak limit, $g(t) \in L_{\beta,l(1/t)}^q$. As $G_\varepsilon(w) \rightarrow G(w)$, $\varepsilon \rightarrow 0$ for any $w, \Re w > 0$, equation (14) implies equation (1) and equation (20) implies equation (18).

Theorems 3, 4 and 5, 6 imply two corollaries; respectively. ■

COROLLARY 1 *Let $\beta < 0$ and let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. Then, the space $A_{-1-\beta,l(t)}^2\{\Re w > 0\}$ is equal to the space of functions*

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt, \quad g \in L_{\beta,l(1/t)}^2,$$

and $\|g\|_{2,\beta,l(1/t)} \asymp \|G(w)\|_{2,-1-\beta,l(t)}$.

COROLLARY 2 *Let $l(x)$ be a slowly varying function in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. If $\mathcal{L}(1) < \infty$, $\lim_{x \rightarrow 0} \mathcal{L}(x) = \infty$, then the space $A_{-1,l(t)}^2\{\Re w > 0\}$ is equal to the space of functions*

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt, \quad g \in L_{0,\mathcal{L}(1/t)}^2$$

and $\|g\|_{2,0,\mathcal{L}(1/t)} \asymp \|G(w)\|_{2,-1,l(t)}$.

5. Zeros of functions from $A_{-1,l(t)}^p\{\Re w > 0\}$

Denote $N_\Lambda(x)$ for a number of elements of sequence Λ in the circle of the radius $(x^2 - 1)^{1/2}$ with $x > 0$ for the center and $N_G(x)$ for the number of zeros of function G in this circle.

THEOREM 7 *Let $l(t)$ be a slowly varying function, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$ and let $\mathcal{L}(1) < \infty$. If $G(w) \in A_{-1,l(t)}^p(\Re w > 0)$, $0 < p < \infty$, then*

1. *The following estimate holds*

$$\limsup_{x \rightarrow \infty} \frac{N_G(x)}{x \log(1/\mathcal{L}(x))} \leq \frac{1}{p}.$$

2. *If for some $\varepsilon > 0$ function l satisfies the condition*

$$\sum_{n=1}^{\infty} \frac{\mathcal{L}(n)^\delta}{n+1} < \infty, \quad \delta = (q-1) \frac{\varepsilon/4}{1+\varepsilon/4}, \quad (21)$$

then for some function $G \in A_{-1,l(t)}^p(\Re w > 0)$

$$\limsup_{x \rightarrow \infty} \frac{N_G(x)}{x \log(1/\mathcal{L}(x))} > \frac{1}{p(1+\varepsilon)}.$$

Proof 1. Let $\Lambda = \{\lambda_n\}$ be a sequence of zeros of function $G(w)$ and let $\{z_n\}$ be an image of Λ under the mapping $z = (w - 1)/(w + 1)$ of the right half-plane into the unit circle. By Lemma 3, $\{z_n\}$ is a sequence of zeros for some function $f(z) \in A_{-1, l(t)}^p \{|z| < 1\}$. Let $n(r)$ be a number of zeros of function f in the circle of radius r . Then, according to ref. [5], the following estimate holds:

$$\limsup_{r \rightarrow 1} \frac{n(r)}{(1/1 - r) \log(1/\mathcal{L}(1/1 - r))} \leq \frac{1}{p}. \quad (22)$$

■

Let C_r be a counter image of the circle $|z| = r < 1$ under the mapping $z = (w - 1)/(w + 1)$. The circle C_r is symmetric with respect to the real axis and passes through the points $(1 - r)/(1 + r)$ and $(1 + r)/(1 - r)$ (counter images of $-r$ and r , respectively). Hence, if we denote x and R for its center and radius, then

$$x = \frac{1 - r}{2(1 + r)} + \frac{1 + r}{2(1 - r)} = \frac{1 + r^2}{1 - r^2}, \quad R = 2r(1 - r^2)^{-1}. \quad (23)$$

Expressing r through x and applying the formula for R , we find that $R = (x^2 - 1)^{1/2}$. Thus, $N_\Lambda(x) = n(r)$. It follows from the left side in equation (23) that $x \sim (1 - r)^{-1}$, $r \rightarrow 1 - 0$. Hence,

$$\limsup_{x \rightarrow \infty} \frac{N_\Lambda(x)}{x \log(1/\mathcal{L}(x))} = \limsup_{r \rightarrow 1-0} \frac{n(r)}{(1/1 - r) \log(1/\mathcal{L}(1/1 - r))}, \quad (24)$$

and the first statement follows from equation (22).

2. It is proved in ref. [5] that if $\varepsilon > 0$ and if a slowly varying function l satisfies equation (21), then for some function $f \in A_{-1, l(t)}^p \{|z| < 1\}$

$$\limsup_{r \rightarrow 1} \frac{n(r)}{(1/1 - r) \log(1/\mathcal{L}(1/1 - r))} > \frac{1}{p(1 + \varepsilon)}.$$

Let z_n be a sequence of zeros of f . By Lemma 3, Λ is a sequence of zeros of some function $G(w) \in A_{-1, l(t)}^p \{\Re w > 0\}$. Now the second statement follows immediately from equation (24).

6. Completeness of systems of exponentials

THEOREM 8 *Let L be a differentiable function, bounded outside some neighborhood of zero, and let $L(x) \rightarrow \infty$, $x \rightarrow 0$, $L(x) \rightarrow 0$, $x \rightarrow \infty$. Let function*

$$l(x) = -\frac{q}{px} L\left(\frac{1}{x}\right)^{-q} L'\left(\frac{1}{x}\right)$$

be slowly varying in zero and in the infinity, bounded and separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$. Then,

(1) *If $1 < p \leq 2$ and the sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$ satisfy*

$$\delta(\Lambda) = \limsup_{x \rightarrow \infty} \frac{N_\Lambda(x)}{x \log(L(1/x)^{q/p})} > \frac{1}{q}, \quad (25)$$

then the system $\{e^{-\lambda_n t}\}_{n=1}^\infty$ is complete in $L_{p-2, L(x)}^p$.

(2) If $p \geq 2$, then for any $\varepsilon > 0$ and for any function L , such that

$$\sum_{n=1}^{\infty} \frac{L^{-\delta q/p}(1/n)}{n+1} < \infty, \quad (26)$$

there exists a sequence $\Lambda = \{\lambda_n\}$ such that the system $\{e^{-\lambda_n t}\}_{n=1}^{\infty}$ is not complete in $L_{p-2, L(x)}^p$ and $\delta(\Lambda) \geq (1/q) - \varepsilon$.

The second proposition shows that the constant $1/q$ in the right part of equation (25) is sharp.

Proof (1) Assume the contrary: let the mentioned system be incomplete in $L_{p-2, L(x)}^p$. By Lemma 1, we can choose a nontrivial function

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt, \quad \Re w > 0, \quad g \in L_{q-2, L(x)^{-q/p}}^q$$

vanishing on Λ . It is clear that $N_{\Lambda}(x) \leq N_G(x)$.

The function l is slowly varying in zero and in the infinity, bounded and separated from zero in all finite intervals. As $((L(x))^{-q/p})' = (1/x)l(1/x)$, we apply the Newton–Leibnitz formula and the condition $L(x) \rightarrow \infty$, $x \rightarrow +0$, to obtain

$$L(x)^{-q/p} = \int_0^x \frac{l(1/t)}{t} dt.$$

We have $\lim_{x \rightarrow 0} L(1/x)^{-q/p} = \infty$. Thus, we can apply Theorem 5 with $\mathcal{L}(x) = L(1/x)^{-q/p}$, to obtain $G(w) \in A_{-1, l(t)}^q$.

However, by Theorem 7

$$\delta(\Lambda) \leq \frac{1}{q}.$$

This contradicts the hypothesis and statement (1) is proved.

(2) By Theorem 7, for some $G(w) \in A_{-1, l(t)}^q$, we have

$$\frac{N_G(x)}{x \log (L(1/x)^{q/p})} \geq \frac{1}{q} - \varepsilon.$$

Let Λ be a sequence of zeros of $G(w)$. Then $\delta(\Lambda) \geq (1/q) - \varepsilon$. By Theorem 6, the function $G(w)$ can be represented in the form

$$G(w) = \int_{\mathbb{R}_+} e^{-wt} g(t) dt, \quad g(t) \in L_{q-2, L(x)^{-q/p}}^q.$$

By Lemma 1, the system $\{e^{-\lambda_n t}\}_{n=1}^{\infty}$ is incomplete in $L_{p-2, L(x)}^p$ and the second statement is also proved. ■

COROLLARY 3 *Let L be a differentiable function bounded outside some neighborhood of zero and let $L(x) \rightarrow \infty$, $x \rightarrow 0$, $L(x) \rightarrow 0$, $x \rightarrow \infty$. Let function*

$$l(x) = -\frac{1}{x} L\left(\frac{1}{x}\right)^{-2} L'\left(\frac{1}{x}\right)$$

be slowly varying in zero and in the infinity, bounded and separated from zero in all finite intervals. Then, if the sequence $\Lambda = \{\lambda_n\}_{n=1}^\infty$, satisfies the condition

$$\delta(\Lambda) = \limsup_{x \rightarrow \infty} \frac{N_\Lambda(x)}{x \log(L(1/x))} > \frac{1}{2}, \quad (27)$$

then the system $\{e^{-\lambda_n t}\}_{n=1}^\infty$ is complete in $L^2_{0,L(x)}$, and for some function L and for some sequence $\Lambda = \{\lambda_n\}$, the system $\{e^{-\lambda_n t}\}_{n=1}^\infty$ is incomplete in $L^2_{0,L(x)}$ and $\delta(\Lambda) \geq (1/2) - \varepsilon$.

We give an example of a function, satisfying the hypothesis of the second statement of the theorem. Let $q \leq 2$ and $L(x) = 1/\log x$ for $x \geq 2$ and

$$L(x) = \left(\log^2\left(\frac{1}{x}\right)\right)^{p/q\delta}$$

for $x \leq 1/2$. For $1/2 \leq x \leq 2$, we can use an arbitrary differentiable positive function as $L(x)$. We also demand that $L'(x)$ exists for $x = 1/2$, $x = 2$. In this case, series (26) converges and $l(x) = (2/\delta)(\log x)^{-1-2/\delta}$, $x \geq 2$, and $l(x) = (q/p)(\log(1/x))^{q-2}$, $x \leq 1/2$. It is clear that $l(x)$ is bounded, separated from zero on all intervals of the kind (a, b) , $0 < a < b < \infty$, and slowly varying.

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Characterization of Fourier hyperfunctions by solutions of the Hermite heat equation

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Let $E(x, \xi, t)$ be the Mehler kernel and let $U(x, t)$ be the solution, with some appropriate growth estimate (*), of the Hermite heat equation in $\mathbf{R} \times (0, T)$. Then there exists a unique Fourier hyperfunction u such that $U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle$. Conversely for any Fourier hyperfunction u , $U(x, t) := \langle u(\xi), E(x, \xi, t) \rangle$ is the smooth solution of the Hermite heat equation in $\mathbf{R} \times (0, T)$ satisfying the growth estimate (*).

Keywords: Hermite functions; Mehler kernel; Hermite heat equation

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1. Introduction

In refs. [1, 2], we see that the boundary value of a C^∞ solution of the heat equation in $\mathbf{R}_+^{n+1}$, which does not exceed $\exp(\epsilon t^{-1})$ is a well-defined hyperfunction. But, if a C^∞ solution of the heat equation in $\mathbf{R}_+^{n+1}$, which does not increase faster than $\exp(\epsilon t^{-1} + |x|)$ then its boundary value determines a unique Fourier hyperfunction. This is a result by Kim-Chung-Kim in ref. [3] generalizing the result of refs. [1, 2] for Fourier hyperfunctions.

In both results above one can observe that it is the heat kernel to play the central role for the proof of main theorems. Our effort in this paper is to use the Mehler kernel $E(x, \xi, t)$ and characterize the Fourier hyperfunctions by the solutions of the Hermite heat equation in $\mathbf{R} \times (0, T)$. More precisely

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'For every $\epsilon > 0$ and every Fourier hyperfunction u , the function $U(x, t) := \langle u(\xi), E(x, \xi, t) \rangle$ is the smooth function in $\mathbf{R} \times (0, T)$ such that

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2 \right) U(x, t) = 0, \\ \sup_{x \in \mathbf{R}} |U(x, t)| \leq C \exp(\epsilon t^{-1}), \quad (1)$$

for some constant $C := C(\epsilon) > 0$. Conversely let $U(x, t)$ be the solution of the Hermite heat equation in $\mathbf{R} \times (0, T)$ such that the growth estimate (1) holds, then there exists a unique Fourier hyperfunction u such that $U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle$. Moreover, we show that $\lim_{t \rightarrow 0^+} U(x, t) = u(x)$ in the generalized sense.'

Since $(\mathcal{S}_1^1)'$, the dual space of the Gelfand–Shilov space $\mathcal{S}_1^1$ and the space of Fourier hyperfunctions are topologically the same, we will prove the above main result in the space $(\mathcal{S}_1^1)'$. Furthermore, we provide a generalization of this result even for the space $(\mathcal{S}_r^r)'$, $2r > 1$ by providing a suitable growth on the solution of the Hermite heat equation. The important aspect of our result is that it extends the result on tempered distributions obtained in ref. [4].

To prove the main theorem, we heavily make application of theory of generalized functions. In particular, we largely depend on the characterization of the spaces $\mathcal{S}_r^r$, $(\mathcal{S}_r^r)'$ by the growth on Hermite coefficients and some results on the Mehler kernel prepared in the following section.

2. Preliminaries

DEFINITION 2.1 ([3]) (i) We denote by $\mathcal{F}$ the set of all infinitely differentiable functions ϕ on $\mathbf{R}$ such that for some $B, a > 0$

$$|\phi|_{B,a} = \sup_{x \in \mathbf{R}, \beta \in \mathbf{N}_0} \frac{|\phi^{(\beta)}(x)| \exp(a|x|)}{B^\beta \beta!} < \infty. \quad (2)$$

(ii) We say that $\phi_j \rightarrow 0$ as $j \rightarrow \infty$ if $|\phi_j|_{B,a} \rightarrow 0$ as $j \rightarrow \infty$.

(iii) We denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements Fourier hyperfunctions.

DEFINITION 2.2 ([5]) (i) For $\alpha, \beta \in \mathbf{N}_0$, the space $\mathcal{S}_{r,A}^{r,B}$ for $r \geq 0$; $A, B > 0$ consists of all infinitely differentiable functions ϕ on $\mathbf{R}$ such that

$$|x^\alpha \phi^{(\beta)}(x)| \leq C_\phi A^\alpha B^\beta \alpha^{\alpha r} \beta^{\beta r},$$

for some constant C_ϕ depending on ϕ . This is a Banach space with the norm

$$\|\phi\|_{\mathcal{S}_{r,A}^{r,B}} = \sup_{\alpha, \beta} \frac{|x^\alpha \phi^{(\beta)}(x)|}{A^\alpha B^\beta \alpha^{\alpha r} \beta^{\beta r}}.$$

The space $\mathcal{S}_r^r$ is defined by the inductive limit $\mathcal{S}_r^r = \lim_{A,B} \mathcal{S}_{r,A}^{r,B}$

(ii) We denote by $(\mathcal{S}_r^r)'$ the strong dual of $\mathcal{S}_r^r$.

Remark 2.1 (i) The space $\mathcal{S}_r^r$ is nontrivial if $r \geq 1/2$ otherwise it degenerates to the single function $\phi \equiv 0$.

(ii) Let $\phi \in \mathcal{S}_r^r$, $2r \geq 1$. Then we have the inequality

$$|\phi^{(\beta)}(x)| \leq C B^\beta \beta^{\beta r} \exp(-a|x|^{1/r}), \quad \beta \in \mathbf{N}_0, \quad (3)$$

where the positive constants C , B , and a depend on the function ϕ .

(iii) In view of equations (2) and (3), it is easy to see that the spaces $\mathcal{S}_1^1$ and $\mathcal{F}$ are topologically same.

As a main tool for the proof of the main theorem, we have the following

THEOREM 2.1 ([6]) (i) Let $\phi \in \mathcal{S}_r^r$, $2r \geq 1$. Then there exist some constants $C, L > 0$ such that $\phi = \sum_{k=0}^{\infty} \langle \phi, h_k \rangle h_k$ and $|\langle \phi, h_k \rangle| \leq C \exp(-Lk^{1/2r})$. Conversely the series $\sum_{k=0}^{\infty} a_k h_k$ converges in $\mathcal{S}_r^r$ if the coefficients a_k satisfy the above growth.

(ii) Let $u \in (\mathcal{S}_r^r)'$, $2r \geq 1$. Then for every $L > 0$, there exists a constant $C := C(L) > 0$ such that $|\langle u, h_k \rangle| \leq C \exp(Lk^{1/2r})$. Conversely the series $\sum_{k=0}^{\infty} b_k h_k$ converges in $(\mathcal{S}_r^r)'$ if the coefficients b_k satisfy the above growth. Moreover if $u \in (\mathcal{S}_r^r)'$, then $u = \sum_{k=0}^{\infty} \langle u, h_k \rangle h_k$ in the sense that $\langle u, \phi \rangle = \sum_{k=0}^{\infty} \langle u, h_k \rangle \langle \phi, h_k \rangle$ for all $\phi \in \mathcal{S}_r^r$.

In the above h_k , $k = 0, 1, 2, \dots$ are L^2 -normalized Hermite functions defined by

$$h_k(x) = \frac{(-1)^k e^{x^2/2}}{\sqrt{2^k k! \sqrt{\pi}}} \frac{\partial^k}{\partial x^k} e^{-x^2}.$$

It is well known that h_k , $k = 0, 1, 2, \dots$ are the eigenfunctions of the eigenvalue problem $(-\partial^2/\partial x^2 + x^2)\psi = \lambda\psi$. The corresponding eigenvalues are $2k + 1$, $k = 0, 1, 2, \dots$. In view of the result in ref. ([7], p. 171), it is easy to find a constant $G > 0$ such that

$$|h_k(x)| \leq G \text{ for all } x \in \mathbf{R} \text{ and all } k \in \mathbf{N}_0. \quad (4)$$

Furthermore, for $N = 2k + 1$, h_k satisfies the following estimates (see [8], p. 26):

$$|h_k(x)| \leq \begin{cases} C(N^{1/3} + |x^2 - N|)^{-1/4}, & x^2 \leq 2N \\ C e^{-\gamma x^2}, & x^2 > 2N, \end{cases} \quad (5)$$

where γ is a fixed positive constant.

As introduced in ref. [9], let $E(x, \xi, t)$ be the Mehler kernel defined by

$$E(x, \xi, t) = \begin{cases} \sum_{k=0}^{\infty} e^{-(2k+1)t} h_k(x) h_k(\xi), & t > 0 \\ 0, & t \leq 0 \end{cases}, \quad (6)$$

where $x, \xi \in \mathbf{R}$ and its explicit form for $t > 0$ is

$$E(x, \xi, t) = \frac{e^{-t}}{\sqrt{\pi}(1 - e^{-4t})^{1/2}} e^{-(1/2)(1+e^{-4t}/1-e^{-4t})(x-\xi)^2 - (1-e^{-2t}/1+e^{-2t})x\xi}. \quad (7)$$

We note that for each ξ , $E(x, \xi, t)$ satisfies the Hermite heat equation $(\partial/\partial t - \partial^2/\partial x^2 + x^2)U(x, t) = 0$. Furthermore, $E(x, \xi, t) = \tilde{\eta}(x, t) \tilde{E}(x, \xi, t)$ where

$$\tilde{\eta}(x, t) = \frac{2^{1/2} e^{-t}}{(1 + e^{-4t})^{1/2}} e^{-(1/2)(1-e^{-4t}/1+e^{-4t})x^2}, \quad (8)$$

$$\tilde{E}(x, \xi, t) = (2\pi)^{-1/2} \left(\frac{1 + e^{-4t}}{1 - e^{-4t}} \right)^{1/2} e^{-1/2(1+e^{-4t}/1-e^{-4t})(\xi - (2e^{-2t}/1+e^{-4t})x)^2} \quad (9)$$

for $x, \xi \in \mathbf{R}$ and $t > 0$.

LEMMA 2.1 (Lemma 3.1, [4]) For any $\delta > 0$

$$\int_{\mathbf{R}} \tilde{E}(x, \xi, t) d\xi = 1, \quad (10)$$

$$\int_{|\xi - (2e^{-2t}/1 + e^{-4t})x| \geq \delta} \tilde{E}(x, \xi, t) d\xi \longrightarrow 0 \text{ uniformly for } x \in \mathbf{R} \text{ as } t \longrightarrow 0^+. \quad (11)$$

LEMMA 2.2 Let $g(x)$ be a continuous function on $\mathbf{R}$ satisfying that for every $\epsilon > 0$, there exists a constant $C := C(\epsilon) > 0$ such that

$$|g(x)| \leq C \exp(\epsilon x^2) \quad (12)$$

and let $U(x, t) := \int_{\mathbf{R}} g(\xi) E(x, \xi, t) d\xi$. Then $U(x, t)$ is a well-defined C^∞ function in $\mathbf{R} \times (0, 1/2 \sinh^{-1}(1/4\epsilon))$ and satisfies that

- (i) $(\partial/\partial t - \partial^2/\partial x^2 + x^2)U(x, t) = 0$, $0 < t < 1/2 \sinh^{-1}(1/4\epsilon)$.
- (ii) For some constant $C := C(\epsilon) > 0$, $|U(x, t)| \leq C \exp(2\epsilon x^2)$, $0 < t < 1/2 \sinh^{-1}(1/8\epsilon)$.
- (iii) $U(x, t) \rightarrow g(x)$ uniformly on each compact subset of $\mathbf{R}$ as $t \rightarrow 0^+$.

Proof The assertion (i) is obvious. We put $\sqrt{1/2 \coth 2t}(\xi - (2e^{-2t}/1 + e^{-4t})x) = s$. Then

$$\begin{aligned} |g(\xi)| &\leq \left| g \left(\sqrt{2 \tanh 2t} s + \frac{2e^{-2t}}{1 + e^{-4t}} x \right) \right| \\ &\leq C \exp \left[\epsilon \left(\sqrt{2 \tanh 2t} s + \frac{2e^{-2t}}{1 + e^{-4t}} x \right)^2 \right]. \end{aligned} \quad (13)$$

It then follows from equation (13) that

$$|g(\xi)| \leq C \exp \left[\frac{8\epsilon e^{-4t}}{(1 + e^{-4t})^2} x^2 + 4\epsilon \tanh 2t s^2 \right] \leq C \exp [2\epsilon x^2 + 4\epsilon \sinh 2t s^2],$$

since $e^{-4t}/(1 + e^{-4t})^2 = 1/(4 \cosh^2 2t)$ and $\cosh 2t > 1$ for $t > 0$. For $0 < t < (1/2) \sinh^{-1}(1/8\epsilon)$,

$$\begin{aligned} |U(x, t)| &\leq \tilde{\eta}(x, t) \int_{\mathbf{R}} |g(\xi)| \tilde{E}(x, \xi, t) d\xi \\ &\leq C \pi^{-1/2} \exp [2\epsilon x^2] \int_{\mathbf{R}} \exp [(4\epsilon \sinh 2t - 1)s^2] ds \\ &= \frac{C \exp [2\epsilon x^2]}{(1 - 4\epsilon \sinh 2t)^{1/2}} \\ &\leq C' \exp [2\epsilon x^2], \end{aligned}$$

where $C' > 0$ is a constant depending on ϵ .

To prove (iii), let $\delta > 0$ and $0 < t < (1/2)\sinh^{-1}(1/4\epsilon)$. Then

$$\begin{aligned}
 & |U(x, t) - g(x)| \\
 & \leq \tilde{\eta}(x, t) \int_{\mathbf{R}} |g(\xi) - g(x)| \tilde{E}(x, \xi, t) d\xi + |\tilde{\eta}(x, t) - 1| |g(x)| \\
 & \leq \tilde{\eta}(x, t) \sup_{|\xi - (2e^{-2t}/1 + e^{-4t})x| < \delta} |g(\xi) - g(x)| \int_{|\xi - (2e^{-2t}/1 + e^{-4t})x| < \delta} \tilde{E}(x, \xi, t) d\xi \\
 & \quad + \tilde{\eta}(x, t) |g(x)| \int_{|\xi - (2e^{-2t}/1 + e^{-4t})x| \geq \delta} \tilde{E}(x, \xi, t) d\xi \\
 & \quad + \tilde{\eta}(x, t) \int_{|\xi - (2e^{-2t}/1 + e^{-4t})x| \geq \delta} |g(\xi)| \tilde{E}(x, \xi, t) d\xi + |\tilde{\eta}(x, t) - 1| |g(x)| \\
 & = I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

Let K be a compact subset of $\mathbf{R}$. Since $g(x)$ is uniformly continuous on a δ -neighborhood K_δ of K , it follows that for any $\epsilon > 0$, $|\xi - x| < \delta$ implies $|g(\xi) - g(x)| < \epsilon$ for $\xi, x \in K_\delta$. Let $|g(x)| \leq C(K)$ for every $x \in K$. We note that $\tilde{\eta}(x, t) \rightarrow 1$ in view of equation (8) as $t \rightarrow 0^+$. Then clearly I_4 tends to zero as $t \rightarrow 0^+$. Furthermore, for every $x \in K$, I_1 tends to zero as $t \rightarrow 0^+$ since

$$\left| \xi - \frac{2e^{-2t}}{1 + e^{-4t}}x \right| < \delta \implies |\xi - x| < \delta \text{ as } t \rightarrow 0^+$$

and hence applying the uniform continuity of g on K_δ . In view of Lemma 2.1, I_2 tends to zero as $t \rightarrow 0^+$. For every $x \in K$,

$$|I_3| \leq C \tilde{\eta}(x, t) \pi^{-(1/2)} e^{2\epsilon x^2} \int_{|s| \geq \sqrt{1/2 \coth 2t} \delta} e^{(4\epsilon \sinh 2t - 1)s^2} ds \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

This completes the proof. ■

3. Main Result

LEMMA 3.1 ([10]) *For any $h > 0$, $s > 1$ and a small $\epsilon > 0$ there exist functions $v(t)$, $w(t) \in \mathcal{C}_0^\infty(\mathbf{R})$ and a differential operator $P(d/dt)$ of infinite order such that*

- (i) $P(d/dt)v(t) + w(t) = \delta$,
- (ii) $\text{supp } v \subset [0, \epsilon]$, $\text{supp } w \subset [\epsilon/2, \epsilon]$,
- (iii) $P(d/dt) = \sum_{p=0}^\infty a_p (d/dt)^p$, $|a_p| \leq Ch^p/p!^s$,
- (iv) $|v(t)| \leq C \exp[-(ch/t)^{1/s-1}]$,

where δ is the Dirac measure and c is a constant depending only on s .

Remark 3.1 In the proof of Lemma 2.3 in ref. [10], we see that the inequality $|a_p| \leq Ch^p/p!^s$ is true even for any $h > 0$.

THEOREM 3.1 *Let $\epsilon > 0$ be arbitrary. Then for every $u \in (\mathcal{S}_1^1)'$, the function $U(x, t) := \langle u(\xi), E(x, \xi, t) \rangle$ is the smooth function in $\mathbf{R} \times (0, T)$ such that*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2 \right) U(x, t) &= 0, \\ \sup_{x \in \mathbf{R}} |U(x, t)| &\leq C \exp(\epsilon t^{-1}), \end{aligned} \quad (14)$$

for some constant $C := C(\epsilon) > 0$.

Conversely let $U(x, t)$ be the solution of the Hermite heat equation in $\mathbf{R} \times (0, T)$ such that the growth estimate (14) holds. Then there exists a unique element $u \in (\mathcal{S}_1^1)'$ such that $U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle$. Moreover $\lim_{t \rightarrow 0^+} U(x, t) = u(x)$ in $(\mathcal{S}_1^1)'$.

Proof By using the orthogonality of the normalized Hermite functions on $L^2(\mathbf{R})$, the defining function $U(x, t)$ of u in $\mathbf{R} \times (0, T)$ takes the form

$$U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle = \sum_{k=0}^{\infty} e^{-(2k+1)t} \langle u(\xi), h_k(\xi) \rangle h_k(x). \quad (15)$$

We easily see that it satisfies the Hermite heat equation. Moreover, it is smooth in $\mathbf{R} \times (0, T)$ by the hypoelliptic property of the operator $\partial/\partial t - \partial^2/\partial x^2 + x^2$ (see [11] p. 168).

It follows from equations (4), (15) and Theorem 2.1 that

$$|U(x, t)| \leq CG \sum_{k=0}^{\infty} e^{Lk^{1/2}-2kt} \leq CG \left\{ \sup_k e^{Lk^{1/2}-kt} \right\} \sum_{k=0}^{\infty} e^{-kt}. \quad (16)$$

But

$$\sup_k e^{Lk^{1/2}-kt} = e^{\sup_k \{Lk^{1/2}-kt\}} = e^{(1/4)L^2 t^{-1}}. \quad (17)$$

Also

$$\sum_{k=0}^{\infty} e^{-kt} = 1 + \sum_{k=1}^{\infty} e^{-kt} \leq 1 + \frac{2\pi^2}{3t^2}. \quad (18)$$

It then follows from equations (16)–(18) that

$$|U(x, t)| \leq CG \left(1 + \frac{2\pi^2}{3t^2} \right) e^{(1/4)L^2 t^{-1}}. \quad (19)$$

Let $\epsilon = L^2/2$. Then $\epsilon > 0$ is arbitrary since $L > 0$ is arbitrary. Now it is easy to obtain from (19) that

$$\sup_{x \in \mathbf{R}} |U(x, t)| \leq C \exp(\epsilon t^{-1}),$$

for some positive constant $C := C(\epsilon)$.

We now prove the converse. Let $T_0 > 0$ be such that

$$2T_0 < \min(T, \sinh^{-1}(1/8\epsilon)).$$

We take $s = 2$. Then it follows from Lemma 3.1 that for any $h > 0$ there exist functions $v(t) \in C_0^\infty([0, T_0])$, $w(t) \in C_0^\infty([T_0/2, T_0])$ and a differential operator $P(d/dt)$ of infinite order such

that

$$P\left(\frac{d}{dt}\right)v(t) + w(t) = \delta, \quad (20)$$

$$P\left(\frac{d}{dt}\right) = \sum_{p=0}^{\infty} a_p \left(\frac{d}{dt}\right)^p, \quad |a_p| \leq \frac{Ch^p}{p!^2}, \quad (21)$$

$$|v(t)| \leq C \exp\left[-\left(\frac{ch}{t}\right)\right]. \quad (22)$$

Define two functions $G(x, t)$ and $H(x, t)$ in $\mathbf{R} \times (0, T_0]$ by

$$G(x, t) = \int_0^T U(x, t + \tau)v(\tau) d\tau, \quad H(x, t) = \int_0^T U(x, t + \tau)w(\tau) d\tau.$$

Then clearly the above integrals converge and are continuous functions on $\mathbf{R} \times [0, T_0]$. They clearly satisfy the Hermite heat equation and moreover in view of (22) and the hypothesis, they satisfy the following estimates:

$$|G(x, t)| \leq C \exp(\epsilon x^2),$$

$$|H(x, t)| \leq C \exp(\epsilon x^2),$$

for all $(x, t) \in \mathbf{R} \times [0, T_0]$ and for some constant $C := C(\epsilon) > 0$. If we take $g(x) = G(x, 0)$ and $h(x) = H(x, 0)$, then Lemma 2.2 implies that the functions

$$\int_{\mathbf{R}} g(\xi)E(x, \xi, t) d\xi \quad \text{and} \quad \int_{\mathbf{R}} h(\xi)E(x, \xi, t) d\xi$$

are continuous on $\mathbf{R} \times [0, T_0]$, which converge to $g(x)$ and $h(x)$, respectively, as $t \rightarrow 0^+$. By Theorem 1.1 of ref. [9], we have

$$G(x, t) = \int_{\mathbf{R}} g(\xi)E(x, \xi, t) d\xi \quad \text{and} \quad H(x, t) = \int_{\mathbf{R}} h(\xi)E(x, \xi, t) d\xi \quad \text{in } \mathbf{R} \times [0, T_0]. \quad (23)$$

In view of equation (20), it is easy to see that

$$P\left(-\frac{\partial}{\partial t}\right)G(x, t) = P\left(-\frac{\partial^2}{\partial x^2} + x^2\right)G(x, t) = U(x, t) - H(x, t) \quad \text{in } \mathbf{R} \times [0, T_0]. \quad (24)$$

It follows from equations (23) and (24) that

$$U(x, t) = P\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\langle g(\xi), E(x, \xi, t) \rangle - \langle h(\xi), E(x, \xi, t) \rangle \quad \text{in } \mathbf{R} \times [0, T_0]. \quad (25)$$

But for each $x \in \mathbf{R}$ and $0 < t < T_0$, we have

$$\begin{aligned} & \sum_{p=0}^{\infty} \int_{\mathbf{R}} \left| g(\xi) a_p \left(-\frac{\partial^2}{\partial x^2} + x^2\right)^p E(x, \xi, t) \right| d\xi \\ & \leq \sum_{p=0}^{\infty} \left[\int_{|\xi| \leq \sqrt{2N}} + \int_{|\xi| \geq \sqrt{2N}} \right] |g(\xi)| |a_p| \sum_{k=0}^{\infty} e^{-(2k+1)t} (2k+1)^p |h_k(x)| |h_k(\xi)| d\xi \\ & = Q_1 + Q_2 \end{aligned}$$

where $N = 2k + 1$. In view of equations (4), (21) and the growth estimate of g , we can find a constant $C_1 := C_1(\epsilon) > 0$ such that

$$\begin{aligned} Q_1 &\leq C_1 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \sum_{k=0}^{\infty} e^{-(2k+1)t} (2k+1)^p \int_{|\xi| \leq \sqrt{2N}} e^{\epsilon \xi^2} d\xi \\ &\leq 4C_1 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \sum_{k=0}^{\infty} e^{(2\epsilon-t)(2k+1)} (2k+1)^{p+1}. \end{aligned} \quad (26)$$

Choose ϵ such that $\epsilon < t/4$. Then it follows from equation (26) that

$$\begin{aligned} Q_1 &\leq 4C_1 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \sum_{k=0}^{\infty} e^{-(2k+1)(t/2)} (2k+1)^{p+1} \\ &\leq 4C_1 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \left(1 + 3^{p+1} \sum_{k=1}^{\infty} \frac{k^{p+1}}{e^{kt}} \right), \\ &\leq 4C_1 \left(\sum_{p=0}^{\infty} \frac{h^p}{p!} + \frac{3\pi^2}{6t^3} \sum_{p=0}^{\infty} \frac{3^p h^p (p+3)!}{t^p p!^2} \right). \end{aligned}$$

With the aid of inequality $(p+3)! \leq p!3!2^{p+3}$, we have

$$Q_1 \leq 4C_1 \left(e^h + \frac{24\pi^2}{t^3} \sum_{p=0}^{\infty} \frac{(6h/t)^p}{p!} \right) \leq 4C_1 \left(e^h + \frac{24\pi^2}{t^3} \exp\left(\frac{6h}{t}\right) \right). \quad (27)$$

The right hand side of equation (27) is a finite number depending on ϵ and t . Again in view of equations (4), (5), (21) and the growth estimate of g , we can find some positive constants C_2, C_3 depending on ϵ such that

$$\begin{aligned} Q_2 &\leq C_2 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \sum_{k=0}^{\infty} e^{-(2k+1)t} (2k+1)^p \int_{|\xi| \geq \sqrt{2N}} e^{(\epsilon-\gamma)\xi^2} d\xi \\ &\leq C_3 \sum_{p=0}^{\infty} \frac{h^p}{p!^2} \left(1 + 3^p \sum_{k=1}^{\infty} \frac{k^p}{e^{2kt}} \right) \\ &\leq C_3 \left(\sum_{p=0}^{\infty} \frac{h^p}{p!} + \frac{\pi^2}{6t^2} \sum_{p=0}^{\infty} \frac{3^p h^p (p+2)!}{2^{p+2} t^p p!^2} \right) \end{aligned}$$

by choosing ϵ such that $\epsilon < \min(\gamma, t/4)$. Using the inequality $(p+2)! \leq p!2!2^{p+2}$, we have

$$Q_2 \leq C_3 \left(e^h + \frac{\pi^2}{3t^2} \sum_{p=0}^{\infty} \frac{(3h/t)^p}{p!} \right) \leq C_3 \left(e^h + \frac{\pi^2}{3t^2} \exp\left(\frac{3h}{t}\right) \right). \quad (28)$$

The right hand side of equation (28) is a finite number depending on ϵ and t . Therefore, the Lebesgue dominated convergence theorem implies that

$$P \left(-\frac{\partial^2}{\partial x^2} + x^2 \right) \langle g(\xi), E(x, \xi, t) \rangle = \langle g(\xi), P \left(-\frac{\partial^2}{\partial x^2} + x^2 \right) E(x, \xi, t) \rangle. \quad (29)$$

By the growth estimate of g , we see that $g \in (\mathcal{S}_1^1)'$. We claim that

$$P\left(-\frac{\partial^2}{\partial \xi^2} + \xi^2\right)g(\xi) \in (\mathcal{S}_1^1)'.$$

Let $\phi \in \mathcal{S}_1^1$. Then $\langle P(-\partial^2/\partial \xi^2 + \xi^2)g(\xi), \phi(\xi) \rangle = \langle g(\xi), P(-\partial^2/\partial \xi^2 + \xi^2)\phi(\xi) \rangle$. The claim will be proved once we show that $P(-\partial^2/\partial \xi^2 + \xi^2)\phi(\xi) \in \mathcal{S}_1^1$. It follows from Theorem 2.1 that

$$\begin{aligned} \sum_{p=0}^{\infty} a_p \left(-\frac{\partial^2}{\partial \xi^2} + \xi^2\right)^p \phi &= \sum_{p=0}^{\infty} a_p \sum_{k=0}^{\infty} \left\langle \left(-\frac{\partial^2}{\partial \xi^2} + \xi^2\right)^p \phi, h_k \right\rangle h_k \\ &= \sum_{p=0}^{\infty} a_p \sum_{k=0}^{\infty} \langle \phi, h_k \rangle (2k+1)^p h_k \\ &= \sum_{k=0}^{\infty} \langle \phi, h_k \rangle \left(\sum_{p=0}^{\infty} a_p (2k+1)^p \right) h_k. \end{aligned} \quad (30)$$

The interchange of the sums above follows from the fact that

$$\begin{aligned} \sum_{p=0}^{\infty} \frac{h^p (2k+1)^p}{p!^2} &\leq \exp[2(2hk+h)^{1/2}] \leq \exp[2^{1/2} \{(2hk)^{1/2} + h^{1/2}\}] \\ &\leq \exp[2(2h)^{1/2}] \exp[2(4hk)^{1/2}] \end{aligned} \quad (31)$$

and hence for some constant $C > 0$

$$\begin{aligned} \sum_{k=0}^{\infty} |\langle \phi, h_k \rangle| \sum_{p=0}^{\infty} |a_p| (2k+1)^p |h_k| \\ \leq C \exp[2(2h)^{1/2}] \sum_{k=0}^{\infty} \exp[\{2(4h)^{1/2} - L\} k^{1/2}] < \infty \end{aligned}$$

in view of equations (4), (21), (31), Theorem 2.1 and the fact that $h > 0$ is arbitrary. Again in view of Theorem 2.1 and since $h > 0$ is arbitrary, the Hermite coefficient of the series (30) satisfies the following estimate:

$$\left| \langle \phi, h_k \rangle \left(\sum_{p=0}^{\infty} a_p (2k+1)^p \right) \right| \leq C \exp[2(2h)^{1/2}] \exp[\{2(4h)^{1/2} - L\} k^{1/2}] \leq C' e^{-L' k^{1/2}},$$

for some positive constants C' and L' . This follows that $P(-\partial^2/\partial \xi^2 + \xi^2)\phi(\xi) \in \mathcal{S}_1^1$. It now follows from equation (29) that

$$\begin{aligned} P\left(-\frac{\partial^2}{\partial x^2} + x^2\right) \langle g(\xi), E(x, \xi, t) \rangle &= \left\langle g(\xi), P\left(-\frac{\partial^2}{\partial \xi^2} + \xi^2\right) E(x, \xi, t) \right\rangle \\ &= \left\langle P\left(-\frac{\partial^2}{\partial \xi^2} + \xi^2\right) g(\xi), E(x, \xi, t) \right\rangle. \end{aligned} \quad (32)$$

Then it follows from equations (25) and (32) that

$$\begin{aligned} U(x, t) &= \left\langle P \left(-\frac{\partial^2}{\partial \xi^2} + \xi^2 \right) g(\xi), E(x, \xi, t) \right\rangle + \langle h(\xi), E(x, \xi, t) \rangle \\ &= \left\langle P \left(-\frac{\partial^2}{\partial \xi^2} + \xi^2 \right) g(\xi) + h(\xi), E(x, \xi, t) \right\rangle. \end{aligned} \quad (33)$$

Define $u := P(-\partial^2/\partial \xi^2 + \xi^2)g + h$. Then $u \in (\mathcal{S}'_1)'$. Hence $U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle$. The uniqueness of u follows from the uniqueness of the coefficients of the Hermite series.

We now prove the last part. Let $\phi \in \mathcal{S}'_1$. Using equation (15) and the growth estimates of $\langle u, h_k \rangle$ and $\langle \phi, h_k \rangle$ from Theorem 2.1, we have

$$\lim_{t \rightarrow 0^+} \langle U, \phi \rangle = \lim_{t \rightarrow 0^+} \sum_{k=0}^{\infty} e^{-(2k+1)t} \langle u, h_k \rangle \langle \phi, h_k \rangle = \left\langle \sum_{k=0}^{\infty} \langle u, h_k \rangle h_k, \phi \right\rangle = \langle u, \phi \rangle.$$

This completes the proof. ■

Remark 3.2 (i) Let $2r > 1$ and let $s = 2r$ in Lemma 3.1. Then as an easy generalization of Theorem 3.1 for $(\mathcal{S}'_r)'$ we have the following theorem.

THEOREM 3.1 *Let $\epsilon > 0$ be arbitrary and $2r > 1$. Then for every $u \in (\mathcal{S}'_r)'$, the function $U(x, t) := \langle u(\xi), E(x, \xi, t) \rangle$ is the smooth function in $\mathbf{R} \times (0, T)$ such that*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2 \right) U(x, t) &= 0, \\ \sup_{x \in \mathbf{R}} |U(x, t)| &\leq C \exp(\epsilon t^{-(1/2r-1)}) \end{aligned} \quad (34)$$

for some constant $C := C(\epsilon) > 0$.

Conversely let $U(x, t)$ be the solution of the Hermite heat equation in $\mathbf{R} \times (0, T)$ such that the growth estimate (14) holds. Then there exists a unique element $u \in (\mathcal{S}'_r)'$ such that $U(x, t) = \langle u(\xi), E(x, \xi, t) \rangle$. Moreover, $\lim_{t \rightarrow 0^+} U(x, t) = u(x)$ in $(\mathcal{S}'_r)'$.

(ii) The important aspect of Theorem 3.2 is that it not only generalizes Theorem 3.1 but extends the result on tempered distributions obtained in ref. [4].

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Expansions of multivariable Bessel functions

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First kind multivariable Bessel functions $J_n^{(m)}(\{ax\}_1^m)$ are expanded in terms of $J_K^{(m)}(\{x\}_1^m)J_{n-K}^{(m)}((a-1)x_1^m)$. Parallely we expand many variable Hermite-Bessel functions $B_n^{(m)}(\{ax\}_1^m)$ and draw further consequences.

Keywords: Multivariable Bessel functions; Hermite-Bessel functions; Jacobi–Anger generating functions

2000 Mathematics Subject Classification: 33C30

1. Introduction

First kind multivariable Bessel functions $J_n^{(m)}(\{x\}_1^m)$ Dattoli and Torre [1] are defined through the Jacobi–Anger generating function

$$\sum_{n=-\infty}^{\infty} e^{in\vartheta} J_n^{(m)}(\{x\}_1^m) = e^{i \sum_{s=1}^m x_s \sin(s\vartheta)}, \{x\}_1^m = x_1, \dots, x_m \quad (1)$$

and by the integral representation

$$J_n^{(m)}(\{x\}_1^m) = \frac{1}{\pi} \int_0^\pi \cos \left(n\vartheta - \sum_{s=1}^m x_s \sin(s\vartheta) \right) d\vartheta. \quad (2)$$

For $m = 1$ and $e^{i\vartheta} = t$, equation (1) reduces to the Bessel functions of the first kind $J_n(x)$.

$$\sum_{n=-\infty}^{\infty} J_n(x) t^n = e^{(x/2)(t-(t/2))}, t \neq 0, n \in \mathbb{Z} \quad (3)$$

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Kapteyn type series [2] associated with the above function have been studied in [3] and written as

$$\sum_{n=-\infty}^{\infty} J_n^{(m)}(\{nx\}_1^m) = \frac{1}{1 - \sum_{s=1}^m sx_s} \sum_{s=1}^m |sx_s| < 1. \quad (4)$$

Along with multivariable Bessel functions, Hermite-Bessel functions defined by the following generating function

$$\sum_{n=-\infty}^{\infty} e^{in\vartheta} B_n^{(m)}(\{x\}_1^m) = e^{i \sum_{s=1}^m i^s x_s (\sin(s\vartheta))^s} \quad (5)$$

have been introduced in [4]. The functions $B_n^{(m)}(\{x\}_1^m)$ are linked to the multivariable Hermite polynomials [5] by the series expansion

$$B_n^{(m)}(\{x\}_1^m) = \sum_{r=0}^{\infty} \frac{(-1)^r H_{n+2r}(\{x\}_1^m)}{2^{n+2r} r! (n+r)!}. \quad (6)$$

The polynomials $H_n^{(m)}(\{x\}_1^m)$ are also specified by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(m)}(\{x\}_1^m) = e^{\sum_{s=1}^m x_s t^s}. \quad (7)$$

2. Expansions

In order to obtain expansions of multivariable Bessel functions, are considered the generating function (1) and written as

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{in\vartheta} J_n^{(m)}(\{ax\}_1^m) &= e^{i \sum_{s=1}^m a_s x_s \sin(s\vartheta)} = e^{i \sum_{s=1}^m (a_s - 1)x_s \sin(s\vartheta)} e^{i \sum_{s=1}^m x_s \sin(s\vartheta)} \\ &= \sum_{k=-\infty}^{\infty} e^{ik\vartheta} J_k^{(m)}(\{x\}_1^m) \sum_{l=-\infty}^{\infty} e^{il\vartheta} J_l^{(m)}(\{(a_s - 1)x_s\}_1^m) \end{aligned}$$

where $\{ax\}_1^m = a_1 x_1, a_2 x_2, \dots, a_m x_m$.

Substitution of $l = n - k$ for fixed values of $k \in \mathbb{Z}$ gives

$$\sum_{n=-\infty}^{\infty} e^{in\vartheta} J_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} J_k^{(m)}(\{x\}_1^m) \sum_{n=-\infty}^{\infty} J_{n-k}^{(m)}(\{(a_s - 1)x_s\}_1^m) e^{in\vartheta}$$

Equating the coefficients of powers of $e^{i\vartheta}$ on each side finally gives

$$J_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} J_k^{(m)}(\{x\}_1^m) J_{n-k}^{(m)}(\{(a_s - 1)x_s\}_1^m), \quad k, n \in \mathbb{Z} \quad (8)$$

In analogy to equation (1), the generating function for the multivariable extension of the modified Bessel functions of the first kind is

$$\sum_{n=-\infty}^{\infty} e^{in\vartheta} I_n^{(m)}(\{x\}_1^m) = e^{\sum_{s=1}^{\infty} x_s \cos(s\vartheta)}, \quad \{x\}_1^m = x_1, \dots, x_m \quad (9)$$

With this in mind, one may adopt the same procedure to obtain the following expansion for the above functions.

$$I_n^{(m)}(\{x\}_1^m) = \sum_{k=-\infty}^{\infty} I_k^{(m)}(\{x\}_1^m) I_{n-k}^{(m)}((a_s - 1)x_s)_1^m, \quad k, n \in \mathbb{Z} \quad (10)$$

Next we consider the generating function (5) in place of equation (1) and apply the same technique as before to get the following expansion of multivariable Hermite–Bessel functions $B_n^{(m)}$

$$B_n^{(m)}(\{ax\}_1^m) = \sum_{k=-\infty}^{\infty} B_k^{(m)}(\{x\}_1^m) B_{n-k}^{(m)}((a_s - 1)x_s)_1^m, \quad k, n \in \mathbb{Z}. \quad (11)$$

3. Special cases

The main results (8), (10) and (11) offer many special cases. A few are given below.

Set $m = 1$ in equations (8), (10) and (11), we get the following known results of Predel [6] for ordinary Bessel and modified Bessel functions

$$J_n(ax) = \sum_{k=-\infty}^{\infty} J_k(x) J_{n-k}((a - 1)x), \quad k, n \in \mathbb{Z}$$

$$I_n(ax) = \sum_{k=-\infty}^{\infty} I_k(x) I_{n-k}((a - 1)x), \quad k, n \in \mathbb{Z}$$

Setting $m = 3$ and assume that $x_2 = 0$ in equation (11). Then using [7]

$$B_n^{(3)}(x_1, x_3) = J_n^{(3)}\left(x_1 - \frac{3}{4}x_3, \frac{1}{4}x_3\right)$$

we get,

$$J_n^{(3)}\left(a_1x_1 - \frac{3}{4}a_3x_3, \frac{1}{4}a_3x_3\right) = \sum_{k=-\infty}^{\infty} J_k^{(3)}\left(x_1 - \frac{3}{4}x_3, \frac{1}{4}x_3\right) J_{n-k}^{(3)}\left((a_1 - 1)x_1 - \frac{3}{4}(a_3 - 1)x_3, \frac{1}{4}(a_3 - 1)x_3\right)$$

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Integral type operators from mixed-norm spaces to α -Bloch spaces

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Let $g : B \rightarrow \mathbb{C}^1$ be a holomorphic map of the unit ball B . We study the following integral operators

$$T_g f(z) = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t} \quad \text{and} \quad L_g f(z) = \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}, \quad z \in B.$$

The boundedness and compactness of operators T_g and L_g from mixed-norm spaces to the α -Bloch space $\mathcal{B}^\alpha$ and the little α -Bloch space on the unit ball are discussed in this article.

Keywords: Integral operator; Mixed-norm space; α -Bloch space

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1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in the complex vector space $\mathbb{C}^n$, $\partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k$. Let dv denote the normalized Lebesgue volume measure on the unit ball B such that $v(B) = 1$, and $d\sigma$ be the normalized rotation invariant measure on the boundary S of B such that $\sigma(S) = 1$. Let $H(B)$ denote the class of all holomorphic functions on the unit ball and $H^\infty = H^\infty(B)$ the space of all bounded holomorphic functions on the unit ball.

For $f \in H(B)$ with the Taylor expansion $f(z) = \sum_{|\beta| \geq 0} a_\beta z^\beta$, let $\Re f(z) = \sum_{|\beta| \geq 0} |\beta| a_\beta z^\beta$ be the radial derivative of f , where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$. It is well known that

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z),$$

see, for example [1].

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Let $\alpha > 0$. The α -Bloch space $\mathcal{B}^\alpha = \mathcal{B}^\alpha(B)$ is the space of all holomorphic functions f on B such that

$$b_\alpha(f) = \sup_{z \in B} (1 - |z|^2)^\alpha |\Re f(z)| < \infty.$$

It is clear that $\mathcal{B}^\alpha$ is a normed space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + b_\alpha(f)$, modulo constant functions and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which

$$(1 - |z|^2)^\alpha |\Re f(z)| \longrightarrow 0 \quad \text{as } |z| \longrightarrow 1.$$

This space is called the little α -Bloch space.

Let $p, q \in (0, \infty)$ and $\gamma > -1$, then the mixed norm space $H_{p,q,\gamma}(B)$ consists of all holomorphic functions f on the unit ball such that

$$\|f\|_{H_{p,q,\gamma}}^q = \int_0^1 M_p^q(f, r) (1 - r)^\gamma dr < \infty,$$

where

$$M_p(f, r) = \left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p}$$

Suppose that $g : B \rightarrow \mathbb{C}^1$ is a holomorphic map of the unit ball, for a holomorphic function $f : B \rightarrow \mathbb{C}^1$, define

$$T_g f(z) = \int_0^1 f(tz) \frac{dg(tz)}{dt} = \int_0^1 f(tz) \Re g(tz) \frac{dt}{t}, \quad z \in B. \quad (1)$$

This operator is called extended-Cesàro operator, and was studied in [2–4].

The following integral operator was defined in [5] for the case of holomorphic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Here, we extend operator I_g for the case of holomorphic functions on the unit ball as follows

$$L_g f(z) = \int_0^1 \Re f(tz) g(tz) \frac{dt}{t}, \quad z \in B. \quad (2)$$

The purpose of this article is to study the boundedness and compactness of the operators T_g and L_g from $H_{p,q,\gamma}$ to the α -Bloch space $\mathcal{B}^\alpha(B)$ and little α -Bloch space $\mathcal{B}_0^\alpha(B)$. This article can be considered as a natural continuation of our investigations of integral operators on $\mathbb{C}^n$, see [4, 6–9].

Throughout this article, constants are denoted by C , they are positive and may differ from one occurrence to the other. The notation $a \leq b$ means that there is a positive constant C such that $a \leq Cb$. If both $a \leq b$ and $b \leq a$ hold, then one says that $a \asymp b$.

2. Boundedness and compactness of the operators $T_g, L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$

In this section, we consider the boundedness and compactness of the operators $T_g, L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$. For this purpose, we need some auxiliary results which are incorporated in the following lemmas.

LEMMA 1 *Let $f \in H_{p,q,\gamma}(B)$. Then there is a positive constant C independent of f such that*

$$|f(z)| \leq C \frac{\|f\|_{H_{p,q,\gamma}}}{(1-|z|)^{(\gamma+1)/q+n/p}}. \quad (3)$$

Proof By the monotonicity of the integral means, and Theorem 7.2.5 in [10], we have

$$\begin{aligned} \|f\|_{H_{p,q,\gamma}}^q &\geq \int_{(1+|z|)/2}^1 M_p^q(f, r)(1-r)^\gamma dr \\ &\geq C M_p^q(f, (1+|z|)/2)(1-|z|)^{\gamma+1} \\ &\geq C(1-|z|)^{\gamma+1+(qn/p)} |f(z)|^q, \end{aligned}$$

from which the result follows. ■

Similar to the proof of Lemma 1 and by using the well-known asymptotic formula

$$\int_0^1 M_p^q(f, r)(1-r)^\gamma dr \asymp |f(0)|^q + \int_0^1 M_p^q(\Re f, r)(1-r)^{\gamma+q} dr$$

we obtain the following lemma.

LEMMA 2 *Let $f \in H_{p,q,\gamma}(B)$. Then there is a positive constant C independent of f such that*

$$|\Re f(z)| \leq C \frac{\|f\|_{H_{p,q,\gamma}}}{(1-|z|)^{(\gamma+1)/q+(n/p)+1}}. \quad (4)$$

The following lemma can be found in [11].

LEMMA 3 *For $\beta > -1$ and $m > 1 + \beta$ we have*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \leq C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1.$$

The next lemma is an analog of the following one-dimensional results

$$\left(\int_0^z f(\zeta) g'(\zeta) d\zeta \right)' = f(z) g'(z), \quad \left(\int_0^z f'(\zeta) g(\zeta) d\zeta \right)' = f'(z) g(z).$$

LEMMA 4 *For every $f, g \in H(B)$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z) \quad \text{and} \quad \Re[L_g(f)](z) = \Re f(z)g(z).$$

The first identity was proved in [2], and the second can be proved similarly.

The following criterion for compactness follows by standard arguments similar, for example, to those outlined in Proposition 3.11 of [12].

LEMMA 5 *The operator T_g (or L_g): $H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact if and only if T_g (or L_g) is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $H_{p,q,\gamma}$ which converges to zero uniformly on compact subsets of B as $k \rightarrow \infty$, we have $\|T_g f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $k \rightarrow \infty$ (or $\|L_g f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $k \rightarrow \infty$).*

Now, we are in a position to formulate and prove the main results of this section.

THEOREM 1 Assume that $p, q > 0$ and $\gamma > -1$. Then the operator $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in B} (1 - |z|)^{\alpha - (\gamma+1)/q - (n/p)} |\Re g(z)| < \infty. \quad (5)$$

Moreover, the following relationship holds

$$\|T_g\|_{H_{p,q,\gamma}(B) \rightarrow \mathcal{B}^\alpha} \asymp \sup_{z \in B} (1 - |z|^2)^{\alpha - ((\gamma+1)/q) - (n/p)} |\Re g(z)|. \quad (6)$$

Proof By Lemma 1, we have

$$\begin{aligned} (1 - |z|^2)^\alpha |\Re T_g f(z)| &= (1 - |z|^2)^\alpha |\Re g(z)| |f(z)| \\ &\leq C (1 - |z|^2)^{\alpha - ((\gamma+1)/q) - (n/p)} |\Re g(z)| \|f\|_{H_{p,q,\gamma}}. \end{aligned} \quad (7)$$

From equations (5) and (7), and since $(T_g f)(0) = 0$, it follows that $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded.

Conversely, suppose T_g is a bounded operator from $H_{p,q,\gamma}$ to $\mathcal{B}^\alpha$. For $w \in B$ and $b > (\gamma + 1)/q$, set

$$f_w(z) = \frac{(1 - |w|^2)^{b - ((\gamma+1)/q)}}{(1 - \langle z, w \rangle)^{(n/p) + b}}. \quad (8)$$

It is easy to see that

$$f_w(w) = \frac{1}{(1 - |w|^2)^{((\gamma+1)/q) + (n/p)}}, \quad |\Re f_w(w)| = \left(\frac{n}{p} + b\right) \frac{|w|^2}{(1 - |w|^2)^{((\gamma+1)/q) + (n/p) + 1}}. \quad (9)$$

By [10, Theorem 1.4.10], we know that

$$M_p(f_w, r) \leq C \frac{(1 - |w|^2)^{b - ((\gamma+1)/q)}}{(1 - r|w|)^b}.$$

Applying Lemma 3, we have that

$$\begin{aligned} \|f_w\|_{H_{p,q,\gamma}}^p &= \int_0^1 M_p^q(f_w, r) (1 - r)^\gamma dr \\ &\leq C \int_0^1 \frac{(1 - |w|^2)^{qb - (\gamma+1)}}{(1 - r|w|)^{qb}} (1 - r)^\gamma dr \\ &\leq C. \end{aligned}$$

Therefore $f_w \in H_{p,q,\gamma}$, and moreover $\sup_{w \in B} \|f_w\|_{H_{p,q,\gamma}} \leq C$. Hence

$$\begin{aligned} (1 - |w|^2)^\alpha |f_w(w) \Re g(w)| &\leq \sup_{z \in B} (1 - |z|^2)^\alpha |f_w(z) \Re g(z)| \\ &\leq \|T_g f_w\|_{\mathcal{B}^\alpha} \leq C \|T_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} \end{aligned}$$

which implies that

$$\sup_{w \in B} (1 - |w|^2)^{\alpha - ((\gamma+1)/q) - (n/p)} |\Re g(w)| \leq C \|T_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} < \infty, \quad (10)$$

as desired. From equations (7) and (10), (6) follows. ■

THEOREM 2 Assume that $p, q > 0$ and $\gamma > -1$. Then $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded if and only if

$$\sup_{z \in B} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| < \infty. \quad (11)$$

Moreover, the following relationship holds

$$\|L_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} \asymp \sup_{z \in B} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)|. \quad (12)$$

Proof Assume that equation (11) holds and let $f \in H_{p,q,\gamma}$, then by Lemma 2, we have

$$\begin{aligned} (1 - |z|^2)^\alpha |\Re(L_g f)(z)| &= (1 - |z|^2)^\alpha |\Re f(z)| |g(z)| \\ &\leq C (1 - |z|^2)^\alpha |g(z)| \frac{\|f\|_{H_{p,q,\gamma}}}{(1 - |z|^2)^{(n/p) - ((\gamma+1)/q) + 1}} \\ &\leq C \|f\|_{H_{p,q,\gamma}} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)|. \end{aligned} \quad (13)$$

Notice also that $(L_g f)(0) = 0$. Taking the supremum in equation (13) over $z \in B$, it follows that $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded.

Conversely, if $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded. For $a \in B$ and $r > 0$ the set

$$D(a, r) = \{z \in B : \beta(a, z) < r\}, \quad a \in B$$

is a Bergman metric ball centered at a with radius r . It is well known that [1]

$$\frac{(1 - |a|^2)^{n+1}}{|1 - \langle a, z \rangle|^{2(n+1)}} \asymp \frac{1}{(1 - |z|^2)^{n+1}} \asymp \frac{1}{(1 - |a|^2)^{n+1}} \asymp \frac{1}{|D(a, r)|} \quad (14)$$

when $z \in D(a, r)$ and where $|D(a, r)|$ is the volume of the Bergman ball $D(a, r)$. For $w \in B \setminus \{0\}$, let $f_w(z)$ be defined by equation (8), then by equations (9) and (14), we have that

$$\begin{aligned} &\frac{|w|^4}{(1 - |w|^2)^{2((\gamma+1)/q) + 2(n/p) + 2}} |g(w)|^2 \asymp |\Re f_w(w) g(w)|^2 \\ &\leq \frac{C}{(1 - |w|^2)^{n+1}} \int_{D(w,r)} |\Re f_w(z)|^2 |g(z)|^2 dv(z) \\ &= \frac{C}{(1 - |w|^2)^{n+1}} \int_{D(w,r)} |\Re f_w(z)|^2 |g(z)|^2 (1 - |z|^2)^{2\alpha} \frac{1}{(1 - |z|^2)^{2\alpha}} dv(z) \\ &\leq C \int_{D(w,r)} \frac{dv(z)}{(1 - |z|^2)^{2\alpha + n + 1}} \sup_{z \in D(w,r)} (1 - |z|^2)^{2\alpha} |\Re f_w(z)|^2 |g(z)|^2 \\ &\leq \frac{C}{(1 - |w|^2)^{2\alpha}} \|L_g f_w\|_{\mathcal{B}^\alpha}^2. \end{aligned}$$

Therefore,

$$(1 - |w|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(w)| |w|^2 \leq C \|L_g f_w\|_{\mathcal{B}^\alpha} \leq C \|L_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha}.$$

By the maximal modulus principle, we obtain

$$\sup_{z \in B} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| \leq C \|L_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} < \infty. \quad (15)$$

From equations (13) and (15), (12) easily follows. ■

THEOREM 3 Assume that $p, q > 0$ and $\gamma > -1$. Then the operator $T_g: H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)| = 0. \quad (16)$$

Proof Assume that equation (16) holds, in order to prove that T_g is compact, by Lemma 5, it suffices to show that if $(f_k)_{k \in \mathbb{N}}$ is a bounded sequence in $H_{p,q,\gamma}$ converging to 0 uniformly on compact subsets of B , then $\|T_g f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $k \rightarrow \infty$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $H_{p,q,\gamma}$ such that $\sup_{k \in \mathbb{N}} \|f_k\|_{H_{p,q,\gamma}} \leq L$ and $f_k \rightarrow 0$ uniformly on compact subsets of B as $k \rightarrow \infty$. From equation (16), we have that for every $\varepsilon > 0$, there is a constant $\delta \in (0, 1)$, such that $\delta < |z| < 1$ implies

$$(1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)| < \frac{\varepsilon}{L}.$$

Let $K = \{z \in B : |z| \leq \delta\}$ (K is a compact subset of B). By Lemma 1 we have

$$\begin{aligned} \|T_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z \in B} (1 - |z|^2)^\alpha |\Re(T_g f_k)(z)| \leq \sup_{z \in K} (1 - |z|^2)^\alpha |\Re g(z) f_k(z)| \\ &\quad + C \sup_{z \in B \setminus K} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)| \|f_k\|_{H_{p,q,\gamma}} \\ &\leq M \sup_{z \in K} (1 - |z|^2)^{((\gamma+1)/q) + (n/p)} |f_k(z)| + C\varepsilon, \end{aligned}$$

where

$$M = \sup_{z \in B} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)|.$$

Since $f_k \rightarrow 0$ uniformly on K as $k \rightarrow \infty$, we obtain

$$\|T_g f_k\|_{\mathcal{B}^\alpha} \longrightarrow 0$$

as $k \rightarrow \infty$. Therefore, $T_g: H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact.

Conversely, suppose $T_g: H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact. Let $(z_k)_{k \in \mathbb{N}}$ be a sequence in B such that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$. Let

$$f_k(z) = \frac{(1 - |z_k|^2)^{b - ((\gamma+1)/q)}}{(1 - \langle z, z_k \rangle)^{(n/p) + b}}. \quad (17)$$

Then $\sup_{k \in \mathbb{N}} \|f_k\|_{H_{p,q,\gamma}} \leq C$ and f_k converges to 0 uniformly on compact subsets of B as $k \rightarrow \infty$. Since T_g is compact, by Lemma 5 it follows that

$$\|T_g f_k\|_{\mathcal{B}^\alpha} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

From this and since

$$\begin{aligned} \|T_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z \in B} (1 - |z|^2)^\alpha |\Re(T_g f_k)(z)| = \sup_{z \in B} (1 - |z|^2)^\alpha |f_k(z)| |\Re g(z)| \\ &\geq (1 - |z_k|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z_k)| \end{aligned}$$

we obtain

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)| = 0,$$

finishing the proof of the theorem. ■

THEOREM 4 Assume that $p, q > 0$ and $\gamma > -1$. Then $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| = 0. \quad (18)$$

Proof Assume that condition (18) holds. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $H_{p,q,\gamma}$ with $\sup_{k \in \mathbb{N}} \|f_k\|_{H_{p,q,\gamma}} \leq K$ and $f_k \rightarrow 0$ uniformly on compact subsets of B as $k \rightarrow \infty$. From equation (18) we have that for every $\varepsilon > 0$, there is a constant $\delta \in (0, 1)$, such that

$$(1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| < \varepsilon, \quad (19)$$

whenever $\delta < |z| < 1$. Similar to the proof of Theorem 3, we can prove that

$$\lim_{k \rightarrow \infty} \|L_g f_k\|_{\mathcal{B}^\alpha} = 0.$$

Therefore, $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact.

Conversely, suppose $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is compact. Assume that $(z_k)_{k \in \mathbb{N}}$ is a sequence in B such that $|z_k| \rightarrow 1$ as $k \rightarrow \infty$, and $(f_k)_{k \in \mathbb{N}}$ is defined by equation (17).

Then $\sup_{k \in \mathbb{N}} \|f_k\|_{H_{p,q,\gamma}} \leq C$ and f_k converges to 0 uniformly on compact subsets of B as $k \rightarrow \infty$. Since L_g is compact, we have

$$\|L_g f_k\|_{\mathcal{B}^\alpha} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

In addition

$$\begin{aligned} \|L_g f_k\|_{\mathcal{B}^\alpha} &= \sup_{z \in B} (1 - |z|^2)^\alpha |\Re(L_g f_k)(z)| \\ &\geq \left(\frac{n}{p} + b \right) (1 - |z_k|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z_k)| |z_k|^2, \end{aligned}$$

which implies that

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| = 0.$$

■

3. Boundedness and compactness of the operators $T_g, L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$

In this section, we characterize the compactness of $T_g, L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$. For this purpose, we need the following lemma, which can be proved in a standard way [13].

LEMMA 6 A closed set K in $\mathcal{B}_0^\alpha$ is compact if and only if it is bounded and satisfies

$$\lim_{|z| \rightarrow 1} \sup_{f \in K} (1 - |z|^2)^\alpha |\Re f(z)| = 0. \quad (20)$$

THEOREM 5 Assume that $p, q > 0$ and $\gamma > -1$. Then the following statements hold.

- (i) $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $g \in \mathcal{B}_0^\alpha$ and $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded.
- (ii) $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q)} |\Re g(z)| = 0. \quad (21)$$

Proof

(i) It is clear that $g \in \mathcal{B}_0^\alpha$ and $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded, if $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is bounded. Conversely, suppose that $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded and $g \in \mathcal{B}_0^\alpha$. Since $g \in \mathcal{B}_0^\alpha$ and

$$(1 - |z|^2)^\alpha \Re(T_g p)(z) = (1 - |z|^2)^\alpha p(z) \Re g(z),$$

we have that $T_g p \in \mathcal{B}_0^\alpha$, for any polynomial $p(z)$. Since the set of all polynomials is dense in $H_{p,q,\gamma}$ we have that for each $f \in H_{p,q,\gamma}$, there exist a sequence of polynomials $(p_k)_{k \in \mathbb{N}}$ such that $\|f - p_k\|_{H_{p,q,\gamma}} \rightarrow 0$, as $k \rightarrow \infty$. From the boundedness of the operator $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$, we have

$$\|T_g f - T_g p_k\|_{\mathcal{B}^\alpha} \leq \|T_g\|_{H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha} \|f - p_k\|_{H_{p,q,\gamma}}.$$

From this and since $\mathcal{B}_0^\alpha$ is closed, we obtain

$$T_g f = \lim_{k \rightarrow \infty} T_g p_k \in \mathcal{B}_0^\alpha.$$

In addition, the boundedness of $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ obviously implies the boundedness of $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$.

(ii) If $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is compact, then by Theorem 3 we have that equation (21) holds.

Conversely, assume that equation (21) holds. From Lemma 6, it follows that $T_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} \sup_{\|f\|_{H_{p,q,\gamma}} \leq 1} (1 - |z|^2)^{\alpha - ((\gamma+1)/q) - (n/p)} \|f\|_{H_{p,q,\gamma}} = 0.$$

By Lemma 1, we have

$$(1 - |z|^2)^\alpha |\Re(T_g f)(z)| \leq C(1 - |z|^2)^{\alpha - ((\gamma+1)/q) - (n/p)} \|f\|_{H_{p,q,\gamma}}.$$

Taking the supremum in the last inequality over the set $\{f \in H(B) : \|f\|_{H_{p,q,\gamma}} \leq 1\}$ and letting $|z| \rightarrow 1$ the result follows. ■

Similar to the proof of Theorem 5, the following result can be proven. The details will be omitted.

THEOREM 6 *Assume that $p, q > 0$ and $\gamma > -1$. Then the following statements hold.*

(i) $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is bounded if and only if $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}^\alpha$ is bounded and

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |g(z)| = 0.$$

(ii) $L_g : H_{p,q,\gamma} \rightarrow \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha - (n/p) - ((\gamma+1)/q) - 1} |g(z)| = 0.$$

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Asymptotic expansion of a class of one-dimensional Fourier integrals

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The method of stationary phase is used to develop an asymptotic expansion of a class of one-dimensional Fourier integrals whose phase function is a general second-degree polynomial. Because of its form, the phase function contains one stationary point that may occur anywhere on the real line. The asymptotic expansion is obtained using the theory of distributions and Fourier transform analysis. In addition to the expansion, a bound on the remainder term is derived. Finally, it is shown that the resulting expansion is, indeed, asymptotic.

Keywords: Asymptotic expansions; Fourier integrals; Fourier transforms; Method of Stationary Phase; Distributions

AMS Subject Classification: 41A60; 41A80; 42A38; 42A85; 46F12; 60E05

1. Introduction

There are many problems that exist in mathematical physics and engineering for which a solution is obtained as a Fourier integral, having the form

$$I(\omega) = \int_{-\infty}^{\infty} g(x) e^{i\omega f(x)} dx, \quad (1)$$

where $g(x)$ is called the coefficient function and $f(x)$ represents the phase function. Most often one is interested in such integrals when the parameter ω approaches a limiting value, say $\omega \rightarrow \infty$. Although this type of integral cannot be solved in closed form, these solutions are important because there exist methods by which we can approximate Fourier integrals.

In this paper, the method of stationary phase[†] is used to develop an asymptotic expansion of a Fourier integral having the form

$$I(\omega) = \int_{-\infty}^{\infty} g(x) e^{i\omega(ax^2+bx+c)} dx, \quad (2)$$

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[†]For a complete discussion on the method of stationary phase, the reader is referred to [2].

where $g(x)$ represents a distribution belonging to the space of Schwartz' tempered distributions, $S'(R)$ (a subset of $D'(R)$ – the space of all distributions on R). Previous results [4, 6] concerning Fourier integrals similar to equation (2) have been obtained for the case where the phase function is a monomial. For such a case, the phase function contains one stationary point found at the origin. In what follows, one obtains results for the case where the phase function is a polynomial – specifically, a general second-degree polynomial. For this case, the phase function contains one stationary point that can exist anywhere on the real line. In addition, we provide a bound on the remainder term and also prove that the expansion is asymptotic.

2. Asymptotic expansion

For what follows we assume that $\tilde{g}(\xi)$, the inverse Fourier transform of $g(x)$, is an integrable function, continuous at $\xi = 0$, satisfying $|\tilde{g}(\xi)| \leq C_n |\xi|^{-n}$ for all integers $n \geq 0$, and $0 < |\xi| < \infty$, where the constant C_n is such that $C_n \leq Ce^n$ for some constant $0 < C < \infty$ and all $n \geq 0^\dagger$.

One may begin by factoring equation (2) to obtain

$$I(\omega) = e^{i\omega c} \int_{-\infty}^{\infty} g(x) e^{i\omega(ax^2+bx)} dx \quad (3)$$

and by letting $\tilde{g}(\xi)$ represent the inverse transform of $g(x)$ and $\wedge$ denote Fourier transformation, one may write equation (3) as,

$$I(\omega) = e^{i\omega c} \int_{-\infty}^{\infty} [\tilde{g}(\xi)]^\wedge(x) e^{i\omega(ax^2+bx)} dx \quad (4)$$

If we consider $g(x)$ and $e^{i\omega(ax^2+bx)}$ to be distributions, then one may use Parseval's formula [1], and express equation (4) as,

$$I(\omega) = e^{i\omega c} \int_{-\infty}^{\infty} \tilde{g}(\xi) (e^{i\omega(ax^2+bx)})^\wedge(\xi) d\xi, \quad (5)$$

where $(e^{i\omega(ax^2+bx)})^\wedge(\xi)$ is the Fourier transform of $e^{i\omega(ax^2+bx)}$.

Using the convolution, one may write,

$$(e^{i\omega(ax^2+bx)})^\wedge(\xi) = ((e^{i\omega ax^2})^\wedge(\xi) * (e^{i\omega bx})^\wedge(\xi))(\xi) \quad (6)$$

and substituting equation (6) into equation (5), one obtains

$$I(\omega) = e^{i\omega c} \int_{-\infty}^{\infty} \tilde{g}(\xi) ((e^{i\omega ax^2})^\wedge(\xi) * (e^{i\omega bx})^\wedge(\xi))(\xi) d\xi \quad (7)$$

From [6], one obtains

$$(e^{i\omega ax^2})^\wedge(\xi) = \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i\xi^2/4\omega a)}, \quad (8)$$

and from [5], one obtains

$$(e^{i\omega bx})^\wedge(\xi) = \delta(\xi + \omega b). \quad (9)$$

Since the class of Schwartz' tempered distributions, $S'(R)$, is closed under Fourier transformation [9], *i.e.* the Fourier transform of a tempered distribution is itself a tempered distribution,

[†]It can be shown, by induction, that C and C_n can always be found to exist.

the author has two distributions in ξ , namely,

$$G(\xi) = \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i\xi^2)/4\omega a} \quad \text{and} \quad H(\xi) = \delta(\xi + \omega b).$$

A sufficient condition for the convolution of two distributions[†] to exist is that at least one must have a bounded support [9]. Noting that the support of $\delta(\xi + \omega b)$ is the single point $\xi = -\omega b$, the convolution of G and H must exist, and the following is obtained

$$(e^{i\omega a x^2})^\wedge(\xi)^* (e^{i\omega b x})^\wedge(\xi) = G(\xi)^* H(\xi) = \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i\xi^2)/4\omega a} * \delta(\xi + \omega b).$$

Now since $e^{i\omega a x^2} \in S'$, its Fourier transform, $\sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i\xi^2)/4\omega a}$, must also be in S' , hence, the following formula may be used,

$$f(t)^* \delta(t - a) = \delta(t - a)^* f(t) = f(t - a),$$

which is valid for every distribution f in D' , and thus also in S' , since $S' \subset D'$, [9].

Therefore, the convolution of the two distributions represented by equation (8) and equation (9) is given by

$$(e^{i\omega a x^2})^\wedge(\xi)^* (e^{i\omega b x})^\wedge(\xi) = \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i(\xi + \omega b)^2)/4\omega a} \quad (10)$$

Substituting equation (10) into equation (7) one obtains

$$I(\omega) = e^{i\omega c} \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} e^{(-i(\xi + \omega b)^2)/4\omega a} \tilde{g}(\xi) d\xi$$

By expanding the exponential term, and converting to infinite series, one may write

$$I(\omega) = e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi, \quad (11)$$

where $k = (4ac - b^2)/4a$, and where the series

$$e^{(i^3 \xi^2)/4\omega a} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n$$

converges in D' [5] and hence in S' .

[†]In [6], the phase function was the monomial x^2 , hence those results were obtained by taking advantage of a situation unique to that problem—namely that $e^{i\omega x^2}$ is essentially its own transform. Since the phase function is a polynomial in this case, it is necessary to use the method of convolution to obtain the Fourier transform of $e^{i\omega(ax^2 + bx)}$.

To switch the order of integration and summation in equation (11) one must be able to show that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n e^{i(-b/2a)\xi} \tilde{g}(\xi)$$

is a uniformly convergent series in ξ . To show this, consider the following

$$\begin{aligned} \left| \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n e^{i(-b/2a)\xi} \tilde{g}(\xi) \right| &\leq \frac{1}{n!} \left(\frac{|\xi|^2}{4\omega a} \right)^n |e^{i(-b/2a)\xi}| |\tilde{g}(\xi)| \\ &= \frac{1}{n!} \left(\frac{|\xi|^2}{4\omega a} \right)^n |\tilde{g}(\xi)| \leq \frac{1}{n!} \left(\frac{|\xi|^2}{4\omega a} \right)^n (4\omega a)^n C e^n |\xi|^{-2n} = C e^n \frac{1}{n!} \end{aligned}$$

where one only considers values of ω such that[†] $\omega \geq 1/4a$, and since also by assumption $|\tilde{g}(\xi)| \leq C_n |\xi|^{-n}$, where $0 < |\xi| < \infty$, and $C_n \leq C e^n$ for all $n \geq 0$.

Now since

$$\sum_{n=0}^{\infty} C e^n \frac{1}{n!} = C e^e,$$

where C is a constant, the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n e^{i(-b/2a)\xi} \tilde{g}(\xi)$$

is uniformly convergent by the Weierstrass comparison test [7]. Hence, one may switch the order of integration and summation in equation (11) obtaining

$$\begin{aligned} I(\omega) &= e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{i\pi/4} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left(\frac{i^3 \xi^2}{4\omega a} \right)^n \frac{1}{n!} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi \\ &= e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{i\pi/4} \sum_{n=0}^{\infty} \left(\frac{i}{4\omega a} \right)^n \frac{1}{n!} \int_{-\infty}^{\infty} (i\xi)^{2n} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi \end{aligned} \quad (12)$$

If $g(x)$ is a distribution, then for any positive integer k , one obtains

$$[(ix)^k g(x)]^{\wedge}(\xi) = \frac{d^k}{d\xi^k} g^{\wedge}(\xi)$$

[9], which implies

$$[(i\xi)^k \tilde{g}(\xi)]^{\wedge}(x) = \frac{d^k}{dx^k} g(x)$$

and thus

$$\int_{-\infty}^{\infty} (i\xi)^k e^{ix\xi} \tilde{g}(\xi) d\xi = g^{(k)}(x).$$

By letting $x = -b/2a$ and $k = 2n$, one obtains

$$\int_{-\infty}^{\infty} (i\xi)^{2n} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi = g^{2n} \left(\frac{-b}{2a} \right) \quad (13)$$

[†]As previously stated, in applications, one is often interested in equation (1) for large values of ω , namely $\omega \rightarrow \infty$, therefore, it is reasonable to consider $\omega \geq 1/4a$ ($a \neq 0$).

Since $g(x) \in S'(R)$, its derivatives of all orders must exist [8], and by substituting equation (13) into equation (12), one obtains

$$I(\omega) = e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} \times \left[\sum_{n=0}^N \left(\frac{i}{4\omega a} \right)^n \frac{1}{n!} g^{2n} \left(\frac{-b}{2a} \right) + \xi_{N+1} \right],$$

where

$$\xi_{N+1} = \frac{1}{(N+1)!} \left(\frac{i}{4\omega a} \right)^{N+1} \sum_{n=0}^{\infty} \left(\frac{i}{4\omega a} \right)^n \frac{1}{(N+2)_n} \int_{-\infty}^{\infty} (i\xi)^{2(N+1+n)} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi$$

and $(N+2)_n = (N+2)(N+3)(N+4) \cdots (N+1+n)$ and $(N+2)_0 = 1$ Now, let

$$R_N(\omega, g) = \sum_{n=0}^{\infty} \left(\frac{i}{4\omega a} \right)^n \frac{1}{(N+2)_n} \int_{-\infty}^{\infty} (i\xi)^{2(N+1+n)} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi$$

and since by assumption $\tilde{g}(\xi)$ is continuous at $\xi = 0$, there exist constants $s \geq 0$ and $M \geq 0$ such that $|\tilde{g}(\xi)| \leq M$ for $-s \leq \xi \leq s$. Thus one obtains[†]

$$\begin{aligned} |R_N(\omega, g)| &= \left| \sum_{n=0}^{\infty} \left(\frac{i}{4\omega a} \right)^n \frac{1}{(N+2)_n} \int_{-\infty}^{\infty} (i\xi)^{2(N+1+n)} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi \right| \\ &\leq \sum_{n=0}^{\infty} \left(\frac{1}{4\omega a} \right)^n \frac{1}{(N+2)_n} \int_{-\infty}^{\infty} |\xi|^{2(N+1+n)} |\tilde{g}(\xi)| d\xi \leq \sum_{n=0}^{\infty} \left(\frac{1}{4\omega a} \right)^n \frac{1}{n!} \\ &\quad \times \left[M \int_{-s}^s |\xi|^{2(N+1+n)} d\xi + 2 \int_s^{\infty} |\xi|^{2(N+1+n)} (4\omega a)^n C e^n |\xi|^{-2(N+2+n)} d\xi \right] \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{4\omega a} \right)^n \frac{1}{n!} \left(2M \frac{s^{2N+3+2n}}{2N+3+2n} \right) + 2C \sum_{n=0}^{\infty} \frac{e^n}{n!} \int_s^{\infty} \xi^{-2} d\xi \\ &= 2M \sum_{n=0}^{\infty} \left(\frac{1}{4\omega a} \right)^n \frac{1}{n!} \left(\frac{s^{2N+3+2n}}{2N+3+2n} \right) + \frac{2C}{s} \sum_{n=0}^{\infty} \frac{e^n}{n!} \\ &\leq 2M \sum_{n=0}^{\infty} \left(\frac{1}{4\omega a} \right)^n \frac{1}{n!} (s^{2N+3} s^{2n}) + \frac{2C}{s} \sum_{n=0}^{\infty} \frac{e^n}{n!} \\ &\leq M_1 \sum_{n=0}^{\infty} \left(\frac{s^2}{4\omega a} \right)^n \frac{1}{n!} + \frac{2C}{s} \sum_{n=0}^{\infty} \frac{e^n}{n!} = M_1 e^{(s^2/4\omega a)} + \frac{2Ce^e}{s} \end{aligned}$$

for some positive constants C , and M_1 such that $M_1 > 2Ms^{2N+3}$ for fixed s , M and N . To summarize the results shown above, one may say the following:

Let $g(x)$ be a function, which when considered as a distribution, belongs to the space of Schwartz' tempered distributions, $S'(R)$. Suppose also that $\tilde{g}(\xi)$, the inverse Fourier transform of $g(x)$ is an integrable function, continuous at $\xi = 0$, satisfying $|\tilde{g}(\xi)| \leq C_n |\xi|^{-n}$ for all

[†]The method used here to derive a bound on the remainder is the same as that used in [4]. In that work results were obtained for the case, where the phase function was the monomial x^α ($\alpha \geq 2$)—a situation similar to the one discussed in [6].

integers $n \geq 0$, and $0 < |\xi| < \infty$, where the constant C_n is such that $C_n \leq Ce^n$ for some constant $0 < C < \infty$ and all $n \geq 0$. Now, let

$$I(\omega) = \int_{-\infty}^{\infty} g(x) e^{i\omega(ax^2+bx+c)} dx$$

where the integral is taken in the sense of distributions. Then for all $\omega \geq 1/4a$, $a \neq 0$, one may write

$$I(\omega) = e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} \sum_{n=0}^N \left(\frac{i}{4\omega a} \right)^n \frac{1}{n!} g^{2n} \left(\frac{-b}{2a} \right) + \frac{1}{(N+1)!} \left(\frac{i}{4\omega a} \right)^{N+1} R_N(\omega, g), \quad (14)$$

where $k = (4ac - b^2)/4a$ and

$$|R_N(\omega, g)| \leq M_1 e^{s^2/4\omega a} + \frac{2Ce^e}{s}$$

for suitable constants C , M_1 and s .

The reader will notice that equation (14) represents an expansion involving the derivatives of $g(x)$ evaluated at the point $x = -b/2a$. This is consistent with the method of stationary phase—the idea that the value of $I(\omega)$, ω large, depends primarily on the behavior of the coefficient function $g(x)$ at (or near) the stationary points of the phase function.

3. Proof the expansion is asymptotic

The final result is to show that the expansion represented by equation (14) may be written as

$$e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} \sum_{n=0}^N \left(\frac{i}{4\omega a} \right)^n \frac{1}{n!} g^{2n} \left(\frac{-b}{2a} \right) + O(\omega^{-N-1})$$

is indeed asymptotic by demonstrating that

$$\phi_n = \left(\frac{i}{4\omega a} \right)^n \frac{1}{n!} g^{2n} \left(\frac{-b}{2a} \right)$$

is an asymptotic sequence [3]; this is equivalent to showing that $\phi_{n+1} = o(\phi_n)$ as $\omega \rightarrow \infty$. Consider

$$\lim_{\omega \rightarrow \infty} \left| \frac{\phi_{n+1}}{\phi_n} \right| = \lim_{\omega \rightarrow \infty} \left(\left(\frac{1}{n+1} \right) \left(\frac{1}{4\omega a} \right) \frac{|g^{2n+2}(-b/2a)|}{|g^{2n}(-b/2a)|} \right)$$

and recall that

$$\int_{-\infty}^{\infty} (i\xi)^{2n} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi = g^{2n} \left(\frac{-b}{2a} \right)$$

Now, following the same procedure as before, one obtains

$$\begin{aligned} \left| g^{2n} \left(\frac{-b}{2a} \right) \right| &= \left| \int_{-\infty}^{\infty} (i\xi)^{2n} e^{i(-b/2a)\xi} \tilde{g}(\xi) d\xi \right| \leq \int_{-\infty}^{\infty} |\xi|^{2n} |\tilde{g}(\xi)| d\xi \\ &\leq M \int_{-s}^s |\xi|^{2n} d\xi + 2 \int_2^{\infty} |\xi|^{2n} C e^n |\xi|^{-2(n+1)} d\xi \\ &\leq 2M \frac{s^{2n+1}}{2n+1} + \frac{2Ce^n}{s}, \end{aligned}$$

hence,

$$\begin{aligned}
 \frac{|g^{2n+2}(-b/2a)|}{|g^{2n}(-b/2a)|} &\leq \frac{2M(s^{2n+3}/2n+3) + (2Ce^{n+1}/s)}{2M(s^{2n+1}/2n+1) + (2Ce^n/s)} \\
 &= \frac{Ms^{2n+4} + Ce^{n+1}(2n+3)}{s(2n+3)} \cdot \frac{s(2n+1)}{Ms^{2n+2} + Ce^n(2n+1)} \\
 &= \frac{Ms^{2n+4} + Ce^{n+1}(2n+3)}{Ms^{2n+2} + Ce^n(2n+1)} \cdot \frac{2n+1}{2n+3} \\
 &\leq \frac{Ms^{2n+4} + 3eMs^{2n+2} + s^2Ce^n(2n+1) + Ce^{n+1}(6n+3)}{Ms^{2n+2} + Ce^n(2n+1)} \cdot \frac{2n+1}{2n+3} \\
 &= \frac{[Ms^{2n+2} + Ce^n(2n+1)](s^2 + 3e)}{Ms^{2n+2} + Ce^n(2n+1)} \cdot \frac{2n+1}{2n+3} = (s^2 + 3e) \cdot \left(\frac{2n+1}{2n+3}\right)
 \end{aligned}$$

for constants $C > 0$, and n, s and $M \geq 0$. Finally, one has

$$\begin{aligned}
 &\lim_{\omega \rightarrow \infty} \left(\left(\frac{1}{n+1}\right) \left(\frac{1}{4\omega a}\right) \frac{|g^{2n+2}(-b/2a)|}{|g^{2n}(-b/2a)|} \right) \\
 &\leq \lim_{\omega \rightarrow \infty} \left(\left(\frac{1}{n+1}\right) \left(\frac{1}{4\omega a}\right) (s^2 + 3e) \left(\frac{2n+1}{2n+3}\right) \right) \\
 &= \left(\frac{s^2 + 3e}{n+1}\right) \left(\frac{2n+1}{2n+3}\right) \lim_{\omega \rightarrow \infty} \left(\frac{1}{4\omega a}\right) = 0
 \end{aligned}$$

and so

$$\lim_{\omega \rightarrow \infty} \left| \frac{\phi_{n+1}}{\phi_n} \right| = 0$$

which proves the assertion that ϕ_n is an asymptotic sequence and

$$e^{i\omega k} \sqrt{\frac{\pi}{\omega a}} e^{(i\pi/4)} \sum_{n=0}^N \left(\frac{i}{4\omega a}\right)^n \frac{1}{n!} g^{2n}\left(\frac{-b}{2a}\right) + O(\omega^{-N-1})$$

is an asymptotic expansion.

4. Conclusions

In most cases Fourier Integrals cannot be expressed in closed form. Although these integrals cannot be evaluated explicitly, the knowledge of their behavior is very important for many application problems. By developing an asymptotic expansion, one is able to study the behavior of these integrals for some range of parameters.

There are many advantages in using the theory of distributions to study this type of problem – one being the fact that the Fourier transform of many types of distributions (namely tempered distributions) is guaranteed to exist, and another is that all distributions are infinitely differentiable. Hence, using distributions allows one to perform operations that would otherwise be too difficult – or even inadmissible – with functions defined in the classical sense.

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Three classes of logarithmically completely monotonic functions involving gamma and psi functions

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By a simple approach, two classes of functions involving Euler's gamma function and originating from certain problems of traffic flow are proved to be logarithmically completely monotonic and a class of functions involving the psi function is showed to be completely monotonic.

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1. Introduction

Recall [1–3] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $0 \leq (-1)^n f^{(n)}(x) < \infty$ for $x \in I$ and $n \geq 0$. The set of the completely monotonic functions on I is denoted by $\mathcal{C}[I]$. The well-known Bernstein's Theorem [3, p. 161] states that $f \in \mathcal{C}[(0, \infty)]$ if and only if $f(x) = \int_0^\infty e^{-xs} d\mu(s)$, where μ is a nonnegative measure on $[0, \infty)$ such that the integral converges for all $x > 0$. This expresses that $f \in \mathcal{C}[(0, \infty)]$ if and only if f is a Laplace transform of the measure μ .

Recall [1, 4–9] also that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \leq (-1)^k [\ln f(x)]^{(k)} < \infty$ for all $k \in \mathbb{N}$ on I . The set of the logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. In refs. [5, 9] and [10, Theorem 1.1] it is pointed out that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterised as the infinitely divisible completely monotonic functions studied by Horn in ref. [11, Theorem 4.4].

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It was proved in [1, 2, 8–10] that $\mathcal{L}[I] \subset \mathcal{C}[I]$, but not conversely. Stimulated by the papers [7, 8], among other things, it was further revealed in ref. [10] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms.

The Kershaw's inequality in ref. [12] states that the double inequality

$$\left(x + \frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left(x - \frac{1}{2} + \sqrt{s + \frac{1}{4}}\right)^{1-s} \quad (1)$$

holds for $0 < s < 1$ and $x \geq 1$, where Γ denotes the classical Euler's gamma function and $\psi = \Gamma'/\Gamma$, the logarithmic derivative of Γ , the psi function. If taking $s = 1/2$ in equation (1), then, for $x > 1$,

$$\sqrt{x + \frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma(x+1/2)} < \sqrt{x + \frac{\sqrt{3}-1}{2}}. \quad (2)$$

Let s and t be nonnegative numbers and $\alpha = \min\{s, t\}$. For $x \in (-\alpha, \infty)$, define

$$z_{s,t}(x) = \begin{cases} \left[\frac{\Gamma(x+t)}{\Gamma(x+s)} \right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s = t. \end{cases} \quad (3)$$

In order to establish the best bounds in Kershaw's inequality (1), among other things, the papers [13–17] established the following monotonicity and convexity property of $z_{s,t}(x)$: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s| < 1$ or concave and increasing for $|t-s| > 1$. This result was further generalized in the papers [18, 19]

In ref. [20, p. 123] and [21], while studying certain problems of traffic flow, a double inequality below was obtained for $n \in \mathbb{N}$:

$$2\Gamma\left(n + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(n+1) \leq 2^n \Gamma\left(n + \frac{1}{2}\right), \quad (4)$$

In ref. [22], inequality (4) was extended and refined for $x > 0$ as

$$\sqrt{x} \leq \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \sqrt{x + \frac{1}{2}}. \quad (5)$$

It is clear that the double inequality (5) is weaker than (2).

Observe that inequality (4) can be rearranged for $n > 1$ as

$$1 \leq \left[\frac{\Gamma(1/2)\Gamma(n+1)}{2\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (6)$$

Hinted by this, the following function $g(x)$ was defined in ref. [23] for $x \in (-1/2, \infty)$:

$$g(x) = \begin{cases} \left[\frac{\Gamma(1/2)\Gamma(x+1)}{2\Gamma(x+1/2)} \right]^{1/(x-1)}, & x \neq 1, \\ \exp\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right], & x = 1, \end{cases} \quad (7)$$

where $\gamma = 0.57721566\dots$ is Euler–Mascheroni's constant, and, among other things, it was proved in ref. [23] that the function $g(x)$ is logarithmically completely monotonic

in $(-1/2, \infty)$: $g(x) \in \mathcal{L}[-1/2, \infty]$ with $\lim_{x \rightarrow -1/2} g(x) = \infty$ and $\lim_{x \rightarrow \infty} g(x) = 1$. As consequences of this result, it is deduced that

$$2\Gamma\left(x + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(x+1) \leq 2\Gamma\left(x + \frac{1}{2}\right) \exp\left\{(x-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} \quad (8)$$

for $x \in [1, \infty)$ and

$$2\Gamma\left(x + \frac{1}{2}\right) \leq \Gamma\left(\frac{1}{2}\right) \Gamma(x+1) \leq 2^x \Gamma\left(x + \frac{1}{2}\right) \quad (9)$$

for $x \in (0, \infty)$. It was remarked in ref. [23] that inequalities (8) and (9) extend (4) and (6), the right-hand side inequality of (8) refines the right-hand side inequality of (4) and (6), and the right-hand side inequalities in equations (2) and (5) and the following inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leq \frac{2}{\Gamma(1/2)} \exp\left\{(x-1)\left[1 - \gamma - \psi\left(\frac{3}{2}\right)\right]\right\} \quad (10)$$

for $x \in [1, \infty)$, which is deduced from the right-hand side inequality of (8), are not included with each other respectively.

Now rewrite inequality (4) or (6) for $n > 1$ as

$$1 \leq \left[\frac{\Gamma(1+1/2)}{\Gamma(1+1)} \cdot \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \right]^{1/(n-1)} \leq 2. \quad (11)$$

The definition (7) of $g(x)$ and inequality (11) motivate us to introduce a new function $h_\beta(x)$ as follows: Let s and t be two real numbers with $s \neq t$, $\alpha = \min\{s, t\}$ and $\beta > -\alpha$. For $x \in (-\alpha, \infty)$, define

$$h_\beta(x) = \begin{cases} \left[\frac{\Gamma(\beta+t)}{\Gamma(\beta+s)} \cdot \frac{\Gamma(x+s)}{\Gamma(x+t)} \right]^{1/(x-\beta)}, & x \neq \beta, \\ \exp[\psi(\beta+s) - \psi(\beta+t)], & x = \beta. \end{cases} \quad (12)$$

The first aim of this paper is to consider the logarithmically completely monotonic property of $h_\beta(x)$ by a simple approach. Our first main result is the following Theorem 1.1.

THEOREM 1.1 *The following two conclusions are valid:*

- (i) *If $s > t$, then $h_\beta(x) \in \mathcal{L}[-\alpha, \infty]$ with $\lim_{x \rightarrow -\alpha} h_\beta(x) = \infty$ and $\lim_{x \rightarrow \infty} h_\beta(x) = 1$.*
- (ii) *If $s < t$, then $[h_\beta(x)]^{-1} \in \mathcal{L}[-\alpha, \infty]$ with $\lim_{x \rightarrow -\alpha} h_\beta(x) = 0$ and $\lim_{x \rightarrow \infty} h_\beta(x) = 1$.*

Remark 1 It is noted that taking $s = 1$, $t = \alpha = 1/2$ and $\beta = 1$ in Theorem 1.1 can deduce one of the results obtained in ref. [23], the logarithmically complete monotonicity of the function $g(x)$ defined by (7).

Since $h_\beta(x)$ is decreasing (or increasing) for $s > t$ (or $s < t$), $h_\beta(\beta) = \exp[\psi(\beta+s) - \psi(\beta+t)]$ and $\lim_{x \rightarrow \infty} h_\beta(x) = 1$, then the following double inequality (13), as a direct consequence of Theorem 1.1, is established easily.

COROLLARY 1.2 *Let s and t be nonnegative numbers, $\alpha = \min\{s, t\}$ and $\beta > -\alpha$. If $s > t$, inequality*

$$\frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \leq \frac{\Gamma(x+s)}{\Gamma(x+t)} \leq \frac{\Gamma(\beta+s)}{\Gamma(\beta+t)} \exp\{(x-\beta)[\psi(\beta+s) - \psi(\beta+t)]\} \quad (13)$$

holds for $x \in [\beta, \infty)$. If $s < t$, inequality (13) reverses.

Remark 2 If taking $\beta = 1, s = 1$ and $t = 1/2$ in equation (13), then inequality (8) is deduced. So, it can be said that inequality (13) is a generalization of (8).

In ref. [24], it was showed that the function $\ln x - 1/(2x) - \psi(x) \in \mathcal{C}[(0, \infty)]$. In order to prove Theorem 1.4 below, this result need to be generalized. Our second main result is the following Theorem 1.3.

THEOREM 1.3 *Let $\alpha \in \mathbb{R}$. Then $\psi(x) - \ln x + \alpha/x \in \mathcal{C}[(0, \infty)]$ if and only if $\alpha \geq 1$ and $\ln x - \alpha/x - \psi(x) \in \mathcal{C}[(0, \infty)]$ if and only if $\alpha \leq 1/2$.*

For $x \in (0, \infty)$, define

$$p(x) = \begin{cases} \left[\frac{x^x}{\Gamma(x+1)} \right]^{1/(1-x)}, & x \neq 1, \\ e^{-\gamma}, & x = 1. \end{cases} \quad (14)$$

In ref. [23], among other things, the logarithmically complete monotonicity of $p(x)$ was proved: $p(x) \in \mathcal{L}[(0, \infty)]$ with $\lim_{x \rightarrow 0+} p(x) = 1$ and $\lim_{x \rightarrow \infty} p(x) = 1/e$. Motivated by inequality (11) and the definition of $h_\beta(x)$ in equation (12), a more general function than $p(x)$ can be introduced: For $x \in (0, \infty)$ and $\alpha > 0$, let

$$p_\alpha(x) = \begin{cases} \left[\frac{\Gamma(\alpha+1)}{\alpha^\alpha} \cdot \frac{x^x}{\Gamma(x+1)} \right]^{1/(\alpha-x)}, & x \neq \alpha, \\ \frac{\exp[\psi(\alpha+1) - 1]}{\alpha}, & x = \alpha. \end{cases} \quad (15)$$

It is clear that $p_1(x) = p(x)$.

The third aim of this paper is to show the logarithmically complete monotonicity of the function $p_\alpha(x)$ for any fixed $\alpha > 0$ by a simple approach. Our third main result is the following Theorem 1.4.

THEOREM 1.4 *For any fixed $\alpha > 0$, $p_\alpha(x) \in \mathcal{L}[(0, \infty)]$ with*

$$\lim_{x \rightarrow 0+} p_\alpha(x) = \frac{\sqrt[\alpha]{\Gamma(\alpha+1)}}{\alpha} \quad \text{and} \quad \lim_{x \rightarrow \infty} p_\alpha(x) = \frac{1}{e}. \quad (16)$$

2. Proofs of theorems

It is well known (see [1, 6–9, 16, 17]) that, for $x > 0$ and $\omega > 0$,

$$\frac{1}{x^\omega} = \frac{1}{\Gamma(\omega)} \int_0^\infty t^{\omega-1} e^{-xt} dt, \quad (17)$$

and that, for $k \in \mathbb{N}$ and $x > 0$,

$$\psi(x) = \ln x + \int_0^\infty \left(\frac{1}{u} - \frac{1}{1-e^{-u}} \right) e^{-xu} du, \quad (18)$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}} dt. \quad (19)$$

2.1 Proof of Theorem 1.1

Without loss of generality, assume $s > t$. For $x \neq \beta$, taking logarithm of the function $h_\beta(x)$ gives

$$\begin{aligned} \ln h_\beta(x) &= \frac{1}{x - \beta} \left[\ln \frac{\Gamma(x + s)}{\Gamma(\beta + s)} - \ln \frac{\Gamma(x + t)}{\Gamma(\beta + t)} \right] \\ &= \frac{\ln \Gamma(x + s) - \ln \Gamma(\beta + s)}{x - \beta} - \frac{\ln \Gamma(x + t) - \ln \Gamma(\beta + t)}{x - \beta} \\ &= \frac{1}{x - \beta} \int_\beta^x \psi(u + s) du - \frac{1}{x - \beta} \int_\beta^x \psi(u + t) du \\ &= \frac{1}{x - \beta} \int_\beta^x [\psi(u + s) - \psi(u + t)] du \\ &= \frac{1}{x - \beta} \int_\beta^x \int_t^s \psi'(u + v) dv du \\ &\triangleq \frac{1}{x - \beta} \int_\beta^x \Phi_{s,t}(u) du \\ &= \int_0^1 \Phi_{s,t}((x - \beta)u + \beta) du \end{aligned}$$

and, by differentiating $\ln h_\beta(x)$ with respect to x ,

$$[\ln h_\beta(x)]^{(k)} = \int_0^1 u^k \Phi_{s,t}^{(k)}((x - \beta)u + \beta) du \quad (20)$$

for $k \in \mathbb{N}$.

If $x = \beta$, formula (20) is also valid.

Formula (19) implies that $\psi' \in \mathcal{C}[(0, \infty)]$ and $\Phi_{s,t}(u) \in \mathcal{C}[(-t, \infty)]$. This means that $(-1)^i [\Phi_{s,t}(u)]^{(i)} \geq 0$ holds in $u \in (-t, \infty)$ for any nonnegative integer i . Thus,

$$(-1)^k [\ln h_\beta(x)]^{(k)} = \int_0^1 u^k \{(-1)^k \Phi_{s,t}^{(k)}((x - \beta)u + \beta)\} du \geq 0 \quad (21)$$

in $(-t, \infty)$ for $k \in \mathbb{N}$. The proof of Theorem 1.1 is complete. ■

2.2 Proof of Theorem 1.3

Formulas (17) and (18) imply that

$$\begin{aligned} \psi(x) - \ln x + \frac{\alpha}{x} &= \int_0^\infty \left[\frac{(1 - u)e^u - 1}{u(e^u - 1)} + \alpha \right] e^{-xu} du \triangleq \int_0^\infty [\theta(u) + \alpha] e^{-xu} du, \\ \theta'(u) &= \frac{(u^2 + 2)e^u - 1 - e^{2u}}{u^2(e^u - 1)^2} \triangleq \frac{\theta_1(u)}{u^2(e^u - 1)^2}, \\ \theta'_1(u) &= [u^2 + 2u + 2 - 2e^u]e^u \triangleq e^u \theta_2(u), \\ \theta'_2(u) &= 2(1 + u - e^u) < 0. \end{aligned}$$

It is clear that the function $\theta_2(u)$ is decreasing in $(0, \infty)$. From $\theta_2(0) = 0$, it follows that $\theta_2(u) < 0$ and $\theta'_1(u) < 0$ in $(0, \infty)$ and that $\theta'_1(u)$ decreases in $(0, \infty)$. Since $\theta'_1(0) = 0$, then

$\theta_1'(u) < 0$ and $\theta_1(u)$ is decreasing in $(0, \infty)$. From $\theta_1(0) = 0$, it follows that $\theta_1(u) < 0$ and $\theta'(u) < 0$ in $(0, \infty)$, and then the function $\theta(u)$ is decreasing in $(0, \infty)$. L'Hôpital's rule yields that $\lim_{u \rightarrow 0^+} \theta(u) = -1/2$ and $\lim_{u \rightarrow \infty} \theta(u) = -1$. Thus,

$$\theta(u) + \alpha \begin{cases} \geq 0, & \alpha \geq 1 \\ \leq 0, & \alpha \leq \frac{1}{2} \end{cases}$$

on $[0, \infty]$. Since $[\psi(x) - \ln x + \alpha/x]^{(k)} = (-1)^k \int_0^\infty [\theta(u) + \alpha] u^k e^{-xu} du$ for nonnegative integer k , then $\psi(x) - \ln x + \alpha/x \in \mathcal{C}[(0, \infty)]$ if $\alpha \geq 1$ (or $\ln x - \alpha/x - \psi(x) \in \mathcal{C}[(0, \infty)]$ if $\alpha \leq 1/2$).

If $\psi(x) - \ln x + \alpha/x \in \mathcal{C}[(0, \infty)]$ (or $\ln x - \alpha/x - \psi(x) \in \mathcal{C}[(0, \infty)]$), then $\psi(x) - \ln x + \alpha/x > 0$ (or < 0), this leads to $\alpha > x[\ln x - \psi(x)]$ (or $\alpha < x[\ln x - \psi(x)]$) in $(0, \infty)$. Since $\lim_{x \rightarrow 0^+} \{x[\ln x - \psi(x)]\} = 1$ (or $\lim_{x \rightarrow \infty} \{x[\ln x - \psi(x)]\} = 1/2$), then $\alpha \geq 1$ (or $\alpha \leq 1/2$). The proof of Theorem 1.3 is complete. ■

2.3 Proof of Theorem 1.4

From the well-known differences equation $\Gamma(x+1) = x\Gamma(x)$, it follows easily that

$$\psi(x+1) - \psi(x) = \frac{1}{x} \quad (22)$$

for $x > 0$. For $x \neq \alpha$, taking logarithm of $p_\alpha(x)$ and using (22) gives

$$\begin{aligned} \ln p_\alpha(x) &= \frac{1}{x-\alpha} \left[\ln \frac{\Gamma(x+1)}{\Gamma(\alpha+1)} - \frac{x^x}{\alpha^\alpha} \right] \\ &= \frac{\ln \Gamma(x+1) - \ln \Gamma(\alpha+1)}{x-\alpha} - \frac{x \ln x - \alpha \ln \alpha}{x-\alpha} \\ &= \frac{1}{x-\alpha} \int_\alpha^x \psi(u+1) du - \frac{1}{x-\alpha} \int_\alpha^x [1 + \ln x] du \\ &= \frac{1}{x-\alpha} \int_\alpha^x \left[\psi(u) - \ln u + \frac{1}{u} - 1 \right] du \\ &\triangleq \frac{1}{x-\alpha} \int_\alpha^x [\Psi(u) - 1] du \\ &= \int_0^1 \Psi((x-\alpha)u + \alpha) du - 1 \end{aligned}$$

and, by differentiating $\ln p_\alpha(x)$ with respect to x ,

$$[\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k \Psi^{(k)}((x-\alpha)u + \alpha) du \quad (23)$$

for $k \in \mathbb{N}$.

If $x = \alpha$, formula (23) is also valid.

From Theorem 1.3, it follows that $\Psi(u) \in \mathcal{C}[(0, \infty)]$. This implies $(-1)^i \Psi^{(i)}(u) \geq 0$ for nonnegative integer i . As a result, for $k \in \mathbb{N}$,

$$(-1)^k [\ln p_\alpha(x)]^{(k)} = \int_0^1 u^k \{(-1)^k \Psi^{(k)}((x-\alpha)u + \alpha)\} du \geq 0 \quad (24)$$

holds in $x \in (0, \infty)$. The proof of Theorem 1.4 is complete. ■

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Hypergeometric functions in the parabolic starlike and uniformly convex domains

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For a sufficiently adequate special case of the Dziok–Srivastava linear operator defined by means of the Hadamard product (or convolution) with a generalized hypergeometric function, the authors investigate several mapping properties involving various subclasses of analytic and univalent functions, which are introduced here. Relevant connections of the definitions and results presented in this paper with those in several earlier works on the subject are also pointed out.

Keywords: Generalized hypergeometric function; Dziok–Srivastava linear operator; Parabolic starlike functions; Uniformly starlike functions; Uniformly convex functions; Hadamard product (or convolution)

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1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are *analytic* in the open unit disk

$$\mathbb{U} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also *univalent* in $\mathbb{U}$.

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A function $f \in \mathcal{A}$ is said to be *starlike of order α* in $\mathbb{U}$ ($0 \leq \alpha < 1$), if and only if

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1). \quad (2)$$

This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write

$$\mathcal{S}^*(0) =: \mathcal{S}^*,$$

where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{K}(\alpha)$ of *convex univalent functions of order α* in $\mathbb{U}$ ($0 \leq \alpha < 1$), if

$$zf'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1),$$

that is, if

$$\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1). \quad (3)$$

Clearly, we have

$$\mathcal{K}(0) =: \mathcal{K}$$

for the class $\mathcal{K}$ of convex univalent functions in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be in the class UCV of *uniformly convex functions* in $\mathbb{U}$ if $f(z)$ is a normalized convex function in $\mathbb{U}$ and has the property that, for every circular arc γ contained in the unit disk $\mathbb{U}$, with center ζ also in $\mathbb{U}$, the image curve $f(\gamma)$ is a convex arc. The function class UCV was introduced by Goodman [1]. Subsequently, Kanas and Wiśniowska [2] defined the class k -UCV as follows:

$$k\text{-UCV} := \left\{ f: f \in \mathcal{A} \quad \text{and} \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right| \right\}. \quad (4)$$

$$(z \in \mathbb{U}; 0 \leq k < \infty).$$

The class k -UCV, which is an extension of the class UCV studied by Goodman [1, 3], describes geometrically the domain of values of the function

$$1 + \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U})$$

as a parabolic region Ω given by

$$\Omega = \{ \omega : \omega \in \mathbb{C} \quad \text{and} \quad [\Im(\omega)]^2 < 2\Re(\omega) - 1 \}.$$

A characterization for the case $k = 1$ of the class k -UCV was given *independently* by Rønning [4], and Ma and Minda [5], as follows:

$$f \in \text{UCV} \iff \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}). \quad (5)$$

Using the analytic condition in equation (5), Rønning [4] defined a certain class $\mathcal{S}_p$ of functions $f \in \mathcal{A}$ satisfying the following inequality:

$$\Re \left(\frac{zf'(z)}{f(z)} \right) \geq \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}). \quad (6)$$

Subsequently, Kanas and Wiśniowska [6] extended the function class $\mathcal{S}_p$ as follows:

$$k\text{-ST} := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re \left(\frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \right\} \quad (7)$$

$$(z \in \mathbb{U}; \quad 0 \leq k < \infty).$$

The various properties of the function classes k -UCV and k -ST were investigated extensively by Kanas and Srivastava [7] and by Srivastava and Mishra [8]. On the other hand, Bharati *et al.* [9] studied a certain class $\text{UCV}(\alpha, \beta)$ defined by

$$\text{UCV}(\alpha, \beta) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re \left(1 + \frac{zf''(z)}{f'(z)} \right) \geq \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \beta \right\} \quad (8)$$

$$(z \in \mathbb{U}; \quad \alpha \geq 0; \quad 0 \leq \beta < 1).$$

Bharati *et al.* [9] gave a sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\text{UCV}(\alpha, \beta)$ in terms of the coefficients of the function f . The function class $\mathcal{S}_p(\alpha, \beta)$ was also defined in [9] as follows:

$$\mathcal{S}_p(\alpha, \beta) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re \left(\frac{zf'(z)}{f(z)} \right) \geq \alpha \left| \frac{zf'(z)}{f(z)} - 1 \right| + \beta \right\} \quad (9)$$

$$(z \in \mathbb{U}; \quad \alpha \geq 0; \quad 0 \leq \beta < 1).$$

For parameters τ , A and B , where

$$\tau \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad -1 \leq B < A \leq 1,$$

we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{R}^\tau(A, B)$ if it satisfies the following inequality:

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1 \quad (10)$$

$$(z \in \mathbb{U}; \quad \tau \in \mathbb{C} \setminus \{0\}; \quad -1 \leq B < A \leq 1).$$

The function class $\mathcal{R}^\tau(A, B)$ was introduced by Dixit and Pal [10]. Indeed, in its special case when

$$\tau = e^{-i\eta} \cos \eta \quad \left(-\frac{\pi}{2} < \eta < \frac{\pi}{2} \right),$$

$$A = 1 - 2\gamma \quad (0 \leq \gamma < 1) \quad \text{and} \quad B = -1,$$

the class $\mathcal{R}^\tau(A, B)$ reduces to the function class $\mathcal{R}_\eta(\gamma)$ investigated by Ponnusamy and Rønning [11], where

$$\mathcal{R}_\eta(\gamma) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re(e^{i\eta}[f'(z) - \gamma]) > 0 \right\} \quad (11)$$

$$\left(z \in \mathbb{U}; \quad -\frac{\pi}{2} < \eta < \frac{\pi}{2}; \quad 0 \leq \gamma < 1 \right).$$

Finally, for functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and

$$g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

we recall that the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written symbolically as $(f * g)(z)$, is defined (as usual) by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n =: (g * f)(z). \quad (12)$$

In terms of the Hadamard product (or convolution), we now introduce a sufficiently adequate *special* case of the familiar Dziok–Srivastava linear operator involving generalized hypergeometric functions, which was introduced and studied systematically in a series of earlier papers by Dziok and Srivastava [12–14]. For parameters

$$\alpha_j \in \mathbb{C} \quad (j = 1, \dots, p)$$

and

$$\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (j = 1, \dots, q; \mathbb{Z}_0^- := \{0, -1, -2, \dots\}),$$

the generalized hypergeometric function ${}_pF_q(z)$ (with p numerator and q denominator parameters) is defined by

$$\begin{aligned} {}_pF_q(z) &= {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \end{aligned} \quad (13)$$

$$(p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \quad \mathbb{N} := \{1, 2, 3, \dots\}; \quad p < q + 1 \text{ and } z \in \mathbb{C};$$

$$p = q + 1 \text{ and } z \in \mathbb{U}; \quad p = q + 1, \quad z \in \partial\mathbb{U}, \text{ and } \Re(\omega) > 0),$$

where an empty product is to be interpreted as 1, $(\lambda)_n$ is the Pochhammer symbol (or the *shifted* factorial, since $(1)_n = n!$) defined by

$$(\lambda)_n := \begin{cases} 1 & (n = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}), \end{cases}$$

and

$$\omega := \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j.$$

From the work of Dziok and Srivastava [12, p. 3] (see also [13] and [14]), we thus have

$$\begin{aligned} \left(I_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_p} f \right)(z) &:= z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q, z) * f(z) \\ &(f \in \mathcal{A}; \quad p \leq q + 1; \quad z \in \mathbb{U}), \end{aligned} \quad (14)$$

so that, for a function $f \in \mathcal{A}$ of the form (1),

$$\left(I_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_p} f \right)(z) = z + \sum_{n=2}^{\infty} \Omega_n a_n z^n, \quad (15)$$

where, for convenience,

$$\Omega_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1}} \cdot \frac{1}{(n-1)!} \quad (n \in \mathbb{N} \setminus \{1\}). \quad (16)$$

Several mapping properties of the above-defined function classes k -UCV and k -ST, involving hypergeometric functions, were studied recently by Gangadharan *et al.* [15]. Moreover, just as it was observed by Dziok and Srivastava [12, pp. 3–4], the convolution operator defined by equation (14) includes, as its *further* special cases, various other linear operators, which were considered in many earlier works. In particular, for

$$p = 2 \quad (\alpha_1 = a; \alpha_2 = b) \quad \text{and} \quad q = 1 \quad (\beta_1 = c),$$

we obtain the linear operator $I_{a,b;c}$ defined by (see [16])

$$I_{a,b;c}(f)(z) := z {}_2F_1(a, b; c; z) * f(z) \quad (17)$$

$$(f \in \mathcal{A}; a, b \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}_0^-)$$

involving the Gauss hypergeometric function. Many univalence, starlikeness, and convexity properties of the hypergeometric functions

$$z {}_2F_1(a, b; c; z)$$

and

$$z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$$

were investigated in a number of earlier works (see, for details, [15, p. 1518]; see also the references cited therein). In particular, Parvatham and Prabhakaran [17] obtained sufficient condition for the Gauss hypergeometric function $z {}_2F_1(a, b; c; z)$ to be in the class $\mathcal{S}_p$. More recently, the class $\mathcal{S}_p(\alpha, \beta)$ was investigated systematically by Swaminathan [18]. Motivated essentially by the aforementioned works, we obtain sufficient conditions for the *special* Dziok–Srivastava linear operator $I_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_p} f$ to be in the function classes $UCV(\alpha, \beta)$ and $\mathcal{S}_p(\alpha, \beta)$.

2. Preliminary results

In this section, we state each of the following results, which are potentially useful in our investigation. In particular, Lemma 3 provides a collection of some of the *well-known* (rather *classical*) situations in which a generalized hypergeometric series of the type relevant to our investigation can be expressed in a closed form. These *well-known* hypergeometric identities were *rederived* recently (*cf.*, *e.g.*, [19, p. 767, Lemma 2]).

LEMMA 1 (see [9, p. 21, Theorem 2.3]) *A function $f \in \mathcal{A}$ of the form (1) is in the class $UCV(\alpha, \beta)$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} n [n(1 + \alpha) - (\alpha + \beta)] |a_n| \leq 1 - \beta. \quad (18)$$

LEMMA 2 (see [9, p. 23, Theorem 2.6]) *A function $f \in \mathcal{A}$ of the form (1) is in the class $\mathcal{S}_p(\alpha, \beta)$, if it satisfies the following condition:*

$$\sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)] |a_n| \leq 1 - \beta. \quad (19)$$

LEMMA 3 Each of the following well-known (or easily derivable) hypergeometric identities holds true:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{1}{(n+1)!} &=: {}_3F_2(a, b, 1; c, 2; 1) \\ &= \frac{c-1}{(a-1)(b-1)} \left(\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right) \\ &\quad (a \neq 1; b \neq 1; c \neq 1; c > \max\{0, a+b-1\}). \end{aligned} \quad (20)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{(n+1)^3}{n!} &=: {}_5F_4(a, b, 2, 2, 2; c, 1, 1, 1; 1) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ &\quad \cdot \left(\frac{(a)_3(b)_3}{(c-a-b-3)_3} + 6 \frac{(a)_2(b)_2}{(c-a-b-2)_2} + 7 \frac{ab}{(c-a-b)} + 1 \right) \\ &\quad (a > 0; b > 0; c > a+b+3). \end{aligned} \quad (21)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{(n+1)^2}{n!} &=: {}_4F_3(a, b, 2, 2; c, 1, 1; 1) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{(a)_2(b)_2}{(c-a-b-2)_2} + 3 \frac{ab}{c-a-b} \right) \\ &\quad (a > 0; b > 0; c > a+b+2). \end{aligned} \quad (22)$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \cdot \frac{n+1}{n!} &=: {}_3F_2(a, b, 2; c, 1; 1) \\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left(1 + \frac{ab}{c-a-b-1} \right) \\ &\quad (a > 0; b > 0; c > a+b+3). \end{aligned} \quad (23)$$

LEMMA 4 (see [10]). If $f \in \mathcal{R}^{\tau}(A, B)$ is of form (1), then

$$|a_n| \leq \left(\frac{A-B}{1+|B|} \right) \frac{|\tau|}{n} \leq (A-B) \frac{|\tau|}{n} \quad (n \in \mathbb{N} \setminus \{1\}). \quad (24)$$

3. Main results and their consequences

We begin by proving a mapping and inclusion property of the *special* Dziok–Srivastava operator defined by equation (14) involving the function class UCV.

THEOREM 1 Suppose that

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \dots, p) \quad \text{and} \quad \Re(\beta_j) > 0 \quad (j = 1, \dots, q),$$

and that

$$\Re \left(\sum_{j=1}^q \beta_j \right) > \sum_{j=1}^p |\alpha_j| + p - q.$$

If $f \in \mathcal{R}^\tau(A, B)$ and the following hypergeometric inequality:

$$\begin{aligned} (1 + \alpha) \frac{|\alpha_1 \cdots \alpha_p|}{\Re(\beta_1) \cdots \Re(\beta_q)} \\ \cdot {}_pF_q(1 + |\alpha_1|, \dots, 1 + |\alpha_p|; 1 + \Re(\beta_1), \dots, 1 + \Re(\beta_q); 1) \\ + (1 - \beta) {}_pF_q(|\alpha_1|, \dots, |\alpha_p|; \Re(\beta_1), \dots, \Re(\beta_q); 1) \\ \leq (1 - \beta) \left(1 + \frac{1}{(A - B)|\tau|} \right) \end{aligned} \quad (25)$$

holds true, then

$$I_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_p} f \in \text{UCV}(\alpha, \beta).$$

Proof In view of Lemma 1 and the operational formula (15), we only need to show that

$$S_1 := \sum_{n=2}^{\infty} n [n(1 + \alpha) - (\alpha + \beta)] |\Omega_n a_n| \leq 1 - \beta, \quad (26)$$

where the coefficients Ω_n ($n \in \mathbb{N} \setminus \{1\}$) are given by equation (16). Indeed, by applying Lemma 4, we find from equations (16) and (26) that

$$\begin{aligned} S_1 &\leq (A - B) |\tau| \sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)] \left| \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1}} \cdot \frac{1}{(n-1)!} \right| \\ &\leq (A - B) |\tau| \left[(1 + \alpha) {}_{p+1}F_{q+1}(|\alpha_1|, \dots, |\alpha_p|, 2; \Re(\beta_1), \dots, \Re(\beta_q), 1; 1) \right. \\ &\quad \left. - (\alpha + \beta) {}_pF_q(|\alpha_1|, \dots, |\alpha_p|; \Re(\beta_1), \dots, \Re(\beta_q); 1) - (1 - \beta) \right] \\ &= (A - B) |\tau| \left[(1 + \alpha) \frac{|\alpha_1 \cdots \alpha_p|}{\Re(\beta_1) \cdots \Re(\beta_q)} \right. \\ &\quad \cdot {}_pF_q(1 + |\alpha_1|, \dots, 1 + |\alpha_p|; 1 + \Re(\beta_1), \dots, 1 + \Re(\beta_q); 1) \\ &\quad \left. + (1 - \beta) {}_pF_q(|\alpha_1|, \dots, |\alpha_p|; \Re(\beta_1), \dots, \Re(\beta_q); 1) - (1 - \beta) \right] \\ &\leq 1 - \beta, \end{aligned}$$

which, by virtue of the hypothesis (25), completes the proof of Theorem 1. ■

Remark 1 Each of the hypergeometric series in the hypothesis (25) of Theorem 1 can be expressed in a closed form in their special case when (for instance)

$$p - 1 = q = 1 \quad (\alpha_1 = a; \alpha_2 = b; \gamma_1 = c).$$

In this special case, by applying the celebrated *Gauss summation theorem*:

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

$$(a \in \mathbb{C}; b \in \mathbb{C}; c \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(c - a - b) > 0),$$

which happens to be remarkably instrumental in the derivation of the *well-known* hypergeometric identities asserted by Lemma 3 of the preceding section, we arrive at Corollary 1 below involving the Hohlov operator $I_{a,b;c}$ defined by equation (17).

COROLLARY 1 *Suppose that*

$$a \in \mathbb{C} \setminus \{0\}, \quad b \in \mathbb{C} \setminus \{0\}, \quad c \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad \text{and} \quad \Re(c) > |a| + |b| + 1.$$

If $f \in \mathcal{R}^\tau(A, B)$ and the following inequality:

$$\begin{aligned} & \frac{\Gamma(\Re(c))\Gamma(\Re(c) - |a| - |b|)}{\Gamma(\Re(c) - |a|)\Gamma(\Re(c) - |b|)} \left(1 + \frac{(1 + \alpha)|ab|}{(1 - \beta)[\Re(c) - |a| - |b| - 1]}\right) \\ & \leq 1 + \frac{1}{(A - B)|\tau|} \end{aligned} \quad (27)$$

holds true, then

$$I_{a,b;c}(f) \in \text{UCV}(\alpha, \beta).$$

Remark 2 A further consequence of Corollary 1 would emerge when we set

$$\begin{aligned} \tau &= e^{-i\eta} \cos \eta \quad \left(-\frac{\pi}{2} < \eta < \frac{\pi}{2}\right), \\ A &= 1 - 2\gamma \quad (0 \leq \gamma < 1), \quad B = -1 \quad \text{and} \quad \alpha - 1 = \beta = 0. \end{aligned}$$

Corollary 1 can thus be reduced *eventually* to the following known result due to Kim and Ponnusamy [19].

COROLLARY 2 *Let*

$$a \in \mathbb{C} \setminus \{0\}, \quad b \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad c > |a| + |b| + 1.$$

If $f \in \mathcal{R}_\eta(\gamma)$ and the following inequality:

$$\begin{aligned} & \frac{\Gamma(c)\Gamma(c - |a| - |b|)}{\Gamma(c - |a|)\Gamma(c - |b|)} \left(1 + \frac{2|ab|}{c - |a| - |b| - 1}\right) \\ & \leq 1 + \frac{\sec \eta}{2(1 - \gamma)} \end{aligned} \quad (28)$$

holds true, then

$$I_{a,b;c}(f) \in \text{UCV}.$$

Our next inclusion relationship (given by Theorem 2 below) involves the function class $\mathcal{S}_p(\alpha, \beta)$ defined by equation (9).

THEOREM 2 Suppose that

$$\alpha_j \in \mathbb{C} \setminus \{0\} \quad (j = 1, \dots, p) \quad \text{and} \quad \Re(\beta_j) > 0 \quad (j = 1, \dots, q),$$

and that

$$\Re \left(\sum_{j=1}^q \beta_j \right) > \sum_{j=1}^p |\alpha_j|.$$

If $f \in \mathcal{R}^\tau(A, B)$ and the following hypergeometric inequality:

$$\begin{aligned} & (1 + \alpha) {}_pF_q(|\alpha_1|, \dots, |\alpha_p|; \Re(\beta_1), \dots, \Re(\beta_q); 1) \\ & - (\alpha + \beta) {}_{p+1}F_{q+1}(|\alpha_1|, \dots, |\alpha_p|, 1; \Re(\beta_1), \dots, \Re(\beta_q), 2; 1) \\ & \leq (1 - \beta) \left(1 + \frac{1}{(A - B)|\tau|} \right) \end{aligned} \quad (29)$$

holds true, then

$$I_{\beta_1, \dots, \beta_q}^{\alpha_1, \dots, \alpha_p} f \in \mathcal{S}_p(\alpha, \beta).$$

Proof Proceeding as in the proof of Theorem 1, in light of Lemma 2 and the operational formula (15), we only need to show that

$$S_2 := \sum_{n=2}^{\infty} [n(1 + \alpha) - (\alpha + \beta)] |\Omega_n a_n| \leq 1 - \beta, \quad (30)$$

where the coefficients Ω_n ($n \in \mathbb{N} \setminus \{1\}$) are given (as before) by equation (16). Indeed, by applying Lemma 4 again in conjunction with the hypothesis (29) of Theorem 2, we readily observe from equation (30) that

$$\begin{aligned} S_2 & \leq (A - B)|\tau| \sum_{n=2}^{\infty} \left((1 + \alpha) - (\alpha + \beta) \frac{1}{n} \right) \left| \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_q)_{n-1} (1)_{n-1}} \right| \\ & \leq (A - B)|\tau| \left((1 + \alpha) [{}_pF_q(|\alpha_1|, \dots, |\alpha_p|; \Re(\beta_1), \dots, \Re(\beta_q); 1) - 1] \right) \\ & \quad - ((\alpha + \beta) [{}_{p+1}F_{q+1}(|\alpha_1|, \dots, |\alpha_p|, 1; \Re(\beta_1), \dots, \Re(\beta_q), 2; 1) - 1]) \\ & \leq 1 - \beta, \end{aligned}$$

which evidently completes the proof of Theorem 2. ■

Remark 3 The hypothesis (29) of Theorem 2 can be simplified considerably whenever the hypergeometric series occurring in it can be expressed in closed forms (such as in some of the hypergeometric reducible cases exemplified in Lemma 3). We choose to leave the details involved in these and other related developments as an exercise for the interested reader (see also [20] and [21]).

We conclude this paper by observing that, if we appeal to the *first* inequality in equation (24) asserted by Lemma 4, we can modify each of the hypergeometric inequalities (25), (27) and

(29) required in the hypotheses of Theorem 1, Corollary 1 and Theorem 2, respectively, by simply making the following replacement:

$$(A - B) \longmapsto \left(\frac{A - B}{1 + |B|} \right).$$

In fact, in this situation, the factor 2 occurring in the denominator on the right-hand side of the hypothesis (28) of Corollary 2 can be dropped.

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Convexity of the incomplete beta functions

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In this paper the author finds the condition on b and c such that the incomplete beta function $\phi(b, c; z)$ defined by $\phi(b, c; z) := {}_2F_1(1, b; c; z)$ and the normalized confluent hypergeometric function ${}_1F_1(1; c; z) = \sum_{k=1}^{\infty} (b, k-1)/(c, k-1)z^k$ are convex in Δ . Conditions on a, b , and c for the convexity of ${}_2F_1(a, b; c; z)$ and ${}_1F_1(a; c; z)$ for $a \neq 1$ remains an open problem.

Keywords: Gaussian hypergeometric functions; Incomplete beta functions

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1. Introduction and Main Results

Let $\mathcal{A}$ denote the class of all normalized analytic functions f ($f(0) = 0 = f'(0) - 1$) defined on the unit disc $\Delta = \{z : |z| < 1\}$ and $\mathcal{S} = \{f \in \mathcal{A} \mid f \text{ is one-to-one in } \Delta\}$. A function $f \in \mathcal{A}$ is called starlike, denoted by $f \in \mathcal{S}^*$, if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. A function $f \in \mathcal{A}$ that maps the unit disc Δ onto a convex domain is called a convex function. Let $\mathcal{K}$ denote the class of all functions $f \in \mathcal{A}$ that are convex. Each $f \in \mathcal{K}$ has the analytic characterization [1]

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \Delta.$$

A well-known fact is that $\mathcal{K} \subsetneq \mathcal{S}^* \subsetneq \mathcal{S}$. $f \in \mathcal{A}$ is said to be close-to-convex with respect to a convex function g if $\operatorname{Re}(f'(z)/g'(z)) > 0$ for $z \in \Delta$. Let $\mathcal{C}$ denotes the union of all such close-to-convex functions. There has been a considerable interest to find conditions on the parameters a, b , and c such that the normalized hypergeometric functions $(c/ab)(F(a, b; c; z) - 1)$ or ${}_2F_1(a, b; c; z)$ belong to one of the known subclasses of $\mathcal{S}$. Here $F(a, b; c; z)$ denotes the Gaussian hypergeometric function defined by the series

$$F(a, b; c; z) := {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} z^n, \quad (1)$$

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where $(a, 0) = 1$, $(a, n + 1) = (a + n)(a, n)$, $n = 0, 1, 2, \dots$, and $c \neq 0, -1, -2, -3, \dots$. We are interested in

Problem 1 Find conditions on a, b , and c so that $zF(a, b; c; z)$ is convex in Δ .

The problem of starlikeness and close-to-convexity of $zF(a, b; c; z)$ has been discussed by a number of authors (e.g. ref. [2–5]). But the exact range on the triplet (a, b, c) leading to solutions to Problem 1 remains open for each of these classes. The aim of this article is to find conditions on b and c so that the incomplete beta function ϕ defined by $\phi(b, c; z) \equiv zF(1, b; c; z)$ is in $\mathcal{K}$. It is well known that $\mathcal{K}$ is included in $\mathcal{S}^*(1/2)$, i.e. functions $f \in \mathcal{A}$ such that $\operatorname{Re}(zf'(z)/f(z)) > 1/2$ for $z \in \Delta$. We recall the following results concerning the starlikeness and the starlikeness of order $1/2$ for $\phi(b, c; z)$:

1. If b and c are related by either $b \in (0, 1/3]$ and $c \geq b + 1$ or $b \in [1/3, \infty)$ and $c \geq (5b + 1)/2$, then $\phi(b, c; z)$ is in $\mathcal{S}^*$ (see [4]).
2. If b and c are related by either $b \in (0, 1]$ and $c \geq 3 - b$ or $b \in [1, \infty)$ and $c \geq b + 1$, then $\phi(b, c; z)$ is in $\mathcal{S}^*$ see [3, 4]
3. If $b \geq 0$ and $c \geq \max\{1, b\}$, then $\phi(b, c; z)$ is in $\mathcal{S}^*$ (see [6]).
4. If $b > 0$ and $c = b$, then $\phi(b, c; z)$ is in $\mathcal{S}^*(1/2)$ (see [2, 7]).

For additional information concerning the starlikeness, the close-to-convexity and the starlikeness of order $1/2$ for $zF(a, b; c; z)$, we refer to ref. [2–4, 6, 7]. Although the convexity of $F(a, b; c; z)$ is addressed by Miller and Mocanu [3] and extended later by Ponnusamy and Vuorinen [4], very little on the results concerning the convexity of the normalized function $zF(a, b; c; z)$ is available in the literature. However, the non-convexity condition of $zF(a, b; c; z)$ is discussed in ref. [4]. A condition for $zF(a, b; c; z)$ to be convex in Δ is given in ref. [8] which we state as follows.

LEMMA [8] If $a, b > 0$ and $c > a + b + 2$, then a sufficient condition for $zF(a, b; c; z)$ to be convex in Δ is that

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[1 + \frac{3ab}{c-a-b-1} + \frac{(a, 2)(b, 2)}{(c-a-b-2, 2)} \right] \leq 2. \quad (2)$$

Because of equation (2), it can be easily seen that the condition on the parameters a, b , and c become very restrictive. We note that $1 + z = F(-1, -1; 1; z)$ is convex in Δ , whereas $z(1 + z) = zF(-1, -1; 1; z)$ is not even univalent. Further, for $f \in \mathcal{A}$ to be in $\mathcal{K}$, it is necessary that $|f''(0)| \leq 1$. In view of this fact, we easily have the following.

LEMMA 1 The function $zF(a, b; c; z)$ is not convex in Δ for $|ab| > |c|$, $c \neq 0, -1, -2, \dots$

The condition $|ab| > |c|$ in the above lemma may be weakened if one of a or b becomes a negative integer. We state our main result in the following form, which is proved in section 2

THEOREM 1 Let b and c be real and satisfy any one of the following conditions:

- (i) $0 < b \leq 3 = c$,
- (ii) $c \geq 3 + |b|$, $b \neq 0$.

Then the incomplete beta function $\phi(b, c; z) = zF(1, b; c; z)$ is convex in Δ , whenever $\phi'(b, c; z) \neq 0$.

${}_2F_1(1, b; c; z)$ has the integral representation

$$F(1, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-1} dt \quad \operatorname{Re}(c) > 0, \operatorname{Re}(b) > 0.$$

The Carlson–Shaffer operator $\mathcal{L}(b, c)(f)(z)$ is defined as

$$\mathcal{L}(b, c)(f)(z) = \phi(b, c; z) * f(z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \frac{f(tz)}{t} dt$$

and $\mathcal{L}(b, c)(\mathcal{F}) = \{\mathcal{L}(b, c)(f) : f \in \mathcal{F}\}$. Here $*$ denotes the usual Hadamard product or convolution. Several basic geometric properties of this operator is known in the literature (for example see [9, 10]).

COROLLARY 1 Assume the hypothesis of Theorem 1. Then,

1. $\mathcal{L}(b, c)(\mathcal{F}) \subseteq \mathcal{F}$ whenever $\mathcal{F}$ is $\mathcal{S}^*(\alpha)$ or $\mathcal{C}(\alpha)$ or $\mathcal{K}(\alpha)$,
2. $\mathcal{L}(b, c)(f)$ is univalent in Δ whenever $\operatorname{Re} f'(z) > \alpha$ ($0 \leq \alpha < 1$).

Here $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$ and $\mathcal{K}(\alpha)$ denote, respectively the functions that are starlike of order α , close-to-convex of order α , and convex of order α (for details, see [1] and references therein).

Proof Let b and c satisfy the conditions (i) and (ii) of Theorem 1.

1. Since $\phi(b, c; z)$ has been shown to be convex, the Pólya–Schoenberg conjecture proved in ref. [7] implies that $f(z) * \phi(b, c; z) = \mathcal{L}(b, c)(f)(z)$ is in $\mathcal{F}$ whenever f is in $\mathcal{F}$.
2. We know that $\phi(b, c; z)$ is convex, which implies that $\operatorname{Re} \phi(b, c; z)/z > 1/2$. We have from ref. [9] that if $\operatorname{Re} f'(z) > \alpha$, then $\operatorname{Re} \mathcal{L}'(b, c)(f) > \alpha$ ($0 \leq \alpha < 1$), which implies that $\mathcal{L}(b, c)(f)$ is univalent in Δ , as $\alpha \geq 0$. ■

If we replace b by $1/\rho$ and z by $z\rho$ in the series representation of ${}_2F_1(a, b; c; z)$ and allow $\rho \rightarrow 0$, then we get the confluent hypergeometric function ${}_1F_1(a; c; z)$ given in series form by

$${}_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)}{(c, n)n!} z^n, \quad z \in \mathbb{C}.$$

We find the condition on the convexity of $z{}_1F_1(1; c; z)$ in the following form, which is proved in section 2.

THEOREM 2 For $c \geq 5/2$, $z{}_1F_1(1; c; z)$ is convex in Δ .

Using the integral representation of the confluent hypergeometric function, one can easily have

$$C(f) = z{}_1F_1(1; c; z) * f(z) = (c-1) \int_0^1 (1-t)^{c-2} [ze^{tz} * f(z)] dt$$

and $C(\mathcal{F}) = \{C(f) : f \in \mathcal{F}\}$. The same reasoning as in the proof of Corollary 1 immediately shows the following result.

COROLLARY 2 Assume the hypothesis of the Theorem 1. Then,

1. $C(\mathcal{F}) \subseteq \mathcal{F}$ whenever $\mathcal{F}$ is $\mathcal{S}^*(\alpha)$ or $\mathcal{C}(\alpha)$ or $\mathcal{K}(\alpha)$,
2. For $\operatorname{Re} f'(z) > \alpha$ ($0 \leq \alpha < 1$), $C(f)$ is univalent in Δ .

Let $h(z) = \phi(z^2)/z$. Then, we have

$$z \frac{h'(z)}{h(z)} = 2 \frac{z^2 \phi'(z^2)}{\phi(z^2)} - 1$$

and, as $f \in \mathcal{A}$ is convex, which implies that f is starlike of order $1/2$, it is clear that the odd function $zF(1, b; c; z^2)$ is starlike under the hypothesis of Theorem 1. The same remark applies for the odd confluent hypergeometric function ${}_1F_1(1; c; z^2)$. We need the following lemma for the proof of our main theorems.

LEMMA 2 [11] *Let $\Omega \subset \mathbb{C}$. Suppose that $\psi : \mathbb{C}^2 \times \Delta \rightarrow \mathbb{C}$ satisfies the condition $\psi(ir_2, s_1; z) \notin \Omega$ when r_2 is real and $s_1 \leq -(1 + r_2^2)/2$. If p is analytic in Δ , with $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$ for $z \in \Delta$, then $\text{Re} p(z) > 0$ in Δ .*

2. Proofs of Theorems 1 and 2

We note that $\phi(b, c; z) \neq 0$, under the hypothesis of Theorem 1, follows from ref. [2, p. 8]. Throughout the proof, we assume that $z = x + iy \in \Delta$. This information will be required in the proof of Theorem 1.

Proof of Theorem 1 Let $\phi(b, c; z) = zF(1, b; c; z)$ and

$$1 + \frac{z\phi''(b, c; z)}{\phi'(b, c; z)} = p(z)$$

so that

$$(p(z) - 1)\phi'(b, c; z) = z\phi''(b, c; z)$$

or, equivalently,

$$[p(z) - 1](zF'(z) + F(z)) = z^2F''(z) + 2zF'(z), \quad z \in \Delta. \quad (3)$$

By hypothesis, we have $\phi'(z) \neq 0$ in Δ . Thus, $p(z)$ is analytic in Δ , and to complete the proof, it suffices to prove that $\text{Re} p(z) > 0$ in Δ . Since $F(z)$ satisfies the differential equation

$$z(1 - z)F''(z) + [c - (2 + b)z]F'(z) - bF(z) = 0, \quad (4)$$

it follows from equations (3) and (4) that

$$zF'(z) [(p(z) - 1)(1 - z) + c - 2 - bz] + [(1 - z)(p(z) - 1) - bz] F(z) = 0. \quad (5)$$

Substituting equation (5) in equation (3), we get

$$z^2F''(z) [(p(z) - 1)(1 - z) + c - 2 - bz] = (p(z) - 1)[2(1 - z) + c - 2]F(z) - 2bzF(z). \quad (6)$$

Differentiating equation (5) with respect to z gives

$$\begin{aligned} & z^2F''(z) [(p(z) - 1)(1 - z) + c - 2 - bz] + F(z) [(1 - z)zp'(z) - zp(z) + z(1 - b)] \\ & + zF'(z) [zp'(z)(1 - z) + 2(p(z) - 1)(1 - z) - z(p(z) - 1) - 3bz + c - 2] = 0. \end{aligned}$$

Comparing the last equation with equations (5) and (6) and using the fact that $F(z) \neq 0$, we get after simplification

$$(zp'(z) + p^2(z))(1-z)(c-2) - (c-2)[bz - (c-3)]p(z) + z(c-2)(1-b) - (c-2)^2 = 0. \quad (7)$$

If $c = 2$ and $0 \leq b < 2$, we easily see that

$$\phi(b, 2; z) = zF(1, b; 2; z) = \begin{cases} \frac{(1-z)^{1-b} - 1}{b-1} & \text{if } b \neq 1, \\ -\log(1-z) & \text{if } b = 1, \end{cases}$$

which is known to be convex, as it is an extremal function for the class of convex functions of order $b/2$. We note from Lemma 1 that for $b > 1 = c$, $\phi(b, 1; z)$ is not convex. For $c \neq 2$, we denote the left-hand side of the differential equation (7) as

$$(c-2)\psi(p(z), zp'(z); z).$$

This gives

$$\psi(p(z), zp'(z); z) = (1-z)(zp'(z) + p^2(z)) - bzp(z) + (c-3)p(z) + z(1-b) - (c-2).$$

Hence,

$$\psi(r, s; z) = (1-z)(s + r^2) + r(c-3-bz) + z(1-b) - (c-2). \quad (8)$$

According to Lemma 2, it suffices to show that $\psi(ir_2, s_1; z) \neq 0$ for $z \in \Delta$ whenever r_2 is real and $s_1 \leq -(1+r_2^2)/2$. From equation (8), we get

$$\operatorname{Re} \psi(ir_2, s_1; z) = (1-x)(s_1 - r_2^2) + br_2y + x(1-b) - (c-2),$$

where $z = x + iy$. Using $s_1 \leq -(1+r_2^2)/2$, it is enough if we can prove

$$-\frac{1}{2}(3r_2^2 + 1)(1-x) + br_2y + x(1-b) - (c-2) < 0,$$

which is equivalent to proving $g(r_2, b, c) > 0$, where

$$g(r_2, b, c) := 3(1-x)r_2^2 - 2yr_2b + x(2b-3) + 2c-3. \quad (9)$$

Case (i): If $b = c = 3$, we have $\phi(b, b; z) = z/(1-z)$ and there is nothing to prove. If $c = 3$ and $0 < b < c$, then equation (9) is positive if its discriminant is less than or equal to zero. This means that we need to verify the inequality $4b^2y^2 \leq 12(1-x)[3 + x(2b-3)]$. Since $y^2 < 1-x^2$, it suffices to show that

$$b^2(1-x^2) \leq 3(1-x)[3 + x(2b-3)],$$

which after simplification is seen to be equivalent to $(3-b)[3 + b - (3-b)x] \geq 0$. As $0 < b < c = 3$, the last inequality becomes $3(1-x) + b(1+x) > 0$, which is clearly true as $x \in (-1, 1)$, and hence $\operatorname{Re}(ir_2, s_1; z) < 0$.

Case (ii): Assume that $c \geq 3 + |b|$ and $b \neq 0$ so that $c - 3 - bz \neq 0$ for $z \in \Delta$. We rewrite equation (8) as

$$\psi(r, s; z) = (c - 3 - bz) \left[J(z)(s + r^2) + r + \frac{J(z) - H(z)}{2} \right], \quad (10)$$

where

$$J(z) = \frac{1 - z}{c - 3 - bz} = \frac{1}{c - 3} \left(\frac{1 - z}{1 - (b/c - 3)z} \right)$$

and

$$H(z) = \frac{2c - 3 + (2b - 3)z}{c - 3 - bz} = \frac{2c - 3}{c - 3} \left(\frac{1 + (2b - 3/2c - 3)z}{1 - (b/c - 3)z} \right).$$

We show that $\operatorname{Re} \psi(ir_2, s_1; z) \neq 0$ using the method followed in ref. [4]. To do this, let us first consider the situation where $b > 0$. We know that the function

$$w(z) = \frac{1 + Az}{1 + Bz}, \quad -1 \leq A, \quad B \leq 1,$$

maps the unit disc Δ conformally onto the disc

$$\left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{|B - A|}{1 - B^2} \quad \text{if } B \neq \pm 1.$$

We also note that

$$\operatorname{Re}(w) > \begin{cases} \frac{1 + A}{1 + B} & \text{if } B > A, \\ \frac{1 + B}{1 - A} & \text{if } B < A. \end{cases}$$

If we let $A = -1$ and $B = -b/(c - 3)$, then one has $A < B$ so that the function $J(z)$ satisfies the inequality $\operatorname{Re} J(z) > 0$ for $z \in \Delta$. Similarly, it is a simple exercise to see that $\operatorname{Re} H(z) > 0$ for $z \in \Delta$. For $b < 0$, we have $c + b > 3$. Then, by an argument similar to the case $b > 0$, we get that $\operatorname{Re} J(z) > 0$ and $\operatorname{Re} H(z) > 0$ in Δ . Let $c = 3 - b$ and $b < 0$. Then

$$\operatorname{Re} J(z) = -\frac{1}{b} \operatorname{Re} \left(\frac{1 - z}{1 + z} \right) > 0 \quad \text{and} \quad \operatorname{Re} H(z) = \operatorname{Re} \frac{2b - 3}{b} \left(\frac{1 - z}{1 + z} \right) > 0.$$

Now, for all real r_2 and $s_1 \leq -(1 + r_2^2)/2$ and $z \in \Delta$, it suffices to show that the square bracketed term in equation (10) is non-zero. Indeed,

$$\begin{aligned} \operatorname{Re} \left[J(z)(s_1 - r_2^2) + \frac{J(z) - H(z)}{2} \right] &= \operatorname{Re} J(z)[s_1 - r_2^2] + \operatorname{Re} \frac{J(z) - H(z)}{2} \\ &\leq \left[-\frac{1 + r_2^2}{2} - r_2^2 \right] \operatorname{Re} J(z) + \operatorname{Re} \left(\frac{J(z) - H(z)}{2} \right) \\ &= -\frac{3r_2^2}{2} \operatorname{Re} J(z) - \operatorname{Re} \left(\frac{H(z)}{2} \right) \\ &< 0, \end{aligned}$$

since $\operatorname{Re} J(z) > 0$ and $\operatorname{Re} H(z) > 0$.

Let $c = b + 3$ and $b > 0$. Then

$$J(z) = \frac{1-z}{c-3-bz} = \frac{1}{b} > 0$$

and

$$\operatorname{Re} \frac{J(z) - H(z)}{2} = \operatorname{Re} \frac{(2-c) + (1-b)z}{c-3-bz} = -\frac{1}{b} - \operatorname{Re} \left(\frac{1+z}{1-z} \right) < -\frac{1}{b}$$

so that, in this case too, one has

$$\operatorname{Re} \left[J(z)(s+r^2) + r + \frac{J(z) - H(z)}{2} \right] = \frac{1}{b}(s_1 - r_2^2) - \frac{1}{b} - \operatorname{Re} \left(\frac{1+z}{1-z} \right) < 0$$

whenever $s_1 \leq -(1+r_2^2)/2$ and $r_2 \in \mathbb{R}$. The proof of the theorem is complete. $\blacksquare$

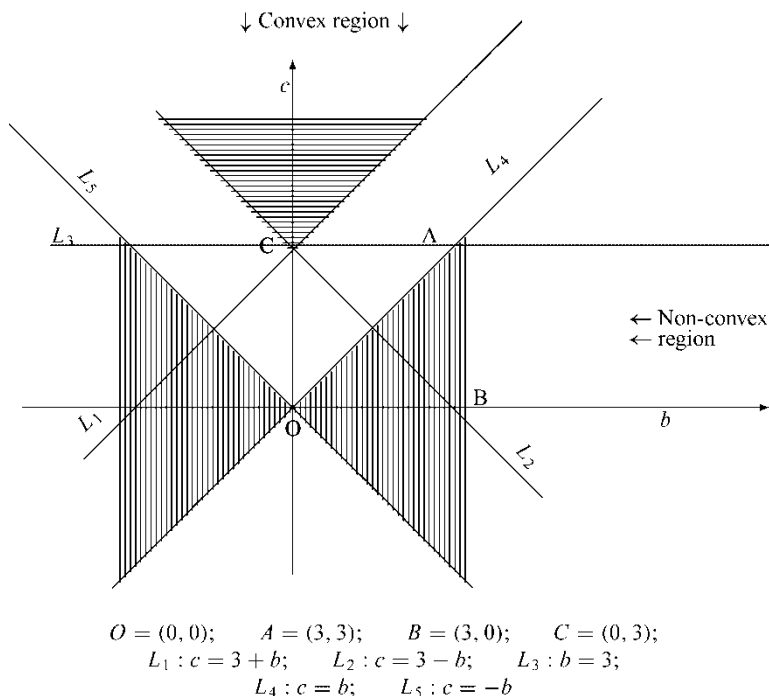


Figure 1. Regions of convexity and non-convexity for the incomplete beta function.

Proof of Theorem 2 In equation (7), we first substitute b by $1/\rho$ and z by $z\rho$. Now, if allow $\rho \rightarrow 0$, then we see that the function p defined by

$$p(z) = 1 + \frac{z\phi''(b, c; z)}{\phi'(b, c; z)},$$

where $\phi(b, c; z) = {}_1F_1(1; c; z)$, satisfies the differential equation

$$(c-2)[zp'(z) + p^2(z) + (c-3-z)p(z) - z - (c-2)] = 0. \quad (11)$$

If $c = 2$, then $z {}_1F_1(1; 2; z) = e^z - 1$, which is convex. Indeed, if $g(z) = e^z - 1$, then

$$\left| \frac{zg''(z)}{g'(z)} \right| < 1 \quad \text{for } z \in \Delta,$$

and so g is convex in Δ . If $c \neq 2$, the above differential equation is equivalent to

$$\psi(p(z), zp'(z); z) = 0,$$

where

$$\psi(r, s; z) = (s + r^2) + r(c - 3 - z) + 2 - c - z. \quad (12)$$

As usual, we need to prove that $\psi(ir_2, s_1; z) \neq 0$ for $z \in \Delta$ whenever r_2 is real and $s_1 \leq -(1 + r_2^2)/2$. As $s_1 \leq -(1 + r_2^2)/2$, we have

$$\operatorname{Re} \psi(ir_2, s_1; z) = (s_1 - r_2^2) + r_2 y + 2 - c - x \leq -\frac{1}{2} [3r_2^2 - 2r_2 y + 2x + 2c - 3].$$

Thus, $\operatorname{Re} \psi(ir_2, s_1; z) < 0$ if the inequality $y^2 < 3(2x + 2c - 3)$ holds for $x^2 + y^2 < 1$. The desired inequality follows if we can prove $1 - x^2 \leq 3(2x + 2c - 3)$, which is clearly true because $c \geq 5/2$. ■

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A weighted and exponential generalization of Wilker's inequality and its applications

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In this paper, the authors first prove a weighted and exponential generalization of Wilker's inequality. The main result presented here is then applied with a view to deriving an improved version of the Sándor–Bencze conjectured inequality. Some other closely-related inequalities are also considered.

Keywords: Wilker's inequality; Huygens's inequality; Weighted generalization; Sándor–Bencze conjectured inequality; Weighted inequalities; Exponential generalization

AMS Subject Classification: Primary: 26D15; Secondary: 26D05; 33B20

1. Introduction

The following inequality is known in the literature as Wilker's inequality [1]:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \quad \left(0 < x < \frac{\pi}{2}\right). \quad (1)$$

Wilker's inequality (1) has attracted remarkable interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations and improvements ([1–8]; see also the references therein). Recently, the following similar inequality (proved by Huygens [9]) was considered by Sándor and Bencze [10]:

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3 \quad \left(0 < x < \frac{\pi}{2}\right). \quad (2)$$

Huygens's inequality (2) prompts us to ask a natural question: Does there exist an inequality which unifies (and possibly also extends) Wilker's inequality (1) and Huygens's inequality (2)? The following theorem gives an affirmative answer to this question.

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THEOREM 1 *Let*

$$0 < x < \frac{\pi}{2}, \quad \lambda > 0, \quad \mu > 0 \quad \text{and} \quad p \leq \frac{2q\mu}{\lambda}.$$

Then, for

$$q > 0 \quad \text{or} \quad q \leq \min \left\{ -\frac{\lambda}{\mu}, -1 \right\},$$

the following inequality holds true:

$$\frac{\lambda}{\mu + \lambda} \left(\frac{\sin x}{x} \right)^p + \frac{\mu}{\mu + \lambda} \left(\frac{\tan x}{x} \right)^q > 1. \quad (3)$$

2. A set of lemmas

In order to prove Theorem 1, we need the following lemmas.

LEMMA 1 (see [11, p. 17]). *If*

$$x_i > 0, \quad \lambda_i > 0 \quad (i = 1, \dots, n) \quad \text{and} \quad \sum_{i=1}^n \lambda_i = 1,$$

then

$$\sum_{i=1}^n \lambda_i x_i \geq \prod_{i=1}^n x_i^{\lambda_i}. \quad (4)$$

LEMMA 2 (see [12, p. 238]). *The following two-sided trigonometric inequality holds true:*

$$\cos x < \left(\frac{\sin x}{x} \right)^3 < 1 \quad \left(0 < x < \frac{\pi}{2} \right). \quad (5)$$

LEMMA 3 *The following trigonometric inequality holds true:*

$$\left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x} > 2 \quad \left(0 < x < \frac{\pi}{2} \right). \quad (6)$$

Proof Define a function

$$f : \left(0, \frac{\pi}{2} \right) \longrightarrow \mathbb{R}$$

by

$$f(x) = \left(\frac{x}{\sin x} \right)^2 + \frac{x}{\tan x}.$$

Then, upon differentiating $f(x)$ with respect to x , we get

$$f'(x) = \frac{1}{\sin^3 x} (\sin^2 x \cos x - 2x^2 \cos x + x \sin x).$$

Next, by applying Lemma 2 followed by a simple calculation, we find that

$$\begin{aligned}
 f'(x) &= \frac{x^2}{\sin^3 x} \left[\cos x \left(\left(\frac{\sin x}{x} \right)^2 - 2 \right) + \frac{\sin x}{x} \right] \\
 &= \frac{x^2}{\sin^3 x} \left[\left(\cos x - \left(\frac{\sin x}{x} \right)^3 \right) \left(\left(\frac{\sin x}{x} \right)^2 - 2 \right) \right. \\
 &\quad \left. + \left(\frac{\sin x}{x} \right)^3 \left(\left(\frac{\sin x}{x} \right)^2 - 2 \right) + \frac{\sin x}{x} \right] \\
 &= \frac{x^2}{\sin^3 x} \left[\left(\cos x - \left(\frac{\sin x}{x} \right)^3 \right) \left(\left(\frac{\sin x}{x} \right)^2 - 2 \right) \right. \\
 &\quad \left. + \left(\frac{\sin x}{x} \right) \left(\left(\frac{\sin x}{x} \right)^2 - 1 \right)^2 \right] \\
 &> 0 \quad \left(0 < x < \frac{\pi}{2} \right).
 \end{aligned}$$

This means that $f(x)$ is *strictly increasing* on the open interval $(0, \pi/2)$. Consequently, we can deduce from the following observation:

$$\lim_{x \rightarrow 0^+} f(x) = 2$$

that

$$f(x) > 2 \quad \left(0 < x < \frac{\pi}{2} \right),$$

which leads us to the inequality (6) asserted by Lemma 3. ■

3. Proof of the main result (Theorem 1)

In our proof of Theorem 1, we consider the following two cases.

Case 1 Let

$$\lambda > 0, \quad \mu > 0, \quad p \leq \frac{2q\mu}{\lambda} \quad \text{and} \quad q > 0.$$

Then, by applying Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}
 &\frac{\lambda}{\mu + \lambda} \left(\frac{\sin x}{x} \right)^p + \frac{\mu}{\mu + \lambda} \left(\frac{\tan x}{x} \right)^q \\
 &\geq \left(\frac{\sin x}{x} \right)^{p\lambda/(\mu+\lambda)} \left(\frac{\tan x}{x} \right)^{q\mu/(\mu+\lambda)} \\
 &= \left(\frac{\sin x}{x} \right)^{p\lambda/(\mu+\lambda)} \left(\frac{\sin x}{x} \right)^{q\mu/(\mu+\lambda)} \left(\frac{1}{\cos x} \right)^{q\mu/(\mu+\lambda)}
 \end{aligned}$$

$$\begin{aligned}
&> \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)} \left(\frac{\sin x}{x}\right)^{q\mu/(\mu+\lambda)} \left(\frac{\sin x}{x}\right)^{-3q\mu/(\mu+\lambda)} \\
&= \left(\frac{\sin x}{x}\right)^{(p\lambda-2q\mu)/(\mu+\lambda)} \\
&\geq 1 \quad \left(0 < x < \frac{\pi}{2}\right),
\end{aligned}$$

which is the desired inequality (3).

Case 2 Let

$$\lambda > 0, \quad \mu > 0, \quad p \leq \frac{2q\mu}{\lambda} \quad \text{and} \quad q \leq \min \left\{ -\frac{\lambda}{\mu}, -1 \right\}.$$

Then it follows from the hypothesis of Theorem 1 that

$$\begin{aligned}
&\frac{\lambda}{\mu+\lambda} \left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda} \left(\frac{\tan x}{x}\right)^q \\
&\geq \frac{\lambda}{\mu+\lambda} \left(\frac{\sin x}{x}\right)^{2q\mu/\lambda} + \frac{\mu}{\mu+\lambda} \left(\frac{\tan x}{x}\right)^q \\
&= \frac{\lambda}{\mu+\lambda} \left(\frac{x}{\sin x}\right)^{-2q\mu/\lambda} + \frac{\mu}{\mu+\lambda} \left(\frac{x}{\tan x}\right)^{-q} \quad \left(-\frac{q\mu}{\lambda} \geq 1; -q \geq 1\right). \quad (7)
\end{aligned}$$

Moreover, we find from Lemma 3 that

$$\left(\frac{x}{\sin x}\right)^2 > 2 - \frac{x}{\tan x} > 0 \quad \left(0 < x < \frac{\pi}{2}\right).$$

By combining the inequality (7) with the above trigonometric inequality, we obtain

$$\begin{aligned}
&\frac{\lambda}{\mu+\lambda} \left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda} \left(\frac{\tan x}{x}\right)^q \\
&> \frac{\lambda}{\mu+\lambda} \left(2 - \frac{x}{\tan x}\right)^{-q\mu/\lambda} \\
&\quad + \frac{\mu}{\mu+\lambda} \left(\frac{x}{\tan x}\right)^{-q} \quad \left(0 < x < \frac{\pi}{2}\right). \quad (8)
\end{aligned}$$

We now define a function

$$f : (0, 1) \longrightarrow \mathbb{R}$$

by

$$f(x) = \frac{\lambda}{\mu+\lambda} (2-x)^{-q\mu/\lambda} + \frac{\mu}{\mu+\lambda} x^{-q},$$

which, upon differentiating with respect to x , yields

$$f'(x) = \frac{q\mu}{\mu+\lambda} [(2-x)^{-(q\mu/\lambda)-1} - x^{-q-1}].$$

For

$$-\frac{q\mu}{\lambda} \geq 1, \quad -q \geq 1 \quad \text{and} \quad 0 < x < 1,$$

it is easy to verify that

$$(2-x)^{-(q\mu/\lambda)-1} \geq 1 \geq x^{-q-1}.$$

We thus conclude that

$$f'(x) \leq 0 \quad (0 < x < 1),$$

which immediately implies that $f(x)$ is *decreasing* on the open interval $(0, 1)$. Hence we have

$$f(x) \geq f(1) = 1 \quad (0 < x < 1).$$

Now, making use of the inequality (8) together with the following well-known trigonometric inequality:

$$0 < \frac{x}{\tan x} < 1 \quad \left(0 < x < \frac{\pi}{2}\right),$$

we deduce that

$$\frac{\lambda}{\mu + \lambda} \left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu + \lambda} \left(\frac{\tan x}{x}\right)^q > f\left(\frac{x}{\tan x}\right) \geq 1,$$

which proves the inequality (3). This completes the proof of Theorem 1.

By setting

$$(p, q) = (2, 1) \quad \text{and} \quad (\lambda, \mu) = (2, 1)$$

in Theorem 1, we obtain a weighted generalization of Wilker's inequality (1) and an exponential generalization of Huygens's inequality (2), given by Corollary 1 and Corollary 2, respectively.

COROLLARY 1 *Let*

$$0 < x < \frac{\pi}{2} \quad \text{and} \quad 0 < \lambda \leq \mu.$$

Then

$$\frac{\lambda}{\mu + \lambda} \left(\frac{\sin x}{x}\right)^2 + \frac{\mu}{\mu + \lambda} \left(\frac{\tan x}{x}\right) > 1. \quad (9)$$

COROLLARY 2 *Let*

$$0 < x < \frac{\pi}{2} \quad \text{and} \quad p \leq q.$$

Then, for

$$q > 0 \quad \text{or} \quad q \leq -2,$$

the following inequality holds true:

$$2 \left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q > 3. \quad (10)$$

Remark 1 It is obvious that Wilker's inequality (1) would follow as a special case of the inequality (9) when

$$\lambda = \mu = 1.$$

Furthermore, in its special case when

$$p = q = 1,$$

the inequality (10) reduces to Huygens's inequality (2).

4. Applications of Theorem 1

As an *open problem*, Sándor and Bencze [10] asked to prove that, for all

$$x \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \alpha \in (0, \infty),$$

the following inequality holds true:

$$\left(\frac{\sin x}{x}\right)^\alpha > \frac{\cos^\alpha x}{1 + \cos^\alpha x}. \quad (11)$$

The Sándor–Bencze conjectured inequality (11) provides a good opportunity to illustrate the application of the foregoing results. Based upon the improved Wilker inequality (3) asserted by Theorem 1, we give here a sharp and generalized version of the Sándor–Bencze conjectured inequality (11).

THEOREM 2 *Let*

$$0 < x < \frac{\pi}{2}.$$

Then, for

$$\alpha > 0 \quad \text{or} \quad \alpha \leq -1,$$

the following inequality holds true:

$$\left(\frac{\sin x}{x}\right)^\alpha > \frac{4 \cos^\alpha x}{1 + \sqrt{1 + 8 \cos^{2\alpha} x}}. \quad (12)$$

Proof Putting

$$\lambda = \mu = 1, \quad p = 2\alpha \quad \text{and} \quad q = \alpha$$

in Theorem 1, we get

$$\left(\frac{\sin x}{x}\right)^{2\alpha} + \cos^{-\alpha} x \left(\frac{\sin x}{x}\right)^\alpha - 2 > 0 \quad \left(0 < x < \frac{\pi}{2}; \alpha > 0 \quad \text{or} \quad \alpha \leq -1\right),$$

which is equivalent to the following inequality:

$$\left[\left(\frac{\sin x}{x}\right)^\alpha + \frac{\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] \cdot \left[\left(\frac{\sin x}{x}\right)^\alpha + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] > 0.$$

We can now deduce from the above inequality that

$$\left(\frac{\sin x}{x}\right)^\alpha + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2} > 0,$$

this is, that

$$\left(\frac{\sin x}{x}\right)^\alpha > \frac{-\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2} = \frac{4 \cos^\alpha x}{1 + \sqrt{1 + 8 \cos^{2\alpha} x}}. \quad (13)$$

The proof of Theorem 2 is thus completed. ■

As a consequence of Theorem 2, we immediately obtain the following refinement of the Sándor–Bencze conjectured inequality (11):

COROLLARY 3 *If*

$$0 < x < \frac{\pi}{2} \quad \text{and} \quad \alpha > 0,$$

then

$$\left(\frac{\sin x}{x}\right)^\alpha > \frac{4 \cos^\alpha x}{1 + \sqrt{1 + 8 \cos^{2\alpha} x}} > \frac{2 \cos^\alpha x}{1 + \cos^\alpha x} > \frac{\cos^\alpha x}{1 + \cos^\alpha x}. \quad (14)$$

In addition, upon replacing α by $-\alpha$ in Theorem 2, a *reversed* version of the Sándor–Bencze conjectured inequality (11) is derived as follows.

COROLLARY 4 *If*

$$0 < x < \frac{\pi}{2} \quad \text{and} \quad \alpha \geq 1,$$

then

$$\left(\frac{\sin x}{x}\right)^\alpha < \frac{\cos^\alpha x + \sqrt{\cos^{2\alpha} x + 8}}{4}. \quad (15)$$

Remark 2 Corollary 3 and Corollary 4 show that the inequality (12) is sharper and more general than the Sándor–Bencze conjectured inequality (11).

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Continuous and discrete inversion formulas for the Stockwell transform

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The Stockwell transform of a signal, *i.e.*, a function in $L^2(\mathbb{R})$, is defined as a hybrid of the Gabor transform and the wavelet transform. The continuous and discrete reconstruction formulas for a signal from its Stockwell transform are established.

Keywords: Fourier transform; Gabor transform; Wavelet transform; Stockwell transform; Spectrum; Inversion formula; Wavelet frame

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1. Introduction

Let us recall that for a signal f in $L^2(\mathbb{R})$, the Gabor transform or the short-time Fourier transform $G_\varphi f$ of f with respect to a window φ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(x-b)} dx, \quad b, \xi \in \mathbb{R}.$$

Let us note that

$$(G_\varphi f)(b, \xi) = (2\pi)^{-1/2} (f, M_\xi T_{-b} \varphi)_{L^2(\mathbb{R})}, \quad b, \xi \in \mathbb{R},$$

where $(\cdot, \cdot)_{L^2(\mathbb{R})}$ is the inner product in $L^2(\mathbb{R})$, M_ξ and T_{-b} are the modulation operator and the translation operator given by

$$(M_\xi h)(x) = e^{ix\xi} h(x)$$

and

$$(T_{-b} h)(x) = h(x - b)$$

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for all measurable functions h on $\mathbb{R}$ and all x in $\mathbb{R}$. We call the function $M_\xi T_{-b}\varphi$ the Gabor wavelet generated from φ by translation T_{-b} and modulation M_ξ .

The usefulness of the Gabor wavelets in signal analysis is enhanced by the following resolution of the identity formula, which allows the reconstruction of a signal from its Gabor transform.

THEOREM 1.1 *Suppose that $\|\varphi\|_2 = 1$, where $\|\cdot\|_2$ is the norm in $L^2(\mathbb{R})$, then for all f and g in $L^2(\mathbb{R})$,*

$$(f, g)_{L^2(\mathbb{R})} = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R})} (M_\xi T_{-x}\varphi, g)_{L^2(\mathbb{R})} dx d\xi.$$

Another way of looking at Theorem 1.1 is that for all f in $L^2(\mathbb{R})$,

$$f = (2\pi)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, M_\xi T_{-x}\varphi)_{L^2(\mathbb{R})} M_\xi T_{-x}\varphi dx d\xi.$$

The formula in Theorem 1.1 is also known as a continuous inversion formula for the Gabor transform.

In signal analysis, $(G_\varphi f)(b, \xi)$ gives the time-frequency content of a signal f at time b and frequency ξ by placing the window φ at time b . The drawback here is that a window of fixed width is used for all time b . It is more accurate and desirable if we can have an adaptive window that gives a wide window for low frequency and a narrow window for high frequency. That this can be done comes from familiarity with the wavelet transform, which we now recall.

Let $\varphi \in L^2(\mathbb{R})$ be such that

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty, \quad (1)$$

where $\hat{\varphi}$ is the Fourier transform of φ . The convention that we use in this paper is that the Fourier transform $\hat{f}$ of a function f in $L^1(\mathbb{R})$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-(1/2)} \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx, \quad \xi \in \mathbb{R}.$$

Such a function φ is said to satisfy the admissibility condition (1) and is sometimes called the mother affine wavelet. The adjective *affine* comes from the connection with the affine group. See Chapter 18 of [1] in this connection.

Let $\varphi \in L^2(\mathbb{R})$ be a mother affine wavelet. Then for all b in $\mathbb{R}$ and a in $\mathbb{R} \setminus \{0\}$, we define the affine wavelet $\varphi_{b,a}$ by

$$\varphi_{b,a}(x) = \frac{1}{\sqrt{|a|}} \varphi\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}.$$

We note that $\varphi_{b,a}$ is generated from the function φ by translation and dilation. To put things in perspective, let $b \in \mathbb{R}$ and let $a \in \mathbb{R} \setminus \{0\}$. Then we see that the wavelet $\varphi_{b,a}$ can be expressed as

$$\varphi_{b,a} = T_{-b} D_{1/a} \varphi,$$

where $D_{1/a}$ is the dilation operator defined by

$$(D_{1/a}h)(x) = \frac{1}{\sqrt{|a|}} h\left(\frac{x}{a}\right)$$

for all measurable functions h on $\mathbb{R}$ and all x in $\mathbb{R}$.

Let φ be a mother affine wavelet. Then the wavelet transform $\Omega_\varphi f$ of a function f in $L^2(\mathbb{R})$ is defined to be the function on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ by

$$(\Omega_\varphi f)(b, a) = (f, \varphi_{b,a})_{L^2(\mathbb{R})}$$

for all b in $\mathbb{R}$ and a in $\mathbb{R} \setminus \{0\}$. The nucleus of the analysis of the wavelet transform is the following resolution of the identity formula, which is a continuous inversion formula.

THEOREM 1.2 *Let φ be a mother affine wavelet. Then for all functions f and g in $L^2(\mathbb{R})$,*

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} (\varphi_{b,a}, g)_{L^2(\mathbb{R})} \frac{db da}{a^2},$$

where

$$c_\varphi = 2\pi \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi.$$

Remark It can also be proved that a necessary condition for the continuous inversion formula to hold is that φ has to be a mother affine wavelet. Indeed, suppose that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} (\varphi_{b,a}, g)_{L^2(\mathbb{R})} (db da/a^2)$ exists for all f and g in $L^2(\mathbb{R})$. Then, letting $f = g = \varphi$, we get

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\varphi, \varphi_{b,a})_{L^2(\mathbb{R})}|^2 \frac{db da}{a^2} < \infty,$$

which can be shown to be the same as

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Details can be found in [2] and [1].

The resolution of the identity formula leads to the reconstruction formula, which says that

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi_{b,a})_{L^2(\mathbb{R})} \varphi_{b,a} \frac{db da}{a^2}$$

for all f in $L^2(\mathbb{R})$. In other words, we have a continuous inversion formula for the signal f from a knowledge of its time-scale spectrum.

As in the case of the Gabor transform, there is a window $\varphi_{b,a}$ in the wavelet transform. Unlike the case of the Gabor transform, the window $\varphi_{b,a}$ is adjustable, in the sense that it is narrow if the scale a is small and wide if the scale a is big.

Now, let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then, combining the merits of the Gabor transform and the wavelet transform, the Stockwell transform $S_\varphi f$ with window φ of a signal f in $L^2(\mathbb{R})$ is defined by

$$(S_\varphi f)(b, \xi) = (2\pi)^{-(1/2)} |\xi| \int_{-\infty}^{\infty} e^{-ix\xi} f(x) \overline{\varphi(\xi(x-b))} dx, \quad b, \xi \in \mathbb{R}.$$

We note that for all f in $L^2(\mathbb{R})$, all b in $\mathbb{R}$ and all ξ in $\mathbb{R} \setminus \{0\}$,

$$(S_\varphi f)(b, \xi) = (f, \varphi^{b,\xi})_{L^2(\mathbb{R})},$$

where

$$\varphi^{b,\xi} = (2\pi)^{-(1/2)} M_\xi T_{-b} \tilde{D}_\xi \varphi.$$

Here, the dilation operator $\tilde{D}_\xi$ is defined by

$$(\tilde{D}_\xi h)(x) = |\xi| h(\xi x)$$

for all x in $\mathbb{R}$ and all measurable functions h on $\mathbb{R}$. Besides the modulation with respect to frequency ξ , a notable feature in the Stockwell transform is the normalizing factor in the dilation operator, which is $|\cdot|$ in lieu of $|\cdot|^{1/2}$, as in the case of the wavelet transform. These features distinguish the Stockwell transform from the wavelet transform.

The Stockwell transform has recently been successfully used in geophysics [3,4] and in medical imaging [5,6]. An attempt in understanding the mathematical underpinnings of the Stockwell transform is underway. The following continuous inversion formula for the Stockwell transform can be found in [4] and [7] for the case when

$$\varphi(x) = e^{-\pi x^2}, \quad x \in \mathbb{R}.$$

THEOREM 1.3 *Let $\varphi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that*

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

Then for all f in $L^2(\mathbb{R})$,

$$f = \mathcal{F}^{-1} A S_\varphi f,$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform and A is the time average operator given by

$$(AF)(\xi) = \int_{-\infty}^{\infty} F(b, \xi) db$$

for all ξ in $\mathbb{R}$ and all measurable functions F on $\mathbb{R} \times \mathbb{R}$, provided that the integral exists.

The aim of this paper is to obtain another continuous inversion formula for the Stockwell transform akin to the continuous inversion formulas for the Gabor transform and the wavelet transform. Such a formula is formulated and proved in section 2. In section 3, we give a discrete inversion formula for the Stockwell transform using techniques from the wavelet frames.

Background information about the Gabor transform and the wavelet transform in this section can be found in the books by Cohen [8], Daubechies [2] and Wong [1].

2. A continuous inversion formula

The starting point is the formula for the Fourier transform $\widehat{g^{b,\xi}}$ of $g^{b,\xi}$, where $g \in L^2(\mathbb{R})$ and

$$g^{b,\xi} = (2\pi)^{-(1/2)} M_\xi T_{-b} \tilde{D}_\xi g.$$

The following lemma can be proved by easy computations.

LEMMA 2.1 *For all η in $\mathbb{R}$,*

$$\widehat{g^{b,\xi}}(\eta) = (2\pi)^{-(1/2)} e^{-i(\eta-\xi)b} \hat{g}\left(\frac{\eta-\xi}{\xi}\right).$$

We need another lemma, which is an easy exercise in integration.

LEMMA 2.2 For all η in $\mathbb{R} \setminus \{0\}$,

$$\int_{-\infty}^{\infty} \left| \hat{g} \left(\frac{\eta - \xi}{\xi} \right) \right|^2 \frac{d\xi}{|\xi|} = \int_{-\infty}^{\infty} \frac{|\hat{g}(t-1)|^2}{|t|} dt.$$

The interesting point in Lemma 2.2 is that the integral on the left hand side is independent of η in $\mathbb{R} \setminus \{0\}$.

We can now state and prove a continuous inversion formula for the Stockwell transform.

THEOREM 2.3 Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_{L^2(\mathbb{R})} = 1$ and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty.$$

Then for all f and g in $L^2(\mathbb{R})$,

$$(f, g)_{L^2(\mathbb{R})} = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})_{L^2(\mathbb{R})} (\varphi^{b,\xi}, g)_{L^2(\mathbb{R})} \frac{db d\xi}{|\xi|},$$

where

$$c_\varphi = \int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi.$$

Remark Theorem 2.3 tells us that every signal f can be reconstructed from its Stockwell spectrum by means of the formula

$$f = \frac{1}{c_\varphi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})_{L^2(\mathbb{R})} \varphi^{b,\xi} \frac{db d\xi}{|\xi|}.$$

Using exactly the same argument given in Remark 1.3 for the wavelet transform, we see that a necessary condition for the continuous inversion formula to hold for the Stockwell transform is that

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi-1)|^2}{|\xi|} d\xi < \infty.$$

This means that $\hat{\varphi}(-1) = 0$ whenever $\hat{\varphi}$ is continuous at -1 . So, it is important to observe that the Gaussian window used exclusively for the Stockwell transform in the literature is *not admissible*.

Proof By Plancherel's formula and Lemma 2.1,

$$(f, \varphi^{b,\xi})_{L^2(\mathbb{R})} = e^{-i\xi b} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{i b \eta} f_\xi(\eta) d\eta = e^{-i\xi b} (\mathcal{F}^{-1} f_\xi)(b),$$

where

$$f_\xi(\eta) = \hat{f}(\eta) \overline{\hat{\varphi} \left(\frac{\eta - \xi}{\xi} \right)}, \quad \eta \in \mathbb{R}.$$

Similarly,

$$(\varphi^{b,\xi}, g)_{L^2(\mathbb{R})} = e^{i\xi b} \overline{(\mathcal{F}^{-1}g_\xi)(b)},$$

where

$$g_\xi(\eta) = \hat{g}(\eta) \overline{\hat{\varphi}\left(\frac{\eta - \xi}{\xi}\right)}, \quad \eta \in \mathbb{R}.$$

So, by Plancherel's formula again and Lemma 2.2,

$$\begin{aligned} & \int_{-\infty}^{\infty} (f, \varphi^{b,\xi})_{L^2(\mathbb{R})} (\varphi^{b,\xi}, g)_{L^2(\mathbb{R})} \frac{db d\xi}{|\xi|} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\mathcal{F}^{-1}f_\xi)(b) \overline{(\mathcal{F}^{-1}g_\xi)(b)} \frac{db d\xi}{|\xi|} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_\xi(\eta) \overline{g_\xi(\eta)} \frac{d\eta d\xi}{|\xi|} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta) \overline{g(\eta)} \left| \hat{\varphi}\left(\frac{\eta - \xi}{\xi}\right) \right|^2 \frac{d\eta d\xi}{|\xi|} \\ &= \int_{-\infty}^{\infty} \hat{f}(\eta) \overline{\hat{g}(\eta)} \left(\int_{-\infty}^{\infty} \left| \hat{\varphi}\left(\frac{\eta - \xi}{\xi}\right) \right|^2 \frac{d\xi}{|\xi|} \right) d\eta \\ &= c_\varphi(f, g)_{L^2(\mathbb{R})} \end{aligned}$$

and the proof is complete. ■

3. A discrete inversion formula

Let $\varphi \in L^2(\mathbb{R})$ be such that $\|\varphi\|_2 = 1$,

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi)|^2}{|\xi|} d\xi < \infty$$

and

$$\int_{-\infty}^{\infty} \frac{|\hat{\varphi}(\xi - 1)|^2}{|\xi|} d\xi < \infty.$$

This is the same as saying that φ satisfies the admissibility conditions for the wavelet transform and the Stockwell transform.

We have the following connection between the Stockwell transform and the wavelet transform, which follows immediately from the definitions of the transforms.

THEOREM 3.1 *For all b in $\mathbb{R}$ and all ξ in $\mathbb{R} \setminus \{0\}$,*

$$\varphi^{b,\xi}(x) = (2\pi)^{-(1/2)} |\xi|^{(1/2)} e^{ix\xi} \varphi_{b,(1/\xi)}(x), \quad x \in \mathbb{R}.$$

Following the technique on page 54 in [2], we pick a frequency step ξ_0 and a time step b_0 such that $\xi_0 > 1$ and $b_0 > 0$. Then we discretize the frequency ξ and time b by

$$\xi = \xi_0^m$$

and

$$b = nb_0\xi_0^m$$

for all m and n in $\mathbb{Z}$. For all m and n in $\mathbb{Z}$, we let $\Phi_{m,n}$ and $\Phi^{m,n}$ be defined by

$$\Phi_{m,n} = \varphi_{nb_0\xi_0^m, \xi_0^m}$$

and

$$\Phi^{m,n} = \varphi^{nb_0\xi_0^m, \xi_0^{-m}}.$$

Then, we have the following discrete inversion formula for a signal from its Stockwell transform.

THEOREM 3.2 *Suppose that $\{\Phi_{m,n} : m, n \in \mathbb{Z}\}$ is a tight wavelet frame for $L^2(\mathbb{R})$ with frame bounds A, A , then for all f in $L^2(\mathbb{R})$, we get*

$$f = 2\pi \sum_{m,n \in \mathbb{Z}} \xi_0^m (f, M_{-\xi_0^{-m}} \Phi^{m,n})_{L^2(\mathbb{R})} M_{-\xi_0^{-m}} \Phi^{m,n}.$$

Good conditions on φ to ensure that $\{\Phi_{m,n} : m, n \in \mathbb{Z}\}$ forms a tight frame for $L^2(\mathbb{R})$ abound and can be found in Chapter 3 of the book [2] by Daubechies.

Remark Although the formula in Theorem 3.2 is essentially the affine wavelet frame expansion of a signal f in the disguise of the Stockwell transform, it is very valuable for practitioners using the time-frequency representations of signals obtained by Stockwell transforms.

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Besov spaces associated with Dunkl's operator

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The purpose of this paper is to investigate the harmonic analysis associated with Dunkl's operator on the real line to introduce a Besov type spaces generalizing the classical one. Different characterizations of these spaces are given using generalized translation operator and also by mean modulus of continuity.

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1. Introduction

Throughout this paper we fix $\alpha > -(1/2)$, $1 \leq p, q \leq \infty$ and $\gamma \in \mathbb{R}$ and we introduce Besov type spaces $\dot{B}_{p,q}^\gamma(\mathbb{R})$ ([1–4]) using the harmonic analysis associated with Dunkl's operator:

$$D_\alpha u(x) := \frac{d}{dx} u(x) + \frac{2\alpha + 1}{x} \left(\frac{u(x) - u(-x)}{2} \right).$$

We recall here that the function $\Psi_\lambda^\alpha(x) = j_\alpha(\lambda x) + (i\lambda x)/(2(\alpha + 1))j_{\alpha+1}(\lambda x)$ is the unique solution of the following system (see ref. [5])

$$\begin{cases} D_\alpha u(x) = i\lambda u(x), \\ u(0) = 1 \end{cases}$$

where j_α is the normalized Bessel function of first kind and order α .

The Fourier Dunkl transform for a reasonable function f is given by [5]:

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_D(f)(\lambda) := \int_{\mathbb{R}} \Psi_{-\lambda}^\alpha(x) f(x) dm_\alpha(x) \quad (1)$$

where $dm_\alpha(x) = |x|^{2\alpha+1}/2^{\alpha+1} \Gamma(\alpha + 1) dx$.

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It has been proved in ref. [6] that the Dunkl transform is a topological isomorphism from $S(\mathbb{R})$ onto itself where $S(\mathbb{R})$ is the Schwartz space. We briefly summarize the contents of this paper. In section 1, we collect some harmonic analysis properties related to Dunkl's operator and we include the needed definitions and background. In the second section, we introduce the homogenous Besov–Dunkl type spaces $\dot{B}_{p,q}^\gamma(\mathbb{R})$ and we give some properties and inclusion results with respect to its parameters p, q and γ . The third section deals with the characterization of the space $B_{p,q}^\gamma(\mathbb{R}) = \dot{B}_{p,q}^\gamma(\mathbb{R}) \cap L^p$ under some restriction on γ by mean of the difference $\Delta_x f = T_x^\alpha f - f$ and the generalized modulus of continuity $m_p(f, x) = \sup_{0 \leq y \leq x} \|\Delta_y f\|_p$ associated with the translation Operator T_x^α .

2. Preliminaries

In what follows we collect some basic results in harmonic analysis related to Dunkl's Operator. We first begin by some notations.

Notations. Let $\alpha > -(1/2)$. We denote by

- $S_0(\mathbb{R})$ the subspace of functions ψ in $S(\mathbb{R})$ such that the support of $\mathcal{F}(\psi)$ is a compact subset of $\mathbb{R} \setminus \{0\}$.
- $S_0^1(\mathbb{R})$ the subset of functions ψ in $S_0(\mathbb{R})$ such that

$$\int_0^\infty |\mathcal{F}_D \psi(r\lambda)|^2 \frac{dr}{r} = 1, \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\}.$$

These functions are known as generalized wavelets on $\mathbb{R}$.

- $L^p(dm_\alpha)$, $1 \leq p \leq \infty$, the Lebesgue space associated with the measure dm_α , and $\|\cdot\|_p$ its usual norm given by:

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dm_\alpha(x) \right)^{(1/p)}, \quad \text{if } p \in [1, \infty[$$

$$\|f\|_\infty = \text{esssup}_{x \in \mathbb{R}} |f(x)|.$$

- $f_r(x) = r^{-(2\alpha+2)} f(x/r)$; $r > 0$, the dilated of the function f defined on $\mathbb{R}$ preserving the integral of f with respect to the measure dm_α , in the sense that

$$\int_{\mathbb{R}} f_r(x) dm_\alpha(x) = \int_{\mathbb{R}} f(x) dm_\alpha(x), \quad \text{for all } r > 0 \quad \text{and } f \in L^1(dm_\alpha). \quad (2)$$

•

$$\sigma_{x,y,z} = \frac{x^2 + y^2 - z^2}{2xy}, \quad \text{for all } x, y \in \mathbb{R}^* \quad \text{and } z \in \mathbb{R}.$$

•

$$\rho(x, y, z) = \frac{1}{2} \{1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x}\}.$$

•

$$K_\alpha(x, y, z) = \begin{cases} \frac{2^{1-2\alpha} \Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+(1/2))} \frac{[(x+y)^2 - z^2](z^2 - (x-y)^2)^{\alpha-(1/2)}}{(xyz)^{2\alpha}}, \\ \text{if } |x-y| < z < x+y \\ 0, \quad \text{otherwise} \end{cases}$$

- $W_\alpha(x, y, z) = 2^{\alpha+1}\Gamma(\alpha+1)\rho(x, y, z)K_\alpha(|x|, |y|, |z|)$,
- For all $x, y \in \mathbb{R}$, $\nu_{x,y}^\alpha$ are the signed measures given by

$$d\nu_{x,y}^\alpha(z) := \begin{cases} W_\alpha(x, y, z)dm_\alpha(z) & \text{if } x, y \in \mathbb{R}^* \\ d\delta_x & \text{if } y = 0, \\ d\delta_y & \text{if } x = 0, \end{cases} \quad (3)$$

where δ_x is the Dirac measure at x .

- Finally $\mathbb{R}_+^*$ will be always the subset $]0, \infty[$ and $\mathbb{R}_+ = [0, \infty[$.

DEFINITION 2.1 For all $x, y \in \mathbb{R}$ and for a suitable function f , we put

$$T_x^\alpha f(y) = \nu_{x,y}^\alpha(f) := \int_{\mathbb{R}} f(z)d\nu_{x,y}^\alpha(z). \quad (4)$$

The Operator T_x^α ; $x \in \mathbb{R}$, are called Dunkl translation operator on $\mathbb{R}$ and it can be expressed in the following form (see ref. [7])

$$\begin{aligned} T_x^\alpha f(y) = & C_\alpha \int_0^\pi f_e(\sqrt{x^2 + y^2 - 2|xy|\cos\theta})h_1(x, y, \theta)(\sin\theta)^{2\alpha}d\theta \\ & + C_\alpha \int_0^\pi f_o(\sqrt{x^2 + y^2 - 2|xy|\cos\theta})h_2(x, y, \theta)(\sin\theta)^{2\alpha}d\theta \end{aligned} \quad (5)$$

where $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f , with $C_\alpha = (\Gamma(\alpha+1)/\sqrt{\pi}\Gamma(\alpha+1/2))$,

$$h_1(x, y, \theta) = 1 - \operatorname{sgn}(xy)\cos\theta \quad \text{and} \quad h_2(x, y, \theta) = \begin{cases} \frac{(x+y)[1 - \operatorname{sgn}(xy)\cos\theta]}{\sqrt{x^2 + y^2 - 2|xy|\cos\theta}}, \\ 0, \text{ if } (x, y) = (0, 0). \end{cases}$$

The convolution product for a pair of reasonable functions f and g is given by

$$f \# g(x) := \int_{\mathbb{R}} T_x^\alpha f(y)g(-y)dm_\alpha(y), \quad \forall x, y \in \mathbb{R}.$$

Remark 1 Let $p \geq 1$. Then for all $f \in L^p(dm_\alpha)$ and $\psi \in S(\mathbb{R})$, $\langle f, \psi \rangle$ means the value of $f \in S'(\mathbb{R})$ on ψ and it is given by:

$$\langle f, \psi \rangle = \int_{\mathbb{R}} f(x)\psi(x)dm_\alpha(x). \quad (6)$$

Hence, we identify $L^p(\mathbb{R})$ with a subspace of $S'(\mathbb{R})$ via the formula (6).

It might be observed that a long list of properties of the classical distributions on $\mathbb{R}^n$ remain valid also in our context, while $\mathcal{F}_D$ and $\mathcal{D}_\alpha$ satisfy the following properties

$$\begin{aligned} \int_{\mathbb{R}} f\mathcal{F}_D g dm_\alpha &= \int_{\mathbb{R}} \mathcal{F}_D f g dm_\alpha, \quad \text{and} \\ \int_{\mathbb{R}} \mathcal{D}_\alpha f g dm_\alpha &= - \int_{\mathbb{R}} \mathcal{D}_\alpha g f dm_\alpha, \quad \text{for all } f, g \in S(\mathbb{R}). \end{aligned}$$

Therefore, we extend naturally the Dunkl transform and the operator $\mathcal{D}_\alpha$ on $S'(\mathbb{R})$ as follows:

For all $T \in S'(\mathbb{R})$ and $\psi \in S(\mathbb{R})$ we put

$$\langle \mathcal{F}_D(T), \psi \rangle = \langle T, \mathcal{F}_D(\psi) \rangle \quad \text{and} \quad \langle \mathcal{D}_\alpha T, \psi \rangle = -\langle T, \mathcal{D}_\alpha \psi \rangle.$$

In addition, for all $f \in \mathcal{D}'(\mathbb{R})$ (resp. $S'(\mathbb{R})$) and $\psi \in \mathcal{D}(\mathbb{R})$ (resp. $S(\mathbb{R})$), the generalized convolution product associated with Dunkl operator of f and ψ is the $\mathcal{C}^\infty$ function given by $\psi \# f(x) = \langle f, T_x^\alpha \psi \rangle$.

3. Homogenous Besov–Dunkl spaces

In what follows we endow the group $\mathbb{R}_+^*$ with the invariant measure (dr/r) , we introduce the homogenous Besov–Dunkl space $\dot{B}_{p,q}^\gamma(\mathbb{R})$ and we give some properties therein.

DEFINITION 3.1 Let $1 \leq p, q \leq \infty$, $\psi \in S_0^1(\mathbb{R})$ and $\gamma \in \mathbb{R}$. We define the homogenous Besov–Dunkl space $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$ as the set of all $f \in S'(\mathbb{R})$ such that

$$f = \int_0^\infty f \# \psi_r \# \psi_r \frac{dr}{r} \quad (7)$$

and $\mathbf{N}_{p,q}^{\gamma,\psi}(f) < \infty$, where

$$\mathbf{N}_{p,q}^{\gamma,\psi}(f) = \begin{cases} \left(\int_0^\infty \left(\frac{\|f \# \psi_r\|_p}{r^\gamma} \right)^q \frac{dr}{r} \right)^{(1/q)}, & \text{if } 1 \leq q < \infty, \\ \text{ess sup}_{r>0} \left(\frac{\|f \# \psi_r\|_p}{r^\gamma} \right), & \text{if } q = \infty. \end{cases}$$

Remark 1 In what follows we point out some remarks.

(1) We begin by mentioning here that the above definition of the homogeneous Besov–Dunkl type space $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$ is the same than that introduced by Chemin in the classical case (see ref. [8]) which was generalized by Bahouri, Gérard and Xu on the Heisenberg group [9] and by Assal and Ben Abdallah on the Laguerre hypergroup [4]. We do not choose the classical definition introduced by Peetre (see ref. [10]) in which $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$ is defined as a set of distributions modulo polynomials.

(2) Using Plancherel’s formula (see ref. [5]) one can prove easily that (7) holds for all f in $L^2(\mathbb{R})$.

(3) The expression (7) can be not true in $S'(\mathbb{R})$, hence for all polynomial f in $\mathbb{R}[X]$ we have $f \# \psi_r = 0$ for all $r > 0$. So using equation (5) one gets for $f(y) = a_k y^{2k} + b_k y^{2k+1}$

$$T_x^\alpha f(y) = C_\alpha \int_0^\pi (a_k + b_k(x+y))(x^2 + y^2 - 2|xy| \cos \theta)^k (1 - \text{sgn}(xy) \cos \theta) (\sin \theta)^{2\alpha} d\theta$$

which is a polynomial in the form $T_x^\alpha f(y) = \sum_{i,j} C_{i,j} x^i y^j$; with $C_{i,j}$ being real constants. The result turns out using the fact all moments of $\psi \in S_0^1$ vanish.

The following proposition gives the independence of the space $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$ on the choice of ψ in S_0^1 .

PROPOSITION 3.2 Let $1 \leq p, q \leq \infty$. Then the space $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$ is independent of the choice of the function ψ in $S_0^1(\mathbb{R})$.

Proof of Proposition 3.2. Assume ψ and ϕ be a pair of functions belonging to $S_0^1(\mathbb{R})$. To get the desired result it suffices to prove that $\dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R}) \subseteq \dot{B}_{p,q}^{\gamma,\phi}(\mathbb{R})$. Since $\mathcal{F}_D\phi$ and $\mathcal{F}_D\psi$ are supported by some compact subsets of $\mathbb{R}$, then there exist α, β ($0 < \alpha < \beta$) such that $\phi_\rho \# \psi_r = 0$ if $(r/\rho) \notin [\alpha, \beta]$. Moreover we have $\|\phi_\rho \# \psi_r\|_1 \leq \|\phi\|_1 \|\psi\|_1$ and we obtain for all $f \in \dot{B}_{p,q}^{\gamma,\psi}(\mathbb{R})$

$$f \# \phi_\rho = \int_{\alpha\rho}^{\beta\rho} (f \# \psi_r) \# \psi_r \# \phi_\rho \frac{dr}{r}.$$

and by Minkowski's inequality we get

$$\begin{aligned} \frac{\|f \# \phi_\rho\|_p}{\rho^\gamma} &\leq C \int_{\alpha}^{\beta} \frac{\|f \# \psi_{r\rho}\|_p}{(r\rho)^\gamma} \frac{dr}{r} \leq C \int_0^\infty r^\gamma \mathbb{I}_{[\alpha,\beta]}(r) \frac{\|f \# \psi_{r\rho}\|_p}{(r\rho)^\gamma} \frac{dr}{r} \\ &= C(H \star G)(\rho), \end{aligned}$$

with $H(s) = s^\gamma \mathbb{I}_{[\alpha,\beta]}(s)$, $G(s) = (\|f \# \psi_s\|_p / s^\gamma)$ and $H \star G$ is the convolution of H and G on the group $(\mathbb{R}_+^*, (dr/r))$. Now, by Young's inequality, it holds

$$\mathbf{N}_{p,q}^{\gamma,\phi}(f) \leq \|H\|_{(L^1, d\rho/\rho)} \mathbf{N}_{p,q}^{\gamma,\psi}(f).$$

This completes the proof of the proposition. ■

Remark 2 In the sequel the Dunkl Besov spaces will be denoted $\dot{B}_{p,q}^\gamma(\mathbb{R})$; $1 \leq p, q \leq \infty$.

Remark 3 Let $1 \leq p, q \leq \infty$, and $\gamma \in \mathbb{R}$. Then the Besov–Dunkl type space $\dot{B}_{p,q}^\gamma(\mathbb{R})$ is homogeneous of degree $d(p, \gamma) = ((2\alpha + 2)/p) - \gamma$ in the sense that, for all $f \in \dot{B}_{p,q}^\gamma(\mathbb{R})$

$$\mathbf{N}_{p,q}^\gamma(d_r f) = r^{((2\alpha+2)/p)-\gamma} \mathbf{N}_{p,q}^\gamma(f), \quad \text{for all } r > 0,$$

where $d_r f := f((\cdot/r))$ is the distribution given by $\langle d_r f, \psi \rangle = \langle f, \psi_{1/r} \rangle$ for all $\psi \in S(\mathbb{R})$.

PROPOSITION 3.3 Let $1 \leq p, q \leq \infty$ and $\gamma < ((2\alpha + 2)/p)$. Then $\dot{B}_{p,q}^\gamma(\mathbb{R})$ is a Banach space.

To prove the proposition above we need the following lemmas.

LEMMA 3.4 Let $h \in S(\mathbb{R})$ and $\psi \in S_0^1(\mathbb{R})$. Then, for all $k \in \mathbb{N}$, there exists $\psi_{(k)} \in S_0(\mathbb{R})$ such that

$$h \# \psi_r = r^k (\mathcal{D}_\alpha^k h) \# (\psi_{(k)})_r \quad (8)$$

where $\mathcal{D}_\alpha$ is the operator given in the introduction part. Furthermore there exists $C > 0$ such that

$$\|h \# \psi_r\|_1 \leq Cr^k \quad \text{for all } 0 \leq r \leq 1 \quad (9)$$

and

$$\|h \# \psi_r\|_1 \leq C \quad \text{for all } r \geq 1. \quad (10)$$

Proof We obtain equation (8) by induction on k . The inequalities (9) and (10) follow from equation (8). ■

LEMMA 3.5 Let $1 \leq p, q \leq \infty$ and $\gamma < ((2\alpha + 2)/p)$. For ϕ in S_0^1 , put

$$\Phi(g) = \int_0^\infty r^\gamma g(r) \# \phi_r \frac{dr}{r}. \quad (11)$$

Then Φ define a linear and continuous mapping from $L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$ to $\dot{B}_{p,q}^\gamma(\mathbb{R})$.

Proof Let us first prove that, for $g \in L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$, $\Phi(g)$ define an element of $S'(\mathbb{R})$, that is for all $h \in S(\mathbb{R})$,

$$\int_0^\infty |r^\gamma \langle g(r) \# \phi_r, h \rangle| \frac{dr}{r} < \infty.$$

Take $\psi \in S(\mathbb{R})$ such that $\mathcal{F}_D(\psi) = 1$ on $\text{Supp } \mathcal{F}_D \phi$. Then, using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} |\langle g(r) \# \phi_r, h \rangle| &= |\langle g(r) \# \phi_r, h \# \psi_r \rangle| \\ &\leq \|g(r) \# \phi_r\|_\infty \|h \# \psi_r\|_1 \\ &\leq Cr^{-(2\alpha+2)/p} \|g(r)\|_p \|h \# \psi_r\|_1. \end{aligned}$$

On the other hand, using Lemma 3.4, we get

$$\begin{aligned} &\int_0^\infty |r^\gamma \langle g(r) \# \phi_r, h \rangle| \frac{dr}{r} \\ &\leq C \left\{ \int_0^1 r^\gamma r^{k-((2\alpha+2)/p)} \|g(r)\|_p \frac{dr}{r} + \int_1^\infty r^\gamma r^{-(2\alpha+2)/p} \|g(r)\|_p \frac{dr}{r} \right\} \\ &\leq C \left(\int_0^1 \|g(r)\|_p^q \frac{dr}{r} \right)^{1/q} \left(\int_0^1 (r^\gamma r^{k-((2\alpha+2)/p)})^{\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}} \\ &\quad + C \left(\int_1^\infty \|g(r)\|_p^q \frac{dr}{r} \right)^{1/q} \left(\int_1^\infty (r^\gamma r^{-(2\alpha+2)/p})^{\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}}, \end{aligned}$$

where $\bar{q}$ is the conjugate exponent of q . For k sufficiently large we obtain

$$\int_0^\infty |r^\gamma \langle g(r) \# \phi_r, h \rangle| \frac{dr}{r} \leq C \|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))} < \infty.$$

Now, let ψ in S_0^1 . From Minkowski's inequality together with Young's inequality and by a similar reasoning as in Proposition 3.2 we get

$$\frac{\|\Phi(g) \# \psi_\rho\|_p}{\rho^\gamma} \leq C(H \star G)(\rho)$$

with $H(s) = \mathbb{1}_{[\alpha, \beta]}(s)$, $G(s) = \|g(s)\|_p$ and $H \star G$ is the convolution of H and G on the group $(\mathbb{R}_+^*, (dr/r))$. Now, by Young's inequality, it holds

$$\|\Phi(g)\|_{\dot{B}_{p,q}^\gamma(\mathbb{R})} \leq C \|H \star G\|_{L^q(\mathbb{R}_+^*, (d\rho/\rho))} \leq C \|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))}.$$

The lemma is proved. ■

Proof of Proposition 3.3. Let ψ in S_0^1 and take $\phi = \psi$ in Lemma 3.5. Then Φ defined by equation (11) is a continuous linear mapping from $L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$ to $\dot{B}_{p,q}^\gamma(\mathbb{R})$. On the other hand the operator Ψ associating to f in $\dot{B}_{p,q}^\gamma(\mathbb{R})$ the function $\Psi(f)$ defined on $\mathbb{R}_+^*$ by:

$$\Psi(f)(r) = \frac{f \# \psi_r}{r^\gamma}$$

is obviously a linear isometry from $\dot{B}_{p,q}^\gamma(\mathbb{R})$ to $L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$ and using the decomposition (7), we obtain $\Phi \circ \Psi = Id_{\dot{B}_{p,q}^\gamma(\mathbb{R})}$. This implies

$$(\Psi \circ \Phi) \circ \Psi = \Psi \quad \text{on } \dot{B}_{p,q}^\gamma(\mathbb{R}).$$

So $\Psi(\dot{B}_{p,q}^\gamma(\mathbb{R})) = \text{Ker}(\Psi \circ \Phi - Id_{L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))})$ is a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$. Since Ψ is an isometry, then $\dot{B}_{p,q}^\gamma(\mathbb{R})$ can be identified with a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{R}), (dr/r))$. The completeness of $\dot{B}_{p,q}^\gamma(\mathbb{R})$ follows. ■

PROPOSITION 3.6 *Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Then the subspace $\dot{B}_{p,q}^\gamma(\mathbb{R}) \cap C_*^\infty(\mathbb{R})$ is dense in $\dot{B}_{p,q}^\gamma(\mathbb{R})$.*

Proof Let $\phi \in S_0^1(\mathbb{R})$ and $f \in \dot{B}_{p,q}^\gamma(\mathbb{R})$. Then for $\varepsilon > 1$, the function

$$f_\varepsilon = \int_{1/\varepsilon}^\varepsilon f \# \phi_r \# \phi_r \frac{dr}{r}$$

is obviously C^∞ and belongs to $\dot{B}_{p,q}^\gamma(\mathbb{R})$. Moreover, the same reasoning given in Proposition 3.2 leads to

$$\|f_\varepsilon - f\|_{\dot{B}_{p,q}^\gamma(\mathbb{R})} \leq C \left\| \frac{\|f \# \phi_r\|_p}{r^\gamma} \mathbb{1}_\varepsilon(r) \right\|_{L^q(\mathbb{R}_+^*, (dr/r))}$$

where $\mathbb{1}_\varepsilon$ is the characteristic function of the set $\mathbb{R}_+ \setminus [1/\varepsilon, \varepsilon]$. And the right-hand side of the inequality above tends to zero as ε tends to ∞ . ■

PROPOSITION 3.7 *Let $1 \leq q \leq \infty$. Then, for $1 \leq p_1 \leq p_2 \leq \infty$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $r^{(2\alpha+2/p_1)} - \gamma_1 = r^{(2\alpha+2/p_2)} - \gamma_2$, we have*

$$\dot{B}_{p_1,q}^{\gamma_1}(\mathbb{R}) \subseteq \dot{B}_{p_2,q}^{\gamma_2}(\mathbb{R}) \quad (\text{with continuous embedding}).$$

Proof Let ψ in S_0^1 and let $1 \leq p_3 \leq \infty$ such that $(1/p_1) + (1/p_3) = 1 + (1/p_2)$. We consider $\phi \in S(\mathbb{R})$ satisfying $\mathcal{F}\phi = 1$ on $\text{supp}(\mathcal{F}\psi)$. Using Young's inequality, a direct calculation together with the condition given above we get for all $r > 0$ and $f \in \dot{B}_{p_1,q}^\gamma(\mathbb{R})$

$$\|f \# \psi_r\|_{p_2} = \|f \# \psi_r \# \phi_r\|_{p_2} \leq C \|f \# \psi_r\|_{p_1} r^{(2\alpha+2)((1/p_3)-1)}.$$

That is $\mathbf{N}_{p_2,q}^{\gamma_2}(f) \leq C \mathbf{N}_{p_1,q}^{\gamma_1}(f)$. ■

To obtain more general properties we can introduce an equivalent discrete norm on $\dot{B}_{p,q}^\gamma(\mathbb{R})$ replacing the group $\mathbb{R}_+^*$ by the 2-powers group $\mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\}$ equipped with the invariant counting measure which leads to compare these spaces with respect to their parameter q . More precisely we obtain

$$\dot{B}_{p,q_1}^\gamma(\mathbb{R}) \subseteq \dot{B}_{p,q_2}^\gamma(\mathbb{R}); \quad 1 \leq q_1 \leq q_2 < \infty \quad (\text{with continuous embedding}). \quad (12)$$

4. Characterization of the space $B_{p,q}^\gamma(\mathbb{R}) = \dot{B}_{p,q}^\gamma(\mathbb{R}) \cap L^p$

In this section we study the space $B_{p,q}^\gamma(\mathbb{R}) = \dot{B}_{p,q}^\gamma(\mathbb{R}) \cap L^p(dm_\alpha)$ endowed with the norm $\mathbf{N}_{p,q}^\gamma$ and we give some characterizations.

THEOREM 4.1 (*First characterization theorem*)

Let $1 \leq p, q \leq \infty$ and $0 < \gamma < 2$. For f in $B_{p,q}^\gamma(\mathbb{R})$ put

$$\mathbf{M}_{p,q}^\gamma(f) = \begin{cases} \left(\int_{\mathbb{R}} \left(\frac{\|\Delta_x f\|_p}{|x|^\gamma} \right)^q \frac{dx}{x} \right)^{(1/q)}, & \text{if } 1 \leq q < \infty, \\ \text{ess sup}_{x \in \mathbb{R}} (\|\Delta_x f\|_p |x|^\gamma), & \text{if } q = \infty \end{cases}$$

where $\Delta_x f = T_x^\alpha f - f$, for all $x \in \mathbb{R}$. Then $\mathbf{M}_{p,q}^\gamma$ is a norm on $B_{p,q}^\gamma(\mathbb{R})$ equivalent to $\mathbf{N}_{p,q}^\gamma$.

Remark 1 Note that the case $1 < q < \infty$ could have been shown by interpolation with the extreme cases ($q = 1$ and $q = \infty$), but a direct proof is presented in this paper.

Let us first start with some basic lemmas that will be useful for our purpose.

LEMMA 4.2 Let $\psi, \phi \in S_0(\mathbb{R})$. Then there exists $C > 0$ such that

$$\|\Delta_x \phi_\rho\|_1 \leq C \min \left(1, \frac{|x|}{\rho} \right). \quad (13)$$

and

$$\|\phi_\rho \# \psi_r\|_1 \leq C \min \left(r\rho, \frac{\rho}{r} \right), \quad \text{for all } r, \rho > 0. \quad (14)$$

The main idea in the proof of the Lemma above is based on the following Theorem [11, p. 349].

THEOREM 4.3 [11] Let f a C^∞ complex-valued function on $\mathbb{R}$ and $a \in \mathbb{R}$. Then for any $n = 0, 1, 2, \dots$, we have the following generalized Taylor formula with integral remainder:

$$T_y^\alpha f(x) = \sum_{p=0}^n b_p(x) D_\alpha^p f(a) + \int_{-|x|}^{|x|} \omega_n(x, z) T_z^\alpha D_\alpha^{n+1} f(z) dm_\alpha(z).$$

where $\omega_n(x, y) = u_n(x, y) + v_n(x, y)$ and $\{b_p\}$ are the family of real functions given by $b_0 = 1$, $b_{2p} = (1/(\alpha + 1))_p (x/2)^{2p}$ and $b_{2p+1} = (1/(\alpha + 1))_{p+1} (x/2)^{2p+1}$. $\{u_p(x, y)\}$ and $\{v_p(x, y)\}$, $p = 0, 1, 2, \dots$, are two sequences of functions defined via the following recursive integral formulae:

$$\begin{aligned} u_0(x, y) &= \frac{\text{sgn}(x)}{2|x|^{2\alpha+1}}, & v_0(x, y) &= \frac{\text{sgn}(y)}{2|y|^{2\alpha+1}} \\ u_{p+1}(x, y) &= \int_{|y|}^{|x|} v_p(x, z) dz, & v_{p+1}(x, y) &= \frac{\text{sgn}(y)}{|y|^{2\alpha+1}} \int_{|y|}^{|x|} u_p(x, z) |z|^{2\alpha+1} dz. \end{aligned}$$

Proof of lemma 4.2. Using the theorem above and the contraction property of the translation operator we obtain $T_x^\alpha \phi(y) - \phi(y) = \int_{-|x|}^{|x|} \omega_0(x, z) T_y^\alpha D_\alpha \phi(z) dm_\alpha(z)$.

And hence,

$$\begin{aligned}
 \|\Delta_x \phi\|_1 &\leq \int_{-|x|}^{|x|} \omega_0(x, z) \|T_y^\alpha D_\alpha \phi(z)\|_1 dm_\alpha(z) \\
 &\leq 4 \int_{-|x|}^{|x|} \omega_0(x, z) \|D_\alpha \phi(z)\|_1 dm_\alpha(z) \\
 &\leq C \int_{-|x|}^{|x|} \omega_0(x, z) dm_\alpha(z) \\
 &\leq C|x|.
 \end{aligned}$$

Once again, the contraction property of the translation operator gives

$$\|\Delta_x \phi\|_1 \leq \|\phi\|_1 + \|T_x^\alpha \phi\|_1 \leq 5\|\phi\|_1 \leq C. \quad (15)$$

The estimate above on $\|\Delta_x \phi\|_1$ together with equation (15) lead to

$$\|\Delta_x \phi\|_1 \leq C \min(1, |x|).$$

On the other hand a straightforward calculation on the kernel of the translation operator W_α given by equation (3) yields

$$W_\alpha(x, y, \rho z) = \rho^{-(2\alpha+2)} W_\alpha\left(\frac{x}{\rho}, \frac{y}{\rho}, z\right)$$

which leads to

$$\|\Delta_x \phi_\rho\|_1 = \|\Delta_{(x/\rho)} \phi\|_1.$$

And hence

$$\|\Delta_x \phi_\rho\|_1 \leq C \min\left(1, \frac{|x|}{\rho}\right).$$

To prove equation (14), let us take $r \leq \rho$. Since $\int_{\mathbb{R}} \psi_r(x) dm_\alpha(x) = 0$ we get

$$\phi_\rho \# \psi_r(y) = \int_{\mathbb{R}} \psi_r(-x) \Delta_x \phi_\rho(y) dm_\alpha(x).$$

From Minkowski's inequality we obtain

$$\begin{aligned}
 \|\psi_r \# \phi_\rho\|_1 &\leq \int_{\mathbb{R}} |\psi_r(-x)| \|\Delta_x \phi_\rho\|_1 dm_\alpha(x) \\
 &= C \int_{\mathbb{R}} \left(\frac{|x|}{r}\right)^{2\alpha+2} \left|\psi\left(\frac{-x}{r}\right)\right| \|\Delta_x \phi_\rho\|_1 \frac{dx}{x}.
 \end{aligned} \quad (16)$$

Taking into account that ψ belongs to $S(\mathbb{R})$, then there exists $C > 0$ such that

$$\left(\frac{|x|}{r}\right)^{2\alpha+2} \left|\psi\left(\frac{-x}{r}\right)\right| \leq C \left(\frac{|x|}{r}\right)^{-3} \quad \text{and} \quad \left|\psi\left(\frac{-x}{r}\right)\right| \leq C.$$

So it holds

$$\left(\frac{|x|}{r}\right)^{2\alpha+2} \left|\psi\left(\frac{-x}{r}\right)\right| \leq C \min\left(\left(\frac{|x|}{r}\right)^{2\alpha+2}, \left(\frac{r}{|x|}\right)^3\right). \quad (17)$$

Hence from equations (16), (17) and (13) it follows

$$\|\psi_r \# \phi_\rho\|_1 \leq C \int_{\mathbb{R}} \frac{|x|}{\rho} \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^3 \right) \frac{dx}{x} = C \left(\frac{r}{\rho} \right).$$

The lemma is proved. ■

One of the main tool used in this paper is the Calderón reproducing formulae that we recall in the following lemma (see ref. [12]).

LEMMA 4.4 *Let $\psi \in S_0^1$, being even and $1 < \varepsilon < \infty$. For $f \in L^p(dm_\alpha)$ ($1 < p < \infty$) define*

$$f_\varepsilon = \int_{1/\varepsilon}^\varepsilon f \# \psi_r \# \psi_r \frac{dr}{r}.$$

Then f_ε converges to f in $L^p(dm_\alpha)$ as $\varepsilon \rightarrow \infty$.

Note that the measure $\mu = \psi \# \psi$ with ψ being even function, satisfies all conditions of Theorem 3. p. 59 of ref. [12]. And thanks to Proposition 3.2, the choice of ψ even does not affect the result in our context. Now we are going to apply the Calderón reproducing formulae above to prove the following lemma

LEMMA 4.5 (i) *Let $1 \leq p \leq \infty$ and $\psi \in S_0$. Then we have for all $\beta > 0$ and f such that $\mathbf{M}_{p,q}^\gamma(f) < \infty$*

$$\|f \# \psi_r\|_p \leq C \int_{\mathbb{R}} \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^\beta \right) \|\Delta_x f\|_p \frac{dx}{x}, \quad a.e. \ r > 0. \quad (18)$$

(ii) *For all $f \in B_{p,q}^\gamma(\mathbb{R})$ it holds*

$$\|\Delta_x f\|_p \leq C \int_0^\infty \min \left(1, \frac{|x|}{r} \right) \|f \# \psi_r\|_p \frac{dr}{r}, \quad a.e. \ x \in \mathbb{R}. \quad (19)$$

Proof Statement (18) follows by an analogous reasoning of (14).

To prove equation (19) we may use the above Calderón reproducing formulae, say for all f in $B_{p,q}^\gamma(\mathbb{R})$ and $1 < \varepsilon < \infty$ one has

$$\Delta_x f_\varepsilon(y) = \int_{1/\varepsilon}^\varepsilon ((\Delta_x \psi_r) \# f \# \psi_r)(y) \frac{dr}{r} \quad \text{and} \quad \Delta_x f_\varepsilon \longrightarrow \Delta_x f \quad \text{in } L^p \quad \text{as } \varepsilon \longrightarrow \infty. \quad (20)$$

Minkowski's inequality together with Young's inequality imply

$$\|\Delta_x f_\varepsilon\|_p \leq \int_{1/\varepsilon}^\varepsilon \|\Delta_x \psi_r\|_1 \|f \# \psi_r\|_p \frac{dr}{r}.$$

Now using equation (20) we obtain

$$\|\Delta_x f\|_p \leq \int_0^\infty \|\Delta_x \psi_r\|_1 \|f \# \psi_r\|_p \frac{dr}{r}.$$

Inequality (19) follows immediately from equation (13). ■

Proof of Theorem 4.1. (i) Let us start with the case $q = \infty$. Assume f such that $\mathbf{M}_{p,\infty}^\gamma(f) < \infty$. Using (18) with $\beta \geq 2$, we get

$$\|f \# \psi_r\|_p \leq C \mathbf{M}_{p,\infty}^\gamma(f) \int_{\mathbb{R}} |x|^\gamma \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^\beta \right) \frac{dx}{x}$$

That is $\mathbf{N}_{p,\infty}^\gamma(f) \leq C \mathbf{M}_{p,\infty}^\gamma(f)$.

Take now $f \in B_{p,\infty}^\gamma(\mathbb{R})$. Then, from equation (19) it holds

$$\|\Delta_x f\|_p \leq C \mathbf{N}_{p,\infty}^\gamma(f) \int_0^\infty r^\gamma \min \left(1, \left(\frac{|x|}{r} \right) \right) \frac{dr}{r} \leq C \mathbf{N}_{p,\infty}^\gamma(f) |x|^\gamma.$$

This completes the proof of statment (i). ■

(ii) Let us prove the case $q = 1$. Assume f such that $\mathbf{M}_{p,1}^\gamma(f) < \infty$. We shall prove that $\mathbf{N}_{p,1}^\gamma(f) \leq C \mathbf{M}_{p,1}^\gamma(f)$. From equation (18), with $\beta \geq 2$ we obtain

$$\mathbf{N}_{p,1}^\gamma(f) \leq C \int_{\mathbb{R}} \|\Delta_x f\|_p \int_0^\infty \frac{1}{r^\gamma} \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^\beta \right) \frac{dr}{r} \frac{dx}{x} \leq C \mathbf{M}_{p,1}^\gamma(f).$$

Conversely, let us take f in $B_{p,1}^\gamma(\mathbb{R})$. Then using equation (19), one gets

$$\mathbf{M}_{p,1}^\gamma(f) \leq C \int_0^\infty \|f \# \psi_r\|_p \int_{\mathbb{R}} \frac{1}{|x|^\gamma} \min \left(1, \left(\frac{|x|}{r} \right)^2 \right) \frac{dx}{x} \frac{dr}{r} \leq C \mathbf{N}_{p,1}^\gamma(f).$$

In order to prove the main result of Theorem 4.1, let us start with the chief tool in our investigation, that is Schur lemma.

LEMMA 4.6 (*Schur lemma*). (see [13]) *Let $1 < q < \infty$ and q' its conjugate exponent. Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be a pair of σ -finite measure spaces and let $F: \Omega_1 \times \Omega_2 \rightarrow \mathbb{R}_+$ be a measurable function. Define $T_F f$ for all measurable positive function f on Ω_1 by*

$$T_F f(\omega_2) = \int_{r^{\gamma_2}} F(\omega_1, \omega_2) f(\omega_1) d\mu_1(\omega_1), \quad \text{for all } \omega_2 \in \Omega_2.$$

If there exist $C > 0$ and measurable functions $h_i: \Omega_i \rightarrow]0, +\infty[$; ($i = 1, 2$) such that

$$\begin{aligned} \int_{\Omega_1} F(\omega_1, \omega_2) h_1^{q'}(\omega_1) d\mu_1(\omega_1) &\leq C h_2^{q'}(\omega_2) \quad \mu_2 - a.e. \\ \int_{\Omega_2} F(\omega_1, \omega_2) h_2^q(\omega_2) d\mu_2(\omega_2) &\leq C h_1^q(\omega_1) \quad \mu_1 - a.e. \end{aligned}$$

Then T_F can be extended as a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.

LEMMA 4.7 *Let $1 < q, q' < \infty$ such that $(1/q) + (1/q') = 1$ and for $0 \leq \varepsilon < \gamma < \delta < \infty$ and $a, b \in \mathbb{R}_+^*$. Put*

$$R^{\varepsilon, \delta}(a, b) = \left(\frac{a}{b} \right)^\gamma \min \left(\left(\frac{a}{b} \right)^\varepsilon, \left(\frac{b}{a} \right)^\delta \right).$$

Then there exist $C > 0$ and a measurable function $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for all $a, b > 0$,

$$\int_0^\infty R^{\varepsilon, \delta}(a, b) g^{q'}(a) \frac{da}{a} \leq C g^{q'}(b) \quad \text{and} \quad \int_0^\infty R^{\varepsilon, \delta}(a, b) g^q(b) \frac{db}{b} \leq C g^q(a).$$

(iii) Let us now prove the statement $1 < q < \infty$ of Theorem 4.1. Using the inequality (18) we obtain, for $\beta \geq 2$

$$\begin{aligned} \frac{\|f \# \psi_r\|_p}{r^\gamma} &\leq C \int_{\mathbb{R}} \frac{|x|^\gamma}{r^\gamma} \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^\beta \right) \frac{\|\Delta_x f\|_p}{|x|^\gamma} \frac{dx}{x} \\ &= C \int_{\mathbb{R}} R^{2\alpha+2,\beta}(|x|, r) \frac{\|\Delta_x f\|_p}{|x|^\gamma} \frac{dx}{x} \\ &= CT_{R^{2\alpha+2,\beta}} \left(\frac{\|\Delta_x f\|_p}{|x|^\gamma} \right) (r) \end{aligned}$$

where $R^{2\alpha+2,\beta}(|x|, r) = (|x|^\gamma/r^\gamma) \min((|x|/r)^{2\alpha+2}, (r/|x|)^\beta)$. By Lemma 4.6 and Lemma 4.7, we obtain the boundedness of the operator $T_{R^{2\alpha+2,\beta}}$ from $L^q(dx/x) := L^q(\mathbb{R}, , (dx/x))$ into $L^q(dr/r) := L^q(\mathbb{R}_+^*, (dr/r))$. Moreover the hypothesis $\mathbf{M}_{p,q}^\gamma(f) < \infty$ means that the function $x \mapsto (\|\Delta_x f\|_p/|x|^\gamma)$ belongs to $L^q(dx/x)$, hence

$$\mathbf{N}_{p,q}^\gamma(f) \leq C \|T_F(\|\Delta_x f\|_p |x|^\gamma)\|_{L^q(dr/r)} \leq C \left\| \frac{\|\Delta_x f\|_p}{|x|^\gamma} \right\|_{L^q(dx/x)} = C \mathbf{M}_{p,q}^\gamma(f).$$

Conversely, let us take $f \in B_{p,q}^\gamma(\mathbb{R})$. From equation (19) we get

$$\begin{aligned} \frac{\|\Delta_x f\|_p}{|x|^\gamma} &\leq C \int_0^\infty \frac{r^\gamma}{|x|^\gamma} \min \left(1, \frac{|x|}{r} \right) \frac{\|f \# \psi_r\|_p}{r^\gamma} \frac{dr}{r} \\ &= C \int_0^\infty R^{0,1}(r, |x|) \frac{\|f \# \psi_r\|_p}{r^\gamma} \frac{dr}{r} \\ &= CT_{R^{0,1}}(\|f \# \psi_r\|_p r^\gamma)(|x|) \end{aligned}$$

where $R^{0,1}(r, x) = (r^\gamma/|x|^\gamma) \min(1, (|x|/r))$. We proceed as above with adequate changes to obtain

$$\mathbf{M}_{p,q}^\gamma(f) \leq C \left\| T_{R^{0,1}} \left(\frac{\|f \# \psi_r\|_p}{r^\gamma} \right) \right\|_{L^q(dx/x)} \leq C \left\| \frac{\|f \# \psi_r\|_p}{r^\gamma} \right\|_{L^q(dr/r)} = C \mathbf{N}_{p,q}^\gamma(f)$$

which gives the desired result.

THEOREM 4.8 (Second characterization theorem)

Let $1 \leq p, q \leq \infty$ and $0 < \gamma < 2$. For f in $B_{p,q}^\gamma(\mathbb{R})$ put

$$\mathcal{M}_{p,q}^\gamma(f) = \begin{cases} \left(\int_{\mathbb{R}} \left(\frac{m_p(f, x)}{|x|^\gamma} \right)^q \frac{dx}{x} \right)^{(1/q)}, & \text{if } 1 \leq q < \infty, \\ \text{ess sup}_{x \in \mathbb{R}} \left(\frac{m_p(f, x)}{|x|^\gamma} \right), & \text{if } q = \infty \end{cases}$$

where $m_p(f, x) = \sup_{0 \leq y \leq x} \|\Delta_y f\|_p$ is the generalized modulus of continuity associated with the generalized translation Operator T_x^α . Then $\mathcal{M}_{p,q}^\gamma$ is a norm on $B_{p,q}^\gamma(\mathbb{R})$ equivalent to $\mathbf{N}_{p,q}^\gamma$.

Proof To compare $\mathbf{N}_{p,q}^\gamma(f)$ and $\mathcal{M}_{p,q}^\gamma(f)$ we proceed as in the proof of the Theorem 4.1 using the following lemma instead of Lemma 4.5. ■

LEMMA 4.9 (i) Let $1 \leq p \leq \infty$ and $\psi \in S_0$. Then we have for all $\beta > 0$ and f such that $\mathbf{M}_{p,q}^\gamma(f) < \infty$

$$\|f \# \psi_r\|_p \leq C \int_{\mathbb{R}} \min \left(\left(\frac{|x|}{r} \right)^{2\alpha+2}, \left(\frac{r}{|x|} \right)^\beta \right) m_p(f, x) \frac{dx}{x}, \quad a.e. \, r > 0. \quad (21)$$

(ii) For all $f \in B_{p,q}^\gamma(\mathbb{R})$ it holds

$$m_p(f, x) \leq C \int_0^\infty \min \left(1, \frac{|x|}{r} \right) \|f \# \psi_r\|_p \frac{dr}{r}, \quad a.e. \, x \in \mathbb{R}. \quad (22)$$

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Several results on the commutative neutrix product of distributions

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Let $H(x)$ denote Heaviside's function. The goal of this paper is to evaluate the commutative neutrix products $f_+(x) \diamond \delta^{(r)}(x)$ and $f_-(x) \diamond \delta^{(r)}(x)$ for $r = 0, 1, 2, \dots$, where $f(x)$ is only the r -th differentiable on an open interval containing the origin and $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$. We also obtain the products $(\psi(x)/x) \diamond \delta(x)$, including a few examples as well as $x_+^{-r-(1/2)} \diamond x_-^{-r-(1/2)}$.

Keywords: Product; δ -sequence; Neutrix and distribution

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1. Introduction

One of the problems in distribution theory is the lack of definitions for products and powers, in general. In physics (see *e.g.* [1], p. 141), one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interactions. Embacher, Grübl and Oberguggenberger [2] studied the parameter products of distributions in several variables arising from the computations of the one-loop vacuum polarization of zero-mass QED₂ to avoid the occurrence of renormalization ambiguities. In [3], a definition for product of distributions was given by delta sequences. However, δ^2 as a product of δ with itself was shown not to exist. Bremermann [4] used the Cauchy representations of distributions with compact supports to define $\sqrt{\delta_+}$ and $\log \delta_+$. Unfortunately, his definition did not carry over to $\sqrt{\delta}$ and $\log \delta$.

Now let $\mathcal{D}$ be the testing function space of infinitely differentiable functions of a single variable with compact support, and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The definition of the product of a distribution and an infinitely differentiable function is the following (see for example [5]).

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DEFINITION 1.1 *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. Then the product fg is defined by*

$$(fg, \phi) = (f, g\phi)$$

for all functions ϕ in $\mathcal{D}$.

An extension of the product of a distribution and an infinitely differentiable function was given by Fisher in [6].

DEFINITION 1.2 *Let f and g be distributions in $\mathcal{D}'$ for which, on the interval (a, b) , f is the k -th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $(1/p) + (1/q) = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^k \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

To further extend Definition 1.2, Fisher, with his collaborators [6–12], has actively used Temples' δ -sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, where $\rho(x)$ is a fixed infinitely differentiable function on R with four properties:

- (i) $\rho(x) \geq 0$,
- (ii) $\rho(x) = 0$ for $|x| \geq 1$,
- (iii) $\rho(x) = \rho(-x)$,
- (iv) $\int_{-1}^1 \rho(x) dx = 1$

and the concept of neutrix limit of van der Corput [13] (in order to abandon unwanted infinite quantities from asymptotic expressions) to deduce numerous products, powers, convolutions and compositions of distributions on R . One of Fisher's definitions for the commutative product of distributions was given as follows.

Let f be an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x - t))$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

DEFINITION 1.3 *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \cdot g$ of f and g exists and is equal to h if*

$$N - \lim_{n \rightarrow \infty} (f_n g_n, \phi) = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [13]) having domain $N' = \{1, 2, \dots\}$ and range the real numbers, with negligible functions that are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \dots)$$

and all functions of n that converge to zero in the normal sense as n tends to infinity.

Recently, Fisher, Özçağ and Gülen [14] employed the above definition to imply the following theorem.

THEOREM 1.1 *Let f be a function, which is infinitely differentiable on an open interval containing the origin and let $f_+(x) = H(x)f(x)$ and $f_-(x) = H(-x)f(x)$. Then, the commutative neutrix products $f_+(x) \cdot \delta^{(r)}(x)$ and $f_-(x) \cdot \delta^{(r)}(x)$ exist and*

$$f_+(x) \cdot \delta^{(r)}(x) = f_-(x) \cdot \delta^{(r)}(x) = \sum_{k=0}^r \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0) \delta^{(k)}(x),$$

where $r = 0, 1, 2, \dots$

We begin to compute the commutative neutrix products $f_+(x) \diamond \delta^r(x)$ and $f_-(x) \diamond \delta^r(x)$ under the weaker condition that f is only the r -th differentiable with a new definition. Then we obtain several interesting corollaries. Furthermore, we derive the commutative product $(\psi(x)/x) \diamond \delta(x)$, as well as the product $x_+^{-r-(1/2)}$ and $x_-^{-r-(1/2)}$.

2. New results

Let $\delta_n(x) = n\rho(nx)$ be defined as in the introduction. We use the following definition for the commutative neutrix product of a single variable (see [15]).

DEFINITION 2.1 *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f * \delta_n$ and $g_n = g * \delta_n$. We say that the commutative neutrix product $f \diamond g$ of f and g exists and is equal to h if*

$$N - \lim_{n \rightarrow \infty} \frac{1}{2} \{ (f_n g, \phi) + (f g_n, \phi) \} = (h, \phi)$$

for all functions ϕ in $\mathcal{D}$. If the normal limit exists, then it is simply called the commutative product.

THEOREM 2.1 *Definition 2.1 extends Definitions 1.1 and 1.2.*

Proof Let g be a C^∞ function. Obviously, $g_n \phi$ has an uniform support and converges to $g \phi$ in $\mathcal{D}$. For any f in $\mathcal{D}'$, we imply that

$$\begin{aligned} (f \diamond g, \phi) &= N - \lim_{n \rightarrow \infty} \frac{1}{2} \{ (f_n g, \phi) + (f g_n, \phi) \} \\ &= N - \lim_{n \rightarrow \infty} \frac{1}{2} \{ (f_n, g \phi) + (f, g_n \phi) \} = (f, g \phi). \end{aligned} \quad \blacksquare$$

To see Definition 2.1 is an extension of Definition 1.2, we let f and g be functions given in Definition 1.2. Then we have

$$\lim_{n \rightarrow \infty} (f g_n, \phi) = \lim_{n \rightarrow \infty} (f_n g, \phi) = \left(\sum_{i=0}^k \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}, \phi \right)$$

by Theorem 2 in [8]. This completes the proof.

At this moment, it is not quite clear if Definition 2.1 is equivalent to Definition 1.3, although both define the commutative neutrix products. In fact,

$$\begin{aligned} N - \lim_{n \rightarrow \infty} \frac{1}{2} \{ (f_n g, \phi) + (f g_n, \phi) \} &- N - \lim_{n \rightarrow \infty} (f_n g_n, \phi) \\ &- N - \lim_{n \rightarrow \infty} \frac{1}{2} \{ (f_n (g - g_n) + g_n (f - f_n), \phi) \}. \end{aligned}$$

One can neither prove the above being zero nor non-zero by a counter example. The author welcomes and appreciates any discussion from interested readers.

However, we are able to claim that the two definitions are different if we remove the neutrix limits from both. Indeed, we have $\delta'(x) \diamond \delta'(x) = 0$ on subspace $\mathcal{D}_0(R) = \{\phi(x) \in \mathcal{D}(R) \mid \phi(0) = 0\}$ because

$$\begin{aligned} (\delta'_n(x)\delta'(x), \phi(x)) &= -(\delta'_n(x)\phi(x))'|_{x=0} \\ &= -(\delta''_n(0)\phi(0) + \delta'_n(0)\phi'(0)) = 0 \end{aligned}$$

for all n . But it is impractical to compute the product $\delta'(x) \cdot \delta'(x)$ without recourse to neutrix limit, since $(\delta'_n(x)\delta'_n(x), \phi(x))$ diverges.

THEOREM 2.2 *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative neutrix products $f \diamond g$ and $f' \diamond g$ exist. Then the commutative neutrix product $f \diamond g'$ exists and*

$$(f \diamond g)' = f' \diamond g + f \diamond g'.$$

Proof The result follows immediately from Definition 2.1. ■

THEOREM 2.3 *Let f be a function which is the r -th differentiable on an open interval containing the origin. Then the commutative neutrix products $f_+(x) \diamond \delta^{(r)}(x)$ and $f_-(x) \diamond \delta^{(r)}(x)$ exist and*

$$f_+(x) \diamond \delta^{(r)}(x) = f_-(x) \diamond \delta^{(r)}(x) = \sum_{k=0}^r \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0) \delta^{(k)}(x),$$

where $r = 0, 1, 2, \dots$

Proof For any $\phi \in \mathcal{D}$, we use integration by parts repeatedly

$$\begin{aligned} & (f_+(x)(\delta^{(r)}(x) * \delta_n(x)), \phi(x)) \\ &= \int_0^\infty \delta_n^{(r)}(x) f(x) \phi(x) dx \\ &= f(x) \phi(x) \delta_n^{(r-1)}(x) \Big|_0^\infty - \int_0^\infty (f(x) \phi(x))' \delta_n^{(r-1)}(x) dx \\ &= -f(0) \phi(0) \delta_n^{(r-1)}(0) - (f(x) \phi(x))' \delta_n^{(r-2)}(x) \Big|_0^\infty \\ &\quad + \int_0^\infty (f(x) \phi(x))'' \delta_n^{(r-2)}(x) dx \\ &= -f(0) \phi(0) \delta_n^{(r-1)}(0) - (f(0) \phi(0))' \delta_n^{(r-2)}(0) \\ &\quad + \int_0^\infty (f(x) \phi(x))'' \delta_n^{(r-2)}(x) dx \\ &= \dots \dots \\ &= -f(0) \phi(0) \delta_n^{(r-1)}(0) + (f(0) \phi(0))' \delta_n^{(r-2)}(0) + \dots \\ &\quad + (-1)^r \int_0^\infty (f(x) \phi(x))^{(r)} \delta_n(x) dx. \end{aligned}$$

It follows that

$$\begin{aligned} N - \lim_{n \rightarrow \infty} (f_+(x)(\delta^{(r)}(x) * \delta_n(x)), \phi(x)) \\ = \frac{1}{2}(-1)^r (f(0)\phi(0))^{(r)} \\ = \frac{1}{2}(-1)^r \sum_{k=0}^r \binom{r}{k} f^{(r-k)}(0)\phi^{(k)}(0) \end{aligned}$$

by noting that

$$\int_0^\infty \delta_n(x) dx = \frac{1}{2}.$$

On the other hand, we have

$$\begin{aligned} ((f_+(x) * \delta_n(x))\delta^{(r)}(x), \phi(x)) \\ = \left(\delta^{(r)}(x), \int_{-\infty}^\infty H(t)f(t)\delta_n(x-t)dt \phi(x) \right) \\ = (-1)^r \left(\int_0^\infty f(t)\delta_n(x-t)dt \phi(x) \right)^{(r)} \Big|_{x=0} \\ = (-1)^r \sum_{k=0}^r \binom{r}{k} \int_0^\infty f(t)\delta_n^{(r-k)}(-t)dt \phi^{(k)}(0). \end{aligned}$$

Making the substitution $u = -t$, we arrive at

$$\lim_{n \rightarrow \infty} \int_0^\infty f(t)\delta_n^{(r-k)}(-t)dt = \lim_{n \rightarrow \infty} \int_{-\infty}^0 f(-u)\delta_n^{(r-k)}(u)du = \frac{f^{(r-k)}(0)}{2}$$

due to

$$\int_{-\infty}^0 \delta_n(x) dx = \frac{1}{2}.$$

It follows that

$$\lim_{n \rightarrow \infty} ((f_+(x) * \delta_n(x))\delta^{(r)}(x), \phi(x)) = \frac{1}{2}(-1)^r \sum_{k=0}^r \binom{r}{k} f^{(r-k)}(0)\phi^{(k)}(0).$$

Adding the above two terms clearly gives us

$$f_+(x) \diamond \delta^{(r)}(x) = \sum_{k=0}^r \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0)\delta^{(k)}(x).$$

To obtain the product $f_-(x) \diamond \delta^r(x)$, note that

$$(H(x)f(-x)) \diamond \delta^{(r)}(x) = \sum_{k=0}^r \frac{1}{2} \binom{r}{k} f^{r-k}(0)\delta^k(x)$$

and replace x by $-x$ to get

$$f_-(x) \diamond \delta^r(x) = \sum_{k=0}^r \frac{(-1)^{r-k}}{2} \binom{r}{k} f^{(r-k)}(0)\delta^{(k)}(x)$$

by using $\delta^{(k)}(x) = (-1)^k \delta^{(k)}(-x)$ for $k = 0, 1, \dots, r$. This completes the proof. ■

COROLLARY 2.1 *Let f be a function, which is the r -th differentiable on an open interval containing the origin. Then the commutative neutrix product $f(x) \diamond \delta^{(r)}(x)$ exists and*

$$f(x) \diamond \delta^{(r)}(x) = \sum_{k=0}^r (-1)^{r-k} \binom{r}{k} f^{(r-k)}(0) \delta^{(k)}(x),$$

where $r = 0, 1, 2, \dots$

COROLLARY 2.2 *$H(x) \diamond \delta^{(r)}(x) = H(-x) \diamond \delta^{(r)}(x) = (1/2)\delta^{(r)}(x)$ for any non-negative integer r .*

COROLLARY 2.3 $\delta^2(x) = \delta(x) \diamond \delta(x) = 0$.

Proof From Corollary 2.2, we have $H(x) \diamond \delta(x) = \delta(x)/2$ and differentiating both sides by Theorem 2.2 we get

$$\delta(x) \diamond \delta(x) + H(x) \diamond \delta'(x) = \frac{1}{2}\delta'(x),$$

which shows that $\delta^2(x) = \delta(x) \diamond \delta(x) = 0$.

Similarly, we can derive $\delta(x) \diamond \delta'(x) = \delta'(x) \diamond \delta(x) = 0$ by noting that

$$\delta'(x) \diamond \delta(x) + \delta(x) \diamond \delta'(x) = 0. \quad \blacksquare$$

THEOREM 2.4 *The commutative product $(1/x) \diamond \delta(x)$ exists and*

$$\frac{1}{x} \diamond \delta(x) = -\frac{1}{2}\delta'(x).$$

Proof By the mean value theorem, we get

$$\begin{aligned} \left(\frac{1}{x} \delta_n(x), \phi(x) \right) &= \int_0^\infty \frac{\delta_n(x)\phi(x) - \delta_n(-x)\phi(-x)}{x} dx \\ &= \int_0^\infty \delta_n(x) \frac{\phi(x) - \phi(-x)}{x} dx = \frac{\phi(\zeta) - \phi(-\zeta)}{2\zeta}, \end{aligned}$$

where $0 < \zeta < 1/n$.

It follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1}{x} \delta_n(x), \phi(x) \right) &= \lim_{\zeta \rightarrow 0} \frac{\phi(\zeta) - \phi(-\zeta)}{2\zeta} \\ &= \phi'(0). \end{aligned}$$

On the other hand, we have

$$\frac{1}{x} * \delta_n(x) = \left(\frac{1}{t}, \delta_n(x-t) \right)$$

and

$$\begin{aligned} &\left(\left(\frac{1}{x} * \delta_n(x) \right) \delta(x), \phi(x) \right) \\ &= \left(\delta(x), \left(\frac{1}{x} * \delta_n(x) \right) \phi(x) \right) \\ &= \left(\frac{1}{t}, \delta_n(-t) \right) \phi(0) = \int_0^\infty \frac{\delta_n(t) - \delta_n(-t)}{t} dt \phi(0) = 0. \end{aligned}$$

By Definition 2.1, we complete the proof. \blacksquare

COROLLARY 2.4 *The commutative product $(\psi(x)/x) \diamond \delta(x)$ exists and*

$$\frac{\psi(x)}{x} \diamond \delta(x) = -\frac{\psi(0)}{2} \delta'(x) + \psi'(0) \delta(x).$$

Proof Clearly we have

$$\frac{\psi(x)}{x} = \frac{\psi(0)}{x} + \psi'(\zeta x),$$

where $0 < \zeta < 1$ and $\psi'(\zeta x)$ is continuous. Hence,

$$\frac{\psi(x)}{x} \diamond \delta(x) = -\frac{\psi(0)}{2} \delta'(x) + \psi'(\zeta x) \diamond \delta(x) = -\frac{\psi(0)}{2} \delta'(x) + \psi'(0) \delta(x),$$

by Theorem 2.4 and Corollary 2.1. ■

In particular, we obtain the following products from Corollary 2.4.

$$\begin{aligned} \frac{1}{x} \diamond \delta(x) &= -\frac{1}{2} \delta'(x) \\ \frac{\sin x}{x} \diamond \delta(x) &= \delta(x) \\ \frac{\cos x}{x} \diamond \delta(x) &= -\frac{1}{2} \delta'(x) \\ \frac{e^x}{x} \diamond \delta(x) &= -\frac{1}{2} \delta'(x) + \delta(x). \end{aligned}$$

THEOREM 2.5 *The commutative product $x_+^{-r-(1/2)}$ and $x_-^{-r-(1/2)}$ exists and*

$$x_+^{-r-(1/2)} \diamond x_-^{-r-(1/2)} = \frac{(-1)^r \pi}{2(2r)!} \delta^{(2r)}(x),$$

where $r = 0, 1, 2, \dots$

Proof Using

$$\begin{aligned} x_-^{-r-(1/2)} &= \frac{2^r}{1 \cdot 3 \cdot \dots \cdot (2r-1)} \frac{d^r}{dx^r} x_-^{-(1/2)} = c_1 \frac{d^r}{dx^r} x_-^{-(1/2)} \quad \text{and} \\ x_+^{-r-(1/2)} &= \frac{(-1)^r 2^r}{1 \cdot 3 \cdot \dots \cdot (2r-1)} \frac{d^r}{dx^r} x_+^{-(1/2)} = (-1)^r c_1 \frac{d^r}{dx^r} x_+^{-(1/2)}, \end{aligned}$$

we have

$$\begin{aligned} x_-^{-r-(1/2)} * \delta_n &= c_1 (x_-^{-(1/2)} * \delta_n^{(r)}) = c_1 \int_x^{(1/n)} (t-x)^{-(1/2)} \delta_n^{(r)}(t) dt \quad \text{and} \\ x_+^{-r-(1/2)} * \delta_n &= (-1)^r c_1 (x_+^{-(1/2)} * \delta_n^{(r)}) = (-1)^r c_1 \int_{-(1/n)}^x (x-t)^{-(1/2)} \delta_n^{(r)}(t) dt. \quad \blacksquare \end{aligned}$$

By Definition 2.1, we need to compute

$$\begin{aligned} I &= \frac{1}{2} \{ (x_+^{-r-(1/2)}, (x_-^{-r-(1/2)} * \delta_n) \phi(x)) + (x_-^{-r-(1/2)}, (x_+^{-r-(1/2)} * \delta_n) \phi(x)) \} \\ &= \frac{1}{2} (I_1 + I_2). \end{aligned}$$

Consider each term separately,

$$\begin{aligned} I_1 &= (x_+^{-r-(1/2)}, (x_-^{-r-(1/2)} * \delta_n) \phi(x)) \\ &= c_1^2 \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} x^{-(1/2)} \int_x^{(1/n)} (t-x)^{-(1/2)} \delta_n^{(r+j)}(t) dt \phi^{(r-j)}(x) dx \\ &= c_1^2 \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)} (t-x)^{-(1/2)} \phi^{(r-j)}(x) dx \quad \text{and} \\ I_2 &= (x_-^{-r-(1/2)}, (x_+^{-r-(1/2)} * \delta_n) \phi(x)) \\ &= c_1^2 \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)} (t-x)^{-(1/2)} (-1)^{r+j} \phi^{(r-j)}(-x) dx. \end{aligned}$$

Adding I_1 and I_2 to get,

$$\begin{aligned} I &= \frac{c_1^2}{2} \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)} (t-x)^{-(1/2)} \\ &\quad \times (\phi^{(r-j)}(x) + (-1)^{(r+j)} \phi^{(r-j)}(-x)) dx. \end{aligned}$$

By Taylor series,

$$\begin{aligned} &\phi^{(r-j)}(x) + (-1)^{(r+j)} \phi^{(r-j)}(-x) \\ &= \sum_{i=0}^{j+r-1} \frac{1}{i!} [\phi^{(r-j+i)}(0) + (-1)^{r+j+i} \phi^{(r-j+i)}(0)] x^i \\ &\quad + \frac{2\phi^{(2r)}(0)}{(j+r)!} x^{j+r} + \frac{1}{(r+j+1)!} [\phi^{(2r+1)}(\eta x) - \phi^{(2r+1)}(-\eta x)] x^{r+j+1}, \end{aligned}$$

where $0 < \eta < 1$. Integrating by parts, we come to conclude

$$\begin{aligned} &\int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)} (t-x)^{-(1/2)} \frac{2\phi^{(2r)}(0)}{(j+r)!} x^{j+r} dx \\ &= \frac{2\phi^{(2r)}(0)}{(j+r)!} \int_0^{(1/n)} \delta_n^{(r+j)}(t) t^{j+r} dt \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right) \\ &= (-1)^{(r+j)} \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right) \phi^{(2r)}(0) \end{aligned}$$

by using

$$\int_0^t x^{r+j-(1/2)} (t-x)^{-(1/2)} dx = t^{(r+j)} \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right),$$

where β is the Beta function. Similarly, we can derive that

$$\lim_{n \rightarrow \infty} \frac{c_1^2}{2} \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)}(t-x)^{-(1/2)}.$$

$$\frac{1}{(r+j+1)!} [\phi^{(2r+1)}(\eta x) - \phi^{(2r+1)}(-\eta x)] x^{r+j+1} dx = 0.$$

On the other hand, we see that

$$\int_0^{(1/n)} \delta_n^{(r+j)}(t) t^i dt = 0,$$

if $(r+j+i)$ is even and positive. This implies that

$$\frac{c_1^2}{2} \sum_{j=0}^r \binom{r}{j} \int_0^{(1/n)} \delta_n^{(r+j)}(t) dt \int_0^t x^{-(1/2)}(t-x)^{-(1/2)}.$$

$$\sum_{i=0}^{j+r-1} \frac{1}{i!} [\phi^{(r-j+i)}(0) + (-1)^{r+j+i} \phi^{(r-j+i)}(0)] x^i dx = 0.$$

By the following identity in [16]

$$\sum_{j=0}^r \binom{r}{j} (-1)^j \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right) = \beta\left(r+\frac{1}{2}, r+\frac{1}{2}\right),$$

we obtain

$$x_+^{-r-(1/2)} \diamond x_-^{-r-(1/2)} = \frac{c_1^2}{2} \sum_{j=0}^r \binom{r}{j} (-1)^{(r+j)} \beta\left(r+j+\frac{1}{2}, \frac{1}{2}\right) \delta^{(2r)}(x)$$

$$= \frac{c_1^2}{2} (-1)^r \beta\left(r+\frac{1}{2}, r+\frac{1}{2}\right) \delta^{(2r)}(x) = \frac{(-1)^r \pi}{2(2r)!} \delta^{(2r)}(x)$$

due to the fact that

$$\beta\left(r+\frac{1}{2}, r+\frac{1}{2}\right) = \frac{\Gamma^2(r+1/2)}{(2r)!} = \frac{\pi}{(2r)!c_1^2}.$$

Remark 1 This result is obviously identical with one obtained by Fisher in [12]. It is interesting to point out that both $(x_+^{-r-(1/2)}, (x_-^{-r-(1/2)} * \delta_n) \phi(x))$ and $(x_-^{-r-(1/2)}, (x_+^{-r-(1/2)} * \delta_n) \phi(x))$ are divergent in the normal sense $n \rightarrow \infty$, but the addition is convergent.

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Closed-form evaluation of some families of definite tangent and secant integrals

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Several families of definite integrals involving the tangent or secant function have been evaluated in closed form by making use of the elementary arguments. Some of them are expressible in terms of a finite combination of values of trigonometric functions and the Hurwitz zeta function.

Keywords: Hurwitz zeta function; Riemann zeta function; Definite integration; Tangent integral; Secant integral

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1. Introduction

Studies of integral representations of the Riemann zeta function have led us to examine families of definite integrals of the form

$$\int_0^\delta \left\{ \begin{matrix} B_n(t) \tan(\pi t) \\ E_n(t) \sec(\pi t) \end{matrix} \right\} dt \quad (1)$$

when $\delta = 1$ and $1/2$, and where $B_n(t)$ and $E_n(t)$ are the n th degree Bernoulli and Euler polynomials of odd degree [1]. The field appears to be insufficiently investigated and, surprisingly, it has turned out that altogether very few definite integrals involving tangent or secant function can be found in standard tables of integrals. For instance, in section 2.5.26 of the standard reference text by Prudnikov *et al.* [2, pp. 436–438], among not more than half a dozen definite tangent integrals, the following

$$I_n = \int_0^{\pi/4} x^n \tan x dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^n \sum_{k=1}^{\infty} \frac{(4^k - 1)\zeta(2k)}{4^{2k-1}(n + 2k)}, \quad (2)$$

where $\zeta(s)$ is the Riemann zeta function (see section 2) and $n \in \mathbb{N}$, is the most general. All these integrals are classical results and they were recorded in the literature as earlier as 1867 [3, pp. 306–310, tables 204–206] and 1891 [4, tables 204–206].

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In this sequel to our earlier work, by making use of elementary arguments, we will present a systematic closed-form derivation of integrals, which include arbitrary n th degree polynomial and tangent or secant function. As a straightforward corollary we obtain that I_n is expressible in terms of a finite combination of values of trigonometric functions and the Hurwitz zeta function (defined below) and the Riemann zeta function (see equation 22). In addition, many (presumably new) results follow as special cases.

2. Preliminaries and statement of the results

Let us denote by $\zeta(s)$ and $\zeta(s, a)$, respectively, the Riemann zeta and the Hurwitz (or generalized) zeta function defined usually by (see, for instance, [5, p. 19] and [6, p. 96])

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\Re(s) > 1) \quad (3)$$

and by ([5, p. 22] and [6, p. 88])

$$\zeta(s, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1, a \neq 0, -1, -2, \dots). \quad (4)$$

For $\Re(s) \leq 1, s \neq 1$, the function $\zeta(s)$ and $\zeta(s, a)$ are defined as the analytic continuations of the foregoing series. Both are analytic over the whole complex plane, except at $s = 1$, where they have a simple pole.

Define the alternating counterpart of $\zeta(s, a)$ by

$$\eta(s, a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+a)^s} \quad (\Re(s) > 0) \quad (5)$$

and observe that there exists a relationship between $\eta(s, a)$ and $\zeta(s, a)$

$$\eta(s, a) = \frac{1}{2^s} \left[\zeta\left(s, \frac{a}{2}\right) - \zeta\left(s, \frac{a+1}{2}\right) \right], \quad (6)$$

since

$$\begin{aligned} \eta(s, a) &= \sum_{k=0}^{\infty} \frac{1}{(2k+a)^s} - \sum_{k=0}^{\infty} \frac{1}{(2k+1+a)^s} \\ &= \frac{1}{2^s} \left[\sum_{k=0}^{\infty} \frac{1}{(k+(a/2))^s} - \sum_{k=0}^{\infty} \frac{1}{(k+(a+1)/2)^s} \right]. \end{aligned}$$

In what follows we also use the following functions

$$\eta(s) = (1 - 2^{1-s})\zeta(s) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^s}, \quad (7)$$

$$\beta(s) = \frac{1}{4^s} \left[\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right) \right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s}. \quad (8)$$

Further, we shall also make use of the functions defined by

$$\begin{Bmatrix} s_\nu(x) \\ c_\nu(x) \end{Bmatrix} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k\pi)^\nu} \begin{Bmatrix} \sin(2k\pi x) \\ \cos(2k\pi x) \end{Bmatrix}, \quad (9)$$

and

$$\begin{Bmatrix} s_\nu^*(x) \\ c_\nu^*(x) \end{Bmatrix} = \sum_{k=0}^{\infty} \frac{(-1)^k}{[(2k+1)\pi]^\nu} \begin{Bmatrix} \sin[(2k+1)\pi x] \\ \cos[(2k+1)\pi x] \end{Bmatrix}, \quad (10)$$

where x is a real and ν is a complex, $0 \leq x \leq 1$ when $\Re(\nu) > 0$.

Observe that throughout the text an empty sum is (as usual) interpreted as zero and the notation $f^{(n)}(t)$ ($n \in \mathbb{N}_0$) is used to mean the n th-order derivative of f with respect to t . In addition, $\lfloor x \rfloor$ denotes the floor function, often called the greatest integer function or integer value, which gives the largest integer less than or equal to x .

Now we are ready to state our results.

THEOREM 2.1 *Consider the functions defined as in equations (9) and (10), assume that n is a positive integer and let $P_n(x)$ be an arbitrary, but fixed polynomial of degree n in x . Then,*

$$\begin{aligned} \frac{1}{2} \int P_n(t) \begin{Bmatrix} \tan(\pi t) \\ \sec(\pi t) \end{Bmatrix} dt &= \frac{1}{2\pi} P_n(t) \begin{Bmatrix} -\log[2 \cos(\pi t)] \\ \log \left[\tan \left(\frac{\pi t}{2} + \frac{\pi}{4} \right) \right] \end{Bmatrix} \\ &+ \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} P_n^{(2k-1)}(t) \begin{Bmatrix} s_{2k}(t) \\ c_{2k}^*(t) \end{Bmatrix} + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} P_n^{(2k)}(t) \begin{Bmatrix} c_{2k+1}(t) \\ -s_{2k+1}^*(t) \end{Bmatrix}. \end{aligned} \quad (11)$$

COROLLARY 2.1 *Let $P_n(x)$, $n \in \mathbb{N}$, be a fixed polynomial of degree n in x . If $0 < x < 1/2$ then, in terms of the functions defined as in equations (9) and (10), we have:*

$$\begin{aligned} \text{(i)} \quad \frac{1}{2} \int_0^x P_n(t) \tan(\pi t) dt &= \frac{1}{2\pi} \{ P_n(0) \log 2 - P_n(x) \log[2 \cos(\pi x)] \} \\ &+ \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} P_n^{(2k-1)}(x) s_{2k}(x) + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} [P_n^{(2k)}(x) c_{2k+1}(x) - P_n^{(2k)}(0) c_{2k+1}(0)], \end{aligned} \quad (12)$$

$$\begin{aligned} \text{(ii)} \quad \frac{1}{2} \int_0^x P_n(t) \sec(\pi t) dt &= \frac{1}{2\pi} P_n(x) \log[\tan((\pi x/2) + \pi/4)] \\ &+ \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} [P_n^{(2k-1)}(x) c_{2k}^*(x) - P_n^{(2k-1)}(0) c_{2k}^*(0)] - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} P_n^{(2k)}(x) s_{2k+1}^*(x). \end{aligned} \quad (13)$$

COROLLARY 2.2 *Let $P_n(x)$, $n \in \mathbb{N}$, be a fixed n th order polynomial. If $0 < p/q < 1/2$ ($p, q \in \mathbb{N}$), then, in terms of the functions $\zeta(s)$, $\zeta(s, a)$ and $\eta(s, a)$, respectively given by equations (3),*

(4) and (5), we have:

$$\begin{aligned}
 \text{(i)} \quad & \frac{1}{2} \int_0^{p/q} P_n(t) \tan(\pi t) dt = \frac{1}{2\pi} \left\{ P_n(0) \log 2 - P_n\left(\frac{p}{q}\right) \log \left[2 \cos\left(\frac{\pi p}{q}\right) \right] \right\} \\
 & + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}\left(\frac{p}{q}\right)}{(2\pi q)^{2k}} \sum_{j=1}^q (-1)^{j-1} \sin\left(\frac{2j\pi p}{q}\right) \Omega_{2k}\left(\frac{j}{q}\right) \\
 & + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k)}\left(\frac{p}{q}\right)}{(2\pi q)^{2k+1}} \sum_{j=1}^q (-1)^{j-1} \cos\left(\frac{2j\pi p}{q}\right) \Omega_{2k+1}\left(\frac{j}{q}\right) \\
 & - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k)}(0)(1-2^{-2k})}{(2\pi)^{2k+1}} \zeta(2k+1), \tag{14}
 \end{aligned}$$

where

$$\Omega_m(n/q) = \begin{cases} \eta\left(m, \frac{n}{q}\right) & \text{if } q \text{ is odd} \\ \zeta\left(m, \frac{n}{q}\right) & \text{if } q \text{ is even} \end{cases}.$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{1}{2} \int_0^{p/q} P_n(t) \sec(\pi t) dt = \frac{1}{2\pi} P_n\left(\frac{p}{q}\right) \log \left[\tan\left(\frac{\pi p}{2q} + \frac{\pi}{4}\right) \right] \\
 & + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}\left(\frac{p}{q}\right)}{(2\pi q)^{2k}} \sum_{j=1}^q (-1)^{j-1} \cos\left(\frac{(2j-1)\pi p}{q}\right) \Omega_{2k}^*\left(\frac{2j-1}{2q}\right) \\
 & - \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k-1)}(0)}{(2\pi)^{2k}} \frac{1}{2^{2k}} [\zeta(2k, 1/4) - \zeta(2k, 3/4)] \\
 & - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{P_n^{(2k)}(x)}{(2\pi q)^{2k+1}} \sum_{j=1}^q (-1)^{j-1} \sin\left(\frac{(2j-1)\pi p}{q}\right) \Omega_{2k+1}^*\left(\frac{2j-1}{2q}\right), \tag{15}
 \end{aligned}$$

where

$$\Omega_m^*\left(\frac{n}{2q}\right) = \begin{cases} \eta\left(m, \frac{n}{2q}\right) & \text{if } q \text{ is odd} \\ \zeta\left(m, \frac{n}{2q}\right) & \text{if } q \text{ is even} \end{cases}.$$

Remark 1 Observe that $\zeta(s) = \zeta(s, 1)$ and that there exists a relationship (equation 6) between $\eta(s, a)$ and $\zeta(s, a)$. Thus, any definite integral described by Corollary 2.2 (and Corollary 2.4 below) can be evaluated in closed form and is expressible in terms of a finite combination of values of trigonometric functions and the Hurwitz zeta function.

COROLLARY 2.3 Assume that $n \in \mathbb{N}$. Then, in terms of the functions given by equations (9) and (10), we have:

$$(i) \quad \frac{1}{2} \int_0^x t^n \tan(\pi t) dt = \Delta_n^t - \frac{x^n}{2\pi} \log[2 \cos(\pi x)] + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{n! x^{n-2k+1}}{(n-2k+1)!} s_{2k}(x) \\ + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{n! x^{n-2k}}{(n-2k)!} c_{2k+1}(x), \quad (16)$$

where

$$\Delta_n^t = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{\lfloor n/2 \rfloor} n! (2\pi)^{-(n+1)} (1-2^{-n}) \zeta(n+1) & \text{if } n \text{ is even} \end{cases}. \quad (17)$$

$$(ii) \quad \frac{1}{2} \int_0^x t^n \sec(\pi t) dt = \Delta_n^s + \frac{x^n}{2\pi} \log[\tan((\pi x/2) + \pi/4)] \\ + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{n! x^{n-2k+1}}{(n-2k+1)!} c_{2k}^*(x) - \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{n! x^{n-2k}}{(n-2k)!} s_{2k+1}^*(x), \quad (18)$$

where

$$\Delta_n^s = \begin{cases} (-1)^{\lfloor (n+1)/2 \rfloor} n! (4\pi)^{-(n+1)} \left[\zeta\left(n+1, \frac{1}{4}\right) - \zeta\left(n+1, \frac{3}{4}\right) \right] & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \quad (19)$$

COROLLARY 2.4 Assume that $n \in \mathbb{N}$. Let Δ_n^t and Δ_n^s be given by equations (17) and (19), respectively. If $0 < p/q < 1/2$ ($p, q \in \mathbb{N}$), then

$$(i) \quad \frac{1}{2} \int_0^{p/q} t^n \tan(\pi t) dt = \Delta_n^t - \frac{p^n}{2\pi q^n} \log \left[2 \cos \left(\frac{\pi p}{q} \right) \right] \\ + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1} p^{n-2k+1} n!}{(n-2k+1)! q^{n+1} (2\pi)^{2k}} \sum_{j=1}^q (-1)^{j-1} \sin \left(\frac{2j\pi p}{q} \right) \Omega_{2k} \left(\frac{j}{q} \right) \\ + \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1} p^{n-2k} n!}{(n-2k)! q^{n+1} (2\pi)^{2k+1}} \sum_{j=1}^q (-1)^{j-1} \cos \left(\frac{2j\pi p}{q} \right) \Omega_{2k+1} \left(\frac{j}{q} \right) \quad (20)$$

$$(ii) \quad \frac{1}{2} \int_0^{p/q} t^n \sec(\pi t) dt = \Delta_n^s + \frac{p^n}{2\pi q^n} \log \left[\tan \left(\frac{\pi p}{2q} + \frac{\pi}{4} \right) \right] \\ + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{(-1)^{k-1} p^{n-2k+1} n!}{(n-2k+1)! q^{n+1} (2\pi)^{2k}} \sum_{j=1}^q (-1)^{j-1} \cos \left[(2j-1) \frac{\pi p}{q} \right] \Omega_{2k}^* \left(\frac{2j-1}{2q} \right) \\ - \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^{k-1} p^{n-2k} n!}{(n-2k)! q^{n+1} (2\pi)^{2k+1}} \sum_{j=1}^q (-1)^{j-1} \sin \left[(2j-1) \frac{\pi p}{q} \right] \Omega_{2k+1}^* \left(\frac{2j-1}{2q} \right) \quad (21)$$

with

$$\Omega_m \left(\frac{n}{q} \right) = \begin{cases} \eta \left(m, \frac{n}{q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(m, \frac{n}{q} \right) & \text{if } q \text{ is even} \end{cases} \quad \text{and} \quad \Omega_m^* \left(\frac{n}{2q} \right) = \begin{cases} \eta \left(m, \frac{n}{2q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(m, \frac{n}{2q} \right) & \text{if } q \text{ is even} \end{cases},$$

where $\zeta(s)$, $\zeta(s, a)$ and $\eta(s, a)$ are the above defined functions.

Remark 2 We have failed to find in the literature our integrals in equations (11)–(16), (18) and (20), (21). Regarding the integral in the example below it is a well-known classical result [3, pp. 306–310, tables 204–206; 4, tables 204–206] that it could be evaluated by means of infinite series (see equation 2). However, it seems that it has not been noticed before that its finite series evaluation like in our example is possible.

Finally, as an immediate consequence of Corollary 2.4, we have the next example.

Example Let $n \in \mathbb{N}$. Then

$$\begin{aligned} \int_0^{1/4} t^n \tan(\pi t) dt = 2 \left\{ \begin{array}{ll} 0 & \text{if } n \text{ is odd} \\ (-1)^{\lfloor n/2 \rfloor} n! (2\pi)^{-(n+1)} (1 - 2^{-n}) \zeta(n+1) & \text{if } n \text{ is even} \end{array} \right\} \\ + \frac{n!}{2^{2n+1}} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} \frac{\zeta(2k, 1/4) - \zeta(2k, 3/4)}{(2\pi)^{2k} (n - 2k + 1)!} \\ + \frac{n!}{2^{2n}} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} \frac{(2^{2k} - 1) \zeta(2k + 1)}{(2\pi)^{2k+1} (n - 2k)!} - \frac{\log 2}{2^{2n+1} \pi}. \end{aligned} \quad (22)$$

3. Proof of the results

We begin by recalling the definitions of $s_\nu(t)$ and $c_\nu(t)$ in equation (9) and then define

$$l_\nu(t) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{e^{2k\pi i t}}{(2k\pi)^\nu} = c_\nu(t) + i s_\nu(t) \quad \Re(\nu) > 0. \quad (23)$$

Similarly, define

$$l_\nu^*(t) = \sum_{k=0}^{\infty} (-1)^k \frac{e^{(2k+1)\pi i t}}{[(2k+1)\pi]^\nu} = c_\nu^*(t) + i s_\nu^*(t) \quad \Re(\nu) > 0, \quad (24)$$

where $c_\nu^*(t)$ and $s_\nu^*(t)$ are given by equation (10). For the proof of the main results, we need the following three lemmas.

LEMMA 3.1 For any polynomial $p_n(t)$ in t of degree n , $n \in \mathbb{N}_0$, and for any function $f(t)$ we have

$$\int p_n(t) f(t) dt = \sum_{k=0}^n (-1)^k p_n^{(k)}(t) f_{k+1}(t) + C,$$

where

$$\begin{aligned} p_n^{(k)}(t) &= \frac{d}{dt} p_n^{(k-1)}(t), \quad p_n^{(0)}(t) := p_n(t) \quad \text{and} \\ f_k(t) &= \int f_{k-1}(t) dt, \quad f_0(t) := f(t) \quad (k \in \mathbb{N}). \end{aligned}$$

Proof of Lemma 3.1 The following formula

$$\int u(t)v(t)dt = \sum_{k=0}^n (-1)^k u^{(k)}(t)v_{k+1}(t) + (-1)^{n+1} \int u^{(n+1)}(t)v_{n+1}(t) + C,$$

where

$$u^{(k)}(t) = \frac{d}{dt} u^{(k-1)}(t), \quad u^{(0)}(t) := u(t) \quad \text{and} \\ v_k(t) = \int v_{k-1}(t)dt, \quad v_0(t) := v(t) \quad (k \in \mathbb{N})$$

can be found in the literature (see, for instance, [7, p. 643, equation 1]), and it is clearly readily derivable by repeated use of integration by parts. Now, the above sought formula follows without difficulty.

The summations given by Lemma 3.2 are well known and can be, for instance, found in Prudnikov *et al.* [2, p. 726, equation 5.4.2.10 and p. 732, equation 5.4.6.8]. ■

LEMMA 3.2 *We have*

$$(i) \quad c_1(t) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\cos(2k\pi t)}{2k\pi} = \frac{1}{2\pi} \log[2 \cos(\pi t)], \quad \left(-\frac{1}{2} < t < \frac{1}{2}\right) \quad (25)$$

$$(ii) \quad s_1^*(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\sin[(2k+1)\pi t]}{(2k+1)\pi} = \frac{1}{2\pi} \log \left[\tan \left(\frac{\pi t}{2} + \frac{\pi}{4} \right) \right] \quad \left(-\frac{1}{2} < t < \frac{1}{2}\right). \quad (26)$$

LEMMA 3.3 *Assume that v is a complex with $\Re(v) > 1$ and let $\zeta(s, a)$ and $\eta(s, a)$ be the functions defined as in equations (4) and (5), respectively. If $p \in \mathbb{N}_0$, $q \in \mathbb{N}$ with $0 \leq p \leq q$, then*

$$(i) \quad s_v \left(\frac{p}{q} \right) = \frac{1}{(2\pi q)^v} \sum_{k=1}^q (-1)^{k-1} \sin \frac{2k\pi p}{q} \begin{cases} \eta \left(v, \frac{k}{q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(v, \frac{k}{q} \right) & \text{if } q \text{ is even} \end{cases}, \quad (27)$$

$$(ii) \quad c_v \left(\frac{p}{q} \right) = \frac{1}{(2\pi q)^v} \sum_{k=1}^q (-1)^{k-1} \cos \frac{2k\pi p}{q} \begin{cases} \eta \left(v, \frac{k}{q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(v, \frac{k}{q} \right) & \text{if } q \text{ is even} \end{cases}, \quad (28)$$

$$(iii) \quad s_v^* \left(\frac{p}{q} \right) = \frac{1}{(2\pi q)^v} \sum_{k=1}^q (-1)^{k-1} \sin \left(\frac{(2k-1)\pi p}{q} \right) \begin{cases} \eta \left(v, \frac{2k-1}{2q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(v, \frac{2k-1}{2q} \right) & \text{if } q \text{ is even} \end{cases}, \quad (29)$$

$$(iv) \quad c_v^* \left(\frac{p}{q} \right) = \frac{1}{(2\pi q)^v} \sum_{k=1}^q (-1)^{k-1} \cos \left(\frac{(2k-1)\pi p}{q} \right) \begin{cases} \eta \left(v, \frac{2k-1}{2q} \right) & \text{if } q \text{ is odd} \\ \zeta \left(v, \frac{2k-1}{2q} \right) & \text{if } q \text{ is even} \end{cases}, \quad (30)$$

where $s_v(t)$ (and $c_v(t)$) and $s_v^*(t)$ (and $c_v^*(t)$) are given by equations (9) and (10), respectively.

Proof of Lemma 3.3 We first prove parts (i) and (ii) of this lemma. We begin by recalling equation (23) which, upon setting $t = p/q$ ($0 \leq p \leq q$ if $\Re(v) > 0$; $p \in \mathbb{N}_0$; $q \in \mathbb{N}$), and in view of the following elementary series identity (see, for instance, [8, p. 79, equation 2.2])

$$\sum_{m=1}^{\infty} f(m) = \sum_{j=1}^q \sum_{m=0}^{\infty} f(mq + j)$$

as well as of the definitions of $\zeta(s, a)$ and $\eta(s, a)$ in equations (4) and (5) immediately yields

$$\begin{aligned} l_v\left(\frac{p}{q}\right) &= \sum_{j=1}^q \sum_{k=0}^{\infty} (-1)^{qk+j} \frac{e^{2(qk+j)\pi i p/q}}{[(qk+j)2\pi]^v} = \frac{1}{(2\pi q)^v} \sum_{j=1}^q \sum_{k=0}^{\infty} (-1)^{qk+j} \frac{e^{2(qk+j)\pi i p/q}}{(k+j/q)^v} \\ &= \frac{1}{(2\pi q)^v} \sum_{j=1}^q (-1)^j e^{2\pi i j p/q} \sum_{k=0}^{\infty} \frac{(-1)^{qk}}{(k+j/q)^v} \\ &= \frac{1}{(2q\pi)^v} \sum_{j=1}^q (-1)^{j-1} e^{2\pi i j p/q} \begin{cases} \eta\left(v, \frac{j}{q}\right) & \text{if } q \text{ is odd} \\ \zeta\left(v, \frac{j}{q}\right) & \text{if } q \text{ is even} \end{cases}. \end{aligned} \quad (31)$$

Now, by separating the real and imaginary parts on the left-hand and the right-hand sides of equations (23) and (31), we obtain the sought formulae in equations (27) and (28).

Similarly, starting from equation (24) we obtain equations (29) and (30), i.e., we prove the parts (iii) and (iv). ■

Proof of Theorem 2.1 The proof is based on Lemma 3.1 and Lemma 3.2. We first evaluate the tangent integral. Let $f_0(t) = \tan(\pi t)$ and consider the successive integrals of $f_0(t)$

$$f_k(t) = \int f_{k-1}(t) dt \quad (k \in \mathbb{N}).$$

It is easy to show that

$$f_1(t) = \int \tan(\pi t) dt = -\frac{1}{\pi} \log[2 \cos(\pi t)]$$

so that by Lemma 3.2(i) we have

$$f_1(t) = -2c_1(t). \quad (32)$$

Next, in view of the definitions $s_v(t)$ and $c_v(t)$ in equation (9), it follows from equation (32) by repeated integration that

$$f_{2k}(t) = 2(-1)^k s_{2k}(t) \quad \text{and} \quad f_{2k+1}(t) = 2(-1)^{k+1} c_{2k+1}(t) \quad (k \in \mathbb{N}). \quad (33)$$

Finally, for an arbitrary, but fixed polynomial $p_n(t)$ of degree n , by Lemma 3.1 (with $f(t) = \tan(\pi t)$) and by making use of equations (32) and (33) we obtain

$$\begin{aligned} \frac{1}{2} \int \tan(\pi t) p_n(t) dt &= -p_n(t) c_1(t) + \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{m-1} p_n^{(2m-1)}(t) s_{2m}(t) \\ &\quad + \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^{m-1} p_n^{(2m)}(t) c_{2m+1}(t) \end{aligned}$$

so that the proposed formula for the tangent integral follows upon replacing $c_1(t)$ (see equation 25).

Now we evaluate the secant integral. The successive integrals

$$g_k(t) = \int g_{k-1}(t) dt \quad (k \in \mathbb{N})$$

of the function $g_0(t) = \sec(\pi t)$ are given by

$$g_1(t) = \int \sec(\pi t) dt = \frac{1}{\pi} \log \left[\tan \left(\frac{\pi t}{2} + \frac{\pi}{4} \right) \right] = 2s_1^*(t) \quad (34)$$

(see Lemma 3.2(ii)), and

$$g_{2k}(t) = 2(-1)^k c_{2k}^*(t) \quad \text{and} \quad g_{2k+1}(t) = 2(-1)^k s_{2k+1}^*(t) \quad (k \in \mathbb{N}), \quad (35)$$

where $s_v^*(t)$ and $c_v^*(t)$ are given by equation (10). What remains is to verify that

$$\begin{aligned} \frac{1}{2} \int \sec(\pi t) p_n(t) dt &= p_n(t) s_1^*(t) + \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{m-1} p_n^{(2m-1)}(t) c_{2m}^*(t) \\ &\quad - \sum_{m=1}^{\lfloor n/2 \rfloor} (-1)^{m-1} p_n^{(2m)}(t) s_{2m+1}^*(t) \end{aligned}$$

can be obtained by Lemma 3.1 and from equations (34) and (35). Lastly, upon replacing $s_1^*(t)$, we have the proposed formula. This completes proof of our Theorem 2.1. ■

Proof of Corollary 2.1 By Theorem 2.1, we have

$$\begin{aligned} \frac{1}{2} \int_0^x P_n(t) \left\{ \frac{\tan(\pi t)}{\sec(\pi t)} \right\} dt &= \frac{P_n(t)}{2\pi} \left\{ \frac{-\log[2 \cos(\pi t)]}{\log[\tan((\pi t/2) + \pi/4)]} \right\} \Big|_{t=0}^{t=x} \\ &\quad + \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} (-1)^{k-1} P_n^{(2k-1)}(t) \left\{ \frac{s_{2k}(t)}{c_{2k}^*(t)} \right\} \Big|_{t=0}^{t=x} \\ &\quad + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{k-1} P_n^{(2k)}(t) \left\{ \frac{c_{2k+1}(t)}{-s_{2k+1}^*(t)} \right\} \Big|_{t=0}^{t=x} \end{aligned}$$

and the formulae in equations (12) and (13) follow since all $s_{2k}(t)$ (and all $s_{2k+1}^*(t)$) ($k \in \mathbb{N}$) are zero at $t = 0$. ■

Proof of Corollary 2.2 This corollary is obtained by combining Corollary 2.1 and Lemma 3.3. Note that, in light of the definitions in equations (9) and (10) and by equations (7) and (8), for $k \in \mathbb{N}_0$, we have

$$\begin{aligned} c_{2k+1}(t) &= (2\pi)^{-(2k+1)} (1 - 2^{-2k}) \zeta(2k+1), \\ c_{2k}^*(t) &= (4\pi)^{-2k} \left[\zeta \left(2k, \frac{1}{4} \right) - \zeta \left(2k, \frac{3}{4} \right) \right]. \end{aligned}$$

Proof of Corollaries 2.3 and 2.4 It is clear, that, upon setting $P_n(t) = t^n$ ($n \in \mathbb{N}$), our Corollary 2.3 and Corollary 2.4 are, respectively, special cases of Corollary 2.1 and Corollary 2.2. ■

4. Concluding remarks

We conclude by noting that this work has been greatly facilitated by extensive use of symbolic computation with Mathematica (Version 5.0) for Windows. As an illustration, we give a few examples of the application of our integration formula in equation (22)

$$\begin{aligned}
 \text{(i)} \quad & \int_0^{1/4} t \tan(\pi t) dt = \frac{G}{2\pi^2} - \frac{\log 2}{8\pi}, \\
 \text{(ii)} \quad & \int_0^{1/4} t^2 \tan(\pi t) dt = \frac{G}{4\pi^2} - \frac{\log 2}{32\pi} - \frac{21\zeta(3)}{64\pi^3}, \\
 \text{(iii)} \quad & \int_0^{1/4} t^3 \tan(\pi t) dt = \frac{3G}{32\pi^2} - \frac{\log 2}{128\pi} + \frac{9\zeta(3)}{256\pi^3} - \frac{3\zeta(4, 1/4)}{1024\pi^4} + \frac{3\zeta(4, 3/4)}{1024\pi^4}, \\
 \text{(iv)} \quad & \int_0^{1/4} t^4 \tan(\pi t) dt = \frac{G}{32\pi^2} - \frac{\log 2}{512\pi} + \frac{9\zeta(3)}{512\pi^3} + \frac{1395\zeta(5)}{1024\pi^5} \\
 & - \frac{3\zeta(4, 1/4)}{1024\pi^4} + \frac{3\zeta(4, 3/4)}{1024\pi^4},
 \end{aligned}$$

where G is the Catalan constant, and compare them with the results of symbolic integration

$$\begin{aligned}
 \text{(i)} \quad & \text{Out}[1] = \frac{\text{Catalan}}{2\pi^2} - \frac{\text{Log}[2]}{8\pi} \\
 \text{(ii)} \quad & \text{Out}[2] = \frac{\text{Catalan}}{2\pi^2} - \frac{\text{Log}[4]}{64\pi} - \frac{21\text{Zeta}[3]}{64\pi^3} \\
 \text{(iii)} \quad & \text{Out}[3] = \frac{3\text{Catalan}}{32\pi^2} - \frac{\text{Log}[4]}{256\pi} - \frac{\text{PolyGamma}[3, 1/4]}{2048\pi^4} + \frac{\text{PolyGamma}[3, 3/4]}{2048\pi^4} \\
 & + \frac{9\text{Zeta}[3]}{256\pi^3} \\
 \text{(iv)} \quad & \text{Out}[4] = \frac{\text{Catalan}}{32\pi^2} - \frac{\text{Log}[2]}{512\pi} - \frac{\text{PolyGamma}[3, 1/4]}{2048\pi^4} + \frac{\text{PolyGamma}[3, 3/4]}{2048\pi^4} \\
 & + \frac{9\text{Zeta}[3]}{512\pi^3} + \frac{1395\text{Zeta}[5]}{1024\pi^5}.
 \end{aligned}$$

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Dunkl–Appell d -orthogonal polynomials

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In this paper, we characterize the Dunkl–Appell d -orthogonal polynomials. Various properties of the obtained polynomials are singled out: a generating function, a $(d + 1)$ -order recurrence relation and a differential-difference equation. We also explicitly express the d -dimensional functional for which the d -orthogonality holds. A detailed discussion of the $(d + 1)$ -symmetric solutions is presented. Some relationships between the obtained polynomials and some known orthogonal polynomial families are established.

Keywords: Orthogonal polynomials; d -orthogonal polynomials; Dunkl operator; Generating functions; d -dimensional functional; r -symmetric sequence

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1. Introduction

In the last two decades, an increasing research activity is done to some extensions of the concept of standard orthogonality. One of them is the so-called multiple orthogonality, which appears in a natural way in rational approximation to vector functions, in analytic number theory among other domains [1, 2]. Recently, some works [3–7] have focused on the analysis of structural properties of multiple orthogonality. Other papers deal with explicit examples discussed, for convenience, in a framework of a subclass known as d -orthogonality [8, 9] and defined as follows.

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n \geq 0}$ in $\mathcal{P}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n . We denote the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$ by $\langle u, f \rangle$ and the moments of u by $(u)_n := \langle u, x^n \rangle, n \geq 0$. Let $\{P_n\}_{n \geq 0}$ be a PS in $\mathcal{P}$. The corresponding *dual sequence* $(u_n)_{n \geq 0}$ is defined by $\langle u_n, P_m \rangle = \delta_{n,m}$, $n, m = 0, 1, \dots$, $\delta_{n,m}$, being the Kronecker symbol.

DEFINITION 1.1 Let $u_0, u_1, \dots, u_{d-1}$ be the d -linear functionals and $\{P_n\}_{n \geq 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n \geq 0}$ is called *d -orthogonal polynomial set* (d -OPS, for shorter) with respect to the

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d -dimensional functional $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$, if it fulfils:

$$\begin{cases} \langle u_k, P_m P_n \rangle = 0, & m > nd + k, \quad n \geq 0, \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0, & n \geq 0, \end{cases}$$

for each $k \in \mathbb{N}_d := \{0, 1, \dots, d-1\}$.

For $d = 1$, the d -orthogonality is reduced to the orthogonality.

Most of the known explicit examples of d -OPS were introduced by solving a characterization problem that consists in finding all d -OPS, satisfying a given property [10–13]. An interesting one may be considered where the PS is of Brenke type. That is

(P): Find all d -OPS $\{P_n\}_{n \geq 0}$ generated by

$$A(t)B(xt) = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where A and B are two formal power series satisfying:

$$A(t) = \sum_{n=0}^{\infty} a_n t^n \quad \text{and} \quad B(t) = \sum_{n=0}^{\infty} b_n t^n, \quad a_0 b_n \neq 0 \quad \forall n \in \mathbb{N} := \{0, 1, 2, \dots\}.$$

This problem, for $d = 1$, was completely solved by Chihara [14]. For general positive integer d , it was treated only for the following three particular cases suggested by Chihara characterization: $B(z) = e^z$, $A(t) = e^t$ and B , the q -exponential function is which are considered, respectively, by Douak [15], Ben Cheikh and Douak [11, 16] and Zaghouani [17]. Notice that the functions $B(xt)$ considered by Douak [15] and Zaghouani [17] are, respectively, the eigenfunctions of the derivative operator and Hahn operator L_q [18] associated with the eigenvalue t .

Our purpose in this paper is to solve (P) with a further generalization of the exponential function defined by [19]

$$B(z) = e_{\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\gamma_{\mu}(n)},$$

where the coefficients γ_{μ} are defined by equation (5) and (6) below.

For this case, $B(xt)$ is the eigenfunction of the Dunkl operator defined by equation (4) below. Recall that a polynomial sequence $\{P_n\}_{n \geq 0}$ is called an Appell PS [20] if and only if $(d/dx)P_{n+1}(x) = (n+1)P_n(x)$, $n \geq 0$. A natural generalization of this notion, replacing the derivative operator by T_{μ} the Dunkl operator, may be formulated as follows.

DEFINITION 1.2 A PS $\{P_n\}_{n \geq 0}$ is called Dunkl–Appell or a T_{μ} -Appell PS if and only if

$$T_{\mu} P_{n+1}(x) = \frac{\gamma_{\mu}(n+1)}{\gamma_{\mu}(n)} P_n(x), \quad n \geq 0. \quad (1)$$

With the above notions our problem may be rewritten as follows.

(P₁): Find all PSs which are at the same time d -OPS and T_{μ} -Appell.

Such characterization takes into account the fact that PSs which are obtainable from one another by a linear change of variable are assumed equivalent.

A d -OPS $\{P_n\}_{n \geq 0}$ is called Dunkl-classical or T_{μ} -classical, if $\{T_{\mu} P_n\}_{n \geq 1}$ is also a d -OPS.

Note that the PSs obtained as solutions of the problem $(\mathbf{P}_1)$ must be T_μ -classical d -OPS according to equation (1).

The problem $(\mathbf{P}_1)$ was first solved by Angelosco [21] in the special case $\mu = 0$ and $d = 1$, and later by other authors (see [22] and [23] for additional references). The obtained solution is the Hermite PS. Later, Douak [15] treated the case $\mu = 0$ for a general positive integer d and obtained certain generalization of the Hermite polynomials, containing among others the Gould–Hopper polynomials [24]. In [25], we treated the symmetric case when $\mu > -1/2$ and $d = 1$. The obtained solution is the generalized Hermite PS. In this paper, we solve the problem when $\mu > -1/2$ for a general positive integer d . The main result is

THEOREM 1.3 *The only PSs which are d -OPS and T_μ -Appell at the same time are generated by*

$$\exp(\pi_{d+1}(t)) e_\mu(xt) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{\gamma_\mu(n)}, \quad (2)$$

where π_{d+1} is a polynomial of degree $d + 1$ and (μ, d) fulfils

$$(\mu, d) \in \{0\} \times \mathbb{N}^* \text{ or } (\mu, d) \in]-1/2, +\infty[\setminus \{0\} \times (2\mathbb{N} + 1), \quad \mathbb{N}^* := \mathbb{N} \setminus \{0\}. \quad (3)$$

Remark 1.4 If $(\mu, d) \in]-1/2, +\infty[\setminus \{0\} \times 2\mathbb{N} \setminus \{0\}$, there do not exist PSs which are at the same time d -OPS and T_μ -Appell.

The outline of the paper is as follows. In section 2, we recall some properties of the operator T_μ to be used in the sequel. In section 3, we give a characterization of the Dunkl–Appell polynomials by means of a generating function. In section 4, we prove our main result. The technique is based on a linear operator called the transform operator of X according to a suitable generating function of the involved PS. In section 5, we state a $(d + 1)$ -order recurrence relation and a differential-difference equation satisfied by the obtained polynomials. In section 6, we state the d -dimensional functional for which the d -orthogonality holds. We explicitly express the d -dimensional functional for the Gould–Hopper type polynomials and we give an integral representation of the moments of the generalized Hermite OPS. In section 7, we discuss a symmetry property related to obtained polynomials. Finally, in section 8, we describe the link of the obtained polynomials in this paper with other generalizations of the Hermite polynomials in the literature and, which have a property related to an orthogonality notion.

2. Preliminaries and notations

Let μ be a real number satisfying $\mu > -1/2$. The Dunkl operator T_μ is defined by [26, 27]

$$T_\mu \phi(x) = \phi'(x) + \mu \frac{\phi(x) - \phi(-x)}{x}, \quad (4)$$

ϕ being an entire function. The operator T_0 is reduced to the derivative operator.

Throughout this paper, we shall use the following notations, definitions and formulas related to T_μ that we will need in the sequel.

For positive integers n and k , put [19]

$$\gamma_\mu(2n) = \frac{2^{2n} n! \Gamma(n + \mu + 1/2)}{\Gamma(\mu + 1/2)} = (2n)! \frac{\Gamma(n + \mu + 1/2) \Gamma(1/2)}{\Gamma(n + 1/2) \Gamma(\mu + 1/2)} \quad (5)$$

and

$$\gamma_\mu(2n + 1) = \frac{2^{2n+1} n! \Gamma(n + \mu + 3/2)}{\Gamma(\mu + 1/2)} = (2n + 1)! \frac{\Gamma(n + \mu + 3/2) \Gamma(1/2)}{\Gamma(n + 3/2) \Gamma(\mu + 1/2)}. \quad (6)$$

γ_μ plays the role of generalized factorial. In fact, we have

$$\begin{aligned} \gamma_\mu(0) &= 1, & \gamma_\mu(1) &= 1 + 2\mu, & \gamma_\mu(2) &= (1 + 2\mu)2, \\ \gamma_\mu(3) &= (1 + 2\mu)2(3 + 2\mu), & \gamma_\mu(4) &= (1 + 2\mu)2(3 + 2\mu)4 \end{aligned}$$

and, generally, γ_μ satisfies the recursion relation:

$$\gamma_\mu(n + 1) = (n + 1 + 2\mu\theta_{n+1})\gamma_\mu(n), \quad n \in \mathbb{N},$$

where θ_n is defined to be 0 if $n \in 2\mathbb{N}$ and 1 if $n \in (2\mathbb{N} + 1)$.

That leads us to define the Dunkl-binomial coefficient by

$$\binom{n}{k}_\mu = \frac{\gamma_\mu(n)}{\gamma_\mu(k)\gamma_\mu(n-k)}.$$

Notice that $\gamma_0(n) = n!$ and

$$\binom{n}{k}_0 = \binom{n}{k}. \quad (7)$$

A Dunkl-analogue of the exponential function is defined by

$$e_\mu(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\gamma_\mu(n)}.$$

One easily verifies that

$$T_\mu e_\mu(xt) = t e_\mu(xt), \quad (8)$$

$$T_\mu x^n = \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} x^{n-1}. \quad (9)$$

Furthermore, the Dunkl operator satisfies the product rule

$$T_\mu(\phi\psi)(x) = \phi(x)T_\mu\psi(x) + \psi(-x)T_\mu\phi(x) + \phi'(x)(\psi(x) - \psi(-x)). \quad (10)$$

From which we deduce, if ψ is an even function, then

$$T_\mu(\phi\psi)(x) = \phi(x)T_\mu\psi(x) + \psi(x)T_\mu\phi(x). \quad (11)$$

With $\phi(x) = x$, equation (10) is reduced to

$$T_\mu(x\psi(x)) = xT_\mu\psi(x) + \psi(x) + 2\mu\psi(-x).$$

3. Dunkl–Appell polynomial sets

As an example of a Dunkl–Appell PSs, from equation (9), we mention that the PS $\{x^n\}_{n \geq 0}$ is a T_μ -Appell. Such PS is generated by

$$e_\mu(xt) = \sum_{n=0}^{\infty} \frac{x^n t^n}{\gamma_\mu(n)}.$$

This example may be used to characterize all T_μ -Appell PSs. We have, in fact, the following.

THEOREM 3.1 *Let $\{P_n\}_{n \geq 0}$ be a PS. The following assertions are equivalent.*

- (i) $\{P_n\}_{n \geq 0}$ is a Dunkl–Appell PS.
- (ii) There exists a sequence $(a_k)_{k \geq 0}$; independent of n ; $a_0 \neq 0$; such that

$$P_n(x) = \sum_{k=0}^n \binom{n}{k}_\mu a_{n-k} x^k.$$

- (iii) $\{P_n\}_{n \geq 0}$ is generated by

$$A(t)e_\mu(xt) = \sum_{n=0}^{+\infty} \frac{P_n(x)}{\gamma_\mu(n)} t^n,$$

where

$$A(t) = \sum_{k=0}^{+\infty} \frac{a_k}{\gamma_\mu(k)} t^k, \quad a_0 \neq 0.$$

Proof The implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are evident. Next we prove (i) $\Rightarrow$ (ii). Since $\{P_n\}_{n \geq 0}$ and $\{x^n\}_{n \geq 0}$ are two PSs, it is possible to write

$$P_n(x) = \sum_{k=0}^n c_{n,k} x^k, \quad n \in \mathbb{N}, \quad (12)$$

where the coefficients $c_{n,k}$ depend on n and k and $c_{n,0} \neq 0$. Apply the operator T_μ to each member of equation (12). Taking into account the fact that both $\{P_n\}_{n \geq 0}$ and $\{x^n\}_{n \geq 0}$ are Dunkl–Appell polynomials, we obtain

$$\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} P_{n-1}(x) = \sum_{k=0}^{n-1} c_{n,k+1} \frac{\gamma_\mu(k+1)}{\gamma_\mu(k)} x^k, \quad n \in \mathbb{N}^*. \quad (13)$$

Shift $n \rightarrow n+1$ in equation (13) and equating the coefficients of x^k to obtain

$$c_{n+1,k+1} = \frac{\gamma_\mu(n+1)\gamma_\mu(k)}{\gamma_\mu(n)\gamma_\mu(k+1)} c_{n,k}, \quad k \in \mathbb{N}_{n+1}.$$

Then $c_{n,k} = \binom{n}{k}_\mu c_{n-k,0} = \binom{n}{k}_\mu a_{n-k}$, where $a_{n-k} = c_{n-k,0}$. ■

4. Proof of the main result

In order to prove our main result, we need the following two Lemmas.

LEMMA 4.1 ([28]) *Let $\{P_n\}_{n \geq 0}$ be a PS generated by $G(x, t) = \sum_{n=0}^{\infty} P_n(x)t^n$ and let X be the multiplicative operator by x . Then, there exists a unique operator $\widehat{X}_t \in \mathcal{V}^{-1}$ such that $XG(x, t) = \widehat{X}_t G(x, t)$, where $\mathcal{V}^{-1}$ is the space of linear operators acting on analytic functions that reduce the order of every analytic function by exactly one.*

$\widehat{X}_t$ is called the transform operator of X according to $G(x, t)$.

LEMMA 4.2 ([13]) *Let $\{P_n\}_{n \geq 0}$ be a PS generated by*

$$G(x, t) = A(t)G_0(x, t) = \sum_{n=0}^{\infty} P_n(x)t^n$$

with $G_0(0, t) = 1$ and let $\widehat{X}$ be the transform of X according to $G(x, t)$. Then $\{P_n\}_{n \geq 0}$ is a d -OPS if and only if $\widehat{X}_t \in \mathcal{V}_{d+2}^{-1}$, where $\mathcal{V}_r^{-1}$, $r \geq 2$, is the space of linear operators τ such that there exist r complex sequences $(\alpha_n^{(k)})_{n \geq 0}$, $k \in \mathbb{N}_r$, satisfying

$$\left\{ \begin{array}{l} \alpha_n^{(0)} \alpha_n^{(r-1)} \neq 0, \quad n \geq 0, \\ \tau(1) = \sum_{k=1}^{r-1} \alpha_{k-1}^{(k)} t^{k-1}, \\ \tau(t^n) = \sum_{k=0}^{r-1} \alpha_n^{(k)} t^{n+k-1}, \quad n \geq 1. \end{array} \right.$$

Proof of Theorem 1.3 Let $\{P_n\}_{n \geq 0}$ be a PS generated by equation (2). Applying the operator T_μ to each member of equation (2) viewed as functions of the variable t , we obtain by virtue of equations (8) and (10)

$$\begin{aligned} T_\mu G(x, t) &= A(t)T_\mu e_\mu(xt) + A'(t)e_\mu(xt) + (T_\mu A(t) - A'(t))e_\mu(-xt) \\ &= xG(x, t) + \frac{A'(t)}{A(t)}G(x, t) + \frac{T_\mu A(t) - A'(t)}{A(-t)}G(x, -t). \\ &= xG(x, t) + \frac{A'(t)}{A(t)}G(x, t) + \mu \frac{A(t) - A(-t)}{tA(-t)}G(x, -t). \end{aligned} \quad (14)$$

On the other hand, according to the definition of the operator T_μ ,

$$G(x, -t) = \left[\frac{t}{\mu} \left(\frac{d}{dt} - T_\mu \right) + 1 \right] G(x, t). \quad (15)$$

Substituting equation (15) in equation (14):

$$xG(x, t) = \left[\frac{d}{dt} + \frac{\mu}{t} - \frac{A'(t)}{A(t)} + \frac{A(t)}{A(-t)} \left(T_\mu - \frac{d}{dt} - \frac{\mu}{t} \right) \right] G(x, t).$$

Therefore, the transform operator $\widehat{X}_t$ of X , according to $G(x, t)$, is given by

$$\widehat{X}_t = \frac{d}{dt} + \frac{\mu}{t} - \frac{A'(t)}{A(t)} + \frac{A(t)}{A(-t)} \left(T_\mu - \frac{d}{dt} - \frac{\mu}{t} \right).$$

Put

$$\frac{A'(t)}{A(t)} = \sum_{k=0}^{\infty} \beta_k t^k, \quad (16)$$

$$\frac{A(t)}{A(-t)} = \sum_{k=0}^{\infty} \lambda_k t^k. \quad (17)$$

Notice that $(T_\mu - (d/dt) - (\mu/t))(t^n) = \mu(-t)^{n-1}$, then

$$\hat{X}_t(t^n) = (n + \mu + (-1)^{n-1} \mu \lambda_0) t^{n-1} + \sum_{k=0}^{\infty} ((-1)^{n-1} \mu \lambda_{k+1} - \beta_k) t^{n+k}.$$

Since $\{P_n\}_{n \geq 0}$ is a d -OPS, according to Lemma 4.2, there exist $(d+2)$ complex sequences $(\alpha_n^{(k)})_{n \geq 0}$, $k \in \mathbb{N}_{d+2}$, such that: For $n \geq 1$

$$\begin{aligned} \hat{X}_t(t^n) &= (n + \mu(1 + (-1)^{n-1})) t^{n-1} + \sum_{k=0}^{\infty} ((-1)^{n-1} \mu \lambda_{k+1} - \beta_k) t^{n+k} \\ &= \sum_{k=-1}^d \alpha_n^{(k+1)} t^{n+k}, \end{aligned}$$

with $\alpha_n^{(0)} \alpha_n^{(d+1)} \neq 0$, $n \geq 0$.

Equating coefficients of t^{n+k} , one obtains

$$\begin{cases} \alpha_n^0 = n + \mu(1 + (-1)^{n-1}), \\ (-1)^{n-1} \mu \lambda_{k+1} - \beta_k = 0, \forall n \geq 1, \quad \forall k > d, \\ \alpha_n^{(d+1)} = (-1)^{n-1} \mu \lambda_{d+1} - \beta_d \quad \forall n \geq 1. \end{cases} \quad (18)$$

Next, we consider two cases according to the value of μ .

- If $\mu = 0$, we deduce that $A(t) = \exp(\pi_{d+1}(t))$, where π_{d+1} is a polynomial of degree $d+1$. We meet again the d -Appell polynomials [15].
- If $\mu \neq 0$ and $d \in (2\mathbb{N} + 1)$, from the identities (16) and (18), we deduce that $\beta_k = \lambda_{k+1} = 0 \quad \forall k > d$ and $A(t) = \exp(\pi_{d+1}(t))$, where π_{d+1} is a polynomial of degree at most $d+1$ and equation (17) becomes

$$\frac{A(t)}{A(-t)} = \sum_{k=0}^d \lambda_k t^k.$$

It follows that $\lambda_k = 0$, $\forall k \geq 1$ and π_{d+1} is even. The condition $\alpha_n^{(d+1)} = -\beta_d \neq 0$ shows that the degree of π_{d+1} is $d+1$.

Consequently, if d is an even integer and $\mu \neq 0$, there do not exist PSs, which are d -OPS and T_μ -Appell at the same time. ■

For $d = 1$, equation (2) reduces to the generating function of the *generalized Hermite* polynomials $H_n^\mu(x)$ given by [29]

$$\begin{cases} H_{2m}^\mu(x) = (-1)^m 2^{2m} m! L_m^{\mu-(1/2)}(x^2), \\ H_{2m+1}^\mu(x) = (-1)^m 2^{2m+1} m! x L_m^{\mu+(1/2)}(x^2). \end{cases} \quad (19)$$

The corresponding monic PS is a Dunkl–Appell OPS. Here L_n^α denotes the Laguerre polynomials given by

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{(1+\alpha)_k k!} x^k,$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol.

For $\mu = 0$, the polynomials given by equation (19) are reduced to Hermite polynomials.

5. A recurrence relation and a differential-difference equation

In this section, we state a recurrence relation and a differential-difference equation satisfied by a T_μ -Appell d -OPS, where (μ, d) verifies equation (3).

THEOREM 5.1 *Let $\{P_n\}_{n \geq 0}$ be a T_μ -Appell PS generated by equation (2).*

Put

$$\pi'_{d+1}(t) = \sum_{k=0}^d \beta_k t^k. \quad (20)$$

Then,

(i) *The PS $\{P_n\}_{n \geq 0}$ satisfies the $(d+1)$ -recurrence relation*

$$P_{n+1}(x) = x P_n(x) + \sum_{k=1}^d \beta_k \frac{\gamma_\mu(n)}{\gamma_\mu(n-k)} P_{n-k}(x), \quad (21)$$

where we have taken $P_{-n}(x) = 0$, $n \geq 1$.

(ii) *The polynomials P_n , $n \geq 0$, satisfy the differential-difference equation*

$$\left(\sum_{k=2}^{d+1} \beta_{k-1} T_\mu^k + x T_\mu - \frac{\gamma_\mu(n)}{\gamma_\mu(n-1)} \right) P_n(x) = 0. \quad (22)$$

Proof (i) Taking into account the fact that $A(t)$ is an even function, equations (11) and (8), the application of T_μ to the two members of equation (2), viewed as functions of the variable t , leads to

$$\sum_{n=0}^{+\infty} P_{n+1}(x) \frac{t^n}{\gamma_\mu(n)} = T_\mu(A(t)) e_\mu(xt) + x A(t) e_\mu(xt),$$

which, according to equation (20), may be written as

$$\sum_{n=0}^{+\infty} P_{n+1}(x) \frac{t^n}{\gamma_\mu(n)} = \sum_{k=0}^d \beta_k \sum_{n=0}^{+\infty} P_n(x) \frac{t^{n+k}}{\gamma_\mu(n)} + \sum_{n=0}^{+\infty} x P_n(x) \frac{t^n}{\gamma_\mu(n)}. \quad (23)$$

Finally, by comparing the coefficients of $t^n/\gamma_\mu(n)$ in equation (23), we obtain equation (21).

(ii) By shifting $n \rightarrow n - 1$ and multiplying both sides of equation (21) by $(\gamma_\mu(n))/(\gamma_\mu(n - 1))$, we can deduce that $\{P_n\}_{n \geq 2}$ satisfies equation (22) since $T_\mu^k P_n(x) = (\gamma_\mu(n))/(\gamma_\mu(n - k))P_{n-k}(x)$. ■

Two particular cases are worthy to note.

Case 1 $d = 1$. The corresponding monic PS is generated by

$$e^{-(t^2/4)} e_\mu(xt) = \sum_{n=0}^{+\infty} \hat{H}_n^\mu(x) \frac{t^n}{\gamma_\mu(n)}. \quad (24)$$

Indeed, replacing t by $2t$, we obtain the generating function of the generalized Hermite polynomials $\{H_n^\mu\}_{n \geq 0}$ defined in equation (19). The relation equation (21) (respectively equation (22)) becomes [25]

$$H_{n+1}^\mu(x) = 2xH_n^\mu(x) - 2\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)}H_{n-1}^\mu(x),$$

respectively,

$$T_\mu^2 H_n^\mu(x) - 2xT_\mu H_n^\mu(x) + 2\frac{\gamma_\mu(n)}{\gamma_\mu(n-1)}H_n^\mu(x) = 0.$$

Case 2 $\mu = 0$. We meet the He–Ricci result [30]. The Appell polynomials P_n , $n \geq 0$, generated by equation (2), satisfy the differential equation:

$$\left(\sum_{k=2}^{d+1} \beta_{k-1} D^k + xD - n \right) P_n(x) = 0.$$

6. d -Dimensional functionals

In this Section, we express explicitly the d -dimensional functional for which we have the d -orthogonality of the obtained d -OPSs. We state the following theorem by employing a method used in [31] for obtaining the dual sequence of a given PS.

THEOREM 6.1 *The PS $\{P_n\}_{n \geq 0}$, generated by equation (2) is a d -OPS with respect to the d -dimensional functional $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ given by*

$$\langle u_r, f \rangle = \frac{1}{\gamma_\mu(r)} C(T_\mu) T_\mu^r(f)(0), \quad f \in \mathcal{P}, \quad r \in \mathbb{N}_d, \quad (25)$$

where $C(t) = \exp(-\pi_{d+1}(t))$.

Proof Put $C(t) = \sum_{k=0}^{\infty} c_k t^k$. From the relation $e_{\mu}(xt) = C(t)G(x, t)$, we deduce that

$$\begin{aligned} x^n &= \sum_{k=0}^n c_k \frac{\gamma_{\mu}(n)}{\gamma_{\mu}(n-k)} P_{n-k}(x) = \sum_{k=0}^n c_k T_{\mu}^k P_n(x) \\ &= \left(\sum_{k=0}^{\infty} c_k T_{\mu}^k \right) P_n(x) = C(T_{\mu}) P_n(x). \end{aligned} \quad (26)$$

On the other hand, we have

$$[T_{\mu}^n x^m]_{x=0} = \gamma_{\mu}(n) \delta_{mn}. \quad (27)$$

The identities (26) and (27) allow us to express the dual sequence $\{u_n\}_{n \geq 0}$ of $\{P_n\}_{n \geq 0}$ as

$$\langle u_n, f \rangle = \frac{1}{\gamma_{\mu}(n)} C(T_{\mu}) T_{\mu}^n(f)(0), \quad f \in \mathcal{P}, \quad n \in \mathbb{N}. \quad \blacksquare$$

Next, we consider a particular case where $\pi_{d+1}(t) = at^{d+1}$. The corresponding d -OPS will be called Gould–Hopper type polynomials.

COROLLARY 6.2 *The moments of the d -dimensional functional $\mathcal{U}$ ensuring the d -orthogonality of the PS of the $\{P_n^{(d+1)}(\cdot; a, \mu)\}_{n \geq 0}$ generated by*

$$\exp(at^{d+1})e_{\mu}(xt) = \sum_{n=0}^{\infty} P_n^{(d+1)}(x; a, \mu) \frac{t^n}{\gamma_{\mu}(n)}, \quad (28)$$

are given by

$$\begin{cases} (u_0)_{m(d+1)+k} = 0, & k \in \mathbb{N}_{d+1}^* := \{1, \dots, d\}, \\ (u_0)_{m(d+1)} = (-a)^m \frac{(m(d+1))! \Gamma(m((d+1)/2) + \mu + (1/2)) \Gamma(1/2)}{m! \Gamma(\mu + (1/2)) \Gamma(((d+1)/2) + (1/2))}, \end{cases} \quad (29)$$

and for each (u_r) , $r \in \mathbb{N}_d^*$,

$$(u_r)_n = \binom{n}{r}_{\mu} (u_0)_{n-r}, \quad n \geq 0, \quad (30)$$

where we have taken $(u_0)_{-n} = 0$, $n \geq 1$.

Proof From equations (25) and (28), we deduce

$$(u_0)_n = \langle u_0, x^n \rangle = [C(T_{\mu})(x^n)]_{x=0} = \sum_{k=0}^{\infty} \left[\frac{(-a)^k T_{\mu}^{k(d+1)}}{k!} (x^n) \right]_{x=0}, \quad n \geq 0,$$

where $C(t) = \exp(-at^{d+1})$. It follows then

$$\begin{cases} (u_0)_{m(d+1)+k} = 0, & k \in \mathbb{N}_{d+1}^*, \\ (u_0)_{m(d+1)} = (-a)^m \frac{\gamma_{\mu}(m(d+1))}{m!}. \end{cases} \quad (31)$$

According to equation (5), we deduce that

$$(u_0)_{m(d+1)} = (-a)^m \frac{(m(d+1))! \Gamma(m((d+1)/2) + \mu + (1/2)) \Gamma(1/2)}{m! \Gamma(\mu + (1/2)) \Gamma(((d+1)/2) + (1/2))}.$$

For each $r \in \mathbb{N}_d^*$, we have, according to equation (25),

$$(u_r)_n = \langle u_r, x^n \rangle = \frac{1}{\gamma_\mu(r)} \langle u_r, T_\mu^r x^n \rangle = \frac{\gamma_\mu(n)}{\gamma_\mu(r) \gamma_\mu(n-r)} \langle u_0, x^{n-r} \rangle = \binom{n}{r}_\mu (u_0)_{n-r}.$$

■

Two particular cases are worthy to note.

Case 1 $\mu = 0$ and $a = -(1/(d!(d+1)^2))$. Since $\gamma_0(n) = n!$, the identity (31) becomes

$$\begin{cases} (u_0)_{m(d+1)+k} = 0, & k \in \mathbb{N}_{d+1}^*, \\ (u_0)_{m(d+1)} = \frac{(m(d+1))!}{m!(d!)^m (d+1)^{2m}}, \end{cases}$$

and according to equations (7), (30) becomes

$$(u_r)_n = \binom{n}{r} (u_0)_{n-r}, \quad r \in \mathbb{N}_d^*, \quad n \geq 0.$$

Notice that

$$\frac{(m(d+1))!}{m!(d!)^m (d+1)^m} = \prod_{j=0}^{m-1} \binom{(d+1)j+d}{d}$$

to meet the Douak result [15].

Case 2 $d = 1$. The PS $\{P_n^{(2)}(., a, \mu)\}_{n \geq 0}$ is reduced to the generalized Hermite polynomials generated by equation (24). According to equations (5) and (29), we have

$$\begin{cases} (u_0)_{2n+1} = 0, \\ (u_0)_{2n} = \frac{\Gamma(n + \mu + (1/2))}{\Gamma(\mu + (1/2))} = \frac{1}{\Gamma(\mu + (1/2))} \int_0^{+\infty} x^{2n+\mu} e^{-x^2} dx. \end{cases}$$

This is a Szegő and Chihara result [29, 32].

7. Components of Gould-Hopper type polynomials

Among the properties satisfied by the polynomials defined by equation (24), we mention the symmetry property. That is to say $H_n^\mu(-x) = (-1)^n H_n^\mu(x)$, which leads to the definition (19). This means that the sequences $\{H_{2m}^\mu\}_{m \geq 0}$ and $\{H_{2m+1}^\mu\}_{m \geq 0}$ may be expressed in terms of

Laguerre polynomials L_n^α , which are orthogonal and having hypergeometric representation of the type

$${}_1F_1\left(\begin{matrix} -n \\ \alpha + 1 \end{matrix}; x\right),$$

where the ${}_pF_q$, as usual, denotes the generalized hypergeometric functions defined by:

$${}_pF_q\left(\begin{matrix} \alpha_1, \dots, \alpha_p \\ \beta_1, \dots, \beta_q \end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m \dots (\alpha_p)_m}{(\beta_1)_m \dots (\beta_q)_m} \frac{z^m}{m!},$$

- p and q are positive integers or zero (interpreting an empty product as 1),
- z is a complex variable,
- the numerator parameters $\alpha_1, \dots, \alpha_p$ and the denominator parameters $\beta_1, \dots, \beta_q$ take in complex values. $\beta_j; j \in \mathbb{N}_{q+1}^*$: being non-negative integers.

In this section, we attempt to give analogous properties for the polynomials defined by the generating function (28). To this end, we recall the following.

For any arbitrary positive integer n , we denote by $w_n = \exp(2i\pi/n)$ the complex n -th root of unity.

Let f be a function of the complex variable z holomorphic in a circular neighborhood I in the origin, there exists a unique sequence $(f_{[n,k]})_{k \in \mathbb{N}_n}$, such that [33, 34]

$$f = \sum_{k=0}^{n-1} f_{[n,k]}$$

and

$$f_{[n,k]}(z) := \Pi_{n,k} f(z) = \frac{1}{n} \sum_{l=0}^{n-1} w_n^{-kl} f(w_n^l z), \quad k \in \mathbb{N}_n, \quad z \in I,$$

where $\Pi_{[n,k]}$ is the projection operator.

The functions $f_{[n,k]}$ are called the components of f . It is clear that $f_{[n,k]}(w_n z) = w_n^k f_{[n,k]}(z)$. For $n = 2$, $f_{[2,0]}$ and $f_{[2,1]}$ are, respectively, the even and odd parts of the function f . A useful example is given by the following.

LEMMA 7.1 *If n is an even integer, the components $h_{n,k}^{(\mu)}$, $k \in \mathbb{N}_n$, of e_μ are given by*

$$h_{n,k}^{(\mu)}(x) = \frac{x^k}{\gamma_\mu(k)} {}_0F_{n-1}\left(\begin{matrix} - \\ \Delta_\mu^*(n, k+1) \end{matrix}, \frac{x^n}{n^n}\right) \quad (32)$$

where, for the sake of brevity, the symbol $\Delta_\mu^*(n, l)$ abbreviates the set of $(n-1)$ parameters

$$\left\{ \frac{l+j+2\mu\theta_{l+j}}{n}; j \in \mathbb{N}_n \right\} \setminus \left\{ \frac{n}{n} \right\}.$$

Proof For $n \in \mathbb{N}^*$, the components of e_μ are defined by

$$h_{n,k}^{(\mu)}(x) = \sum_{m=0}^{\infty} \frac{x^{nm+k}}{\gamma_\mu(nm+k)}, \quad k \in \mathbb{N}_n.$$

Notice that

$$\begin{aligned} \gamma_\mu(nm+k) &= \prod_{j=1}^{nm+k} (j + 2\mu\theta_j) \\ &= n^{nm} \gamma_\mu(k) \prod_{j=1}^{nm} \left(\frac{j+k+2\mu\theta_{j+k}}{n} \right) \\ &= n^{nm} \gamma_\mu(k) \prod_{l=1}^n \prod_{j=0}^{m-1} \left(\frac{k+l+jn+2\mu\theta_{k+l+jn}}{n} \right). \end{aligned}$$

Taking into account that, for $j \in \mathbb{N}_m$, jn is even, then

$$\begin{aligned} \gamma_\mu(nm+k) &= n^{nm} \gamma_\mu(k) \prod_{l=1}^n \prod_{j=0}^{m-1} \left(j + \frac{k+l+2\mu\theta_{k+l}}{n} \right) \\ &= n^{nm} \gamma_\mu(k) \prod_{l=1}^n \left(\frac{k+l+2\mu\theta_{k+l}}{n} \right)_m \\ &= n^{nm} \gamma_\mu(k) m! \prod_{l=1, l+k \neq n}^n \left(\frac{k+l+2\mu\theta_{k+l}}{n} \right)_m \end{aligned}$$

which leads to equation (32). ■

DEFINITION 7.2 ([35], p. 83) A PS $\{P_m\}_{m \geq 0}$ is called n -symmetric if it fulfils for all $m \in \mathbb{N}$, $\deg P_m = m$, $P_m(w_n x) = w_n^m P_m(x)$.

LEMMA 7.3 ([35], p. 83) A sequence $\{P_m\}_{m \geq 0}$ is n -symmetric if and only if there exist n sequences $\{P_m^k\}_{m \geq 0}$, $k \in \mathbb{N}_n$, such that $P_{nm+k}(x) = x^k P_m^k(x^n)$, $m \in \mathbb{N}$, $k \in \mathbb{N}_n$.

LEMMA 7.4 ([35], p. 85) Let $\{P_m\}_{m \geq 0}$ be a $(d+1)$ -symmetric sequence. If $\{P_m\}_{m \geq 0}$ is d -OPS then $\{P_m^k\}_{m \geq 0}$ is d -OPS for all k , $k \in \mathbb{N}_{d+1}$.

Next, we state the main result of this section.

THEOREM 7.5 Let $\{P_m^{(d+1)}(\cdot; a, \mu)\}_{m \geq 0}$ be a Gould–Hopper type d -OPS generated by equation (28). Then

- (i) $\{P_m^{(d+1)}(\cdot; a, \mu)\}_{m \geq 0}$ is $(d+1)$ -symmetric.
- (ii) $P_{(d+1)m+k}^{(d+1)}(x; a, \mu) = \gamma_\mu((d+1)m+k) x^k P_m^{(d+1,k)}(x^{d+1}; a, \mu), \quad k \in \mathbb{N}_{d+1}, \quad (33)$

where

$$P_m^{(d+1,k)}(x; a, \mu) = \frac{a^m}{m! \gamma_\mu(k)} {}_1F_d \left(\begin{matrix} -m \\ \Delta_\mu^*(d+1, k+1) \end{matrix}, \frac{-x}{a(d+1)^{(d+1)}} \right). \quad (34)$$

Moreover, the families $\{\gamma_\mu((d+1)m+k)P_m^{(d+1,k)}(x; a, \mu)\}_{m \geq 0}$, $k \in \mathbb{N}_{d+1}$, are d -orthogonal.

The polynomials defined by equation (34) generalize the Laguerre ones $L_m^{(\mu-(1/2))}$ and $L_m^{(\mu+(1/2))}$. In fact, we have

$$\begin{cases} P_m^{(2,0)}(x; a, \mu) = \frac{a^m}{(\mu+1/2)_m} L_m^{(\mu-(1/2))} \left(-\frac{x}{4a} \right), \\ P_m^{(2,1)}(x; a, \mu) = \frac{a^m}{(1+2\mu)(\mu+3/2)_m} L_m^{(\mu+(1/2))} \left(-\frac{x}{4a} \right). \end{cases}$$

With $d=1$ and $a=-1$, the identity (33) is reduced to the classical relation between generalized Hermite polynomials and Laguerre polynomials equation (19).

Proof of Theorem 7.5 (i) Since the generating function $G(x, t) = \exp(at^{d+1})e_\mu(xt)$ satisfies $G(w_{d+1}x, t) = G(x, w_{d+1}t)$, the PS $\{P_n^{(d+1)}(.; a, \mu)\}_{n \geq 0}$ is $(d+1)$ -symmetric.

(ii) From (i) and Lemma 7.3, we deduce that there exist $(d+1)$ sequences $\{P_m^{(d+1,k)}(.; a, \mu)\}_{m \geq 0}$, $k \in \mathbb{N}_{d+1}$, such that

$$P_{nm+k}^{(d+1)}(x; a, \mu) = \gamma_\mu((d+1)m+k)x^k P_m^{(d+1,k)}(x^n; a, \mu), \quad m \in \mathbb{N}, \quad k \in \mathbb{N}_{d+1}.$$

Applying the projection operator $\Pi_{[d+1,k]}$ on both sides of the identity (28), viewed as functions of the variable t and taking into account of equations (32) and (33), we obtain

$$\begin{aligned} [\exp(at^{d+1})e_\mu(xt)]_{[d+1,k]} &= \frac{(xt)^k}{\gamma_\mu(k)} \exp(at^{d+1}) {}_0F_d \left(\begin{matrix} - \\ \Delta_\mu^*(d+1, k+1) \end{matrix}, \left(\frac{xt}{d+1} \right)^{d+1} \right) \\ &= \sum_{m=0}^{\infty} P_{(d+1)m+k}^{(d+1)}(x; a, \mu) \frac{t^{(d+1)m+k}}{\gamma_\mu((d+1)m+k)} \\ &= (xt)^k \sum_{m=0}^{\infty} P_m^{(d+1,k)}(x^{d+1}; a, \mu) t^{(d+1)m}. \end{aligned}$$

Then, the polynomials $P_m^{(d+1,k)}$ are generated by

$$\frac{1}{\gamma_\mu(k)} \exp(at) {}_0F_d \left(\begin{matrix} - \\ \Delta_\mu^*(d+1, k+1) \end{matrix}, \frac{xt}{(d+1)^{d+1}} \right) = \sum_{m=0}^{\infty} P_m^{(d+1,k)}(x; a, \mu) t^m.$$

Now, by using the Brafman identity [36]

$$e^t {}_pF_q \left(\begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix}, -xt \right) = \sum_{m=0}^{\infty} \frac{t^m}{m!} {}_{p+1}F_q \left(\begin{matrix} -m & (\alpha_p) \\ (\beta_q) \end{matrix}, x \right),$$

we obtain an hypergeometric representation of the polynomials $P_m^{(d+1,k)}$:

$$P_m^{(d+1,k)}(x; a, \mu) = \frac{a^m}{m! \gamma_\mu(k)} {}_1F_d \left(\begin{matrix} -m \\ \Delta_\mu^*(d+1, k+1) \end{matrix}, \frac{-x}{a(d+1)^{(d+1)}} \right).$$

The d -orthogonality of the polynomials defined by equation (34) may be deduced from Lemma 7.4 or from two earlier works of Ben Cheikh and Douak [11, 16]. ■

Remark 7.6 The PS generated by equation (28) are the only $(d + 1)$ -symmetric T_μ -Appell PS and d -orthogonal where (μ, d) verifies equation (3).

8. Generalized Hermite polynomials and some orthogonality notions

The literature on generalizations of Hermite polynomials contains several hundreds of references. But only a few ones have property related to an orthogonality notion. Let us give an overview of them to situate the obtained polynomials in this paper.

- The generalization of Szegő–Chihara ([29], p. 380; [32], p. 157) given, for $\mu > -1/2$, by equation (19).

The set $\{H_n^\mu\}_{n \geq 0}$ is an OPS with respect to the weight function

$$|x|^{2\mu} e^{-x^2}; \quad -\infty < x < +\infty.$$

For $\mu = 0$, these polynomials are reduced to Hermite polynomials.

- There are several q -analogues of Hermite PS: q -extensions with the continuous orthogonality property on $\mathbb{R}$ [37], continuous big q -Hermite, continuous q -Hermite, Al Salam–Carlitz I and II, discrete q -Hermite I and II and Stieljes–Wigert (see, for instance, [38–40]).
- The generalization given by Ben Cheikh ([41], p. 215)

$$\Phi_{nm+k}^\alpha(x) = (-1)^m m! x^k L_m^{\alpha+2k/n}(x^n), \quad m \geq 0, \quad k \in \mathbb{N}_n$$

orthogonal on the star $\{r \exp(2ik\pi/n), \quad r \geq 0, \quad k \in \mathbb{N}_n\}$ (OPS/ $\star$, for shorter) for the weight function $w(x) = |x|^{\alpha n + n - 1} \exp(-x)^n$.

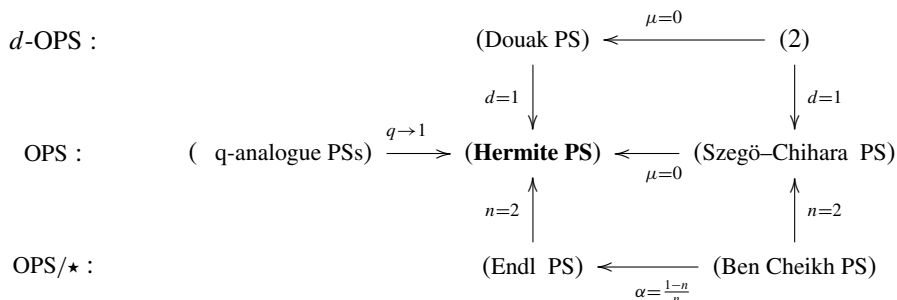
- (i) If $\alpha = (1 - n)/n$, we obtain the Endl PS [42], if moreover $n = 2$, we obtain the Hermite PS.
- (ii) If $n = 2$, we obtain the generalized Hermite PS of Szegő–Chihara.
- The Douak d -OPS generated by [15]

$$\exp(xt - \pi_{d+1}(t)) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!},$$

where π_{d+1} is a polynomial of degree $d + 1$.

If $\pi_{d+1}(t) = at^{d+1}$, these polynomials are reduced to the Gould–Hopper PS and for $d = 1$ they are reduced to Hermite PS.

The link between the aforementioned PSs and the obtained polynomials in this paper may be summarized by the following scheme.



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Some remarks on compact Weyl operators

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We show that, if $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is a tempered distribution and, for some $1 < p, q < \infty$, the Weyl operator L_σ acts as a compact operator $L_\sigma : M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$, then the short time Fourier transform of σ vanishes at infinity. Some results on the eigenvalue distributions of the operators are included, as well as an example of a non-zero function with constant modulus, which is the symbol of a compact localization operator.

Keywords: Localization operators; Weyl operators; Compact operators

AMS Subject Classification: Primary: 47G30; Secondary: 47B07, 47B10

1. Introduction

In 1988, Daubechies [1] introduced localization operators to localize a signal both in time and frequency, although a particular case was already considered by Berezin in 1971. They are also known in some cases as Toeplitz operators or anti-Wick operators. Given a signal f , its short time Fourier transform (STFT) with respect to the window φ is defined by

$$V_\varphi f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t-x)} e^{-2\pi i \omega t} dt.$$

The signal f can be reconstructed from its STFT by the formula

$$f(t) = \int_{\mathbb{R}^{2d}} V_\varphi f(x, \omega) \overline{\psi(t-x)} e^{2\pi i \omega t} dx d\omega,$$

provided that $\langle \varphi, \psi \rangle = 1$. It is often convenient, before reconstructing the signal, to modify $V_\varphi f$ by multiplying it by a suitable function F . In this way, what we recover is a filtered

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version of the original signal f ,

$$L_{\varphi, \psi}^F f(t) := \int_{\mathbb{R}^{2d}} F(x, \omega) V_{\varphi} f(x, \omega) e^{2\pi i t \omega} \psi(t - x) dx d\omega.$$

The modulation spaces $M^{p,q}$ ($1 \leq p, q \leq \infty$) were introduced by Feichtinger as appropriate classes of functions or distributions to discuss certain problems related to time-frequency analysis. For $p = q = 2$, we get $M^{p,q} = L^2$ and their appropriate weighted versions of the modulation spaces produce Sobolev spaces.

In [2], we characterized the symbols F in the modulation class $M^{\infty}(\mathbb{R}^{2d})$ with the property that the localization operator with symbol F and windows φ, ψ , denoted by $L_{\varphi, \psi}^F$, is a compact operator when acting on $L^2(\mathbb{R}^d)$ for every pair of windows φ, ψ in the Schwartz class. This characterization was expressed in terms of the behavior at infinity of the STFT. On the other hand, several authors provided sufficient conditions on the symbol F in order to guarantee the compactness of the corresponding localization operator when acting on some modulation class or some L^p space. We mention, for instance, the work by Boggiatto [3], who proved that each $F \in L^{\infty}(\mathbb{R}^{2d})$ vanishing at infinity defines a compact localization operator on $M^{p,q}(\mathbb{R}^d)$.

We first present an example of a compact localization operator whose symbol F is a bounded function with constant modulus, thus not vanishing at infinity. This complements the above-mentioned result by Boggiatto and emphasizes the fact that the right condition to get compactness on modulation spaces has to be expressed in terms of the behavior at infinity of the STFT.

Localization operators, although of independent interest, can be represented as pseudodifferential operators. Maybe the earliest form of pseudodifferential operators arises in quantum mechanics. In that form they were introduced by Weyl around 1931 as a rule that associates a self-adjoint operator L_{σ} with a real valued symbol $\sigma(x, \xi)$ defined on the space $\mathbb{R}_x^d \times \mathbb{R}_{\xi}^d$. Given a square-summable function $\sigma \in L^2(\mathbb{R}^{2d})$, the Weyl operator with symbol σ is the map $L_{\sigma}: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ given by

$$(L_{\sigma} f)(x) = \int_{\mathbb{R}^{2d}} \sigma\left(\frac{x+y}{2}, \xi\right) f(y) e^{2\pi i (x-y) \cdot \xi} dy d\xi.$$

Connected with Weyl operators is the Wigner transform of a pair of functions $f, g \in L^2(\mathbb{R}^d)$, which is defined as

$$W(f, g)(x, \xi) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i \xi t} dt.$$

In fact, $\langle L_{\sigma} f, g \rangle = \langle \sigma, W(g, f) \rangle$ for any $f, g \in L^2(\mathbb{R}^d)$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$. Since $W(f, g) \in \mathcal{S}(\mathbb{R}^{2d})$ whenever $f, g \in \mathcal{S}(\mathbb{R}^d)$, the previous identity permits to define the Weyl operator $L_{\sigma}: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ for tempered distributions $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.

In 1994/95 Sjöstrand [4] showed that every pseudodifferential operator L_{σ} associated with a symbol σ satisfying that, for some non-zero $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, the function

$$\xi \mapsto \sup_{x \in \mathbb{R}^{2d}} |\mathcal{F}((\sigma_t \cdot \varphi(x - t))(\xi))|$$

is an integrable function in $\mathbb{R}^{2d}$, can be extended as a continuous operator $L_{\sigma}: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$. This class of symbols, known as Sjöstrand class, coincides with the modulation class $M^{\infty,1}(\mathbb{R}^{2d})$, and it contains the Hörmander class $S_{0,0}^0$. Hence, the above-mentioned result about L^2 -boundedness of pseudodifferential operators extends the

Calderón–Vaillancourt theorem. Sjöstrand also showed that $M^{\infty,1}(\mathbb{R}^{2d})$ is a Banach algebra of pseudodifferential operators and that it satisfies the Wiener property (see also [5]).

Continuity of Weyl operators on modulation classes has been studied by Gröchenig, Heil, Toft, Czaja and Labate among others [20], [21]. A very important tool is the study of the continuity of the Wigner transform between different classes of modulation spaces (see [6]). In particular, if $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and $1 \leq p, q \leq \infty$, then the operator $L_\sigma: M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$ is bounded ([7, 14.5.2]) and, if σ can be approximated in $M^{\infty,1}(\mathbb{R}^{2d})$ by functions in the Schwartz class, then L_σ is compact as an operator from $M^{p,q}(\mathbb{R}^d)$ into itself (see for instance [8]).

In this paper, we prove that, if $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and, for some $1 < p, q < \infty$, the Weyl operator L_σ acts as a compact operator $L_\sigma: M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$, then the STFT of σ vanishes at infinity, that is, $\sigma \in M^\infty(\mathbb{R}^{2d})$ and it is the limit in $M^\infty(\mathbb{R}^{2d})$ of a sequence of functions in the Schwartz class. This result is no longer true for p or q equal 1 or ∞ . However, restricting ourselves to symbols in the Sjöstrand class $M^{\infty,1}(\mathbb{R}^{2d})$, we can consider arbitrary values of p and q . The Weyl operators with symbol in the Sjöstrand class can be considered as a small subclass of the class of all bounded operators on $L^2(\mathbb{R}^d)$. Nevertheless, we consider that it can be useful to have at our disposal a characterization of compactness within this class of operators, at least for the following reasons: (i) according to [9, Theorems 5, 6], $M^{\infty,1}(\mathbb{R}^{2d})$ is the only modulation class $M^{p,q}$ with the property that every $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ defines a bounded operator $L_\sigma: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, but there are distributions $\sigma \in M^{p,q}(\mathbb{R}^{2d})$ such that L_σ is not compact; (ii) after the work of Cordero and Gröchenig [10] M^∞ is, in a sense, the ‘optimal’ class for L^2 -boundedness of localization operators, and all this operators can be represented as Weyl operators with symbols in the Sjöstrand class.

Some results on the eigenvalues distribution of the operators are also included, extending previous results by Labate [11].

2. Notations and preliminaries

The modulation and translation operators are defined by

$$M_\omega f(t) = e^{2\pi i \omega t} f(t) \quad \text{and} \quad T_x f(t) = f(t - x).$$

For a non-zero $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$, the STFT of f with respect to the window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is given by

$$V_\varphi f(x, \omega) = \langle f, M_\omega T_x \varphi \rangle,$$

where the bracket denotes the extension of the inner product in $L^2(\mathbb{R}^d)$.

Clearly, $V_\varphi f$ is a continuous function on $\mathbb{R}^{2d}$. If f belongs to $\mathcal{S}(\mathbb{R}^d)$, then $V_\varphi f \in \mathcal{S}(\mathbb{R}^{2d})$.

The modulation spaces are defined as follows. Given a non-zero window $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f \in L^{p,q}(\mathbb{R}^{2d})$, and it is endowed with the norm

$$\|f\|_{M^{p,q}} := \|V_\varphi f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\varphi f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/q}.$$

For $p = q$, we simply write M^p .

$M^{p,q}(\mathbb{R}^d)$ is a Banach space and its definition is independent of the choice of the window φ . Different windows give equivalent norms. M^2 is exactly L^2 , and weighted versions produce,

among others, the Sobolev spaces. In case $p, q < \infty$ the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{p,q}(\mathbb{R}^d)$ and its dual can be identified with $M^{p',q'}(\mathbb{R}^d)$ for $1/p + 1/p' = 1$, $1/q + 1/q' = 1$. We refer to [12] for the properties of modulation spaces.

$M^{0,0}(\mathbb{R}^d)$ or simply $M^0(\mathbb{R}^d)$ denotes the closed subspace of $M^\infty(\mathbb{R}^d)$ consisting of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_\varphi f$ vanishes at infinity. The Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^0(\mathbb{R}^d)$ and $(M^{0,0}(\mathbb{R}^d))' = M^{1,1}(\mathbb{R}^d)$. By $M^{0,1}(\mathbb{R}^d)$, we denote the closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^{\infty,1}(\mathbb{R}^d)$. It is easy to see that $M^{0,1}(\mathbb{R}^d) := M^{0,0}(\mathbb{R}^d) \cap M^{\infty,1}(\mathbb{R}^d)$.

Given $F \in \mathcal{S}(\mathbb{R}^{2d})$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ ($\varphi, \psi \neq 0$), the localization operator $L_{\varphi,\psi}^F$ fulfills

$$\langle L_{\varphi,\psi}^F(g), f \rangle = \langle F, V_\varphi \overline{V_\psi f} \rangle$$

for arbitrary $f, g \in \mathcal{S}(\mathbb{R}^d)$. This identity permits one to consider localization operators defined by symbols $F \in \mathcal{S}'(\mathbb{R}^{2d})$, as continuous linear operators $L_{\varphi,\psi}^F : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$. Localization operators are special pseudodifferential operators. In fact, $L_{\varphi,\psi}^F$ can be interpreted as a Weyl operator L_σ with Weyl symbol $\sigma = F * W(\psi, \varphi)$ [13]

3. An example

It was shown by Boggiatto [3] that each $F \in L^\infty(\mathbb{R}^{2d})$ vanishing at infinity defines a compact localization operator on $M^{p,q}(\mathbb{R}^d)$. The point is that the assumption on the symbol F implies that it can be uniformly approximated by functions in the Schwartz class. Then, it is easy to see that the corresponding localization operator is the uniform limit of a sequence of operators, each of them admitting a continuous linear extension from $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. Compact localization operators on $L^2(\mathbb{R}^d)$ defined by symbols in the modulation class $M^\infty(\mathbb{R}^{2d})$ were characterized in [2] in terms of the behavior at infinity of the STFT of the symbol. In the last section, we will see that this characterization still holds when $L^2(\mathbb{R}^d)$ is replaced by an arbitrary modulation class $M^{p,q}(\mathbb{R}^d)$. Since the condition obtain in [2] works when the symbol of the localization operator is a tempered distribution in M^∞ , it was unclear whether it could be equivalent, when the symbol F is a measurable and essentially bounded function, to the vanishing of F , or at least of its STFT, at infinity. The example we are going to present shows that this is not the case.

THEOREM 3.1 *There is a non-zero function $F \in L^\infty(\mathbb{R}^2)$ with constant modulus and with the property that the localization operator*

$$L_{\varphi,\psi}^F : M^{p,q}(\mathbb{R}) \rightarrow M^{p,q}(\mathbb{R})$$

is compact for every $1 \leq p, q \leq \infty$ and for each pair of windows φ, ψ in the Schwartz class.

Proof Take $f(x) := \exp(i\pi x^2)$. Then, according to [13, pp. 247–248], $f * \mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ and the map

$$\mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}), \quad g \longrightarrow f * g$$

is continuous. Given a real-valued and even function $g \in \mathcal{S}(\mathbb{R})$ and $x, \omega \in \mathbb{R}$, it is clear that

$$V_g f(x, \omega) = \exp(-2\pi i x \omega) (f * M_\omega g)(x).$$

Then, considered as a function of x , it is a function in the Schwartz class, and therefore, for each ω we have $\lim_{x \rightarrow \infty} V_g(x, \omega) = 0$. Since for each $R > 0$ the set $\{M_\omega g : |\omega| \leq R\}$ is

bounded in $\mathcal{S}(\mathbb{R})$, then $\{\exp(-2\pi i x \omega)(f * M_\omega g) : |\omega| \leq R\}$ is also bounded in $\mathcal{S}(\mathbb{R})$. Hence,

$$\sup_{x \in \mathbb{R}, |\omega| \leq R} (1 + x^2) |V_g f(x, \omega)| < \infty$$

from where $\lim_{|x| \rightarrow \infty} \sup_{|\omega| \leq R} |V_g f(x, \omega)| = 0$.

Now, we take $F(x, y) = f(x)f(y)$ and $g_0(x, y) = g(x)g(y)$. Then we have that $V_{g_0} F(x, y, \xi, \omega) = V_g(x, \xi) V_g f(y, \omega)$, thus

$$\lim_{|(x,y)| \rightarrow \infty} \sup_{|(\xi, \omega)| \leq R} |V_{g_0} F(x, y, \xi, \omega)| = 0.$$

It is easy to see that $V_{g_0} F$ does not vanish at infinity. Fix two windows $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ and consider the localization operator $L_{\varphi, \psi}^F$ that coincides with the Weyl operator L_σ for $\sigma = F * W(\psi, \varphi)$. Now, using [2, 3.4], we conclude $\sigma \in M^{0,1}(\mathbb{R}^{2d})$, hence by [8, 2.3], $L_{\varphi, \psi}^F$ is a compact localization operator on $M^{p,q}(\mathbb{R})$, for each pair of windows in $\mathcal{S}(\mathbb{R})$ and every $1 \leq p, q \leq \infty$. However, F does not vanish at infinity since $|F|$ is constant equal 1. ■

The previous result emphasizes the fact that the right condition to get compactness has to be expressed in terms of the STFT of the symbol. Since every localization operator is a Weyl operator, we plan to analyse whether it is possible or not to extend the results in [2] to Weyl operators. As we will see in the next section, a general necessary condition for compactness can be obtained. However, to get a complete characterization that we need to restrict our attention to Weyl operators with symbols in the Sjöstrand class.

4. Compact Weyl operators

The next result improves [11, 3.1(b–c)] and it is related with [6, 4.3].

PROPOSITION 4.1 *Let $1 < p \leq 2$ and $\sigma \in M^p(\mathbb{R}^{2d})$ be given. Then the Weyl operator $L_\sigma: M^{p'}(\mathbb{R}^d) \rightarrow M^p(\mathbb{R}^d)$ is compact. In particular, the operators $L_\sigma: M^p(\mathbb{R}^d) \rightarrow M^p(\mathbb{R}^d)$ and $L_\sigma: M^{p'}(\mathbb{R}^d) \rightarrow M^{p'}(\mathbb{R}^d)$ have the same eigenvectors that the operator $L_\sigma: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ and their eigenvalues are in ℓ^p .*

Proof Let us consider the linear map

$$T: \mathcal{S}(\mathbb{R}^{2d}) \longrightarrow L(M^{p'}(\mathbb{R}^d), M^p(\mathbb{R}^d)), \quad \sigma \longmapsto L_\sigma.$$

Since,

$$\|L_\sigma f\|_{M^p} = \sup_{\|g\|_{M^{p'}} \leq 1} |\langle \sigma, W(g, f) \rangle|$$

and

$$\|W(g, f)\|_{M^{p'}} \leq C \|g\|_{M^{p'}} \|f\|_{M^{p'}}$$

(see for instance [9, page 6]), then

$$\|L_\sigma\| \leq C \|\sigma\|_{M^p} (*)$$

for every $f \in \mathcal{S}(\mathbb{R}^{2d})$. Consequently, T admits a continuous and linear extension

$$\tilde{T}: M^p(\mathbb{R}^d) \longrightarrow L(M^{p'}(\mathbb{R}^d), M^p(\mathbb{R}^d)).$$

It is now easy to prove that $\tilde{T}(\sigma) = L_\sigma$ for all $\sigma \in M^p(\mathbb{R}^{2d})$. In particular, $L_\sigma \in L(M^{p'}(\mathbb{R}^d), M^p(\mathbb{R}^d))$. Moreover, since the Schwartz class is dense in $M^p(\mathbb{R}^{2d})$ and every

symbol in the Schwartz class defines a compact operator, the compactness of $L_\sigma: M^{p'}(\mathbb{R}^d) \rightarrow M^p(\mathbb{R}^d)$ follows from (*). Since $M^p(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset M^{p'}(\mathbb{R}^d)$, the statement about the eigenvectors is immediate. By [9, Theorem 5], $L_\sigma: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is an operator in the Schatten class S_p , hence the sequence of eigenvalues is in l^p , according to [14, 16.31]. ■

In the case that $\sigma \in M^1(\mathbb{R}^{2d})$ something more can be said.

Given two Banach spaces E and F , we denote by $\mathcal{N}(E, F)$ the Banach space of all nuclear operators from E to F . We refer to [15, chapters 17, 18–19] for the background on nuclear operators, compact operators and related topics. We recall that in case $E = F$ is a Hilbert space, the nuclear operators from E to F coincide with the trace-class operators.

Proceeding as in [9, 4.1] or [6, 4.3] (see also [16, 17]), it is easy to prove the following.

THEOREM 4.2 *Let $\sigma \in M^1(\mathbb{R}^{2d})$ be given. Then*

$$L_\sigma: M^\infty(\mathbb{R}^d) \rightarrow M^1(\mathbb{R}^d)$$

is a nuclear operator.

As a consequence it is possible to improve [11, 3.1(a)].

COROLLARY 4.3 *Let $\sigma \in M^1(\mathbb{R}^{2d})$ be given and $1 \leq p, q \leq \infty$. Then*

$$L_\sigma: M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$$

is a nuclear (hence compact) operator and the eigenvalues of L_σ are summable with

$$\sum_{n=1}^{\infty} |\lambda_n| \leq C \|\sigma\|_{M^1}.$$

Proof Since $M^1(\mathbb{R}^d) \subset M^{p,q}(\mathbb{R}^d) \subset M^\infty(\mathbb{R}^d)$ with continuous inclusions and nuclear operators are an operator ideal, the nuclearity follows from the previous result. On the other hand, $L_\sigma(M^{p,q}(\mathbb{R}^d))$ is contained in $L^2(\mathbb{R}^d)$, thus every eigenvalue of $L_\sigma: M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$ is also an eigenvalue of the nuclear operator $L_\sigma: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$. Now the conclusion follows from [14, 16.31]. ■

Now, we restrict ourselves to the case $1 < p, q < \infty$. Therefore, $M^{p,q}$ is reflexive and has an unconditional basis [7, 12.3.7]. In particular, both, the space and its dual, have the approximation property. Therefore, the trace can be extended from the space of finite rank operators on $M^{p,q}(\mathbb{R}^d)$ to a unique continuous linear form on $\mathcal{N}(M^{p,q}(\mathbb{R}^d))$, which we again call trace. Moreover, according to [15, 19.10; Remark 7], the dual of $\mathcal{N}(M^{p,q}(\mathbb{R}^d))$ can be identified with $L(M^{p,q}(\mathbb{R}^d))$ by means of the linear isomorphism

$$L(M^{p,q}(\mathbb{R}^d)) \longrightarrow (\mathcal{N}(M^{p,q}(\mathbb{R}^d)))', \quad R \longmapsto (T \longmapsto \text{Trace}(T \circ R)).$$

Our next result gives a necessary condition for compactness of Weyl operators.

THEOREM 4.4 *Let $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ be given and let us assume that, for some $1 < p, q < \infty$, L_σ can be extended as a compact operator*

$$L_\sigma: M^{p,q}(\mathbb{R}^d) \longrightarrow M^{p,q}(\mathbb{R}^d).$$

Then $\sigma \in M^0(\mathbb{R}^{2d})$.

Proof We consider the following operator

$$T: M^1(\mathbb{R}^{2d}) \longrightarrow \mathcal{N}(M^{p,q}(\mathbb{R}^d)), \quad \sigma \longrightarrow L_\sigma.$$

Its transpose $T^t: L((M^{p,q}(\mathbb{R}^d))) \rightarrow M^\infty(\mathbb{R}^{2d})$ assigns a symbol to each operator. We first show that $T^t(L_\sigma) = \sigma$. To do this, we take $f, g \in \mathcal{S}(\mathbb{R}^d)$ and we evaluate

$$\langle T^t(L_\sigma), W(f, g) \rangle = \text{Trace}(L_\sigma \circ \langle \cdot, f \rangle g) = \langle L_\sigma g, f \rangle = \langle \sigma, W(f, g) \rangle.$$

If we take $f \neq 0$, then $W(f, f) \neq 0$, hence the set of all its time-frequency shifts is dense in $\mathcal{S}(\mathbb{R}^{2d})$ and every shift of $W(f, f)$ can be written, up to a constant, as a Wigner transform of functions in the Schwartz class [7, 4.3.2]. We conclude that $\langle T^t(L_\sigma), F \rangle = \langle \sigma, F \rangle$ for every $F \in \mathcal{S}(\mathbb{R}^{2d})$. As both $T^t(L_\sigma)$ and σ are tempered distributions, we finally obtain $T^t(L_\sigma) = \sigma$. We are now ready to prove that $\sigma \in M^0(\mathbb{R}^{2d})$. In fact, compact operators are the closure in the operator norm of finite rank operators because $M^{p,q}(\mathbb{R}^d)$ has the approximation property. Since $M^0(\mathbb{R}^{2d})$ is closed in $M^\infty(\mathbb{R}^{2d})$, it is enough to prove that $L^t(S) \in M^0(\mathbb{R}^{2d})$ for every finite rank operator $S = \sum_{k=1}^N \langle \cdot, g_k \rangle f_k$ with $f_k \in M^{p,q}(\mathbb{R}^d)$, $g_k \in M^{p',q'}(\mathbb{R}^d)$. But,

$$(T^t(S), \sigma) = \text{Trace}(L_\sigma \circ S) = \sum_{k=1}^N \langle L_\sigma f, g_k \rangle = \left\langle \sigma, \sum_{k=1}^N W(g_k, f_k) \right\rangle.$$

Therefore, for $r = \max(p, q, p', q')$, $T^t(S) \in M^r(\mathbb{R}^{2d}) \subset M^0(\mathbb{R}^{2d})$ ([9, Lemma 7] or [6, Theorem 4.1]), and the theorem is proved. ■

Remark 1 Toft proved in [6, 1.7] that the condition $\sigma \in M^\infty(\mathbb{R}^{2d})$ is necessary for the boundedness of the Weyl operator L_σ on $L^2(\mathbb{R}^d)$. Observe that the argument in the previous proof also gives that $\sigma \in M^\infty$, whenever L_σ is bounded on some $M^{p,q}$.

The previous result is no longer true for the extreme cases p or q equal 1 or ∞ .

Example 4.5 Take $\varphi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ and $f \in M^{\infty,1}(\mathbb{R}^d) \setminus M^0(\mathbb{R}^d)$, then as follows from [7, 14.5.1] $\sigma = W(\varphi, f) \notin M^0(\mathbb{R}^{2d})$. However, for every q , $L_\sigma: M^{1,q}(\mathbb{R}^d) \rightarrow M^{1,q}$ is compact as it is the rank one operator $h \rightarrow \langle h, f \rangle \varphi$. The transpose operator, $L_{\bar{\sigma}}$ is compact from $M^{\infty,q'}(\mathbb{R}^d)$ into itself and clearly $\bar{\sigma} \notin M^0(\mathbb{R}^{2d})$. With obvious modifications one can construct a similar counterexample for $M^{p,1}(\mathbb{R}^d)$.

Now, we restrict our attention to symbols in the Sjöstrand class $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ also to cover this extreme cases. In this case, the corresponding Weyl operator acts continuously on every modulation class and we may use interpolation theory.

THEOREM 4.6 *Let $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and $1 \leq p, q \leq \infty$ be given. The following conditions are equivalent:*

- (i) $L_\sigma: M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$ is compact,
- (ii) $L_\sigma: L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d})$ is compact,
- (iii) $\sigma \in M^{0,1}(\mathbb{R}^{2d})$,
- (iv) $L_\sigma: M^1(\mathbb{R}^d) \rightarrow M^1(\mathbb{R}^d)$ is weakly compact.

Proof It is clear from 4.4 that equation (2) implies equation (3), whereas equation (3) implies that equation (1) is the content of [8, 2.3]. In order to prove that equation (1) implies equation (2), we consider the Banach couples $X := (M^{p,q}, M^{p',q'})$ and

$Y := (L^{p,q}(\mathbb{R}^{2d}), L^{p',q'}(\mathbb{R}^{2d}))$ and let $V : X \rightarrow Y$ and $V^* : Y \rightarrow X$ be continuous and linear operators such that $V^* \circ V = id$ ([14, 11.3.2]). We define $T : Y \rightarrow Y$ by $T := V \circ L_\sigma \circ V^*$. Then, $T : Y \rightarrow Y$ is bounded and $T : L^{p,q}(\mathbb{R}^{2d}) \rightarrow L^{p,q}(\mathbb{R}^{2d})$ is compact. We can apply [18, Corollary 7 (c)] to conclude that $T : L^2(\mathbb{R}^{2d}) \rightarrow L^2(\mathbb{R}^{2d})$ is compact, from where $L_\sigma = V^* \circ T \circ V$ is compact on $L^2(\mathbb{R}^d)$.

Observe that $M^1(\mathbb{R}^d)$ is isometrically isomorphic to ℓ^1 (see [7, 12.2.6] or [19, 9.3]), therefore, weakly compact sets in $M^1(\mathbb{R}^d)$ are compact, hence equation (4) implies equation (1) for $p = q = 1$. ■

Since the localization operator $L_{\varphi,\psi}^F$ can also be interpreted as a Weyl operator L_σ with Weyl symbol $\sigma = F * W(\psi, \varphi)$, the main result in [2] can be generalized as follows.

THEOREM 4.7 *Let $F \in M^\infty(\mathbb{R}^{2d})$, $1 \leq p, q \leq \infty$ and $g_0 \in \mathcal{S}(\mathbb{R}^d)$ be given. Then, the following conditions are equivalent:*

- (i) *The localization operator $L_{\varphi,\psi}^F : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is compact for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.*
- (ii) *The localization operator $L_{\varphi,\psi}^F : M^{p,q}(\mathbb{R}^d) \rightarrow M^{p,q}(\mathbb{R}^d)$ is compact for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$.*
- (iii) *For every $R > 0$ we have*

$$\lim_{|x| \rightarrow \infty} \sup_{|\xi| \leq R} |V_{g_0} F(x, \xi)| = 0.$$

Proof In fact, the condition $F \in M^\infty(\mathbb{R}^{2d})$ implies $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ and the Theorem 4.6 gives the equivalence of equations (1) and (2). The equivalence of equations (1) and (3) is the content of [2, 3.15]. Alternatively, one can prove the equivalence of equations (1) and (3) proceeding as in [2] after replacing [2, 3.8, 3.9, 3.10, 3.11] by the necessary condition obtained in the Theorem 4.4. ■

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Weighted criteria for generalized fractional maximal functions and potentials in Lebesgue spaces with variable exponent

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Necessary and sufficient conditions governing two-weight inequalities with general-type weights for fractional maximal functions and Riesz potentials with variable parameters are established in the Lebesgue spaces with variable exponent. In two-weight inequalities the right-hand side weight to the certain power satisfies the reverse doubling condition. In particular, from the general results we have: generalization of the Sobolev inequality for potentials; criteria governing the trace inequality for fractional maximal functions and potential operators; theorem of Muckenhoupt–Wheeden type (one-weight inequality) for fractional maximal functions defined on a bounded interval when the parameter satisfies the Dini–Lipschitz condition. Sawyer-type two-weight criteria for fractional maximal functions are also derived.

Keywords: Lebesgue spaces with variable exponent; Generalized potentials; Fractional maximal functions; Two-weight inequality; Trace inequality

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1. Introduction

During last two decades a considerable interest of researchers was attracted to the study of mapping properties of classical integral operators in the Lebesgue spaces with non-standard growth $L^{p(\cdot)}$ and to the corresponding generalized Sobolev spaces $W^{n,p(\cdot)}$. Such spaces naturally arise, for example, when one deals with functionals of the form

$$\int_{\Omega} |\nabla f(x)|^{p(x)} dx.$$

Such a functional appears in the study of differential equations of the type

$$\operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u) = |u|^{\sigma(x)-1} u(x) + f(x).$$

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In this case, one deals with the Dirichlet integral

$$\int_{\Omega} (|\nabla f(x)|^{p(x)} + |u(x)|^{\sigma(x)}) dx.$$

Mathematical problems related to the function spaces with variable exponent arise in applications to mechanics of the continuum medium. In some variational problems there arise Lagrangians of the form $|\xi|^{\gamma(x)}$ when the character of non-linearity varies from point-to-point (see, e.g., the survey paper [1] and references therein). Investigations of variational problems with variable exponent started from the papers by V. Zhykov [2–4]. Later M. Ružička studied the problems in the so called rheological and electrorheological fluids, which lead to spaces with variable exponent (see [5, 6]). Many mathematical models in fluid mechanics, elasticity theory, differential equations etc are naturally related to the problems with non-standard local growth.

The spaces $L^{p(\cdot)}([0, 1])$ first appeared in the book by H. Nakano [7] as an example of modular spaces. The basis of the variable exponent Lebesgue and Sobolev spaces were developed by W. Orlicz, J. Musielak [8, 9], H. Hudzik [10], I. I. Sharapudinov [11, 12], S. Samko [13–15], O. Kovačik and J. Rákosník [16], D.E. Edmunds and J. Rákosník [17], L. Pick and M. M. Ružička [18], D.E. Edmunds and A. Nekvinda [19], X. Fan and D. Zhao [20, 21] etc.

It should be stressed that historically, the boundedness of the Hardy–Littlewood maximal operator over bounded domain Ω was established by L. Diening [22]. He showed that the operator

$$Mf(x) = \sup_{r>0} \frac{1}{|Q(x, r)|} \int_{Q(x, r) \cap \Omega} |f(y)| dy, \quad x \in \Omega,$$

is bounded in $L^{p(x)}(\Omega)$ if $p(\cdot)$ satisfies the following conditions:

(i)

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad (1)$$

where $p_- =: \inf_{x \in \Omega} p(x)$ $p_+ =: \sup_{x \in \Omega} p(x)$;

(ii) the Dini–Lipschitz condition on Ω ($p(\cdot) \in \text{DL}(\Omega)$), i.e.,

$$|p(x) - p(y)| \leq \frac{A}{\ln(1/|x - y|)}; \quad |x - y| \leq \frac{1}{2}; \quad x, y \in \bar{\Omega}. \quad (2)$$

The boundedness of the operator M in $L^{p(\cdot)}$ spaces defined on $\mathbb{R}^n$ was proved in the papers [23, 24] under conditions (1)–(2) together with the certain condition on $p(\cdot)$ at the infinity.

The study of mapping properties of fractional integrals in $L^{p(\cdot)}$ spaces was initiated by S. Samko (see [14, 15]). He established the boundedness of the operator $I_{\alpha(\cdot)}$ in Lebesgue spaces with variable exponent provided that Ω is bounded domain, $p \in \text{DL}(\Omega)$ and the Hardy–Littlewood maximal function M is bounded in $L^{p(\cdot)}(\Omega)$.

In his paper L. Diening (see [25]) established the boundedness of I_{α} for constant α on unbounded domains provided that $p(\cdot)$ satisfies (2) and is constant outside of a large ball. The latter result was generalized in [26] for $p(\cdot)$ satisfying the Dini–Lipschitz conditions locally and at infinity. V. Kokilashvili and S. Samko [27, 28] (see also survey papers [1, 29]) derived weighted inequalities in $L^{p(\cdot)}(\Omega)$ with power-type weights for the operators $I_{\alpha(\cdot)}$ and $M_{\alpha(\cdot)}$. They discovered the fact that conditions on powers of weights depend on values of exponents of generalized Lebesgue spaces at the point of singularity of weights (for mapping properties for maximal functions, singular integrals and potentials in $L^{p(\cdot)}$ spaces see also the survey paper [30]).

Weighted inequalities of special type for potentials has been also obtained in refs. [31–33].

Sufficient conditions governing the boundedness of the operator $I_{\alpha(\cdot)}$ in weighted Lebesgue spaces with general weight were derived by D.E. Edmunds, V. Kokilashvili and A. Meskhi [34]. The results of the latter paper are based on the solution of two-weight problem for integral operators with positive kernel obtained in ref. [35] (see also [36], chapter 4). In the paper [34] the domain is bounded and the parameter of fractional integral satisfies condition (2). Finally we mention the paper [37], where weighted inequalities in the case of power-type weights were studied for one-sided potentials with variable exponent in $L^{p(\cdot)}$.

The two-weight problem for classical fractional maximal functions in Lebesgue spaces with constant variable was solved in refs. [38, 39] (see also [40, 36], chapter 4) and for the Riesz potentials in the papers [41–43]. The criteria established by M. Gabidzashvili and V. Kokilashvili are more transparent than those obtained by E. Sawyer. For further results concerning two-weight problem for potential-type operators see refs. [36, 44–47].

Our aim in the present paper is to establish weighted criteria for the operators

$$(I_{\alpha(\cdot)}f)(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha(x)}} dy$$

and

$$(M_{\alpha(\cdot)}f)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| dy$$

in Lebesgue spaces with variable exponent defined on unbounded domains in the case when the parameter $\alpha(\cdot)$ and weights are general functions. Our conditions on the triple $(\alpha(\cdot), v(\cdot), w(\cdot))$ are transparent and we do not require the Dini–Lipschitz condition for parameters and exponents. Unfortunately, in our case the right-hand side space in the classical Lebesgue space and the right-hand side weight raised to the certain power satisfies the (reverse) doubling condition. From the general results it follows criteria on the couple $(\alpha(\cdot), v(\cdot))$ insuring the trace inequality for the operators $M_{\alpha(\cdot)}$ and $I_{\alpha(\cdot)}$. In the paper one-weight inequality of Muckenhoupt–Wheeden type [48] for the operator

$$(M_{\alpha(\cdot)}^J f)(x) := \sup_{I \ni x; I \subseteq J} \frac{1}{|I|^{1-\alpha(x)}} \int_I |f(y)| dy$$

has been also obtained on bounded interval J in $\mathbb{R}$ provided that $\alpha(\cdot)$ satisfies the Dini–Lipschitz condition (2).

The paper is organized as follows: In section 2 we introduce basic notation and definitions. In section 3 we formulate some well-known results concerning Lebesgue spaces with variable exponent and prove some lemmas. Section 3 deals with fractional maximal operators with variable parameter; the same problems are discussed in section 4 for fractional integrals. Section 5 is dedicated to two-weight estimates for $M_{\alpha(\cdot)}$ in the diagonal case. In section 6, we give some further remarks concerning the main results of the paper. Constants (often different constant in the same series of inequalities) will generally be denoted by c or C . Further, we will assume that Ω is a domain in $\mathbb{R}^n$; I and J denote bounded intervals in $\mathbb{R}$; Q is a bounded cube in $\mathbb{R}^n$; $Q(x, r)$ is a cube with center x and edge $l(Q) = 2r$; $B(x, r)$ is a cube with center x and radius r ; $I(x, r)$ is an interval $(x-r, x+r)$, $0 < r < \infty$; $\mu(E)$ (or μE) will denote μ -measure of a set E ; $|E|$ is the Lebesgue measure of E ; if ρ is a weight function (*i.e.*, locally integrable almost everywhere positive function), then sometimes the symbol $\rho(E)$ will be used for $\int_E \rho(x) dx$; f_E will denote an average of a function f on a set E .

2. Preliminaries

Let $q(\cdot)$ be a measurable function on Ω such that $1 \leq q(x) \leq \text{ess sup}_\Omega q(\cdot) =: q_+ < \infty$. Suppose that

$$q_-(E) := \text{ess inf}_{x \in E} q(x); \quad q_+(E) := \text{ess sup}_{x \in E} q(x),$$

where E is a measurable subset of Ω . If $E = \Omega$, then we abbreviate the notation: $q_- := q_-(\Omega)$, $q_+ := q_+(\Omega)$.

Let us recall several basics of the Lebesgue spaces with variable exponent.

Assume that ρ is a weight function on Ω . By $L_\rho^{q(\cdot)}(\Omega)$ we denote the space of all measurable functions f on Ω such that

$$S_{q(\cdot), \rho}(f) := \int_\Omega |f(x)|^{q(x)} \rho(x) dx < \infty.$$

This is a Banach space with respect to the norm

$$\|f\|_{L_\rho^{q(\cdot)}(\Omega)} = \inf\{\lambda > 0 : S_{q(\cdot), \rho}(f/\lambda) \leq 1\}.$$

If $\rho \equiv 1$, then we use the symbol $L^{q(\cdot)}(\Omega)$ instead of $L_\rho^{q(\cdot)}(\Omega)$. Notice that $L^{q(\cdot)}(\Omega)$ coincides with classical Lebesgue space when $q(\cdot)$ is constant.

The next inequality is obvious:

$$\|\rho f\|_{L^{q(\cdot)}(\Omega)} = \|f\|_{L_{\rho^{q(\cdot)}}^{q(\cdot)}(\Omega)}.$$

The Hölder's inequality for $L^{q(\cdot)}(\Omega)$ holds in the form:

$$\int_\Omega |f(x)g(x)| dx \leq k \|f\|_{L^{q(\cdot)}(\Omega)} \|g\|_{L^{q'(\cdot)}(\Omega)},$$

where $k = (1/q_-) + (1/(q)_-)$;

The following properties of $L^{q(\cdot)}(\Omega)$ will be also useful:

$$\|f\|_{L_\rho^{q(\cdot)}(\Omega)}^{q_+} \leq S_{q(\cdot), \rho}(f) \leq \|f\|_{L_\rho^{q(\cdot)}(\Omega)}^{q_-}; \quad \|f\|_{L_\rho^{q(\cdot)}(\Omega)} \leq 1;$$

$$\|f\|_{L_\rho^{q(\cdot)}(\Omega)}^{q_-} \leq S_{q(\cdot), \rho}(f) \leq \|f\|_{L_\rho^{q(\cdot)}(\Omega)}^{q_+}; \quad \|f\|_{L_\rho^{q(\cdot)}(\Omega)} \geq 1;$$

$$\|f\|_{L_\rho^{rq(\cdot)}(\Omega)}^r = \| |f|^r \|_{L^{q(\cdot)}(\Omega)},$$

where r is a constant, $r \geq 1$.

We will also need the next statements:

LEMMA A *Let $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on a bounded domain Ω in $\mathbb{R}^n$. Suppose that $1 \leq q_1(x) \leq q_2(x) \leq C < \infty$. Then for every $f \in L^{q_2(\cdot)}(\Omega)$ the inequality*

$$\|f\|_{L^{q_1(\cdot)}(\Omega)} \leq (1 + \Omega) \|f\|_{L^{q_2(\cdot)}(\Omega)}$$

holds.

LEMMA B *Let Ω be a domain in $\mathbb{R}^n$. Then $p(\cdot) \in \text{DL}(\Omega)$ if and only if*

$$|Q|^{p_-(Q) - p_+(Q)} \leq c$$

for all cubes in $\mathbb{R}^n$ with $|Q \cap \Omega| > 0$.

For the definition and the properties of $L^{q(\cdot)}(\Omega)$ listed above see, e.g., [11, 9, 16, 17, 13, 14, 22].

In the sequel we shall need the following definitions:

DEFINITION A We say that a measure μ on $\mathbb{R}^n$ satisfying the doubling condition ($\mu \in DC(\mathbb{R}^n)$) if there exists a positive constant b such that for all $x \in \mathbb{R}^n$ and all $r > 0$ the inequality

$$\mu B(x, 2r) \leq b\mu B(x, r)$$

holds. Further, a measure μ satisfies the doubling condition on an interval J ($\mu \in DC(J)$) if there exists a positive constant b such that for all $x \in J$ and all $r > 0$

$$\mu(I(x, 2r) \cap J) \leq b\mu(I(x, r) \cap J).$$

DEFINITION B We say that a weight function w belongs to the Muckenhoupt's class $A_p(\Omega)$ ($1 < p < \infty$) if

$$\sup_{\substack{a \in \Omega; \\ r > 0}} A_p^{a,r} := \sup_{\substack{a \in \Omega \\ r > 0}} \left(\frac{w(\Omega \cap B(a, r))}{|\Omega \cap B(a, r)|} \right)^{1/p} \left(\frac{w^{1-p'}(\Omega \cap B(a, r))}{|\Omega \cap B(a, r)|} \right)^{1/p'} < \infty,$$

where $p/(p-1)$.

It is easy to check that if $n = 1$ and Ω is an interval J , then the condition $w \in A_p(J)$ implies $\mu \in DC(J)$, where $d\mu = w(x)dx$.

Let $\mathcal{D}(\mathbb{R}^n)$ be a family of all dyadic cubes in $\mathbb{R}^n$. We denote by $\mathcal{D}(J)$ the family of all dyadic intervals I such that $I \subseteq J$. In the latter case we will assume that the ‘maximal’ dyadic interval is J itself.

DEFINITION C We say that measure μ belongs to the class $RD^{(d)}(\mathbb{R}^n)$ if there exists a constant $\delta > 1$, such that for all dyadic cubes Q and Q' , $Q \subset Q'$, $|Q| = (|Q'|/2^n)$, the inequality

$$\mu(Q') \geq \delta\mu(Q)$$

holds. Further, $\mu \in RD^{(d)}(J)$ if there exists a constant $\delta > 1$ for which the inequality

$$\mu(I') \geq \delta\mu(I)$$

holds for all dyadic intervals I and I' such that $I \subset I' \subseteq J$, $|I| = (|I'|/2)$.

Our main results are based on the following inequalities of Carleson–Hörmander type [49, 50]:

LEMMA C Let $1 < p \leq r < \infty$. Suppose that the weight $\rho^{-p'} \in RD^{(d)}(\mathbb{R}^n)$. Then there exists a positive constant C such that for all non-negative f the inequality

$$\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \left(\int_Q \rho^{-p'} \right)^{-r/p'} \left(\int_Q f \right)^r \leq C \left(\int_{\mathbb{R}^n} (f\rho)^p \right)^{r/p}$$

holds.

LEMMA D Let $1 < p \leq r < \infty$. Suppose that the weight $\rho \in DC^{(d)}(\mathbb{R}^n)$. Then the inequality

$$\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} a_Q \left(\frac{1}{w(Q)} \int_Q f \rho \right)^r \leq C \left(\int_{\mathbb{R}^n} (f)^p \rho \right)^{r/p}, \quad f \geq 0,$$

holds if

$$\sum_{Q \subset Q'; Q, Q' \in \mathcal{D}(\mathbb{R}^n)} a_Q^{r/p} \leq c \rho^{r/p}(Q').$$

For Lemmas C–D we refer to [45, 51].

Remark 1 The latter statements is also true for any family of dyadic cubes (see, e.g., [51]).

LEMMA 2.1 Let $n = 1$ and let p be a constant such that $1 < p < \infty$. Suppose that $\Omega := J$ is an interval in $\mathbb{R}$. Assume that $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < 1/p$; $q(x) = p/(1 - \alpha(x)p)$ and $\alpha \in DL(J)$. If

$$B := \sup_{I \subset J} \|\chi_I |I|^{\alpha(\cdot)-1} w\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} < \infty,$$

then $w^{-p'} \in RD^{(d)}(J)$.

Proof It suffices to see that $w^p \in A_p(J)$. In fact this condition implies $w^{-p'} \in A_{p'}(J)$. From the latter condition we will have that $w^{-p'} \in RD^{(d)}(J)$ (see, e.g., [41, 52–54, p. 21]). By Lemma A we have

$$\left(\int_I w^{q_-(I)} \right)^{1/q_-(I)} \leq c(1 + |I|) \|\chi_I w\|_{L^{q(\cdot)}(J)} \leq c(1 + |J|) \|\chi_I w\|_{L^{q(\cdot)}(J)}.$$

The latter inequality together with Lemma B, the condition $\alpha(\cdot) \in DL(J)$ and the equality $q_-(I) = p/(1 - \alpha_-(I)p)$ gives

$$\begin{aligned} B &\geq \|\chi_I |I|^{\alpha(\cdot)-1} w\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \geq c |I|^{\alpha_+(I)-1} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\geq |I|^{\alpha_-(I)-1} \|\chi_I w\|_{L^{q(\cdot)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)}. \end{aligned}$$

On the other hand, using Hölder's inequality with respect to the exponent $q_-(I)/p$ we find that

$$\begin{aligned} |I|^{-1} \|\chi_I w\|_{L^p(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} &\leq |I|^{-1} |I|^{1/((q_-(I)/p)')} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \\ &\leq |I|^{\alpha_-(I)-1} \|\chi_I w\|_{L^{q_-(I)}(J)} \|\chi_I w^{-1}\|_{L^{p'}(J)} \leq B < \infty. \end{aligned}$$

■

3. Fractional maximal functions

In this section, we derive two-weight criteria for the operator $M_{\alpha(\cdot)}$.

THEOREM 3.1 Let $\Omega := \mathbb{R}^n$. Suppose that $p = \text{const}$; $1 < p < q_- \leq q(x) \leq q_+ < \infty$; $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < n$. Assume that $w^{-p'} \in RD^{(d)}(\mathbb{R}^n)$. Then the following conditions are equivalent:

- (i) $\|v M_{\alpha(\cdot)} f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c \|wf\|_{L^p(\mathbb{R}^n)};$ (3)
- (ii) $A := \sup_{Q \subset \mathbb{R}^n} \|v \chi_Q |Q|^{(\alpha(\cdot)/n)-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|w^{-1} \chi_Q\|_{L^{p'}(\mathbb{R}^n)} < \infty.$

Proof First note that inequality (3) implies the condition $w^{-p'}(Q) < \infty$ for every Q . Indeed, if $w^{-p'}(Q) = \infty$ for some Q , then by duality arguments there exists a non-negative function $g \in L^p(Q)$ such that $\int_Q gw^{-1} = \infty$. Further, let $f = gw^{-1}$. Then it is clear that $\|fw\|_{L^p(Q)} = \|g\|_{L^p(Q)} < \infty$. On the other hand,

$$\begin{aligned} \|vM_{\alpha(\cdot)}(gw^{-1})\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\geq \|\chi_Q vM_{\alpha(\cdot)}(gw^{-1})\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\geq \left(\int_Q gw^{-1} \right) \|\chi_Q |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} = \infty. \end{aligned}$$

Hence, $M_{\alpha(\cdot)}$ is not bounded unless $v = 0$ almost everywhere on Q .

Substituting now the functions $f_Q(x) = w^{-p'}(x)\chi_Q(x)$ in (3) we derive the implication (i) $\Rightarrow$ (ii). Let us now show that (ii) $\Rightarrow$ (i). First we prove that (ii) implies the inequality

$$\|vM_{\alpha(\cdot)}^{(d)}f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|wf\|_{L^p(\mathbb{R}^n)}, \quad (3a)$$

where

$$(M_{\alpha(\cdot)}^{(d)}f)(x) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| dy$$

is the dyadic fractional maximal function with variable parameter.

Let f be a bounded non-negative function with compact support. If we prove the result for such f , then we can pass to arbitrary $f \in L_{w^p}^p(\mathbb{R}^n)$. Indeed, let $f \in L_{w^p}^p(\mathbb{R}^n)$ and let us take the sequence $f_n = f\chi_{Q(0,k_n)}\chi_{\{f < j_n\}}$, where $k_n, j_n \rightarrow \infty$ as $n \rightarrow \infty$. Then $f_n \rightarrow f$ in $L_{w^p}^p(\mathbb{R}^n)$ and also pointwise. Moreover, $f_n(x) \leq f(x)$. On the other hand, $M_{\alpha(\cdot)}f_n$ is a Cauchy sequence in $L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)$, because

$$\begin{aligned} \|M_{\alpha(\cdot)}f_n - M_{\alpha(\cdot)}f_m\|_{L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)} &= \|v[M_{\alpha(\cdot)}f_n - M_{\alpha(\cdot)}f_m]\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leq \|v[M_{\alpha(\cdot)}(f_n - f_m)]\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|w[f_n - f_m]\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Consequently, by completeness of the space $L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)$ there exists $g \in L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)$ such that

$$\|[M_{\alpha(\cdot)}f_n] - g\|_{L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)} \rightarrow 0.$$

Taking into account the properties of the generalized Lebesgue spaces (see section 3) we find that there is a subsequence $M_{\alpha(\cdot)}f_{n_k}$ which converges to g in $L_{v^{q(\cdot)}}^{q(\cdot)}(\mathbb{R}^n)$ and also almost everywhere. But f_{n_k} converges to f in $L_{w^p}^p(\mathbb{R}^n)$ and almost everywhere. This leads to the inequality

$$\|vg\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|wf\|_{L^p(\mathbb{R}^n)}, \quad (4)$$

where c does not depend on f . Now we deal with the sequence f_{n_k} . Since f_{n_k} is non-decreasing, we observe that for fixed $x \in Q$

$$\begin{aligned} |Q|^{\alpha(x)/n-1} \int_Q f(t) dt &= \lim_{k \rightarrow \infty} |Q|^{\alpha(x)/n-1} \int_Q f_{n_k}(t) dt \\ &\leq \lim_{k \rightarrow \infty} \sup_{Q \ni x} |Q|^{\alpha(x)/n-1} \int_Q f_{n_k}(t) dt = \lim_{k \rightarrow \infty} (M_{\alpha(\cdot)}f_{n_k})(x) \end{aligned}$$

and the last limit exists, because it converges a.e. to g . Hence

$$(M_{\alpha(\cdot)}f)(x) \leq \lim_{k \rightarrow \infty} (M_{\alpha(\cdot)}f_{n_k})(x) = g(x)$$

for almost all x . Finally, inequality (4) yields

$$\|vM_{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|wf\|_{L^p(\mathbb{R}^n)}.$$

Now for every $x \in \mathbb{R}^n$ we choose a dyadic cube Q_x so that

$$|Q_x|^{\alpha(x)/n-1} \int_{Q_x} f > \frac{1}{2}(M_{\alpha(\cdot)}^{(d)}f)(x). \quad (5)$$

Since f has a bounded support, for each x we can assume that Q_x is a “maximal” dyadic cube for which (5) holds. Further, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ introduce a set F_Q which is a collection of those x , $x \in Q$, for which (5) holds for Q and Q is ‘maximal’.

It is obvious that if we take arbitrary $x \in \mathbb{R}^n$, then $x \in F_Q$ for some $Q \in \mathcal{D}(\mathbb{R}^n)$. On the other hand, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ we have a set F_Q which might be empty for some Q . It is also clear that $F_Q \subset Q$ and that $F_{Q_1} \cap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$. Let $\mathcal{D}_m := \{Q \in \mathcal{D}(\mathbb{R}^n) : F_Q \neq \emptyset\}$.

Now let us take r so that $p < r < q_-$. Then

$$\|vM_{\alpha(\cdot)}^{(d)}f\|_{L^{q(\cdot)}(\mathbb{R}^n)}^r = \|[vM_{\alpha(\cdot)}^{(d)}f]^r\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} = \sup \int_{\mathbb{R}^n} (vM_{\alpha(\cdot)}^{(d)}f)^r h,$$

where the supremum is taken over all functions h , $\|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \leq 1$. Now for such h using Lemma C we have

$$\begin{aligned} \int_{\mathbb{R}^n} (vM_{\alpha(\cdot)}^{(d)}f)^r h &= \sum_{Q \in \mathcal{D}_m} \int_{F_Q} [vM_{\alpha(\cdot)}^{(d)}f]^r h \\ &\leq c \sum_{Q \in \mathcal{D}_m} \left(\int_{F_Q} v^r(x) |Q|^{\alpha(x)/n-1} h(x) dx \right) \left(\int_Q f \right)^r \\ &\leq c \sum_{Q \in \mathcal{D}_m} \|v^r(\cdot) |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \left(\int_Q f \right)^r \\ &\leq c \sum_{Q \in \mathcal{D}_m} \|v^r(\cdot) |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)/r}(\mathbb{R}^n)} \|h\|_{L^{(q(\cdot)/r)'}(\mathbb{R}^n)} \left(\int_Q f \right)^r \\ &\leq cA^r \sum_{Q \in \mathcal{D}_m} \left(\int_Q w^{-p'} \right)^{-r/p'} \left(\int_Q f \right)^r \leq cA^r \|fw\|_{L^p(\mathbb{R}^n)}^r. \end{aligned}$$

Taking now the supremum over all h we see that (ii) $\Rightarrow$ (3a). Let us now show that (ii) $\Rightarrow$ (i). We use the idea from [55]. Let j and Q be an integer and a cube respectively such that $2^{j-1} < l(Q) \leq 2^j$. Suppose that $k \in \mathbb{Z}$, $k \geq j$. Let E be a set of those $t \in Q(0, 2^{k+2})$ for which there exists a cube Q_1 , $Q_1 \in \mathcal{D}$, $l(Q_1) = 2^{j+1}$, $Q \subset Q_1$. Then (see e.g., [56, p. 431]) $E \geq 2^{(k+2)n}$. Let us take f , $f \geq 0$, and $x \in \mathbb{R}^n$. Then it is obvious that there exists a cube Q , $l(Q) \leq 2^k$, such that Q contains x and

$$|Q|^{\alpha(x)/n-1} \int_Q f > \frac{1}{2}(\bar{M}_{\alpha(\cdot)}^{(2^k)}f)(x),$$

where

$$(\bar{M}_{\alpha(\cdot)}^{(2^k)}f)(x) = \sup_{\substack{Q \ni x \\ l(Q) \leq 2^k}} |Q|^{\alpha(x)/n-1} \int_Q f.$$

Let j be an integer such that $2^{j-1} < |Q| \leq 2^j$, $j \leq k$. Then for every $t \in E$ there exists a cube $Q_1 \in \mathcal{D} - t$, $l(Q_1) = 2^{j+1}$, $Q \subset Q_1$, for which

$$\frac{1}{2}(\bar{M}_{\alpha(\cdot)}^{(2^k)} f)(x) \leq |Q|^{\alpha(x)/n-1} \int_Q f \leq \frac{c_{\alpha,n}}{|Q_1|^{1-\alpha(x)/n}} \int_{Q_1} f \leq c_{\alpha,n} K_t(x),$$

where

$$K_t(x) = \sup_{\substack{Q-t \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} |Q|^{\alpha(x)/n-1} \int_{Q-t} f.$$

At the same time we have $|E| \geq |Q(0, 2^{k+2})|/2^n$. Hence

$$(\bar{M}_{\alpha(\cdot)}^{(2^k)} f)(x) \leq \frac{c}{|E|} \int_E K_t(x) dt \leq \frac{c}{|Q(0, 2^{k+2})|} \int_{Q(0, 2^{k+2})} K_t(x) dt.$$

Let us now take f and h so that $\|fw\|_{L^p(\mathbb{R}^n)} \leq 1$ and let $\|h\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \leq 1$. Then we have

$$\begin{aligned} S_t &:= \int_{\mathbb{R}^n} K_t(x) h(x) v(x) dx = \int_{\mathbb{R}^n} \left(\sup_{\substack{Q-t \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} |Q|^{\alpha(x)/n-1} \int_{Q-t} f(s) ds \right) h(x) v(x) dx \\ &= \int_{\mathbb{R}^n} \left(\sup_{\substack{Q-t \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} |Q|^{\alpha(x)/n-1} \int_Q f(s-t) ds \right) h(x) v(x) dx \\ &= \int_{\mathbb{R}^n} \left(\sup_{\substack{Q \ni y \\ Q \in \mathcal{D}(\mathbb{R}^n)}} |Q|^{\alpha(y-t)/n-1} \int_Q f(s-t) ds \right) h(y-t) v(y-t) dy \\ &\leq \|v(y-t)(M_{\alpha(y-t)}^{(d)} f(\cdot-t))(y)\|_{L^{q(y-t)}(\mathbb{R}^n)} \|h(\cdot-t)\|_{L^{q'(\cdot-t)}(\mathbb{R}^n)}. \end{aligned}$$

Notice that $p < (q(\cdot-t))_- = q_-$; $\alpha_- = (\alpha(\cdot-t))_-$ and

$$\begin{aligned} &\|\chi_{Q-t}(\cdot) |Q-t|^{\alpha(\cdot)/n-1} v(\cdot)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \|w^{-1}(\cdot) \chi_{Q-t}(\cdot)\|_{L^{p'}(\mathbb{R}^n)} \\ &= \|\chi_Q(x) |Q|^{\alpha(x-t)/n-1} v(x-t)\|_{L^{q(x-t)}(\mathbb{R}^n)} \|w^{-1}(\cdot-t) \chi_Q(\cdot)\|_{L^{p'}(\mathbb{R}^n)} \leq c \end{aligned}$$

for all $t \in E$. Hence,

$$S_t \leq cA \|w(\cdot-t) f(\cdot-t)\|_{L^p(\mathbb{R}^n)} = cA \|wf\|_{L^p(\mathbb{R}^n)}.$$

Further, observe that

$$\begin{aligned} \int_{\mathbb{R}^n} (\bar{M}_{\alpha(\cdot)}^{(2^k)} f)(x) h(x) v(x) dx &\leq c \int_{\mathbb{R}^n} (K_t(x))_{Q(0, 2^{k+2})} h(x) v(x) dx \\ &= c |Q(0, 2^{k+1})|^{-1} \int_{Q(0, 2^{k+1})} \left(\int_{\mathbb{R}^n} K_t(x) h(x) v(x) dx \right) dt \\ &\leq cA \|wf\|_{L^p(\mathbb{R}^n)} \leq cA. \end{aligned}$$

Passing now to the limit when $k \rightarrow \infty$ and taking the supremum with respect to h , $\|h\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq 1$, we finally derive the desired result. $\blacksquare$

From the latter statement it follows the trace inequality for $M_{\alpha(\cdot)}$:

COROLLARY 3.2 *Let $\Omega = \mathbb{R}^n$. Suppose that $p = \text{const}$; $1 < p < q_- \leq q(x) \leq q_+ < \infty$; $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < n/p$. Then the inequality*

$$\|vM_{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}$$

holds if and only if

$$\sup_{Q \subset \mathbb{R}^n} \|v\chi_Q|Q|^{(\alpha(\cdot)/(n)-1)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}|Q|^{1/p'} < \infty.$$

The next statement is an restriction version of the latter one.

COROLLARY 3.3 *Let $n = 1$ and let $\Omega := J$ be a bounded interval in $\mathbb{R}$. Suppose that $\alpha(\cdot)$, p and $q(\cdot)$ satisfy the conditions of Theorem 3.1 and that $w^{-p'} \in RD(J)$. Then*

$$\|vM_{\alpha(\cdot)}^Jf\|_{L^{q(\cdot)}(J)} \leq c\|wf\|_{L^p(J)}$$

holds if and only if

$$\sup_{I \subset J} \|v\chi_I|I|^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)}\|\chi_I w^{-1}\|_{L^{p'}(J)} < \infty.$$

Proof For simplicity assume that $J := [0, 1]$. Introduce the following one-sided maximal functions:

$$(M_{\alpha(\cdot)}^+)f(x) = \sup_{0 < h \leq 1-x} \frac{1}{h^{1-\alpha(x)}} \int_x^{x+h} |f|, \quad x \in J;$$

$$(M_{\alpha(\cdot)}^-)f(y) = \sup_{0 < s \leq y} \frac{1}{s^{1-\alpha(y)}} \int_{y-s}^y |f|, \quad y \in J$$

and their dyadic versions:

$$(M_{\alpha(\cdot)}^{(d)+})f(x) = \sup_I \frac{1}{|I|^{1-\alpha(x)}} \int_I |f|;$$

$$(M_{\alpha(\cdot)}^{(d)-})f(y) = \sup_S \frac{1}{|S|^{1-\alpha(y)}} \int_S |f|,$$

where the supremum is taken over all dyadic intervals $I := [a, b] \subseteq J$, $S := [c, d] \subseteq J$, $x < a$, $y > d$, $0 \leq a - x < b - a$, $0 \leq y - d < d - c$.

We show that there exists a positive constant c such that for all $x \in [0, 1]$ the inequality

$$(M_{\alpha(\cdot)}^+f)(x) \leq c(M_{\alpha(\cdot)}^{(d)+}f)(x) \tag{6}$$

holds. We will use the arguments from ref. [57].

First observe that

$$(M_{\alpha(\cdot)}^+)f(x) \leq \frac{2^{\alpha_+-1}}{1 - 2^{\alpha_+-1}} (\bar{M}_{\alpha(\cdot)}^+)f(x) \quad (7)$$

which can be checked immediately, where

$$(\bar{M}_{\alpha(\cdot)}^+)f(x) = \sup_{\substack{h \\ 0 < h \leq 1-x}} \frac{1}{(h/2)^{1-\alpha(x)}} \int_{x+h/2}^{x+h} |f|, \quad x \in J.$$

Further, it is also clear that

$$(\bar{M}_{\alpha(\cdot)}^+)f(x) \leq c(\tilde{M}_{\alpha(\cdot)}^+)f(x), \quad (8)$$

where

$$(\tilde{M}_{\alpha(\cdot)}^+)f(x) = \sup_{\substack{j \in \mathbb{Z} \\ 2^j \leq 1-x}} \frac{1}{2^{(j-1)(1-\alpha(x))}} \int_{x+2^{j-1}}^{x+2^j} |f|, \quad x \in J.$$

Let now $h = 2^j$ for some integer j with $2^j \leq 1 - x$ and let I and I' be dyadic intervals in J such that $I \cup I'$ is dyadic, $|I| = |I'| = 2^{j-1}$ and $[x + h/2, x + h) \subset I \cup I'$. Then

$$\int_{x+h/2}^{x+h} |f| \leq \int_{I \cup I'} |f| \leq c2^{j(1-\alpha(x))} (M_{\alpha(\cdot)}^{(d)+}f)(x),$$

where c does not depend on j . Hence

$$(\tilde{M}_{\alpha(\cdot)}^+)f(x) \leq c(M_{\alpha(\cdot)}^{(d)+}f)(x). \quad (9)$$

If $I \cup I'$ is not dyadic, then we take $I_1 \in \mathcal{D}(J)$ with length 2^j containing I' and, consequently

$$\int_{x+h/2}^{x+h} |f| \leq \int_{I \cup I_1} |f| = \int_I |f| + \int_{I_1} |f| \leq ch^{1-\alpha(x)} (M_{\alpha(\cdot)}^{(d)+}f)(x)$$

with a positive constant c independent of j which gives again (9). Summarizing (7)–(9) we will get (6).

Analogously, it follows that

$$(\tilde{M}_{\alpha(\cdot)}^-)f(x) \leq c(M_{\alpha(\cdot)}^{(d)-}f)(x). \quad (10)$$

Taking into account the argument from the proof of Theorem 3.1 we will find that the inequality

$$\|vM_{\alpha(\cdot)}^{(d)+}f\|_{L^{q(\cdot)}(J)} \leq c\|wf\|_{L^p(J)}$$

holds if

$$\sup_{\substack{a \in J, h > 0 \\ (a-h, a+h) \subseteq J}} \|v\chi_{(a-h, a)}h^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)} \|w^{-1}\chi_{(a, a+h)}\|_{L^{p'}(J)} < \infty$$

provided that $w^{-p'} \in RD^{(d)}(J)$.

Analogously,

$$\|vM_{\alpha(\cdot)}^{(d)-}f\|_{L^{q(\cdot)}(J)} \leq c\|wf\|_{L^p(J)}$$

if

$$\sup_{\substack{a \in J, h > 0 \\ (a-h, a+h) \subseteq J}} \|v\chi_{(a, a+h)}h^{\alpha(\cdot)-1}\|_{L^{q(\cdot)}(J)} \|w^{-1}\chi_{(a-h, a)}\|_{L^{p'}(J)} < \infty$$

provided that $w^{-p'} \in RD^{(d)}(J)$.

Hence, taking into account inequalities (6), (10), the latter observations and the inequality

$$(M_{\alpha(\cdot)}^J f)(x) \leq (M_{\alpha(\cdot)}^+ f)(x) + (M_{\alpha(\cdot)}^- f)(x)$$

we obtain sufficiency. Necessity follows by the standard way. $\blacksquare$

Applying Lemma 2.1 to Corollary 3.3 we have the one-weight inequality for $M_{\alpha(\cdot)}^J$.

THEOREM 3.4 *Let $n = 1$ and let $\Omega := J$ be a bounded interval in $\mathbb{R}$. Suppose that $p = \text{const}$, $1 < p < q_- \leq q(x) \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < 1/p$ and $q(x) = p/(1 - \alpha(x)p)$. Suppose also that $\alpha(\cdot) \in DL(J)$. Then the inequality*

$$\|\rho M_{\alpha(\cdot)}^J f\|_{L^{q(\cdot)}(J)} \leq c\|\rho f\|_{L^p(J)}$$

holds if and only if

$$\sup_{I \subset J} |I|^{\alpha_-(I)-1} \|\chi_I \rho\|_{L^{q(\cdot)}(J)} \|\chi_I \rho^{-1}\|_{L^{p'}(J)} < \infty.$$

4. Fractional integrals

In this section we prove the trace inequality for the operator $I_{\alpha(\cdot)}$. In 1973 D. Adams (see [58]) proved the following statement:

THEOREM A *Let α , p and q be constants such that $1 < p < q < \infty$ and $0 < \alpha < n/p$. Then $I_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n)$ to $L_v^q(\mathbb{R}^n)$ if and only if*

$$\sup_{Q: Q \subset \mathbb{R}^n} v(Q) |Q|^{q(\alpha/n-1/p)} < \infty.$$

The next statement is a generalization of the result by D. Adams:

THEOREM 4.1 *Let $\Omega = \mathbb{R}^n$. Suppose that $p = \text{const}$, $1 < p < q_- \leq q(x) \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < n/p$. Then the inequality:*

$$\|vI_{\alpha(\cdot)}f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq c\|f\|_{L^p(\mathbb{R}^n)} \quad (11)$$

holds if and only if

$$B := \sup_{Q: Q \subset \mathbb{R}^n} \|v\chi_Q|Q|^{(\alpha(\cdot)/n)-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |Q|^{1/p'} < \infty.$$

Proof Necessity follows easily substituting $f_Q = \chi_Q(x)$ in the inequality (11). To prove sufficiency we use the idea similar to that from ref. [59]. In our case we apply the trace

inequality for the operator $M_{\alpha(\cdot)}$ to obtain the similar result for $I_{\alpha(\cdot)}$. For the simplicity assume that $n = 1$. Let β be a number such that $0 < \beta < \alpha_-$. Then $0 < \alpha(x) - \beta < \alpha_+ - \beta < 1/p$. We have

$$I_{\alpha(\cdot)} f(x) = \int_{I(x,r)} \frac{f(y)}{|x-y|^{1-\alpha(x)}} dy + \int_{\mathbb{R} \setminus I(x,r)} \frac{f(y)}{|x-y|^{1-\alpha(x)}} dy := S_1(x, r) + S_2(x, r).$$

Simple calculations show

$$S_1(x, r) \leq cr^\beta (M_{\alpha(\cdot)-\beta} f)(x); \quad S_2(x, r) \leq cr^{\alpha(x)-1/p} \|f\|_{L^p(\mathbb{R})}.$$

Taking $r = [(M_{\alpha(\cdot)-\beta} f(x))^{1/(\alpha(x)-1/p)} \|f\|_{L^p(\mathbb{R})}^{(1/\beta-\alpha(x)+1/p)}]$ in the latter inequalities we will find that

$$\begin{aligned} (I_{\alpha(\cdot)} f)(x) &\leq c(\alpha(x) - 1/p)/(\alpha(x) - \beta - 1/p)^{(\alpha(x)-1/p)/(\alpha(x)-\beta-1/p)} \\ &\quad \times \|f\|_{L^p(\mathbb{R})}^{(\beta-\alpha(x)+1/p+1)/(\beta-\alpha(x)+1/p)}. \end{aligned} \quad (12)$$

Let us now assume $\|f\|_{L^p(\mathbb{R})} \leq 1$ and prove that

$$\|I_{\alpha(\cdot)} f\|_{L_v^{q(\cdot)}(\mathbb{R})} \leq C$$

if

$$\bar{B} := \sup_I \|\chi_I |I|^{\alpha(\cdot)-1}\|_{L_v^{q(\cdot)}(\mathbb{R})} |I|^{1/p'} < \infty.$$

Taking into account (12) and the inequality $\beta - \alpha(x) > \beta - \alpha_+ > -1/p$ we find that

$$\begin{aligned} &\int_{\mathbb{R}} [(I_{\alpha(\cdot)} f)(x)]^{q(x)} v(x) dx \\ &\leq c \int_{\mathbb{R}} q(x) (\alpha(x) - 1/p)/(\alpha(x) - \beta - 1/p)^{(q(x)(\alpha(x)-1/p)/(\alpha(x)-\beta-1/p))} v(x) dx. \end{aligned} \quad (13)$$

Let us now denote $q_1(x) := q(x)(\alpha(x) - 1/p)/(\alpha(x) - \beta - 1/p)$. Then it is easy to see that $q_1(x) > q(x)$. Further, from (13) it follows that

$$\int_{\mathbb{R}} [(I_{\alpha(\cdot)} f)(x)]^{q(x)} v(x) dx \leq c \int_{\mathbb{R}} [(M_{\alpha(\cdot)-\beta} f)(x)]^{q_1(x)} v(x) dx.$$

Further,

$$\| |I|^{\alpha(x)-\beta-1} \chi_I \|_{L_v^{q_1(\cdot)}(\mathbb{R})} |I|^{1/p'} = \| |I|^{\alpha(x)-\beta-1/p} \chi_I \|_{L_v^{q_1(\cdot)}(\mathbb{R})}.$$

Besides, it is easy to see that

$$\int_I |I|^{(\alpha(x)-\beta-1/p)q_1(x)} v(x) dx = \int_I |I|^{(\alpha(x)-1/p)q(x)} v(x) dx \leq C$$

for all I because $\bar{B} < \infty$. In fact, by Corollary 4.1 we have

$$\int_{\mathbb{R}} [I_{\alpha(\cdot)} f(x)]^{q(x)} v(x) dx \leq C$$

for all $\|f\|_{L^p(\mathbb{R})} \leq 1$ if $\bar{B} < \infty$. The latter fact implies that $\|I_{\alpha(\cdot)} f\|_{L_v^{q(\cdot)}(\mathbb{R})} \leq C$ for $f, \|f\|_{L^p(\mathbb{R})} \leq 1$, provided that $\bar{B} < \infty$. ■

Theorem 4.1 yields the following generalization of Sobolev's [60] theorem.

THEOREM 4.2 *Let $\Omega = \mathbb{R}^n$ and let p be a constant. Suppose that $1 < p < q_- \leq q(x) \leq q_+ < \infty$, $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < n/p$. Then $I_{\alpha(\cdot)}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n)$ if and only if*

$$\sup_Q \|\chi_Q |Q|^{\alpha(\cdot)/n-1}\|_{L^{q(\cdot)}(\mathbb{R}^n)} |Q|^{1/p'} < \infty.$$

5. Diagonal case

This section deals with the boundedness of fractional maximal functions with variable parameter from $L_w^p(\mathbb{R}^n)$ to $L_v^p(\mathbb{R}^n)$ provided that w satisfies the doubling condition. In particular, the derived criterion is of Sawyer type [38]. As a corollary we have a transparent necessary and sufficient condition on $\alpha(\cdot)$ and v guaranteeing the trace inequality for $M_{\alpha(\cdot)}$.

THEOREM 5.1 *Let $\Omega = \mathbb{R}^n$ and let $1 < p < \infty$. Suppose that $w \in DC(\mathbb{R}^n)$. Then the inequality*

$$\|v M_{\alpha(\cdot)} f\|_{L^p(\mathbb{R}^n)} \leq c \|wf\|_{L^p(\mathbb{R}^n)}$$

holds if and only if

$$\int_Q v^p(x) (M_{\alpha(\cdot)}(w^{1-p'} \chi_Q))^p dx \leq c \int_Q w^{1-p'} < \infty.$$

Proof Since necessity is trivial we will show sufficiency. In fact we will prove that the inequality

$$\|v M_{\alpha(\cdot),u}^{(d)} f\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L_u^p(\mathbb{R}^n)} \quad (14)$$

holds if

$$\int_Q v^p(x) (M_{\alpha(\cdot),u} \chi_Q)^p dx \leq c \int_Q u < \infty, \quad (15)$$

where

$$(M_{\alpha(\cdot),u}^{(d)} f)(x) := \sup_{\substack{Q \ni x \\ Q \in \mathcal{D}(\mathbb{R}^n)}} \frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q |f(y)| u(y) dy.$$

Arguing as in the proof of Theorem 3.1 suppose that f is non-negative, bounded and has a bounded support. For every $x \in \mathbb{R}^n$ choose a dyadic cube Q_x , $Q_x \ni x$, so that

$$|Q_x|^{\alpha(x)/n-1} \int_{Q_x} f w > \frac{1}{2} (M_{\alpha(\cdot),u}^{(d)} f)(x). \quad (16)$$

Since f has a bounded support, for each x we can assume that Q_x is a 'maximal' dyadic cube satisfying (16). Further, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ introduce a set F_Q consisting of those x which is contained in Q and Q is 'maximal' for which (16) is valid. It is clear that if we take $x \in \mathbb{R}^n$, then $x \in F_Q$ for some $Q \in \mathcal{D}(\mathbb{R}^n)$. On the other hand, for each $Q \in \mathcal{D}(\mathbb{R}^n)$ we have a set F_Q which might be empty for some Q . It is also obvious that $F_Q \subset Q$ and that $F_{Q_1} \cap F_{Q_2} = \emptyset$ if $Q_1 \neq Q_2$. Let $\mathcal{D}_m := \{Q \in \mathcal{D}_m : F_Q \neq \emptyset\}$.

We have

$$\begin{aligned}
 \int_{\mathbb{R}^n} v^p(x) (M_{\alpha(\cdot), u}^{(d)} f)^p(x) dx &\leq 2^p \sum_{Q \in \mathcal{D}_m} \int_{F_Q} v(x) \left(\frac{1}{|Q|^{1-\alpha(x)/n}} \int_Q f u \right)^p dx \\
 &\leq c \sum_{Q \in \mathcal{D}_m} \int_{F_Q} v^p(x) |Q|^{\alpha(x)/n-1} dx \left(\int_Q f u \right)^p \\
 &= c \sum_{Q \in \mathcal{D}_m} \left(\int_{F_Q} v^p(x) |Q|^{\alpha(x)/n-1} dx \right) (u(Q))^p \left(\frac{1}{u(Q)} \int_Q f u \right)^p.
 \end{aligned}$$

At the same time we have

$$\begin{aligned}
 &\sum_{Q \subset Q'; Q, Q' \in D(\mathbb{R}^n)} \left(\int_{F_Q} v^p(x) |Q|^{\alpha(x)/n-1} dx \right) (u(Q))^p \\
 &\leq \sum_{Q \subset Q'; Q, Q' \in D(\mathbb{R}^n)} \int_{F_Q} v^p(x) |Q|^{\alpha(x)/n-1} (u(Q))^p dx \\
 &\leq \sum_{Q \subset Q'; Q, Q' \in D(\mathbb{R}^n)} \int_{F_Q} (M_{\alpha(\cdot)}(\chi_Q u))^p(x) v^p(x) dx \\
 &\leq c \int_{\cup_{Q \subset Q'} F_Q} (M_{\alpha(\cdot)}(\chi_Q u))^p(x) v^p(x) dx \\
 &\leq \int_{F_{Q'}} (M_{\alpha(\cdot)}(\chi_Q u))^p(x) v^p(x) dx \leq \int_Q (M_{\alpha(\cdot)}(\chi_Q u))^p(x) v^p(x) dx \leq c \int_Q u.
 \end{aligned}$$

Here we used the fact $\cup_{Q \subset Q'} F_Q \subset Q'$. Taking into account Lemma D we find that (15) implies (14). Now we can pass to $M_{\alpha(\cdot), u}$ in the same way as in the proof of Theorem 3.1. ■

From the latter statement we have

THEOREM 5.2 *Let $\Omega = \mathbb{R}^n$ and let $1 < p < \infty$. Then the inequality*

$$\|v M_{\alpha(\cdot)}\|_{L^p(\mathbb{R}^n)} \leq c \|f\|_{L^p(\mathbb{R}^n)}$$

holds if and only if

$$\sup_Q \left(\int_Q v^p(x) |Q|^{\alpha(x)p/n} dx \right)^{1/p} |Q|^{-1/p} < \infty.$$

6. Further remarks and observations

1. Note that if the condition

$$|I|^{1/p_+(I)} \leq c |I|^{1/p_-(J)} \quad (17)$$

holds for all $I, I \subset J$, where $\Omega := J$ is a bounded interval in $\mathbb{R}$, then

$$\sum_{I \in \mathcal{D}(J)} \| \chi_I \|_{L^{p'}(J)}^{-r} \left(\int_I f \right)^r \leq c \|f\|_{L^{p(\cdot)}(J)}, \quad r > p_-(J).$$

Indeed, first note that

$$\|\chi_I\|_{L^{p'_+}(J)}^{-r} \leq |I|^{-r/(p'_+ - (I))} = |I|^{-r/(p_+(I))'}.$$

Then

$$\begin{aligned} \sum_{I \subset I'; I, I' \in \mathcal{D}(J)} \|\chi_I\|_{L^{p'_+}(J)}^{-r} |I|^r &\leq c \sum_{I \subset I'; I, I' \in \mathcal{D}(J)} |I|^{r/p_+(I)} \leq \sum_{I \subset I'; I, I' \in \mathcal{D}(J)} |I|^{r/p_-(J)} \\ &\leq \sum_{k=1}^{+\infty} 2^{-k(r/p_-(J)-1)} |I'|^{r/p_-(J)} \leq c |I'|^{r/p_-(J)}. \end{aligned}$$

Therefore by Lemma D we have

$$\sum_{I \in \mathcal{D}(J)} |I|^r \|\chi_I\|_{L^{p'_+}(J)}^{-r} \left(\frac{1}{|I|} \int_I f \right)^r \leq c \left(\int_J f^{p_-(J)} \right)^{r/p_-(J)} \leq c \|f\|_{L^{p(\cdot)}(J)}^r.$$

Summarizing these observations with the arguments used in the proof of Theorem 3.1 we have

THEOREM 6.1 *Let $n = 1$ and let $\Omega := J$ be a bounded interval in $\mathbb{R}$. Suppose that $0 < \alpha_- \leq \alpha(x) \leq \alpha_+ < 1/p$; $1 < p_- \leq p_+(x) \leq p_+ < \infty$. Assume also that condition (17) is satisfied. Then the inequality*

$$\|vM_{\alpha(\cdot)}^J\|_{L^{q(\cdot)}(J)} \leq c \|f\|_{L^{p(\cdot)}(J)}$$

holds if and only if

$$\sup_{I \subset J} \|\chi_I |I|^{\alpha(\cdot)-1} v(\cdot)\|_{L^{q(\cdot)}(J)} \|\chi_I\|_{L^{p'(\cdot)}(J)} < \infty.$$

2. If we require that $\alpha(\cdot) \in DL(J)$, where J is an open interval in $\mathbb{R}$, then we can obtain Sawyer-type two- weight criterion for $M_{\alpha(\cdot)}^J$ without any additional restriction on w . Namely the next statement holds

THEOREM 6.2 *Let $n = 1$ and let $\Omega := J$ be a bounded interval. Suppose that $p = \text{const}$ and $1 < p < \infty$. Assume that $\alpha(\cdot) \in DL(J)$. Then the inequality*

$$\|vM_{\alpha(\cdot)}^J f\|_{L^p(J)} \leq c \|wf\|_{L^p(J)}$$

holds if and only if

$$\int_I v^p(x) (M_{\alpha(\cdot)}^J w^{-p'} \chi_I)(x) dx \leq c \int_I w^{-p'} < \infty$$

for all $I \subseteq J$.

Proof Since necessity is trivial, we show sufficiency. Without loss of generality we assume that J is a dyadic interval itself and all dyadic intervals from $\mathcal{D}(J)$ are contained in J . We begin

again by dyadic maximal operator

$$(M_{\alpha(\cdot)}^{J,(d)} f)(x) := \sup_{\substack{I \ni x \\ I \in \mathcal{D}(J)}} \frac{1}{|I|^{1-\alpha(x)}} \int_I |f(y)| dy.$$

One of the keys in the proof is

$$(M_{\alpha(\cdot)}^{J,(d)} f)(x) \approx (\bar{M}_{\alpha(\cdot)}^{J,(d)} f)(x), \quad (18)$$

where

$$(\bar{M}_{\alpha(\cdot)}^{J,(d)} f)(x) = \sup_{\substack{I \ni x \\ I \in \mathcal{D}(J)}} |I|^{\alpha_-(I)-1} \int_I |f|.$$

Relation (18) holds, because of the condition $\alpha(\cdot) \in \text{DL}(J)$. Let f be non-negative and bounded. We follow to [38] (see also [41]). For each integer k we take a family of ‘minimal’ (instead of ‘maximal’) dyadic intervals $\{I_j^k\}$ such that

$$|I_j^k|^{\alpha_-(I_j^k)-1} \int_{I_j^k} f > 2^k.$$

Such ‘minimal’ dyadic intervals exist because $|I|^{\alpha_-(I)-1} \int_I f \leq c |I|^{\alpha_-(I)} \rightarrow 0$ as $|I| \rightarrow 0$. Further, observe that

- (i) $\{M_{\alpha(\cdot)}^{J,(d)} f > 2^k\} = \cup I_j^k$;
- (ii) I_j^k are non-overlapping for each k ;
- (iii) $2^k < \frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int_{I_j^k} f$

for all j and k .

Let us denote

$$J_k := \{x \in J : 2^k < (\bar{M}_{\alpha(\cdot)}^{J,(d)} f)(x) \leq 2^{k+1}\}; \quad E_j^k := I_j^k \setminus \{(\bar{M}_{\alpha(\cdot)}^{J,(d)} f)(x) > 2^{k+1}\}.$$

We have $(\sigma := w^{-p'})$

$$\begin{aligned} & \int_J v(x) (M_{\alpha(\cdot)}^{J,(d)} f)^p(x) dx \\ & \leq c \sum_{k,j} 2^{kp} \int_{E_j^k} v^p \leq c \left(\frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int_{I_j^k} \sigma \right)^p \left(\frac{1}{\sigma(I_j^k)} \int_{I_j^k} \frac{f}{\sigma} \sigma \right)^p \int_{E_j^k} v^p \\ & := \int_X T(f/\sigma)^p d\omega, \end{aligned}$$

where $\mathbb{X} = \mathbb{N} \times \mathbb{Z}$, the measure ω on $\mathbb{X}$ is

$$\omega(j, k) := \int_{E_j^k} v^p \left(\frac{1}{|I_j^k|^{1-\alpha_-(I_j^k)}} \int_{I_j^k} \sigma \right)^p$$

and the operator T is given by

$$(Tg)(j, k) = \frac{\int_{I_j^k} g \sigma}{\sigma(I_j^k)}.$$

It is obvious that the operator T is bounded in $L^\infty(\mathbb{X})$. Let us check that T is weak-type $(1, 1)$. Fix a bounded function g and a positive number λ . Let

$$F_\lambda = \{(j, k) \in \mathbb{X} : Tg(j, k) > \lambda\}.$$

Denote by I_i the maximal disjoint subcollection of $\{I_j^k : (j, k) \in F_\lambda\}$. Since $E_j^k \subset I_j^k$, using (18) we find that

$$\begin{aligned} \omega(F_\lambda) &:= \sum_{(j,k) \in F_\lambda} \int_{E_j^k} v^p(x) dx \left(\frac{1}{|I_j^k|^{1-\alpha-(I_j^k)}} \int_{I_j^k} \sigma \right)^p \\ &\leq \sum_{(j,k) \in F_\lambda} \int_{E_j^k} (M_{\alpha(\cdot)}^J \sigma \chi_{I_j^k})^p v^p(x) dx \leq \sum_i \sum_{I_j^k \subset I_i} \int_{E_j^k} (M_{\alpha(\cdot)}^J \sigma \chi_{I_j^k})^p v^p(x) dx \\ &\leq c \sum_i \int_{I_i} \sigma \leq c \frac{1}{\lambda} \sum_i \int_{I_i} g \sigma \leq c \frac{1}{\lambda} \int_J g \sigma. \end{aligned}$$

Hence, T is weak-type $(1, 1)$ from which using Marcinkiewicz interpolation we will obtain the desired result for $M_{\alpha(\cdot)}^{J,(d)}(\cdot)$.

Now we can pass to the operator $M_{\alpha(\cdot)}^J$ in the same manner as in the proof of Theorem 3.1 (see also the proof of Corollary 3.2). ■

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On the fourth power moment of the function $\zeta_\lambda(s)$

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We obtain the asymptotics for the fourth power moment of the periodic zeta-function.

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1. Introduction

Let $s = \sigma + it$ be a complex variable. The periodic zeta-function $\zeta_\lambda(s)$ is defined, for $\sigma > 1$, by

$$\zeta_\lambda(s) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \lambda m}}{m^s},$$

where λ is a real parameter, and by analytic continuation elsewhere. If $\lambda \in \mathbb{Z}$, then $\zeta_\lambda(s)$ becomes the Riemann zeta-function $\zeta(s)$. Therefore, we suppose that $0 < \lambda < 1$. Moreover, $\zeta_\lambda(s)$ is closely connected to the Lerch zeta-function $L(\lambda, \alpha, s)$, where α is a fixed parameter, $0 < \alpha \leq 1$. We recall that, for $\sigma > 1$,

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}$$

and if $0 < \lambda < 1$, then $L(\lambda, \alpha, s)$ is analytically continuable to an entire function. For integer λ , the function $L(\lambda, \alpha, s)$ reduces to the Hurwitz zeta-function $\zeta(s, \alpha)$.

Clearly, we have

$$\zeta_\lambda(s) = e^{2\pi i \lambda} L(\lambda, 1, s). \quad (1)$$

Hence, for $0 < \lambda < 1$, the function $\zeta_\lambda(s)$ also is entire.

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In [1], see also [2], it was proved that, as T tends to infinity,

$$\int_1^T \left| L\left(\lambda, \alpha, \frac{1}{2} + it\right) \right|^2 dt = T \log T + T(c(\alpha) + c(\lambda) - 1 - \log 2\pi) + O(T^{(1/2)} \log T),$$

and, for $(1/2) < \sigma < 1$,

$$\begin{aligned} \int_1^T |L(\lambda, \alpha, \sigma + it)|^2 dt &= \zeta(2\sigma, \alpha)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \zeta(2-2\sigma, \lambda)T^{2-2\sigma} \\ &\quad + O(T^{1-\sigma} \log T) + O(T^{(\sigma/2)}). \end{aligned}$$

Here the constant $c(\alpha)$ is defined by

$$\sum_{m=0}^n \frac{1}{m+\alpha} = \log n + c(\alpha) + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty.$$

Since $c(1)$ is the Euler constant γ , hence and from equality (1) we obtain that, as $T \rightarrow \infty$,

$$\int_1^T \left| \zeta_\lambda\left(\frac{1}{2} + it\right) \right|^2 dt = T \log T + T(c(\lambda) + \gamma - 1 - \log 2\pi) + O(T^{(1/2)} \log T),$$

and, for $(1/2) < \sigma < 1$, in view of the equality $\zeta(s, 1) = \zeta(s)$,

$$\begin{aligned} \int_1^T |\zeta_\lambda(\sigma + it)|^2 dt &= \zeta(2\sigma)T + \frac{(2\pi)^{2\sigma-1}}{2-2\sigma} \zeta(2-2\sigma, \lambda)T^{2-2\sigma} \\ &\quad + O(T^{1-\sigma} \log T) + O(T^{(\sigma/2)}). \end{aligned}$$

In this note, we obtain the asymptotics for the fourth power moment of the function $\zeta_\lambda(s)$.

THEOREM 1.1 *Suppose that λ is irrational, $0 < \lambda < 1$. Then, for $(1/2) < \sigma < 1$,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_1^T |\zeta_\lambda(\sigma + it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta^4(4\sigma)} - 2 \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}}.$$

Since, for $\lambda \in \mathbb{Z}$, $\zeta_\lambda(s) = \zeta(s)$ and $\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2) = 0$, the assertion of Theorem 1 is also valid for an integer λ , Theorem 7.5 of [3].

2. Approximate functional equation

For the proof of Theorem 1.1, as usual, we will apply an approximate functional equation for the function $\zeta_\lambda(s)$. Denote by $[u]$ the integer part of u , and define

$$\begin{aligned} m(t) &= \left[\sqrt{\frac{t}{2\pi}} - 1 \right], \quad q(t) = \left[\sqrt{\frac{t}{2\pi}} \right], \\ g(\lambda, t) &= 2\sqrt{\frac{t}{2\pi}} - m(t) - q(t) - \lambda - 1, \\ f(\lambda, t) &= -\frac{t}{2\pi} \log \frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(1 - \lambda^2) + m(t) - q(t) + 2\sqrt{\frac{t}{2\pi}}(q(t) \\ &\quad - m(t) + \lambda - 1) - \frac{1}{2}(q(t) + m(t)) - \lambda(1 + q(t) - m(t)) \end{aligned}$$

and

$$\psi(z) = \frac{\cos \pi((z^2/2) - z - (1/8))}{\cos \pi z}.$$

LEMMA 2.1 *Let $0 < \lambda < 1$ and $0 \leq \sigma \leq 1$. Then, for $t \geq t_0$,*

$$\begin{aligned} \zeta_\lambda(s) &= \sum_{1 \leq m \leq m(t)} \frac{e^{2\pi i \lambda m}}{m^s} + \left(\frac{t}{2\pi}\right)^{(1/2)-\sigma-it} e^{it+(\pi i/4)} \sum_{0 \leq m \leq q(t)} \frac{1}{(m+\lambda)^{1-s}} \\ &\quad + \left(\frac{t}{2\pi}\right)^{-(\sigma/2)} e^{\pi i f(\lambda, t) + 2\pi i \lambda} \psi(g(\lambda, t)) + O(t^{(\sigma/2)-1}). \end{aligned}$$

Proof In [4], see also [2], under the same hypotheses, an approximate functional equation for the Lerch zeta-function has been proved. Using this equation and taking into account equality (1), we obtain the assertion of the lemma. ■

Since the function $\psi(z)$ is bounded, we have by Lemma 2.1

$$\zeta_\lambda(s) = S_1(s) + S_2(s) + O(t^{-(1/4)}), \quad (2)$$

where

$$S_1(s) = \sum_{1 \leq m \leq m(t)} \frac{e^{2\pi i \lambda m}}{m^s}$$

and

$$S_2(s) = \left(\frac{t}{2\pi}\right)^{(1/2)-\sigma-it} e^{it+(\pi i/4)} \sum_{0 \leq m \leq q(t)} \frac{1}{(m+\lambda)^{1-s}}.$$

3. Fourth moment of S_1

In this section, we obtain the asymptotics for the fourth power moment of the sum $S_1(s)$.

LEMMA 3.1 *Let $T \rightarrow \infty$. Then, for $(1/2) < \sigma < 1$,*

$$\begin{aligned} &\int_1^T |S_1(\sigma + it)|^4 dt \\ &= T \left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 = m_2 + n_2}} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) (1 + o(1)). \end{aligned}$$

Proof We use similar arguments as in case of the Riemann zeta-function. We have

$$\begin{aligned} |S_1(\sigma + it)|^4 &= \sum_{m_1} \frac{e^{2\pi i \lambda m_1}}{m_1^{\sigma+it}} \sum_{n_1} \frac{e^{2\pi i \lambda n_1}}{n_1^{\sigma+it}} \sum_{m_2} \frac{e^{-2\pi i \lambda m_2}}{m_2^{\sigma-it}} \sum_{n_2} \frac{e^{-2\pi i \lambda n_2}}{n_2^{\sigma-it}} \\ &= \sum_{m_1, n_1, m_2, n_2} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \left(\frac{m_2 n_2}{m_1 n_1} \right)^{it}, \end{aligned}$$

where in each sum we sum over $[1, m(t)]$. Therefore,

$$\begin{aligned} \int_1^T |S_1(\sigma + it)|^4 dt &= \int_1^T \sum_{m_1, n_1, m_2, n_2} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \left(\frac{m_2 n_2}{m_1 n_1} \right)^{it} dt \\ &= \sum_{1 \leq m_1, n_1, m_2, n_2 \leq m(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1 m_2 n_2)^\sigma} \int_{T_1}^T \left(\frac{m_2 n_2}{m_1 n_1} \right)^{it} dt, \end{aligned}$$

where $T_1 = 2\pi \max((m_1 + 1)^2, (n_1 + 1)^2, (m_2 + 1)^2, (n_2 + 1)^2)$. Hence, we obtain that

$$\begin{aligned} \int_1^T |S_1(\sigma + it)|^4 dt &= \sum_{m_1 n_1 = m_2 n_2}^* \frac{(T - T_1) e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} \\ &\quad + O \left(\sum_{m_1 n_1 \neq m_2 n_2}^* \frac{|\log(m_2 n_2 / m_1 n_1)|^{-1}}{(m_1 n_1 m_2 n_2)^\sigma} \right), \end{aligned} \quad (3)$$

the star $*$ means that the summing runs over $m_1, n_1, m_2, n_2 \in [1, m(T)]$.

Denote by

$$d(k) = \sum_{d|k} 1, \quad k \in \mathbb{N},$$

the divisor function. Let $N(k)$ be the number of solutions of the equation $m_1 n_1 = m_2 n_2 = k$. Then, we have that

$$N(k) = d^2(k), \quad \text{if } k \leq u, \quad m_1, n_1, m_2, n_2 \leq u,$$

and

$$N(k) \leq d^2(k), \quad \text{if } k \geq u, \quad m_1, n_1, m_2, n_2 \leq u.$$

It is well known that, for $\sigma > (1/2)$,

$$\sum_{k=1}^{\infty} \frac{d^2(k)}{k^{2\sigma}} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)},$$

see [3], formula 1.2.10. Therefore, since $d(k) \ll k^\varepsilon$ with arbitrary $\varepsilon > 0$,

$$\begin{aligned}
& \sum_{m_1 n_1 = m_2 n_2}^* \frac{T e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} \\
&= T \sum_{m_1 n_1 = m_2 n_2 \leq m(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} + T \sum_{m(T) < m_1 n_1 = m_2 n_2 \leq m^2(T)}^* \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} \\
&= T \sum_{m_1 n_1 = m_2 n_2 \leq m(T)} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} + O\left(T \sum_{k \geq m(T)} \frac{d^2(k)}{k^{2\sigma}}\right) \\
&= T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 = m_2 + n_2}} \frac{1}{(m_1 n_1)^{2\sigma}} + T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{e^{2\pi i \lambda (m_1 + n_1 - m_2 - n_2)}}{(m_1 n_1)^{2\sigma}} + O(T^{(3/2)-\sigma+\varepsilon}) \\
&= T \sum_{m_1 n_1 = m_2 n_2} \frac{1}{(m_1 n_1)^{2\sigma}} - T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{1 - \cos 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \\
&\quad + iT \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} + O(T^{(3/2)-\sigma+\varepsilon}) \\
&= T \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - T \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{1 - \cos 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \\
&\quad + iT \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin 2\pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} + O(T^{(3/2)-\sigma+\varepsilon}). \tag{4}
\end{aligned}$$

By the definition of T_1 and symmetry

$$\begin{aligned}
& \sum_{m_1 n_1 = m_2 n_2}^* \frac{T_1}{(m_1 n_1)^{2\sigma}} \leq 2\pi \sum_{m_1 n_1 = m_2 n_2}^* \frac{(m_1 + 1)^2 + (n_1 + 1)^2 + (m_2 + 1)^2 + (n_2 + 1)^2}{(m_1 n_1 m_2 n_2)^\sigma} \\
&= O\left(\sum_{m_1 n_1 = m_2 n_2}^* \frac{m_1^2}{(m_1 n_1 m_2 n_2)^\sigma}\right) = O\left(\sum_{m_1, n_1}^* \frac{m_1^2 d(m_1 n_1)}{(m_1 n_1)^{2\sigma}}\right) \\
&= O\left(T^\varepsilon \sum_{m_1 \leq m(T)} \frac{1}{m_1^{2\sigma-2}} \sum_{n_1 \leq m(T)} \frac{1}{n_1^{2\sigma}}\right) \\
&= O(T^\varepsilon (T^{(1/2)(3-2\sigma)}) = O(T^{(3/2)-\sigma+\varepsilon}). \tag{5}
\end{aligned}$$

Using the estimate

$$\sum_{0 < m < n \leq T} \frac{1}{m^\sigma n^\sigma \log(n/m)} = O(T^{2-2\sigma} \log T), \quad \frac{1}{2} \leq \sigma < 1,$$

see [3], section 7.2, we find that

$$\begin{aligned} \sum_{m_1 n_1 \neq m_2 n_2}^* \frac{|\log(m_2 n_2 / m_1 n_1)|^{-1}}{(m_1 n_1 m_2 n_2)^\sigma} &= O\left(\sum_{0 < m < n \leq m^2(T)} \sum \frac{d(m)d(n)}{(mn)^\sigma \log(n/m)}\right) \\ &= O\left(T^\varepsilon \sum_{0 < m < n \leq m^2(T)} \sum \frac{1}{(mn)^\sigma \log(n/m)}\right) = O(T^{2-2\sigma+\varepsilon}). \end{aligned}$$

Since $\sigma > (1/2)$, and the left-hand side of the formula of the lemma is real, this and estimates (3)–(5) show that

$$\begin{aligned} \int_1^T |S_1(\sigma + it)|^4 dt &= T \left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) \\ &\quad + O(T^{(3/2)-\sigma+\varepsilon}), \end{aligned}$$

and the lemma is proved. ■

4. Estimate for the fourth moment of S_2

In this section we will deal with $S_2(s)$. For the proof of Theorem 1.1 it suffices to obtain a suitable estimate for the fourth moment of $S_2(s)$. For us the following statement will be useful.

LEMMA 4.1 *Let $u_1, \dots, u_r$ be complex numbers, $\lambda_1, \dots, \lambda_r$ be distinct real numbers, and*

$$\delta_m = \min_{n \neq m} |\lambda_n - \lambda_m|.$$

Then

$$\sum_{m,n=1}^r u_m \bar{u}_n (\lambda_n - \lambda_m)^{-1} \ll \sum_{m=1}^r |u_m|^2 \delta_m^{-1}.$$

Proof The lemma is a part of the Montgomery–Vaughan theorem, Theorem 2 from [5]. ■

LEMMA 4.2 *Let $T \rightarrow \infty$, $\frac{1}{2} < \sigma < 1$, and let λ be irrational. Then, for arbitrary $\varepsilon > 0$,*

$$\int_1^T |S_2(\sigma + it)|^4 dt \ll_{\lambda} T^{2-2\sigma+\varepsilon}.$$

Proof Denote

$$Z(T) = \int_1^T \left| \sum_{0 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-\sigma-it}} \right|^4 dt.$$

Clearly, we have that

$$Z(T) \ll \int_1^T \left| \frac{1}{\lambda^{1-\sigma-it}} \right|^4 dt + Z_1(T) \ll_\lambda T + Z_1(T), \quad (6)$$

where

$$Z_1(T) = \int_1^T \left| \sum_{1 \leq m \leq q(t)} \frac{1}{(m + \lambda)^{1-\sigma-it}} \right|^4 dt.$$

As in the case of $S_1(s)$, we obtain that

$$\begin{aligned} Z_1(T) &= \int_1^T \sum_{m_1, n_1, m_2, n_2} \frac{1}{((m_1 + \lambda)(n_1 + \lambda)(m_2 + \lambda)(n_2 + \lambda))^{1-\sigma}} \\ &\quad \times \left(\frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right)^{it} dt, \end{aligned}$$

where each m_1, n_1, m_2, n_2 runs over $[1, q(t)]$. Hence

$$\begin{aligned} Z_1(T) &= \sum_{1 \leq m_1, n_1, m_2, n_2 \leq q(T)} \frac{1}{((m_1 + \lambda)(n_1 + \lambda)(m_2 + \lambda)(n_2 + \lambda))^{1-\sigma}} \\ &\quad \times \int_{T_2}^T \left(\frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right)^{it} dt, \end{aligned}$$

where $T_2 = 2\pi \max(m_1^2, n_1^2, m_2^2, n_2^2)$. Since λ is irrational, we have that $(m_1 + \lambda)(n_1 + \lambda) = (m_2 + \lambda)(n_2 + \lambda)$ if and only if $m_1 n_1 = m_2 n_2$ and $m_1 + n_1 = m_2 + n_2$. Therefore,

$$\begin{aligned} Z_1(T) &\ll \sum_{m_1 n_1 = m_2 n_2}^* \frac{T - T_2}{((m_1 + \lambda)(n_1 + \lambda))^{2-2\sigma}} \\ &\quad + \sum_{(m_1 + \lambda)(n_1 + \lambda) \neq (m_2 + \lambda)(n_2 + \lambda)}^* \frac{1}{(m_1 n_1 m_2 n_2)^{1-\sigma} |\log((m_1 + \lambda)(n_1 + \lambda)/(m_2 + \lambda)(n_2 + \lambda))|}. \end{aligned} \quad (7)$$

where the star $*$ means that the summing runs over $m_1, n_1, m_2, n_2 \in [1, q(T)]$. Similarly as above, we find that

$$\sum_{m_1 n_1 = m_2 n_2}^* \frac{T}{((m_1 + \lambda)(n_1 + \lambda))^{2-2\sigma}} \ll T \sum_{k \leq q^2(T)} \frac{d^2(k)}{k^{2-2\sigma}} \ll T^{2\sigma+\varepsilon} \quad (8)$$

and

$$\begin{aligned} \sum_{m_1 n_1 = m_2 n_2}^* \frac{T_2}{((m_1 + \lambda)(n_1 + \lambda))^{2-2\sigma}} &\ll \sum_{m_1, n_1 \leq q(T)} \frac{m_1^2 d(m_1 n_1)}{(m_1 n_1)^{2-2\sigma}} \\ &\ll T^\varepsilon \left(\sum_{m_1 \leq q(T)} m_1^{2\sigma} \sum_{n_1 \leq q(T)} \frac{1}{n_1^{2-2\sigma}} \right) \ll T^{2\sigma+\varepsilon}. \end{aligned} \quad (9)$$

For the estimation of the second sum in estimate (7), we will apply Lemma 4.1.

It is not difficult to see that, for $(m_1 + \lambda)(n_1 + \lambda) \neq (m_2 + \lambda)(n_2 + \lambda)$,

$$\left| \log \frac{(m_1 + \lambda)(n_1 + \lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right| \geq \min \left(\frac{c(\lambda)}{(m_1 + \lambda)(n_1 + \lambda)}, \frac{c(\lambda)}{(m_2 + \lambda)(n_2 + \lambda)} \right),$$

where $c(\lambda)$ is a positive constant. This and Lemma 4.1 show that

$$\sum_{\substack{m_1 n_1 \neq m_2 n_2 \\ (m_1 + \lambda)(n_1 + \lambda) \neq (m_2 + \lambda)(n_2 + \lambda)}}^* \frac{1}{(m_1 n_1 m_2 n_2)^{1-\sigma} |\log((m_1 + \lambda)(n_1 + \lambda)/(m_2 + \lambda)(n_2 + \lambda))|} \\ \ll_{\lambda} T^{\varepsilon} \sum_{1 \leq m \leq q^2(T)} \frac{m}{m^{2-2\sigma}} \ll_{\lambda} T^{2\sigma+\varepsilon},$$

and

$$\sum_{\substack{m_1 n_1 = m_2 n_2 \\ (m_1 + \lambda)(n_1 + \lambda) \neq (m_2 + \lambda)(n_2 + \lambda)}}^* \frac{1}{(m_1 n_1 m_2 n_2)^{1-\sigma} |\log((m_1 + \lambda)(n_1 + \lambda)/(m_2 + \lambda)(n_2 + \lambda))|} \\ \ll_{\lambda} \sum_{k \leq q^2(T)} \frac{k d^2(k)}{k^{2-2\sigma}} \ll T^{2\sigma+\varepsilon}.$$

Two last estimates together with estimates (7)–(9) yield

$$Z_1(T) \ll_{\lambda} T^{2\sigma+\varepsilon}.$$

Hence, in view of estimate (6),

$$Z(T) \ll_{\lambda} T^{2\sigma+\varepsilon}. \quad (10)$$

Now, using the definition of $S_2(s)$ and estimate (10), we obtain

$$\begin{aligned} \int_1^T |S_2(\sigma + it)|^4 dt &\ll \int_1^T t^{2-4\sigma} dZ(t) \\ &= t^{2-4\sigma} Z(t) \Big|_1^T + (4\sigma - 2) \int_1^T Z(t) t^{1-4\sigma} dt \\ &\ll_{\lambda} T^{2-2\sigma+\varepsilon} + \int_1^T t^{1-2\sigma+\varepsilon} dt \ll_{\lambda} T^{2-2\sigma+\varepsilon}. \end{aligned}$$

The lemma is proved. ■

5. Proof of Theorem 1.1

By equality (2),

$$\begin{aligned}
 |\zeta_\lambda(s)|^4 &= |S_1(s) + S_2(s) + O(t^{-(1/4)})|^4 \\
 &= (S_1(s) + S_2(s) + O(t^{-(1/4)}))^2 (\overline{S_1(s)} + \overline{S_2(s)} + O(t^{-(1/4)}))^2 \\
 &= (S_1^2(s) + S_2^2(s) + 2S_1(s)S_2(s) + O(|S_1(s)|t^{-(1/4)}) + O(|S_2(s)|t^{-(1/4)}) \\
 &\quad + O(t^{-(1/2)})) \times (\overline{S_1^2(s)} + \overline{S_2^2(s)} + 2\overline{S_1(s)}\overline{S_2(s)} + O(|S_1(s)|t^{-(1/4)}) \\
 &\quad + O(|S_2(s)|t^{-(1/4)}) + O(t^{-(1/2)})) = |S_1(s)|^4 + S_1^2(s)\overline{S_2^2(s)} \\
 &\quad + 2S_1^2(s)\overline{S_1(s)}\overline{S_2(s)} + \overline{S_1^2(s)}S_2^2(s) + |S_2(s)|^4 \\
 &\quad + 2\overline{S_1(s)}S_2^2(s)\overline{S_2(s)} + 2S_1(s)\overline{S_1^2(s)}S_2(s) + 2S_1(s)\overline{S_2^2(s)}S_2(s) \\
 &\quad + 4|S_1(s)|^2|S_2(s)|^2 + O(|S_1(s)|^3t^{-(1/4)}) + O(|S_1(s)|^2|S_2(s)|t^{-(1/4)}) \\
 &\quad + O(|S_1(s)|^2t^{-(1/2)}) + O(|S_1(s)||S_2(s)|^2t^{-(1/4)}) + O(|S_2(s)|^3t^{-(1/4)}) \\
 &\quad + O(|S_2(s)|^2t^{-(1/2)}) + O(|S_1(s)||S_2(s)|t^{-(1/2)}) + O(|S_1(s)|t^{-(3/4)}) \\
 &\quad + O(|S_2(s)|t^{-(3/4)}) + O(t^{-1}).
 \end{aligned} \tag{11}$$

Clearly, in virtue of the Cauchy–Schwarz inequality, and Lemmas 3.1 and 4.1, we have

$$\begin{aligned}
 &\int_1^T (S_1^2(\sigma + it)\overline{S_2^2(\sigma + it)} + \overline{S_1^2(\sigma + it)}S_2^2(\sigma + it) + 4|S_1^2(\sigma + it)|^2|S_2^2(\sigma + it)|^2) dt \\
 &\ll \left(\int_1^T |S_1(\sigma + it)|^4 dt \int_1^T |S_2(\sigma + it)|^4 dt \right)^{(1/2)} \ll_\lambda T^{(3/2)-\sigma+\varepsilon}, \\
 &\int_1^T (2S_1^2(\sigma + it)\overline{S_1(\sigma + it)}\overline{S_2(\sigma + it)} + 2S_1(\sigma + it)\overline{S_1^2(\sigma + it)}S_2(\sigma + it) \\
 &\quad + 2\overline{S_1(\sigma + it)}S_2^2(\sigma + it)\overline{S_2(\sigma + it)} + 2S_1(\sigma + it)\overline{S_2^2(\sigma + it)}S_2(\sigma + it)) dt \\
 &\ll \left(\int_1^T |S_1(\sigma + it)|^2|S_2(\sigma + it)|^2 dt \right)^{(1/2)} \left(\left(\int_1^T |S_1(\sigma + it)|^4 dt \right)^{(1/2)} \right. \\
 &\quad \left. + \left(\int_1^T |S_2(\sigma + it)|^4 dt \right)^{(1/2)} \right) \ll_\lambda \left(\int_1^T |S_1(\sigma + it)|^4 dt \int_1^T |S_2(\sigma + it)|^4 dt \right)^{(1/4)} \\
 &\quad \times (T^{(1/2)} + T^{1-\sigma+\varepsilon}) \ll_\lambda T^{(3/4)-(\sigma/2)+\varepsilon} (T^{(1/2)} + T^{1-\sigma+\varepsilon}) \ll_\lambda T^{(5/4)-(\sigma/2)+\varepsilon}, \\
 &\int_1^T |S_1(\sigma + it)|^3 t^{-(1/4)} dt \ll \left(\int_1^T |S_1(\sigma + it)|^4 dt \int_1^T |S_1(\sigma + it)|^2 t^{-(1/2)} dt \right)^{(1/2)} \\
 &\ll T^{(3/4)+\varepsilon},
 \end{aligned}$$

$$\begin{aligned}
& \int_1^T |S_2(\sigma + it)|^3 t^{-(1/4)} dt \\
& \ll \left(\int_1^T |S_2(\sigma + it)|^4 dt \int_1^T |S_2(\sigma + it)|^2 t^{-(1/2)} dt \right)^{(1/2)} \\
& \ll_{\lambda} T^{(3/2)-(3/2)\sigma+\varepsilon}, \\
& \int_1^T |S_1(\sigma + it)| |S_2(\sigma + it)|^2 t^{-(1/4)} dt \\
& \ll \left(\int_1^T |S_1(\sigma + it)|^2 t^{-(1/2)} dt \int_1^T |S_2(\sigma + it)|^4 dt \right)^{(1/2)} \ll_{\lambda} T^{(5/4)-\sigma+\varepsilon}, \\
& \int_1^T |S_1(\sigma + it)|^2 |S_2(\sigma + it)| t^{-(1/4)} dt \ll \left(\int_1^T |S_1(\sigma + it)|^4 dt \right. \\
& \quad \times \left. \int_1^T |S_2(\sigma + it)|^2 t^{-(1/2)} dt \right)^{(1/2)} \ll_{\lambda} T^{1-(\sigma/2)+\varepsilon}, \\
& \int_1^T |S_1(\sigma + it)| S_2(\sigma + it) |t^{-(1/2)} dt \ll \left(\int_1^T |S_1(\sigma + it)|^2 t^{-(1/2)} dt \right. \\
& \quad \times \left. \int_1^T |S_2(\sigma + it)|^2 t^{-(1/2)} dt \right)^{(1/2)} \ll_{\lambda} T^{(3/4)-(\sigma/2)+\varepsilon}.
\end{aligned}$$

All these estimates together with Lemmas 3.1 and 4.1 give, for $T \rightarrow \infty$,

$$\begin{aligned}
& \int_1^T |\zeta_{\lambda}(\sigma + it)|^4 dt \\
& = T \left(\frac{\zeta^4(2\sigma)}{\zeta(4\sigma)} - 2 \sum_{\substack{m_1 n_1 = m_2 n_2 \\ m_1 + n_1 \neq m_2 + n_2}} \frac{\sin^2 \pi \lambda (m_1 + n_1 - m_2 - n_2)}{(m_1 n_1)^{2\sigma}} \right) + o(T).
\end{aligned}$$

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Some inclusion properties for certain subclasses of strongly starlike and strongly convex functions involving a family of fractional integral operators

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A known family of fractional integral operators (with the Gauss hypergeometric function in the kernel) is used here to define some new subclasses of strongly starlike and strongly convex functions of order β and type α in the open unit disk $\mathbb{U}$. For each of these new function classes, several inclusion relationships associated with the fractional integral operators are established. Some interesting corollaries and consequences of the main inclusion relationships are also considered.

Keywords: Analytic functions; Starlike functions; Convex functions; Close-to-convex functions; Strongly starlike functions; Strongly convex functions; Gauss hypergeometric function; Fractional integral operators; Inclusion properties

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1. Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Suppose also that $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ and $\mathcal{C}(\alpha)$ denote, respectively, the subclasses of $\mathcal{A}$ consisting of functions, which are starlike of order α ($0 \leq \alpha < 1$) in $\mathbb{U}$, convex of order α ($0 \leq \alpha < 1$) in $\mathbb{U}$ and close-to-convex of order α ($0 \leq \alpha < 1$) in $\mathbb{U}$. If $f(z) \in \mathcal{A}$ satisfies the following

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inequality:

$$\left| \arg \left(\frac{zf'(z)}{f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{U}; 0 \leq \alpha < 1; 0 < \beta \leq 1), \quad (2)$$

then the function $f(z)$ is said to be strongly starlike of order β and type α in $\mathbb{U}$. We denote this subclass of $\mathcal{A}$ by $\mathcal{S}_s^*(\alpha, \beta)$. If, on the other hand, $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\left| \arg \left(1 + \frac{zf''(z)}{f'(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{U}; 0 \leq \alpha < 1; 0 < \beta \leq 1), \quad (3)$$

then the function $f(z)$ is said to be strongly convex of order β and type α in $\mathbb{U}$. We denote this subclass of $\mathcal{A}$ by $\mathcal{K}_c(\alpha, \beta)$. It is obvious that

$$f(z) \in \mathcal{K}_c(\alpha, \beta) \iff zf'(z) \in \mathcal{S}_s^*(\alpha, \beta).$$

We also observe that

$$\mathcal{S}_s^*(0, \beta) = \mathcal{S}_s^*(\beta) \quad \text{and} \quad \mathcal{K}_c(0, \beta) = \mathcal{K}_c(\beta),$$

where $\mathcal{S}_s^*(\beta)$ and $\mathcal{K}_c(\beta)$ are, respectively, the classes of strongly starlike functions of order β ($0 < \beta \leq 1$) in $\mathbb{U}$ and strongly convex functions of order β ($0 < \beta \leq 1$) in $\mathbb{U}$. Furthermore, we have the following relationships:

$$\mathcal{S}_s^*(\alpha, 1) = \mathcal{S}^*(\alpha) \quad \text{and} \quad \mathcal{K}_c(\alpha, 1) = \mathcal{K}(\alpha)$$

(see, for details, [1] and [2]; see also [3]).

We recall here the following family of generalized fractional integral operators $I_{0,z}^{\lambda,\mu,\eta}$ due to Srivastava *et al.* [4] (see also [5]).

DEFINITION 1 Let $\lambda > 0$ and $\mu, \eta \in \mathbb{R}$. Then the fractional integral operator $I_{0,z}^{\lambda,\mu,\eta}$ is defined by

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} {}_2F_1 \left(\lambda + \mu, -\eta; \lambda; 1 - \frac{t}{z} \right) f(t) dt, \quad (4)$$

where the function $f(z)$ is analytic in a simply-connected region of the complex z -plane containing the origin, with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0; \varepsilon > \max\{0, \mu - \eta\} - 1),$$

it being understood that $(z-t)^{\lambda-1}$ denotes the principal value satisfying the following inequality

$$0 \leq \arg(z-t) < 2\pi.$$

The function ${}_2F_1$ occurring in the kernel of equation (4) is the familiar Gauss hypergeometric function (see, for example, [2]).

DEFINITION 2 Under the hypothesis of Definition 1, let

$$\lambda > 0 \quad \text{and} \quad \min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2.$$

Then the fractional integral operator

$$J_{0,z}^{\lambda,\mu,\eta} f(z) : \mathcal{A} \rightarrow \mathcal{A}$$

is defined by

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)} z^\mu I_{0,z}^{\lambda,\mu,\eta} f(z). \quad (5)$$

LEMMA 1 (see [4, p. 415, Lemma 3]). If $\lambda > 0$ and $n > \mu - \eta - 1$, then

$$J_{0,z}^{\lambda,\mu,\eta} z^n = \frac{\Gamma(n+1) \Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1) \Gamma(n+\lambda+\eta+1)} z^{n-\mu}. \quad (6)$$

If $f(z) \in \mathcal{A}$ is of the form (1), it is easily seen from equations (5) and (6) that

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = z + \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)} \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n+\lambda+\eta+1)} a_n z^n. \quad (7)$$

The generalized fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$ satisfies the following three-term recurrence relation:

$$z(J_{0,z}^{\lambda+1,\mu,\eta} f(z))' = (\lambda + \eta + 2)J_{0,z}^{\lambda,\mu,\eta} f(z) - (\lambda + \eta + 1)J_{0,z}^{\lambda+1,\mu,\eta} f(z). \quad (8)$$

We observe from equation (5) that

$$J_{0,z}^{\lambda,-\lambda,\eta} f(z) = \Gamma(2+\lambda)z^{-\lambda} D_z^{-\lambda} f(z) \quad (\lambda > 0), \quad (9)$$

where $D_z^{-\lambda}$ is the relatively more familiar fractional integral operator of order λ ($\lambda > 0$) (cf. [6], [7] and [1]; see also [2]).

Upon setting

$$\mu = 0 \quad \text{and} \quad \eta = \alpha - 1$$

in equations (5) and (7), we obtain the integral operator (or, equivalently, the multiplier transformation) Ω_α^λ , which was introduced and studied by Jung *et al.* [8] (see also [9], [10] and [11]) as follows:

$$\begin{aligned} J_{0,z}^{\lambda,0,\alpha-1} f(z) &=: \Omega_\alpha^\lambda f(z) \\ &= z + \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\alpha + 1)} \sum_{n=2}^{\infty} \frac{\Gamma(n + \alpha)}{\Gamma(n + \lambda + \alpha)} a_n z^n \\ &= \left(\frac{\lambda + \alpha}{\alpha} \right) \frac{\lambda}{z^\alpha} \int_0^z t^{\alpha-1} \left(1 - \frac{t}{z} \right)^{\lambda-1} f(t) dt \\ &\quad (\lambda > 0; \alpha > -1; f \in \mathcal{A}). \end{aligned} \quad (10)$$

We also deduce the following obvious relationship from equation (10):

$$z^{\lambda+\alpha-1} \Omega_\alpha^\lambda f(z) = \frac{\Gamma(\lambda + \alpha + 1)}{\Gamma(\alpha + 1)} Q_\alpha^\lambda f(z), \quad (11)$$

where Q_α^λ denotes the operator (which is essentially a Riemann–Liouville fractional integral operator) studied recently by Dziok [12]. On the other hand, if we set

$$\mu = 0, \quad \lambda = 1 \quad \text{and} \quad \eta = \alpha - 1$$

in equations (5) and (7), we obtain the well-known Bernardi–Libera–Livingston operator $\mathcal{I}_\alpha$ (see [1] and [2]) defined by

$$\begin{aligned} J_{0,z}^{1,0,\alpha-1} f(z) &=: \mathcal{I}_\alpha f(z) \\ &= z + \sum_{n=2}^{\infty} \left(\frac{\alpha + 1}{\alpha + n} \right) a_n z^n \\ &= \frac{\alpha + 1}{z^\alpha} \int_0^z t^{\alpha-1} f(t) dt \quad (\alpha > -1; f \in \mathcal{A}). \end{aligned} \quad (12)$$

Moreover, we have

$$\begin{aligned} J_{0,z}^{1,0,-1} f(z) &=: \mathcal{J}[f](z) \\ &= z + \sum_{n=2}^{\infty} \frac{1}{n} a_n z^n \\ &= \int_0^z \frac{f(t)}{t} dt \quad (f \in \mathcal{A}). \end{aligned} \quad (13)$$

The operator in $\mathcal{J}[f](z)$ is the familiar Alexander integral operator (cf., e.g., [1] and [2]; see also [13]).

Using the generalized fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$, we now introduce the following subclasses of $\mathcal{A}$:

$$\begin{aligned} \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta) \\ := \left\{ f : f(z) \in \mathcal{A}, J_{0,z}^{\lambda,\mu,\eta} f(z) \in \mathcal{S}_s^*(\alpha, \beta) \text{ and } \frac{z(J_{0,z}^{\lambda,\mu,\eta} f(z))'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} \neq \alpha \quad (z \in \mathbb{U}) \right\} \end{aligned} \quad (14)$$

and

$$\begin{aligned} \mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta) \\ := \left\{ f : f(z) \in \mathcal{A}, J_{0,z}^{\lambda,\mu,\eta} f(z) \in \mathcal{K}_c(\alpha, \beta) \text{ and } \frac{(z(J_{0,z}^{\lambda,\mu,\eta} f(z)))'}{(J_{0,z}^{\lambda,\mu,\eta} f(z))'} \neq \alpha \quad (z \in \mathbb{U}) \right\}. \end{aligned} \quad (15)$$

It is obvious from the definitions (14) and (15) that

$$f(z) \in \mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta) \iff z f'(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta).$$

In this paper, we establish some inclusion relationships for the above-mentioned function classes involving the generalized integral operator $J_{0,z}^{\lambda,\mu,\eta}$ defined by equation (5) above. Some corollaries and consequences of our main inclusion relationships are also mentioned.

2. A set of main inclusion relationships

In order to derive our main results, we recall here the following lemma.

LEMMA 2 (see [14]). *Let a function $p(z)$ be analytic in $\mathbb{U}$ with*

$$p(0) = 1 \quad \text{and} \quad p(z) \neq 0 \quad (z \in \mathbb{U}).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(p(z_0))| = \frac{\pi}{2}\beta \quad (0 < \beta \leq 1), \quad (16)$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\beta, \quad (17)$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = \frac{\pi}{2}\beta, \quad (18)$$

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when} \quad \arg(p(z_0)) = -\frac{\pi}{2}\beta, \quad (19)$$

and

$$(p(z_0))^{1/\beta} = \pm ia \quad (a > 0).$$

Our first main inclusion relationship is given by Theorem 1 below.

THEOREM 1 *Let $f \in \mathcal{A}$. Suppose also that*

$$\lambda > 0, \quad \min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2, \quad 0 \leq \alpha < 1 \quad \text{and} \quad 0 < \beta \leq 1.$$

Then

$$\mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta) \subset \mathcal{S}_s^*(\lambda + 1, \mu, \eta, \alpha, \beta).$$

Proof Let $f \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta)$. Then, upon setting

$$p(z) = \frac{1}{1-\alpha} \left(\frac{z(J_{0,z}^{\lambda+1,\mu,\eta} f(z))'}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} - \alpha \right) \quad (z \in \mathbb{U}), \quad (20)$$

we see that the function

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

is analytic in $\mathbb{U}$, with

$$p(0) = 1 \quad \text{and} \quad p(z) \neq 0 \quad (z \in \mathbb{U}).$$

Using the identity (8) in equation (20), we find that

$$\frac{J_{0,z}^{\lambda,\mu,\eta} f(z)}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} = \frac{\alpha + \lambda + \eta + 1 + (1-\alpha)p(z)}{\lambda + \eta + 2}. \quad (21)$$

Differentiating both sides of equation (21) with respect to z , we get

$$\frac{z(J_{0,z}^{\lambda,\mu,\eta} f(z))'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} - \alpha = (1-\alpha)p(z) + \frac{(1-\alpha)zp'(z)}{\alpha + \lambda + \eta + 1 + (1-\alpha)p(z)}.$$

Suppose now that there exists a point $z_0 \in \mathbb{U}$ such that the conditions (16) to (19) of Lemma 2 are satisfied. Thus, if

$$\arg(p(z_0)) = \frac{\pi}{2}\beta \quad (z_0 \in \mathbb{U}),$$

then

$$\begin{aligned} \frac{z_0(J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha &= (1-\alpha)p(z_0) \left(1 + \frac{(z_0 p'(z_0)/p(z_0))}{\alpha + \lambda + \eta + 1 + (1-\alpha)p(z_0)} \right) \\ &= (1-\alpha) a^\beta e^{(i\pi\beta/2)} \left(1 + \frac{ik\beta}{\alpha + \lambda + \eta + 1 + (1-\alpha)a^\beta e^{(i\pi\beta/2)}} \right). \end{aligned}$$

This implies that

$$\arg \left(\frac{z_0 (J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha \right) = \frac{\pi\beta}{2} + \Theta_{k;\alpha,\beta}^{\lambda,\eta}, \quad (22)$$

where

$$\begin{aligned} \Theta_{k;\alpha,\beta}^{\lambda,\eta} &:= \arg \left(1 + \frac{ik\beta}{\alpha + \lambda + \eta + 1 + (1-\alpha)a^\beta e^{(i\pi\beta/2)}} \right) \\ &= \arg \left(\frac{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 a^{2\beta} + 2(\alpha + \lambda + \eta + 1)(1-\alpha)a^\beta \cos(\pi\beta/2) + k\beta(1-\alpha)a^\beta \sin(\pi\beta/2)}{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 a^{2\beta} + 2(\alpha + \lambda + \eta + 1)(1-\alpha)a^\beta \cos(\pi\beta/2)} \right. \\ &\quad \left. + i \frac{k\beta(\alpha + \lambda + \eta + 1 + (1-\alpha)a^\beta \cos(\pi\beta/2))}{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 a^{2\beta} + 2(\alpha + \lambda + \eta + 1)(1-\alpha)a^\beta \cos(\pi\beta/2)} \right) \\ &= \tan^{-1} \left(\frac{k\beta(\alpha + \lambda + \eta + 1 + (1-\alpha)a^\beta \cos(\pi\beta/2))}{(\alpha + \lambda + \eta + 1)^2 + (1-\alpha)^2 a^{2\beta} + 2(\alpha + \lambda + \eta + 1)(1-\alpha)a^\beta \cos(\pi\beta/2) + k\beta(1-\alpha)a^\beta \sin(\pi\beta/2)} \right) \\ &\geq 0, \end{aligned}$$

since

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{and} \quad z_0 \in \mathbb{U}.$$

Thus, in view of equations (2) and (14), this last inequality in conjunction with equation (22) would contradict our assumption that

$$f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta).$$

On the other hand, if we set

$$\arg(p(z_0)) = -\frac{\pi}{2}\beta,$$

then it can *similarly* be shown that

$$\arg \left(\frac{z_0 (J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha \right) \leq -\frac{\pi}{2}\beta,$$

since

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{and} \quad z_0 \in \mathbb{U},$$

which again contradicts the assumption that

$$f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta).$$

Hence, the function $p(z)$ defined by equation (20) satisfies the following inequality:

$$|\arg(p(z))| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}),$$

which implies that

$$\left| \arg \left(\frac{z(J_{0,z}^{\lambda+1,\mu,\eta} f(z))'}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 1. ■

We next prove the following inclusion relationships.

THEOREM 2 *Let $f \in \mathcal{A}$. Suppose also that*

$$\lambda > 0, \quad \min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2, \quad 0 \leq \alpha < 1 \quad \text{and} \quad 0 < \beta \leq 1.$$

Then

$$\mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta) \subset \mathcal{K}_c(\lambda + 1, \mu, \eta, \alpha, \beta).$$

Proof To prove the inclusion relationship asserted by Theorem 2, we observe from Theorem 1 that

$$\begin{aligned} f(z) \in \mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta) &\iff J_{0,z}^{\lambda,\mu,\eta} f(z) \in \mathcal{K}_c(\alpha, \beta) \\ &\iff z(J_{0,z}^{\lambda,\mu,\eta} f(z))' \in \mathcal{S}_s^*(\alpha, \beta) \\ &\iff J_{0,z}^{\lambda,\mu,\eta} (zf'(z)) \in \mathcal{S}_s^*(\alpha, \beta) \\ &\iff zf'(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta) \\ &\implies zf'(z) \in \mathcal{S}_s^*(\lambda + 1, \mu, \eta, \alpha, \beta) \\ &\iff J_{0,z}^{\lambda,\mu,\eta} (zf'(z)) \in \mathcal{S}_s^*(\alpha, \beta) \\ &\iff z(J_{0,z}^{\lambda+1,\mu,\eta} f(z))' \in \mathcal{S}_s^*(\alpha, \beta) \\ &\iff J_{0,z}^{\lambda+1,\mu,\eta} f(z) \in \mathcal{K}_c(\alpha, \beta) \\ &\iff f(z) \in \mathcal{K}_c(\lambda + 1, \mu, \eta, \alpha, \beta), \end{aligned}$$

which establishes Theorem 2. ■

THEOREM 3 *Let $f \in \mathcal{A}$. Suppose also that*

$$\lambda > 0, \quad \min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1 \quad \text{and} \quad c > -\alpha,$$

and that

$$\frac{z(J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z))'}{J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z)} \neq \alpha \quad (z \in \mathbb{U}).$$

Then

$$f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta) \implies \mathcal{I}_c f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta). \quad (23)$$

Proof We begin by assuming that $f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta)$ and defining a function $q(z)$ by

$$q(z) = \frac{1}{1-\alpha} \left(\frac{z(J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z))'}{J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z)} - \alpha \right) \quad (z \in \mathbb{U}), \quad (24)$$

where

$$q(z) = 1 + d_1 z + d_2 z^2 + \dots$$

is analytic in $\mathbb{U}$, with

$$q(0) = 1 \quad \text{and} \quad q(z) \neq 0 \quad (z \in \mathbb{U}).$$

It can easily be verified from equations (7) and (12) that

$$z(J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z))' = (c+1)J_{0,z}^{\lambda,\mu,\eta} f(z) - c J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z). \quad (25)$$

Thus, by using equations (24) and (25), we find that

$$\frac{z(J_{0,z}^{\lambda,\mu,\eta} f(z))'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} - \alpha = (1-\alpha)q(z) + \frac{(1-\alpha)zq'(z)}{c+\alpha+(1-\alpha)q(z)}.$$

Suppose now that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg(q(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \quad \text{and} \quad |\arg(q(z_0))| = \frac{\pi}{2}\beta.$$

Then, by applying Lemma 2, we can write

$$\frac{z_0 q'(z_0)}{q(z_0)} = ik\beta \quad \text{and} \quad (q(z_0))^{1/\beta} = \pm ia \quad (a > 0).$$

If

$$\arg(q(z_0)) = \frac{\pi}{2}\beta \quad (z_0 \in \mathbb{U}),$$

then

$$\begin{aligned} \frac{z_0(J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha &= (1-\alpha)q(z_0) \left(1 + \frac{(z_0 q'(z_0)/q(z_0))}{c+\alpha+(1-\alpha)q(z_0)} \right) \\ &= (1-\alpha)a^\beta e^{(i\pi\beta/2)} \left(1 + \frac{ik\beta}{c+\alpha+(1-\alpha)a^\beta e^{(i\pi\beta/2)}} \right), \end{aligned}$$

which yields

$$\arg \left(\frac{z_0(J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha \right) = \frac{\pi}{2}\beta + \Phi_{k;\alpha,\beta}^c, \quad (26)$$

where

$$\begin{aligned}
 \Phi_{k;\alpha,\beta}^c &:= \arg \left(1 + \frac{ik\beta}{c + \alpha + (1 - \alpha)a^\beta e^{i\pi\beta/2}} \right) \\
 &= \arg \left(\frac{(c + \alpha)^2 + (1 - \alpha)^2 a^{2\beta} + 2(c + \alpha)(1 - \alpha)a^\beta \cos(\pi\beta/2) + k\beta(1 - \alpha)a^\beta \sin(\pi\beta/2)}{(c + \alpha)^2 + (1 - \alpha)^2 a^{2\beta} + 2(c + \alpha)(1 - \alpha)a^\beta \cos(\pi\beta/2)} \right. \\
 &\quad \left. + i \frac{k\beta[c + \alpha + (1 - \alpha)a^\beta \cos(\pi\beta/2)]}{(c + \alpha)^2 + (1 - \alpha)^2 a^{2\beta} + 2(c + \alpha)(1 - \alpha)a^\beta \cos(\pi\beta/2)} \right) \\
 &= \tan^{-1} \left(\frac{k\beta(c + \alpha + (1 - \alpha)a^\beta \cos(\pi\beta/2))}{(c + \alpha)^2 + (1 - \alpha)^2 a^{2\beta} + 2(c + \alpha)(1 - \alpha)a^\beta \cos(\pi\beta/2) + k\beta(1 - \alpha)a^\beta \sin(\pi\beta/2)} \right) \\
 &\geq 0,
 \end{aligned}$$

since

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \geq 1 \quad \text{and} \quad z_0 \in \mathbb{U}.$$

This last inequality in conjunction with equation (26) would obviously contradict our assumption that

$$f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta).$$

Similarly, in the case when

$$\arg(q(z_0)) = -\frac{\pi}{2}\beta \quad (z_0 \in \mathbb{U}),$$

we can prove that

$$\arg \left(\frac{z_0 (J_{0,z}^{\lambda,\mu,\eta} f(z_0))'}{J_{0,z}^{\lambda,\mu,\eta} f(z_0)} - \alpha \right) \leq -\frac{\pi}{2}\beta,$$

since

$$k \leq -\frac{1}{2} \left(a + \frac{1}{a} \right) \leq -1 \quad \text{and} \quad z_0 \in \mathbb{U},$$

contradicting once again the assumption that

$$f(z) \in \mathcal{S}_s^*(\lambda, \mu, \eta, \alpha, \beta).$$

We thus conclude that $q(z)$ must satisfy the following argument inequality:

$$|\arg(q(z))| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}).$$

This shows that

$$\left| \arg \left(\frac{z (J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z))'}{J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z)} - \alpha \right) \right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}),$$

which completes the proof of Theorem 3. ■

The proof of Theorem 4 below is analogous to that of Theorem 3. We, therefore, choose to omit its proof.

THEOREM 4 *Under the parametric constraints stated with Theorem 3, let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad \frac{(z(J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z)))'}{(J_{0,z}^{\lambda,\mu,\eta} \mathcal{I}_c f(z))'} \neq \alpha \quad (z \in \mathbb{U}).$$

Then

$$f(z) \in \mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta) \implies \mathcal{I}_c f(z) \in \mathcal{K}_c(\lambda, \mu, \eta, \alpha, \beta). \quad (27)$$

Remark Upon setting

$$\mu = 0 \quad \text{and} \quad \eta = \alpha - 1,$$

Theorems 1 to 4 would yield the corresponding known results due to Liu [15]. Furthermore, in view of the relationship (11), we can easily apply Theorems 1 to 4 in order to deduce the corresponding inclusion relationships involving the aforementioned operator $\mathcal{Q}_\alpha^\lambda$, which was studied recently by Dziok (see [12]).

3. Corollaries and consequences

In this concluding section, we consider some corollaries and consequences of our main results (Theorems 1 to 4) established in Section 2.

First of all, on setting

$$\lambda = 1, \quad \mu = 0 \quad \text{and} \quad \eta = -1,$$

Theorem 1 would yield the following result.

COROLLARY 1 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad f(z) \neq \alpha \int_0^z \frac{f(t)}{t} dt \quad (z \in \mathbb{U}).$$

If $f(z)$ satisfies the following inequality:

$$\left| \arg \left(\frac{f(z)}{\int_0^z (f(t)/t) dt} - \alpha \right) \right| < \frac{\pi}{2} \beta \quad (0 \leq \alpha < 1; 0 < \beta \leq 1), \quad (28)$$

then

$$2 \int_0^z \left(\frac{1}{t} - \frac{1}{z} \right) f(t) dt \in \mathcal{S}_s^*(\alpha, \beta). \quad (29)$$

In its *further* special case when $\beta = 1$, Corollary 1 reduces to Corollary 2 below.

COROLLARY 2 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad f(z) \neq \alpha \int_0^z \frac{f(t)}{t} dt \quad (z \in \mathbb{U}).$$

If $f(z)$ satisfies the following inequality:

$$\Re(f(z)) > \alpha \Re \left(\int_0^z \frac{f(t)}{t} dt \right) \quad (0 \leq \alpha < 1), \quad (30)$$

then

$$2 \int_0^z \left(\frac{1}{t} - \frac{1}{z} \right) f(t) dt \in \mathcal{S}^*(\alpha). \quad (31)$$

Next, if we set

$$\lambda = 1 \quad \text{and} \quad \mu = \eta = 0$$

in Theorem 1, we get Corollary 3 below.

COROLLARY 3 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad zf(z) \neq (\alpha + 1) \int_0^z f(t) dt \quad (z \in \mathbb{U}).$$

If $f(z)$ satisfies the following inequality:

$$\left| \arg \left(\frac{zf(z)}{\int_0^z f(t) dt} - (\alpha + 1) \right) \right| < \frac{\pi}{2} \beta \quad (0 \leq \alpha < 1; 0 < \beta \leq 1), \quad (32)$$

then

$$\frac{6}{z} \int_0^z \left(1 - \frac{t}{z} \right) f(t) dt \in \mathcal{S}_s^*(\alpha, \beta). \quad (33)$$

A further special case of Corollary 3 when $\beta = 1$ would immediately yield the following result.

COROLLARY 4 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad zf(z) \neq (\alpha + 1) \int_0^z f(t) dt \quad (z \in \mathbb{U}).$$

If $f(z)$ satisfies the following inequality:

$$\Re(zf(z)) > (\alpha + 1) \Re \left(\int_0^z f(t) dt \right) \quad (0 \leq \alpha < 1), \quad (34)$$

then

$$\frac{6}{z} \int_0^z \left(1 - \frac{t}{z} \right) f(t) dt \in \mathcal{S}^*(\alpha). \quad (35)$$

By setting

$$\lambda = 1, \quad \mu = 0 \quad \text{and} \quad \eta = -1$$

in Theorem 2, we arrive at Corollary 5 below.

COROLLARY 5 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad zf'(z) \neq \alpha f(z) \quad (z \in \mathbb{U}).$$

If

$$f(z) \in \mathcal{S}_s^*(\alpha, \beta),$$

then

$$2 \int_0^z \left(\frac{1}{t} - \frac{1}{z} \right) f(t) dt \in \mathcal{K}_c(\alpha, \beta). \quad (36)$$

Upon setting

$$\lambda = 1 \quad \text{and} \quad \mu = \eta = 0,$$

Theorem 2 would yield the following result.

COROLLARY 6 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad z^2 f'(z) \neq (\alpha + 1) \left(zf(z) - \int_0^z f(t) dt \right) \quad (z \in \mathbb{U}).$$

If $f(z)$ *satisfies the following inequality:*

$$\left| \arg \left(\frac{z^2 f'(z)}{zf(z) - \int_0^z f(t) dt} - (\alpha + 1) \right) \right| < \frac{\pi}{2} \beta \quad (0 \leq \alpha < 1; 0 < \beta \leq 1), \quad (37)$$

then

$$\frac{6}{z} \int_0^z \left(1 - \frac{t}{z} \right) f(t) dt \in \mathcal{K}_c(\alpha, \beta). \quad (38)$$

If we set $\beta = 1$ in Corollary 6, we immediately get Corollary 7 below.

COROLLARY 7 *Let*

$$f(z) \in \mathcal{A} \quad \text{and} \quad z^2 f'(z) \neq (\alpha + 1) \left(zf(z) - \int_0^z f(t) dt \right) \quad (z \in \mathbb{U}).$$

If $f(z)$ *satisfies the following inequality:*

$$\Re(z^2 f'(z)) > (\alpha + 1) \Re \left(zf(z) - \int_0^z f(t) dt \right) \quad (0 \leq \alpha < 1), \quad (39)$$

then

$$\frac{6}{z} \int_0^z \left(1 - \frac{t}{z} \right) f(t) dt \in \mathcal{K}(\alpha). \quad (40)$$

Finally, by setting

$$\lambda = 1, \quad \mu = 0, \quad \eta = -1 \quad \text{and} \quad c = 0$$

in Theorem 3, we get the following result.

COROLLARY 8 *If*

$$f(z) \in \mathcal{A} \quad \text{and} \quad z(\mathcal{J}(\mathcal{J}[f](z)))' \neq \alpha \mathcal{J}(\mathcal{J}[f](z)) \quad (z \in \mathbb{U}), \quad (41)$$

then

$$\mathcal{J}[f](z) \in \mathcal{S}_s^*(\alpha, \beta) \implies \mathcal{J}(\mathcal{J}[f](z)) \in \mathcal{S}_s^*(\alpha, \beta). \quad (42)$$

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Approximation properties of modified gamma operators

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We introduce modified gamma operators in polynomial weighted spaces of differentiable functions and give approximation theorems for them.

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1. Preliminaries

1.1 The gamma operators

$$G_n(f; x) := \int_0^\infty g_n(x, y) f\left(\frac{n}{y}\right) dy, \quad x \in I, \quad n \in N, \quad (1)$$

$$g_n(x, y) := \frac{x^{n+1}}{n!} e^{-xy} y^n, \quad (2)$$

$I = (0, \infty)$, $N = \{1, 2, \dots\}$, of functions f continuous and bounded on I were examined in many papers (e.g. refs. [1–4]).

It is known [1, 3] that G_n are well defined for functions $f_r(x) = x^r$, $r \in N_0 = N \cup \{0\}$, and

$$G_n(f_r; x) = f_r(x) \quad \text{for } x > 0, \quad n \geq 1, \quad r = 0, 1. \quad (3)$$

Generally for $r \in N$, we have

$$G_n(f_r(t) = t^r; x) = \frac{n^r x^r}{n(n-1) \cdots (n-r+1)}, \quad \text{for } x > 0, \quad n \geq r, \quad (4)$$

which implies that $G_n(t^2; x) = nx^2/(n-1)$ and

$$G_n((t-x)^2; x) = \frac{x^2}{n-1}, \quad \text{for } x > 0, \quad n \geq 2. \quad (5)$$

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In ref. [1] (Lemma 9.4.4, p. 128), it was proved that

$$G_n((t-x)^{2r}; x) \leq M_1(r) n^{-r} x^{2r} \quad (6)$$

for $x > 0$, $r \in \mathbb{N}$ and $n \geq 2r$, where $M_1(r)$ is a suitable positive constant depending only on r . Moreover, it is known [1, 3] that if f is continuous and bounded on I , then $G_n(f)$, $n \geq n_0$, is also continuous and bounded on I and

$$|G_n(f; x) - f(x)| \leq M_2 \omega_2\left(f; \frac{x}{\sqrt{n}}\right), \quad \text{for } x > 0, \quad n \geq n_0, \quad (7)$$

where $\omega_2(f)$ is the second modulus of continuity of f and $M_2 = \text{const.} > 0$.

1.2. In this paper, we introduce certain modified gamma operators for differentiable functions and show that approximation properties of these operators are better than G_n .

Analogous to ref. [5], we consider the polynomial weighted space C_p , $p \in \mathbb{N}_0$, associated with the weighted function

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1} \text{ if } p \geq 1, \quad x \in I, \quad (8)$$

and C_p is the set of all real-valued functions f defined on I for which $w_p f$ is uniformly continuous and bounded on I and the norm is given by the formula

$$\|f\|_p := \sup_{x \in I} w_p(x) |f(x)|. \quad (9)$$

We shall also consider the class C^p , $p \in \mathbb{N}_0$, of all $f \in C_p$, p times differentiable on I and the derivatives $f^{(k)} \in C_{p-k}$ for $0 \leq k \leq p$. Clearly $C^0 \equiv C_0$, $C_p \subset C_q$ if $p < q$, and $\|f\|_q \leq \|f\|_p$ for $f \in C_p$.

For $f \in C_p$, $p \in \mathbb{N}_0$, we consider the modulus of continuity of the order $k = 1, 2$, i.e.,

$$\omega_k(f; C_p; t) := \sup_{0 \leq h \leq t} \|\Delta_h^k f(\cdot)\|_p, \quad t \geq 0, \quad (10)$$

where $\Delta_h^1 f(x) \equiv \Delta_h f(x) = f(x+h) - f(x)$ and $\Delta_h^2 f(x) = \Delta_h(\Delta_h f(x)) = f(x+2h) - 2f(x+h) + f(x)$. It is known [1, 5] that $\lim_{t \rightarrow 0+} \omega_k(f; C_p; t) = 0$, $k = 1, 2$, for every $f \in C_p$, $p \in \mathbb{N}_0$.

1.3. Let $f \in C^p$, $p \in \mathbb{N}_0$. We define the following modified gamma operators (see [6])

$$G_{n;p}(f; x) := \int_0^\infty g_n(x, y) F_p\left(x, \frac{n}{y}\right) dy, \quad (11)$$

$x \in I$ and $p \leq n \in \mathbb{N}$, where $g_n(x, y)$ is defined by (2) and

$$F_p(x, t) := \sum_{j=0}^p \frac{f^{(j)}(t)}{j!} (x-t)^j, \quad x, t > 0. \quad (12)$$

If $p = 0$ and $f \in C^0$, then we have

$$G_{n;0}(f; x) \equiv G_n(f; x) = \int_0^\infty g_n(x, y) f\left(\frac{n}{y}\right) dy, \quad (13)$$

for $x > 0$ and $n \in \mathbb{N}$. We mention that $G_{n;p}(f(t); x)$ will also be written instead of $G_{n;p}(f; x)$.

In section 2, we show that $G_{n;p}$, $n \geq p$, are operators acting from C^p into C_p . In section 3, we give approximation theorems.

In this paper, we denote by $M_k(\alpha, \beta)$, $k \in \mathbb{N}$, suitable positive constants depending only on indicated parameters α and β .

2. Auxiliary results

2.1. First we shall give some properties of operators G_n .

LEMMA 1 *Let $p \in \mathbb{N}$. Then there exists $M_3(p) = \text{const.} > 0$ such that*

$$\left\| G_n \left(\frac{1}{w_p(t)}; \cdot \right) \right\|_p \leq M_3(p), \quad n \geq p, \quad (14)$$

and for every $f \in C_p$

$$\|G_n(f; \cdot)\|_p \leq M_3(p) \|f\|_p, \quad n \geq p. \quad (15)$$

The inequalities (14) and (15) with $M_3 = 1$ are also true for $p = 0$ and $f \in C_0$.

The formulas (1) and (2) and (15) show that $G_n, n \geq p$ if $p \in \mathbb{N}$ and $n \geq 1$ if $p = 0$, are linear positive operators acting from the space C_p into C_p .

Proof The inequalities (14) and (15) are obvious for $p = 0$ by formulas (1)–(3), (8), and (9). If $p \geq 1$, then by formulas (8), (3), and (4), we have

$$G_n \left(\frac{1}{w_p(t)}; x \right) = 1 + G_n(t^p; x) = 1 + \frac{n^p x^p}{n(n-1) \cdots (n-p+1)}$$

for $x > 0$ and $n \geq p$, which implies that

$$w_1(x) G_n \left(\frac{1}{w_1(t)}; x \right) = \frac{1}{1+x} (1+x) = 1, \quad n \geq 1,$$

and

$$\begin{aligned} w_p(x) G_n \left(\frac{1}{w_p(t)}; x \right) &\leq 1 + \frac{n^{p-1}}{(n-(p-1))^{p-1}} \\ &= 1 + \left(1 + \frac{p-1}{n-p+1} \right)^{p-1} \leq M_3(p) \end{aligned}$$

for $x > 0$, $p \geq 2$, and $n \geq p$. These inequalities and equation (9) yield inequality (14).

From formulas (1), (8), and (9), we get

$$\|G_n(f; \cdot)\|_p \leq \|f\|_p \left\| G_n \left(\frac{1}{w_p}; \cdot \right) \right\|_p, \quad \text{for } n \geq p \geq 1,$$

which by inequality (14) implies thesis (15). ■

From equation (8), we deduce that

$$(w_p(x))^2 \leq w_{2p}(x), \quad (w_p(x))^{-2} \leq \frac{4}{w_{2p}(x)}, \quad \frac{w_{p+r}(x)}{w_p(x)w_r(x)} \leq 3, \quad (16)$$

for $x > 0$ and $p, r \in \mathbb{N}$.

Applying the Hölder inequality and formulas (6), (8), (14), and (16), we immediately obtain the following two lemmas.

LEMMA 2 For every $r \in N$ there exists $M_4(r) = \text{const.} > 0$ such that

$$G_n(|t - x|^r; x) \leq M_4(r) n^{-r/2} x^r, \quad \text{for } x > 0, \quad n \geq 2r.$$

LEMMA 3 Let $p, r \in N$. Then there exists $M_5(p, r) = \text{const.} > 0$ such that

$$w_{p+r}(x) G_n\left(\frac{|t - x|^r}{w_p(t)}; x\right) \leq M_5(p, r) n^{-r/2}$$

for $x > 0$ and $n \geq 2(p + r)$.

2.2. Now we shall prove the analogue of inequality (15) for operators $G_{n;p}$.

LEMMA 4 Let $p \in N$. Then there exists $M_6(p) = \text{const.} > 0$ such that

$$\|G_{n;p}(f)\|_p \leq M_6(p) \sum_{j=0}^p \|f^{(j)}\|_{p-j}, \quad (17)$$

for every $f \in C^p$ and $n \geq 2p$.

The formulas (11) and (12) and the inequality (17) show that $G_{n;p}$ with $p \in N$ and $n \geq 2p$ are linear operators acting from C^p into C_p .

Proof If $f \in C_p$, $p \in N$, then there exist norms $\|f^{(j)}\|_{p-j}$, $0 \leq j \leq p$. By formulas (11)–(13) and (9), we have

$$\begin{aligned} |G_{n;p}(f; x)| &\leq \sum_{j=0}^p G_n(|f^{(j)}(t)| |t - x|^j; x) \\ &\leq \sum_{j=0}^p \|f^{(j)}\|_{p-j} G_n\left(\frac{|t - x|^j}{w_{p-j}(t)}; x\right). \end{aligned}$$

Now applying Lemmas 2 and 3, we obtain

$$w_p(x) |G_{n;p}(f; x)| \leq M_6(p) \sum_{j=0}^p \|f^{(j)}\|_{p-j} n^{-j/2}$$

for $x > 0$ and $n \geq 2p$, which by formula (9) yields inequality (17). ■

3. Theorems

3.1. Arguing as in ref. [5], we shall prove two approximation theorems for operators G_n .

THEOREM 1 Assume that f is a twice differentiable function on I and $f, f', f'' \in C_p$ with a fixed $p \in N_0$. Then there exists $M_7(p) = \text{const.} > 0$ such that

$$w_p(x) |G_n(f; x) - f(x)| \leq M_7(p) \|f''\|_p \frac{x^2}{n}, \quad (18)$$

for $x > 0$ and $n \geq 2p + 4$.

Proof Fix $x > 0$ and f satisfying our assumptions. Analogous to ref. [5], we can write

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t \int_x^s f''(u) du ds, \quad t > 0,$$

which implies that

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t (t-u) f''(u) du,$$

and by formulas (1) and (3), we have

$$G_n(f(t); x) = f(x) + G_n \left(\int_x^t (t-u) f''(u) du; x \right).$$

Using (as in ref. [5]) the inequality

$$\left| \int_x^t (t-u) f''(u) du \right| \leq \|f''\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right) (t-x)^2,$$

we get

$$\begin{aligned} w_p(x) |G_n(f; x) - f(x)| &\leq \|f''\|_p \left(w_p(x) G_n \left(\frac{(t-x)^2}{w_p(t)}; x \right) \right. \\ &\quad \left. + G_n((t-x)^2; x) \right), \quad \text{for } n \geq 2p+4. \end{aligned} \quad (19)$$

Next, by the Hölder inequality and inequalities (6) and (16), we have

$$\begin{aligned} w_p(x) G_n \left(\frac{(t-x)^2}{w_p(t)}; x \right) &\leq \left(4 w_{2p}(x) G_n \left(\frac{1}{w_{2p}(t)}; x \right) \right)^{1/2} \times (G_n((t-x)^4; x))^{1/2} \\ &\leq M_8(p) \frac{x^2}{n}, \quad \text{for } x > 0, \quad n \geq 2p+4. \end{aligned} \quad (20)$$

Using formulas (20) and (5) to inequality (19), we obtain the desired inequality (18). ■

THEOREM 2 Let $p \in N_0$. Then there exists $M_9(p) = \text{const.} > 0$ such that for every $f \in C_p$, $x > 0$, and $n \geq 2p+4$, we have

$$w_p(x) |G_n(f; x) - f(x)| \leq M_9(p) \omega_2 \left(f; C_p; \frac{x}{\sqrt{n}} \right). \quad (21)$$

Proof Similar to ref. [5], we apply the Stiecklov function f_h of $f \in C_p$, i.e.,

$$f_h(x) := \frac{4}{h^2} \int_0^{h/2} \int_0^{h/2} (2f(x+s+t) - f(x+2(s+t))) ds dt$$

for $x, h > 0$. It is known [5] that if $f \in C_p$, then $f_h^{(j)} \in C_p$ for $j = 0, 1, 2$, $h > 0$, and

$$\|f - f_h\|_p \leq \omega_2(f; C_p; h), \quad (22)$$

$$\|f_h''\|_p \leq 9 h^{-2} \omega_2(f; C_p; h), \quad h > 0, \quad (23)$$

where $\omega_2(f)$ is defined by formula (10).

Applying Lemma 1, Theorem 1, inequalities (22), and (23), we get

$$\begin{aligned}
 w_p(x)|G_n(f(t); x) - f(x)| &\leq w_p(x)(|G_n(f(t) - f_h(t); x)| \\
 &\quad + |G_n(f_h(t); x) - f_h(x)| + |f_h(x) - f(x)|) \\
 &\leq M_3(p)\|f - f_h\|_p + M_7(p)\|f_h''\|_p \frac{x^2}{n} + \|f - f_h\|_p \\
 &\leq M_{10}(p)\omega_2(f; C_p; h)(1 + h^{-2} \frac{x^2}{n}),
 \end{aligned}$$

for $x, h > 0$ and $n \geq 2p + 4$. Now setting $h = x/\sqrt{n}$, we obtain the desired inequality (21). ■

3.2. Now we shall prove an analogue of evaluations (21) and (7) for $G_{n;p}$.

THEOREM 3 *Let $p \in \mathbb{N}$. Then there exists $M_{11}(p) = \text{const.} > 0$ such that for every $f \in C^p$, $x > 0$, and $n \geq 2p + 2$, we have*

$$w_p(x)|G_{n;p}(f; x) - f(x)| \leq M_{11}(p)n^{-p/2}\omega_1\left(f^{(p)}; C_0; \frac{x}{\sqrt{n}}\right). \quad (24)$$

Consequently,

$$\|G_{n;p}(f) - f\|_{p+1} \leq 3M_{11}(p)n^{-p/2}\omega_1\left(f^{(p)}; C_0; \frac{1}{\sqrt{n}}\right) \quad (25)$$

for $n \geq 2p + 2$, where ω_1 is defined by formula (10).

Proof From definitions (11)–(13) and (3), we get

$$G_{n;p}(f; x) - f(x) = G_n(F_p(x, t) - f(x); x)$$

for every $f \in C^p$, $x > 0$ and $p \geq 0$, and $n \geq 2p + 2$.

Similar to ref. [4], we use the following Taylor's formula of $f \in C^p$ at a fixed $t > 0$:

$$f(x) = \sum_{j=0}^p \frac{f^{(j)}(t)}{j!}(x-t)^j + \frac{(x-t)^p}{(p-1)!} I_p(x, t), \quad x > 0,$$

where

$$I_p(x, t) := \int_0^1 (1-u)^{p-1} (f^{(p)}(t+u(x-t)) - f^{(p)}(t)) du. \quad (26)$$

From the above, we get

$$f(x) - F_p(x, t) = \frac{(x-t)^p}{(p-1)!} I_p(x, t), \quad x, t > 0,$$

and consequently,

$$\begin{aligned}
 |G_{n;p}(f; x) - f(x)| &\leq G_n(|F_p(x, t) - f(x)|; x) \\
 &\leq \frac{1}{(p-1)!} G_n(|t-x|^p |I_p(x, t)|; x),
 \end{aligned} \quad (27)$$

for $x > 0$ and $n \geq 2p + 2$. Next, by definition (10) and properties of $\omega_1(f^{(p)}; C_0)$ [1] for $f^{(p)} \in C_0$, we derive from equation (26)

$$\begin{aligned} |I_p(x, t)| &\leq \int_0^1 (1-u)^{p-1} \omega_1(f^{(p)}; C_0; u|t-x|) du \\ &\leq \omega_1(f^{(p)}; C_0; |t-x|) \int_0^1 (1-u)^{p-1} du \\ &\leq \frac{1}{p} \omega_1\left(f^{(p)}; C_0; \frac{x}{\sqrt{n}}\right) \left(\frac{\sqrt{n}}{x} |t-x| + 1\right), \end{aligned} \quad (28)$$

for $x > 0$ and $n \in N$. Using evaluation (28) to inequality (27), we obtain

$$\begin{aligned} |G_{n;p}(f; x) - f(x)| &\leq \frac{1}{p!} \omega_1\left(f^{(p)}; C_0; \frac{x}{\sqrt{n}}\right) \\ &\quad \times \left(\frac{\sqrt{n}}{x} G_n(|t-x|^{p+1}; x) + G_n(|t-x|^p; x)\right), \end{aligned}$$

which by Lemma 2 and formula (8) immediately gives thesis (24).

The inequality (25) can be obtained from equation (24) by inequalities (16) and the inequality $\omega_1(f^{(p)}; C_0; x/\sqrt{n}) \leq (x+1) \omega_1(f^{(p)}; C_0; 1/\sqrt{n})$ for $x > 0$ and $n \in N$.

Thus the proof is completed. ■

From Theorems 2 and 3, we can derive the following corollaries.

COROLLARY 1 For every $f \in C_p$, $p \in N_0$, and $x > 0$, we have

$$\lim_{n \rightarrow \infty} (G_n(f; x) - f(x)) = 0.$$

This convergence is uniform on every interval $[x_1, x_2]$, $x_1 > 0$.

If moreover f is twice differentiable function on I and $f', f'' \in C_p$, then

$$|G_n(f; x) - f(x)| = O_x\left(\frac{1}{n}\right) \quad \text{as } n \rightarrow \infty,$$

at every $x > 0$.

COROLLARY 2 If $f \in C^p$, $p \in N$, then

$$\lim_{n \rightarrow \infty} n^{p/2} (G_{n;p}(f; x) - f(x)) = 0 \quad \text{at every } x > 0.$$

Moreover,

$$\lim_{n \rightarrow \infty} n^{p/2} \|G_{n;p}(f) - f\|_{p+1} = 0.$$

COROLLARY 3 The approximation order (24) of $f \in C^p$ with $p \geq 2$ by operators $G_{n;p}$, $n \geq 2p + 2$, is better than order (7) given for classical gamma operators G_n defined by formula (1).

3.3. In this section will be given the Voronovskaya type theorem for operators G_n and $G_{n;p}$.

THEOREM 4 *Let $p \in N_0$ and let f be a function satisfying the assumptions of Theorem 1. Then*

$$\lim_{n \rightarrow \infty} n(G_n(f; x) - f(x)) = \frac{x^2}{2} f''(x) \quad (29)$$

at every $x > 0$.

Proof Choose f and $x > 0$. By properties of f , we can write the Taylor's formula:

$$f(t) = f(x) + f'(x)(t - x) + \frac{1}{2} f''(x)(t - x)^2 + \varphi(t, x)(t - x)^2, \quad t > 0, \quad (30)$$

where $\varphi(t) \equiv \varphi(t, x)$ is a function such that $\varphi(t)t^2$ belongs to C_p and $\lim_{t \rightarrow x} \varphi(t) = \varphi(x) = 0$. From equations (30) and (1)–(3), we can arrive at a result that

$$G_n(f(t); x) = f(x) + \frac{1}{2} f''(x) G_n((t - x)^2; x) + G_n(\varphi(t)(t - x)^2; x). \quad (31)$$

Next by the Hölder inequality, we have

$$|G_n(\varphi(t)(t - x)^2; x)| \leq (G_n(\varphi^2(t); x))^{1/2} (G_n((t - x)^4; x))^{1/2}$$

for $n \geq 2p + 2$. The properties of φ and Corollary 1 imply that

$$\lim_{n \rightarrow \infty} G_n(\varphi^2(t); x) = \varphi^2(x) = 0.$$

From the above and inequality (6), we deduce that

$$\lim_{n \rightarrow \infty} n G_n(\varphi(t)(t - x)^2; x) = 0. \quad (32)$$

Applying equations (5) and (32), we immediately obtain thesis (29) from equation (31). ■

THEOREM 5 *Suppose that $p \in N$ and f is a function belonging to C^p and having derivatives $f^{(p+1)}$ and $f^{(p+2)}$ continuous and bounded on I . Then for every $x > 0$, we have*

$$\begin{aligned} G_{n;p}(f; x) - f(x) &= \frac{(-1)^p f^{(p+1)}(x) G_n((t - x)^{p+1}; x)}{(p + 1)!} \\ &+ \frac{(-1)^p (p + 1) f^{(p+2)}(x) G_n((t - x)^{p+2}; x)}{(p + 2)!} \\ &+ o_x(n^{-(p+2)/2}) \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (33)$$

Proof Fix $x > 0$ and f satisfying our assumptions. Similar to the proof of Theorem 4, we can write the Taylor's formula for every derivative $f^{(j)}$, $0 \leq j \leq p$. We have

$$f^{(j)}(t) = \sum_{i=0}^{p+2-j} \frac{f^{(j+i)}(x)}{i!} (t-x)^i + \varphi_j(t, x)(t-x)^{p+2-j}, \quad t > 0,$$

where $\varphi_j(t) \equiv \varphi_j(t, x)$ is a function such that $\varphi_j(t)t^2$ belongs to C_0 and $\lim_{t \rightarrow x} \varphi_j(t) = \varphi_j(x) = 0$. Hence F_p defined by formula (12) can be written in the following form

$$F_p(x, t) = \sum_{j=0}^p \frac{(-1)^j}{j!} \sum_{i=0}^{p+2-j} \frac{f^{(j+i)}(x)}{i!} (t-x)^{j+i} + \Phi_p(t)(t-x)^{p+2}, \quad t > 0,$$

where

$$\Phi_p(t) := \sum_{j=0}^p \frac{(-1)^j}{j!} \varphi_j(t). \quad (34)$$

Next, by elementary calculations, we get

$$\begin{aligned} F_p(x, t) &= \sum_{j=0}^p (-1)^j \sum_{s=j}^{p+2} \binom{s}{j} \frac{f^{(s)}(x)(t-x)^s}{s!} + \Phi_p(t)(t-x)^{p+2} \\ &= \sum_{s=0}^p \frac{f^{(s)}(x)(t-x)^s}{s!} \sum_{j=0}^s \binom{s}{j} (-1)^j \\ &\quad + \frac{f^{(p+1)}(x)(t-x)^{p+1}}{(p+1)!} \sum_{j=0}^p \binom{p+1}{j} (-1)^j \\ &\quad + \frac{f^{(p+2)}(x)(t-x)^{p+2}}{(p+2)!} \sum_{j=0}^p \binom{p+2}{j} (-1)^j + \Phi_p(t)(t-x)^{p+2}. \end{aligned} \quad (35)$$

It is easily verified that if $m \in N_0$, then

$$\begin{aligned} \sum_{j=0}^m \binom{m}{j} (-1)^j &= \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \geq 1, \end{cases} \\ \sum_{j=0}^m \binom{m+1}{j} (-1)^j &= (-1)^m, \\ \sum_{j=0}^m \binom{m+2}{j} (-1)^j &= (m+1)(-1)^m. \end{aligned}$$

Using these identities and formula (3), we derive the following equality from equation (35):

$$\begin{aligned} G_{n;p}(f; x) &= f(x) + \frac{(-1)^p f^{(p+1)}(x) G_n((t-x)^{p+1}; x)}{(p+1)!} \\ &\quad + \frac{(-1)^p (p+1) f^{(p+2)}(x) G_n((t-x)^{p+2}; x)}{(p+2)!} \\ &\quad + G_n(\Phi_p(t)(t-x)^{p+2}; x), \quad \text{for } n \geq 2(p+2). \end{aligned} \quad (36)$$

The formula (34) and properties of φ_j imply that $\lim_{t \rightarrow x} \Phi_p(t) = \Phi_p(x) = 0$, which by Corollary 1 gives

$$\lim_{n \rightarrow \infty} G_n(\Phi_p^2(t); x) = \Phi_p^2(x) = 0. \quad (37)$$

Applying the Hölder inequality and then equations (37) and (6), we get

$$G_n(\Phi_p(t)(t-x)^{p+2}; x) = o_x(n^{-(p+2)/2}) \quad \text{as } n \longrightarrow \infty. \quad (38)$$

Now equalities (36) and (38) complete the proof of thesis (33). ■

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Research article

d-Orthogonal Faber polynomials

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Douak and Maroni stated the following problem:

P1: For fixed non-negative integer r and positive integer d , find all d -orthogonal polynomial sets $\{P_n\}_{n \geq 0}$ which satisfy

$$P_n^{(r)} = Q_n, \quad \text{where} \quad Q_n = \frac{1}{n+1} P'_{n+1}, \quad n = 0, 1, 2, \dots,$$

$\{P_n^{(r)}\}_{n \geq 0}$ being the r -th associated polynomial set of $\{P_n\}_{n \geq 0}$.

They solved it for the particular cases $(r, d) = (1, 1)$ and $(r, d) = (1, 2)$. In this paper, we show that the d -orthogonal Faber polynomials are solution of **P1** for generals r and d . We also solve **P1** for $r = 1, 2$ and general d with the d -symmetry property as an additional condition. Some properties of the obtained polynomials are singled out.

Keywords: d -orthogonal polynomials; Associated polynomials; Faber polynomials; Generating functions; d -symmetric polynomials; d -orthogonal Chebyshev polynomials

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1. Introduction

Recently, an extension of the concept of standard orthogonality was introduced. That is the multiple orthogonality [1–7]. This notion has many applications in various domains of mathematics as analytic number theory, approximation theory, special functions theory, and spectral theory of operators. A subclass of multiple orthogonality is known as d -orthogonality, d being a positive integer, and it was introduced as follows. Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. A polynomial sequence $\{P_n\}_{n \geq 0}$ in $\mathcal{P}$ is called a polynomial set (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integer n . We denote by $\langle u, f \rangle$ the effect of the linear functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. Let $\{P_n\}_{n \geq 0}$ be a polynomial set in $\mathcal{P}$. The corresponding dual sequence $(u_n)_{n \geq 0}$ is

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defined by

$$\langle u_n, P_m \rangle = \delta_{n,m}, \quad n, m = 0, 1, \dots,$$

$\delta_{n,m}$ being the Kronecker symbol.

DEFINITION 1.1 ([8, 9]) *Let $u_0, u_1, \dots, u_{d-1}$ be d linear functionals and let $\{P_n\}_{n \geq 0}$ be a PS in $\mathcal{P}$. $\{P_n\}_{n \geq 0}$ is called a d -orthogonal polynomial set (d -OPS, for shorter) with respect to the d -dimensional functional $\mathcal{U} = {}^t(u_0, u_1, \dots, u_{d-1})$ if it fulfills for each integer $k \in \{0, 1, \dots, d-1\}$:*

$$\begin{cases} \langle u_k, P_m P_n \rangle = 0 & \text{if } m > nd + k, \quad n \geq 0, \\ \langle u_k, P_n P_{nd+k} \rangle \neq 0, & n \geq 0. \end{cases} \quad (1)$$

For $d = 1$, we meet the well known notion of orthogonality.

The d -orthogonality has many applications in various domains. We quote, for instance, that there exists a closed relationship between the 2-orthogonality and the birth and death process [10]. Also, Frobenius-Padé approximants for series of d -orthogonal polynomials were studied in [11]. Most of the known explicit examples of d -OPSs were obtained by solving characterization problems which consist of finding all d -OPSs satisfying a fixed property satisfied by a known OPS (see for instance [12–18]). In particular, the fixed property considered by Douak and Maroni [12, 17] is related to associated polynomials which are defined as follows:

Let $\{P_n\}_{n \geq 0}$ be a PS. The associated sequence, $\{P_n^{(1)}\}_{n \geq 0}$, of $\{P_n\}_{n \geq 0}$ with respect to u_0 , the first vector of the dual sequence of $\{P_n\}_{n \geq 0}$, is defined by [19]:

$$P_n^{(1)}(z) = \left\langle u_0, \frac{P_{n+1}(z) - P_{n+1}(\xi)}{z - \xi} \right\rangle, \quad n \geq 0. \quad (2)$$

The r -th associated sequence $\{P_n^{(r)}\}_{n \geq 0}$ is, recurrently, defined by:

$$P_n^{(r+1)} = (P_n^{(r)})^{(1)}, \quad r \text{ being a non-negative integer.}$$

The subject of associated polynomials was considered by many authors (see for instance [19–23]). In this paper, we are interested with the following problem related to associated polynomials first posed by Douak and Maroni [12]:

P1: *For fixed non-negative integer r , find all d -OPSs $\{P_n\}_{n \geq 0}$ which satisfy*

$$P_n^{(r)} = Q_n, \quad \text{where} \quad Q_n = \frac{1}{n+1} P'_{n+1}, \quad n = 0, 1, 2, \dots, \quad (3)$$

and P'_n designates the derivative of P_n .

This problem was treated for some special choices of the integers r and d :

1. Case $(r, d) = (0, 1)$: The only resulting polynomials are the Hermite polynomials (see e.g., [24]).
2. Case $(r, d) = (0, 2)$: This case was treated by Douak and Maroni [25].
3. Case $(r = 0, d \geq 2)$: The solution was given by Douak [16]. The resulting polynomials are an extension of the Hermite polynomials. Among such polynomials the d -symmetric ones (see Definition 3.1, below), are singled out. They are the Gould-Hopper polynomials [26].
4. Cases $(r, d) = (1, 1)$ and $(r, d) = (1, 2)$: These cases were solved by Douak and Maroni [12]. The resulting polynomials are, respectively, the Chebyshev polynomials of the first kind and a natural extension of the Chebyshev polynomials of the first kind into

2-OPS generated by [equation (9)]. It was remarked, in the both cases, that the resulting polynomials are particular cases of Faber polynomials which are defined as follows [12, 27]:

Let z and w be two complex variables, $f(z)$ be a mapping and $h(w)$ be its inverse:

$$\begin{aligned} w &= f(z) = z + a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots, \\ z &= h(w) = w + b_0 + b_1 w^{-1} + b_2 w^{-2} + \dots, \end{aligned}$$

$f(z)$ is univalent mapping from a complement of a compact set onto a domain $|w| > \rho$; ρ being the smallest number such that $h(w)$ is univalent in $|w| > \rho$.

The Faber polynomial $F_n(z)$ of degree n , associated with $f(z)$, is the polynomial part of $[f(z)]^n$.

Faber proved that $\{F_n\}_{n \geq 0}$ is generated by [27]:

$$\frac{wh'(w)}{h(w) - z} = \sum_{n=0}^{\infty} F_n(z) w^{-n}, \quad (4)$$

which holds for $|w|$ sufficiently large (depending on z).

Faber polynomials play an important role in different areas of constructive complex analysis. These polynomials are of interest as a basis for polynomial approximation of analytic functions [28–30]. For more explicit properties of Faber polynomials, we refer to [31–34].

Our purpose, in this paper, is to use Faber polynomials to treat problem **P1**. We show that all d -OPSs among the Faber polynomials are solutions of **P1** for general r and general d . We also solve **P1** for $r = 1, 2$ and general d with the d -symmetry property as an additional condition. It is then significant to consider the following characterization problems

P2: Find all d -OPSs among the Faber polynomials.

P3: Find all d -symmetric d -OPSs among the Faber polynomials.

P4: Find all d -symmetric d -OPSs satisfying the condition $P_n^{(r)} = Q_n$.

Such characterizations take into account the fact that PSs which are obtainable from one another by a linear change of the variable are considered equivalent.

The paper is divided into four sections. Following the introduction, section 2 is devoted to the Faber d -OPSs. We give the general solution of **P2**, Theorem 2.1, and we state a $(d + 1)$ -order recurrence relation satisfied by the obtained polynomials. Then, in Theorem 2.4, we show that the Faber d -OPSs are solutions of **P1**. Section 3 deals with the characterization problems **P3** and **P4**. We prove that the solution of **P3** is exactly the d -orthogonal Chebyshev polynomials of the first kind [35] and we solve problem **P4** for $r = 1, 2$ and general d . Some properties of these polynomials are given. In particular, a $(d + 1)$ -order differential equation satisfied by each of these polynomials and some properties of the zeros. Finally, section 4 is a summary of the obtained results.

2. Faber d -OPSs

First, we give the general solution of problem **P2**. Then, we state a $(d + 1)$ -order recurrence relation satisfied by the obtained polynomials and we prove that the Faber d -OPSs are solution of **P1**.

2.1 Solution of problem P2

The main result here is the following:

THEOREM 2.1 *The d -OPSs among the Faber polynomials generated by equation (4) are obtained for*

$$h(\omega) = \omega v\left(\frac{1}{\omega}\right), \quad \text{with} \quad v(t) = 1 + b_2 t^2 + \cdots + b_{d+1} t^{d+1}, \quad b_{d+1} \neq 0. \quad (5)$$

That is to say, they are generated by:

$$\frac{v(t) - tv'(t)}{v(t) - zt} = \sum_{n=0}^{\infty} F_n(z) t^n. \quad (6)$$

These polynomials were first encountered by Verde-Star [36] for some interpolation problems without involving Faber polynomials or the notion of d -orthogonality.

To prove Theorem 2.1, we need the following result.

LEMMA 2.2 ([35, Theorem 1.1]) *Let $\{P_n\}_{n \geq 0}$ be a PS generated by*

$$\frac{A(t)}{1 - zC(t)} = \sum_{n=0}^{\infty} P_n(z) t^n, \quad (7)$$

where A and C are two formal power series satisfying $A(0) = 1$, $C(0) = 0$ and $C'(0) = 1$. Then $\{P_n\}_{n \geq 0}$ is d -orthogonal if and only if

$$A(t) = \frac{u(t)}{v(t)}, \quad C(t) = \frac{t}{v(t)},$$

where $u(t) = 1 + a_1 t + \cdots + a_{d+1} t^{d+1}$, $v(t) = 1 + b_2 t^2 + \cdots + b_{d+1} t^{d+1}$, such that $a_{d+1} \neq b_{d+1} \neq 0$.

Proof of Theorem 2.1 If we put $t = 1/w$ in equation (4), we get

$$\frac{h'(1/t)}{h(1/t)t} \left\{ 1 - \frac{z}{h(1/t)} \right\}^{-1} = \sum_{n=0}^{\infty} F_n(z) t^n.$$

Let

$$C(t) = \frac{1}{h(1/t)} = \frac{t}{1 + b_0 t + b_1 t^2 + b_2 t^3 + \cdots} \quad \text{and} \quad A(t) = \frac{h'(1/t)}{h(1/t)t} = \frac{tC'(t)}{C(t)}.$$

So A and C satisfy: $A(0) = 1$, $C(0) = 0$ and $C'(0) = 1$.

By virtue of Lemma 2.2, we deduce that $\{F_n\}_{n \geq 0}$ is d -orthogonal if and only if

$$h\left(\frac{1}{t}\right) = \frac{v(t)}{t},$$

which gives the desired result. ■

Next, we give two particular cases of Theorem 2.1.

- *Case $d = 1$:* The orthogonal polynomials among the Faber polynomials are generated by

$$\frac{1 - bt^2}{1 - zt + bt^2} = \sum_{n=0}^{\infty} F_n(z)t^n, \quad b \neq 0. \quad (8)$$

The case $b = 1$ corresponds to the Chebyshev PS of the first kind.

- *Case $d = 2$:* The Faber 2-OPSs are generated by

$$\frac{1 - at^2 - 2bt^3}{1 - zt + at^2 + bt^3} = \sum_{n=0}^{\infty} F_n(z)t^n, \quad b \neq 0. \quad (9)$$

These polynomials were obtained by Douak and Maroni [12] as solution of problem **P1** for $(r, d) = (1, 2)$. The case $a = 0$ corresponds to the Chebyshev 2-OPS of the first kind [12].

2.2 $A(d + 1)$ -order recurrence relation

Next, we give a $(d + 1)$ -order recurrence relation satisfied by the Faber d -OPSs obtained in Theorem 2.1.

PROPOSITION 2.3 *The polynomials F_n ; $n \geq 0$ generated by equation (6) satisfy a $(d + 1)$ -order recurrence relation of the type:*

$$\begin{cases} F_0(z) = 1, & F_1(z) = z, & F_2(z) = z^2 - 2b_2, \\ F_{n+1}(z) = zF_n(z) - \sum_{i=2}^n b_i F_{n-i+1}(z) - (n+1)b_{n+1}, & 2 \leq n \leq d, \\ F_{n+d+1}(z) = zF_{n+d}(z) - \sum_{i=2}^{d+1} b_i F_{n+d-i+1}(z), & 1 \leq n. \end{cases} \quad (10)$$

Proof From equation (6) we have

$$\left(1 + \sum_{k=2}^{d+1} (1-k)b_k t^k\right) = \left(1 - zt + \sum_{k=2}^{d+1} b_k t^k\right) \sum_{n=0}^{\infty} F_n(z)t^n. \quad (11)$$

Let $b_0 = 1$, $b_1 = -z$ and $b_n = 0$, $n \geq d + 2$.

Then equation (11) may be rewritten as follows

$$\sum_{k=0}^{d+1} (1-k)b_k t^k = \left(\sum_{k=0}^{d+1} b_k t^k\right) \sum_{n=0}^{\infty} F_n(z)t^n.$$

By using the Cauchy product and by comparing the coefficients of t^n , we obtain the desired result. ■

This result generalizes the third-order recurrence relation satisfied by the 2-orthogonal Chebyshev polynomials of the first kind [12], and it is a particular case of the $(d + 1)$ -order recurrence relation obtained in [35].

Remark. Maroni [8] proved that $\{P_n\}_{n \geq 0}$ is a d -OPS if and only if it satisfies a $(d + 1)$ -order recurrence relation of the form

$$\begin{cases} P_0(z) = 1, \\ P_n(z) = (z - \beta_{n-1})P_{n-1}(z) - \sum_{v=0}^{d-1} \gamma_{n-1-v}^{d-1-v} P_{n-v-2}(z), \quad \forall n \geq 1, \end{cases} \quad (12)$$

with the convention $P_{-n} = 0$, $n \geq 1$ and the regularity condition, $\gamma_{n+1}^0 \neq 0$, $n \geq 0$.

This result, for $d = 1$, is reduced to the so-called Favard Theorem.

It follows that the Faber polynomials satisfying a $(d + 1)$ -order recurrence relation type given in equation (12) are exactly the ones generated by equation (6).

2.3 The associated polynomials problem

Douak and Maroni [12] solved problem **P1** in the particular cases $(r, d) = (1, 1)$ and $(r, d) = (1, 2)$. We note, in the both cases, that the resulting polynomials are, exactly and respectively, the orthogonal Faber polynomials and the Faber 2-OPSs.

Our purpose in this subsection is to prove that the Faber d -OPSs are solutions of problem **P1** for positive integer r and general d . For this, we state the following

THEOREM 2.4 *The r -th associated sequences of the Faber d -OPSs in equation (6) are generated by*

$$\frac{1}{v(t) - zt} = \sum_{n=0}^{\infty} F_n^{(r)}(z)t^n. \quad (13)$$

Moreover, the Faber d -OPSs are solutions of problem **P1** for positive integer r .

To prove this theorem we recall the following two lemmas.

LEMMA 2.5 ([17, 19]) *Let $\{P_n\}_{n \geq 0}$ be a d -OPS satisfying equation (12). Then $\{P_n^{(1)}\}_{n \geq 0}$ is also a d -OPS and satisfies the following $(d + 1)$ -order recurrence relation:*

$$\begin{cases} P_0^{(1)}(z) = 1, \\ P_n^{(1)}(z) = (z - \beta_n)P_{n-1}^{(1)}(z) - \sum_{v=0}^{d-1} \gamma_{n-v}^{d-1-v} P_{n-2-v}^{(1)}(z), \quad n \geq 1. \end{cases} \quad (14)$$

LEMMA 2.6 ([17]) *Let $\{P_n\}_{n \geq 0}$ be a d -OPS. The following statements are equivalent:*

1. $P_n^{(1)} = P_n$.
2. $\{P_n\}_{n \geq 0}$ satisfies the following $(d + 1)$ -order recurrence relation:

$$\begin{cases} P_0(z) = 1, \\ P_n(z) = zP_{n-1}(z) - \sum_{v=2}^{d+1} b_v P_{n-v}(z), \quad n \geq 1, \quad b_{d+1} \neq 0. \end{cases} \quad (15)$$

3. $\{P_n\}_{n \geq 0}$ is generated by

$$\frac{1}{v(t) - zt} = \sum_{n=0}^{\infty} P_n(z)t^n,$$

with $v(t) = 1 + b_2t^2 + \cdots + b_{d+1}t^{d+1}$, $b_{d+1} \neq 0$.

Proof of Theorem 2.4 We divide our proof into two steps. In Step 1, we prove that $\{F_n^{(1)}\}_{n \geq 0}$ is generated by equation (13) and $F_n^{(r)} = F_n^{(1)}$. Then, in Step 2, we show that $F_n^{(1)} = Q_n$.

Step 1: Using equation (10), we deduce that a Faber d -OPS satisfies equation (12) with

$$\begin{aligned} \gamma_n^{d-1-k} &= \{(k+1)\delta_{1,n} + 1\}b_{k+2}, \quad k = 0, 1, \dots, d-1, \quad n \geq 1 \quad \text{and} \\ \beta_n &= 0, \quad n \geq 0. \end{aligned}$$

Thus, by virtue of Lemma 2.5, $\{F_n^{(1)}\}_{n \geq 0}$ satisfies the following $(d+1)$ -order recurrence relation:

$$\begin{cases} F_0^{(1)}(z) = 1, \\ \sum_{v=0}^n b_v F_{n-v}^{(1)}(z) = 0, \quad n \geq 1, \end{cases} \quad (16)$$

where $b_0 = 1$, $b_1 = -z$ and $b_n = 0$, $n \geq d+2$.

Then $\{F_n^{(1)}\}_{n \geq 0}$ satisfies equation (15). So, by virtue of Lemma 2.6, $F_n^{(2)} = F_n^{(1)}$ and consequently $F_n^{(r)} = F_n^{(1)}$.

Step 2: We proceed by induction on n to prove that $\{Q_n\}_{n \geq 0}$ satisfies equation (16).

For $n = 0, 1$ the result holds. We suppose that

$$\sum_{k=0}^p b_k Q_{p-k}(z) = 0, \quad 1 \leq p \leq n-1.$$

It follows from the generating function of equation (6) that

$$\left(\sum_{k=0}^{d+1} b_k t^k \right)^2 \sum_{n=0}^{\infty} (n+1) Q_n(z) t^n = \sum_{k=0}^{d+1} (1-k) b_k t^k. \quad (17)$$

By comparing the coefficients of t^n , we deduce

$$\sum_{k=0}^n \sum_{p=0}^k b_p b_{k-p} (n+1-k) Q_{n-k}(z) = (1-n)b_n, \quad n \geq 0. \quad (18)$$

With sum manipulation in equation (18), we have

$$\begin{aligned} \sum_{p=0}^n \sum_{k=p}^n b_p b_{k-p} (n+1-k) Q_{n-k}(z) &= \sum_{p=0}^n b_p \sum_{k=0}^{n-p} b_k (n+1-k-p) Q_{n-k-p}(z) \\ &= (1-n)b_n. \end{aligned} \quad (19)$$

Then, we use equation (19) to get

$$(n+1) \sum_{k=0}^n b_k Q_{n-k}(z) - \sum_{p=0}^{n-1} b_p \sum_{k=0}^{n-p} k b_k Q_{n-k-p}(z) = -n b_n. \quad (20)$$

By using again sum manipulation and equation (19), we obtain

$$\begin{aligned} \sum_{p=0}^{n-1} b_p \sum_{k=0}^{n-p} k b_k Q_{n-k-p}(z) &= \sum_{k=0}^n k b_k \sum_{p=0}^{n-k} b_p Q_{n-k-p}(z) \\ &= \sum_{k=0}^{n-1} k b_k \sum_{p=0}^{n-k} b_p Q_{(n-k)-p}(z) + n b_n \\ &= n b_n. \end{aligned} \quad (21)$$

Then equation (20) leads to

$$\sum_{k=0}^n b_k Q_{n-k}(z) = 0, \quad n \geq 1, \quad (22)$$

which finishes the proof. ■

3. d -Symmetric Faber d -OPSs

In this section, we are interested by the general solution of problem **P1** with d -symmetry property as an additional condition. For this, in Theorem 3.2 we determine the d -symmetric d -OPSs among the Faber polynomials and we prove that the obtained polynomials are the solution of **P1** with the additional condition.

We recall the following

DEFINITION 3.1 ([13, 19]) A PS $\{P_n\}_{n \geq 0}$ is called d -symmetric if it fulfills

$$P_n(\omega_d z) = \omega_d^n P_n(z) \quad \text{where} \quad \omega_d = \exp\left(\frac{2i\pi}{(d+1)}\right). \quad (23)$$

When $d = 1$, the sequence $\{P_n\}_{n \geq 0}$ is symmetric.

Note here, that the symmetry behavior of this notion is the same as the one of m -fold symmetry, for $m = d + 1$, already used in the theory of univalent functions (see for instance [31, 34]).

3.1 Solution of problem P3

The general solution of problem **P3** (see section 1) is given by the following

THEOREM 3.2 The d -symmetric Faber d -OPSs are exactly the Chebyshev d -OPS of the first kind $\{T_n(., d)\}_{n \geq 0}$ generated by [35]

$$\frac{1 - dbt^{d+1}}{1 + bt^{d+1} - zt} = \sum_{n=0}^{\infty} T_n(z, d) t^n, \quad b \neq 0. \quad (24)$$

$\{T_n(., d)\}_{n \geq 0}$ generalizes the Chebyshev polynomials of the first kind. In particular: for $d = 2$, we meet the polynomials introduced in [12].

For $d = m - 1$ and $b = (1/(m - 1))$, we obtain the Faber polynomials studied by He and Saff [37].

The proof of Theorem 3.2 is easily deduced from the $(d + 1)$ -order recurrence relation satisfied by the Faber d -OPS in equation (6) and from the two following Lemmas.

LEMMA 3.3 ([25]) *Let $\{P_n\}_{n \geq 0}$ be a d -OPS. Then the following statements are equivalent:*

1. $\{P_n\}_{n \geq 0}$ is d -symmetric.
2. $\{P_n\}_{n \geq 0}$ satisfies the $(d + 1)$ -order recurrence relation

$$\begin{cases} P_{n+d+1}(z) = zP_{n+d}(z) - \gamma_{n+1}^0 P_n(z), & n \geq 0, \quad (\gamma_{n+1}^0 \neq 0) \\ P_n(z) = z^n, & 0 \leq n \leq d. \end{cases}$$

LEMMA 3.4 ([35]) *$\{T_n(., d)\}_{n \geq 0}$ satisfies the following recurrence relation*

$$\begin{cases} T_n(z, d) = z^n, & 0 \leq n \leq d, \\ T_{n+d+1}(z, d) = zT_{n+d}(z, d) - (d + 1)^{\delta_{n,0}} b T_n(z, d), & b \neq 0, \quad 0 \leq n. \end{cases} \quad (25)$$

Remark. It is well known that the Chebyshev polynomials of the both kinds, $\{T_n\}_{n \geq 0}$ and $\{U_n\}_{n \geq 0}$, are linked, at the same time, by the two relations

$$T_n^{(r)}(z) = U_n(z) \text{ and } T_{n+1}'(z) = (n + 1)U_n(z), \quad n = 0, 1, \dots \quad (26)$$

Douak and Maroni generalized the Chebyshev polynomials of the second kind into d -OPS $\{U_n(., d)\}_{n \geq 0}$ generated by [17]

$$\frac{1}{1 + bt^{d+1} - zt} = \sum_{n=0}^{\infty} U_n(z, d)t^n, \quad b \neq 0. \quad (27)$$

By applying Theorem 2.4 to $\{T_n(., d)\}_{n \geq 0}$, we deduce that equation (26) may be generalized for $d \geq 1$. We have in fact,

$$T_n^{(r)}(z, d) = U_n(z, d) \text{ and } T_{n+1}'(z, d) = (n + 1)U_n(z, d), \quad n = 0, 1, \dots \quad (28)$$

3.2 General solution of P1 in d -symmetric case

In this subsection, we are interested by problem **P4** cited in section 1.

We treat the two special cases $r = 1, r = 2$.

THEOREM 3.5 *In both the cases $r = 1, r = 2$ and $d \geq 1$, the general solution of problem **P4** is the Chebyshev d -OPS of the first kind $\{T_n(., d)\}_{n \geq 0}$.*

Proof Let $\{P_n\}_{n \geq 0}$ be a d -symmetric d -OPS, then $\{P_n\}_{n \geq 0}$ satisfies the $(d+1)$ -order recurrence relation

$$\begin{cases} P_{n+d+1}(z) = zP_{n+d}(z) - \gamma_{n+1}P_n(z), & n \geq 0, \quad (\gamma_{n+1} \neq 0) \\ P_n(z) = z^n, & 0 \leq n \leq d. \end{cases} \quad (29)$$

By differentiating equation (29) and shifting indices we get, for $n \geq 0$,

$$(n+d+2)Q_{n+d+1}(z) = z(n+d+1)Q_{n+d}(z) - (n+1)\gamma_{n+2}Q_n(z) + P_{n+d+1}(z). \quad (30)$$

Using Lemma 2.5, $\{P_n^{(r)}\}_{n \geq 0}$ satisfies

$$\begin{cases} P_{n+d+1}^{(r)}(z) = zP_{n+d}^{(r)}(z) - \gamma_{n+r+1}P_n^{(r)}(z), & n \geq 0, \\ P_n^{(r)}(z) = z^n, & 0 \leq n \leq d. \end{cases}$$

Since $P_n^{(r)} = Q_n$, we have

$$\begin{cases} Q_{n+d+1}(z) = zQ_{n+d}(z) - \gamma_{n+r+1}Q_n(z), & n \geq 0, \\ Q_n(z) = z^n, & 0 \leq n \leq d. \end{cases} \quad (31)$$

From equations (30) and (31) we deduce

$$P_{n+d+1}(z) = zQ_{n+d}(z) - [(n+d+2)\gamma_{n+r+1} - (n+1)\gamma_{n+2}]Q_n(z), \quad n \geq 0, \quad (32)$$

and then

$$P_{n+d+1}(z) = Q_{n+d+1}(z) - [(n+d+1)\gamma_{n+r+1} - (n+1)\gamma_{n+2}]Q_n(z). \quad (33)$$

Replacing n by 0 in equations (29) and (32), we obtain

$$\begin{cases} P_{d+1}(z) = zQ_d(z) - [(d+2)\gamma_{r+1} - \gamma_2]Q_0(z) = z^{d+1} - [(d+2)\gamma_{r+1} - \gamma_2], & n \geq 0, \\ = zP_d(z) - \gamma_1P_0(z) = z^{d+1} - \gamma_1^0, \end{cases}$$

then

$$\gamma_{r+1} = \frac{1}{d+2}[\gamma_1 + \gamma_2]. \quad (34)$$

Using equations (29) and (33) we get

$$\begin{cases} P_{n+d+1}(z) = zP_{n+d}(z) - \gamma_{n+1}P_n(z), & n \geq 0, \\ = z(Q_{n+d}(z) - [(n+d)\gamma_{n+r} - n\gamma_{n+1}]Q_{n-1}(z)) - \gamma_{n+1}P_n, & n \geq 1. \end{cases} \quad (35)$$

From equations (32) and (35), we deduce for $n \geq 1$

$$[(n+d+2)\gamma_{n+r+1} - (n+1)\gamma_{n+2}]Q_n(z) = [(n+d)\gamma_{n+r} - n\gamma_{n+1}]zQ_{n-1}(z) + \gamma_{n+1}P_n. \quad (36)$$

By comparing the coefficients of t^n , we obtain

$$(n+d+2)\gamma_{n+r+1} - (n+1)\gamma_{n+2} = (n+d)\gamma_{n+r} - (n-1)\gamma_{n+1}, \quad n \geq 1. \quad (37)$$

For the sake of simplicity, in the sequel, we treat the two particular cases $r = 1$ and $r = 2$. We prove, in the both cases that $\{P_n\}_{n \geq 0}$ and $\{T_n(\cdot, d)\}_{n \geq 0}$ satisfy the same recurrence relation.

1. *Case $r = 1$ and $d \geq 1$:* Replacing r by its value in equation (37), we get

$$(n + d + 2)\gamma_{n+2} - (n + 1)\gamma_{n+2} = (n + d)\gamma_{n+1} - (n - 1)\gamma_{n+1}.$$

Then $\gamma_{n+2} = \gamma_{n+1}$, $n \geq 1$.

Using equation (34), we deduce that

$$\gamma_{n+1} = \frac{1}{d+1}\gamma_1, \quad n \geq 1.$$

Replacing γ_n by its value in equation (29) and using Lemma 3.4, we deduce that $P_n = T_n(., d)$, then $P_n^{(r)} = Q_n$.

2. *Case $r = 2$ and $d \geq 1$:* Using equation (37), we get

$$(n + d + 2)\gamma_{n+3} - (n + 1)\gamma_{n+2} = (n + d)\gamma_{n+2} - (n - 1)\gamma_{n+1}.$$

Then

$$(n + d + 2)\gamma_{n+3} - n\gamma_{n+2} = (n + d + 1)\gamma_{n+2} - (n - 1)\gamma_{n+1}, \quad n \geq 1.$$

Let $x_n = (n + d + 1)\gamma_{n+2} - (n - 1)\gamma_{n+1}$, we have $x_n = x_{n+1}$, $n \geq 1$.

Then $(n + d + 1)\gamma_{n+2} - (n - 1)\gamma_{n+1} = (d + 2)\gamma_3$.

From which, we deduce that $\gamma_{n+2} = \gamma_3$, $n \geq 1$.

Using equation (34), we get

$$\gamma_{n+2} = \frac{1}{d+2}(\gamma_1 + \gamma_2), \quad n \geq 1.$$

Using equation (36), and replacing γ_n , $n \geq 3$, by γ_3 , we get

$$(d + 1)\gamma_3 Q_n(z) = [(n + d)\gamma_3 - n\gamma_{n+1}]z Q_{n-1}(z) + \gamma_{n+1}P_n.$$

For $n = d + 1$

$$(d + 1)\gamma_3 Q_{d+1}(z) = d\gamma_3 z Q_d(z) + \gamma_3 P_{d+1}.$$

By comparing the constants, we obtain

$$(d + 1)\gamma_3\gamma_3 = \gamma_3\gamma_1,$$

then

$$\gamma_n = \gamma_3 = \gamma_2 = \frac{1}{d+1}\gamma_1, \quad n \geq 3.$$

It follows that $\{P_n\}_{n \geq 0}$ and $\{T_n(., d)\}_{n \geq 0}$ satisfy the same recurrence relation, so $P_n = T_n(., d)$ which finishes the proof. ■

3.3 Properties of the polynomials $T_n(., d)$

Some properties of the polynomials $\{T_n(., d)\}_{n \geq 0}$ were given in [12, 35]. In particular, the d -dimensional functional for which the d -orthogonality holds and, in the case $d = 2$, an integral representation of the 2-dimensional functional $\mathcal{U} = {}^t(u_0, u_1)$ associated with $\{T_n(., 2)\}_{n \geq 0}$ and a three-order differential equation satisfied by these polynomials.

In the sequel, we give some other properties of these polynomials. In particular, a $(d + 1)$ -order differential equation and some properties of the zeros of these polynomials.

3.3.1 A $(d + 1)$ -order differential equation. Next, we give a $(d + 1)$ -order differential equation satisfied by the d -symmetric Faber d -OPS in equation (24).

PROPOSITION 3.6 *Each polynomial $T_n(., d)$ satisfies a $(d + 1)$ -order differential equation of the form:*

$$\left[D^{d+1} + \frac{1}{(d+1)^{d+1}b} (n - zD) \prod_{j=0}^{d-1} (zD + n + (d+1)j) \right] y = 0, \quad n > d + 1. \quad (38)$$

To prove this proposition, we follow the method used in [13].

Proof Since $U_n(z, d) = (1/n + 1)T'_{n+1}(z, d)$ we deduce from equation (22) that $\{U_n(., d)\}_{n \geq 0}$ satisfies the following recurrence relations

$$\begin{cases} U_n(z, d) = z^n, & 0 \leq n \leq d, \\ U_{n+d+1}(z, d) = zU_{n+d}(z, d) - bU_n(z, d), & 0 \leq n. \end{cases} \quad (39)$$

By differentiating equation (25) and shifting indices we get, for $n \geq 0$,

$$T_{n+d+1}(z, d) = (n + d + 2)U_{n+d+1}(z, d) + (n + 1)bU_n(z, d) - z(n + d + 1)U_{n+d}(z, d). \quad (40)$$

Next, using equations (39) and (40), we obtain

$$\begin{cases} T_{n+1}(z, d) = zU_n(z, d), & n = 0, 1, \dots, d - 1, \\ T_{n+d+1}(z, d) = zU_{n+d}(z, d) - (d + 1)bU_n(z, d), & n \geq 0. \end{cases} \quad (41)$$

By shifting $n \rightarrow n - d - 1$ in equation (41), we obtain

$$\begin{cases} [zD - n]T_n(z, d) = 0, & n = 0, 1, \dots, d, \\ [zD - n]T_n(z, d) = \frac{n}{n-d}(d + 1)bDT_{n-d}(z, d), & n > d. \end{cases} \quad (42)$$

The relation (42) can be rewritten as

$$DT_{n-d}(z, d) = \frac{n - d}{nb(d + 1)}[zD - n]T_n(z, d). \quad (43)$$

Otherwise, by shifting $n \rightarrow n - d - 1$ in equation (40), we obtain

$$DP_{n+1} - [zD + 1]T_n(z, d) = -bDT_{n-d}(z, d). \quad (44)$$

Substituting equation (43) in (44) yields

$$DT_{n+1}(z, d) = \left[\frac{d(n + 1)}{(d + 1)n}(zD - n) + n + 1 \right] T_n(z, d). \quad (45)$$

Now, let r be a positive integer. Differentiate equation (45) r times and use the identity $D^r(zD) = (zD + r)D^r$, to obtain

$$D^{r+1}T_{n+1}(z, d) = \left[\frac{d(n + 1)}{(d + 1)n}(zD - n + r) + n + 1 \right] D^rT_n(z, d). \quad (46)$$

Now, replace in the latest equation, (r, n) by $(d - j, n - 1 - j)$; $0 \leq j \leq d - 1$, to construct the following system:

$$D^{d+1-j}T_{n-j}(z, d) = \left[\frac{d(n-j)}{(d+1)(n-j-1)}(zD - n + d + 1) + n - j \right] D^{d-j}T_{n-1-j}(z, d). \quad (47)$$

From which we deduce a relation linking $T_n(., d)$ and $T_{n-d}(., d)$, that is

$$D^{d+1}T_n(z, d) = \left[\prod_{j=0}^{d-1} \left\{ \frac{d(n-j)}{(d+1)(n-j-1)}(zD - n + d + 1) + n - j \right\} \right] DT_{n-d}(z, d). \quad (48)$$

Then, substitute equation (43) in (48) to obtain for $n > d + 1$

$$\left[D^{d+1} - \frac{n-d}{n(d+1)b}(zD - n) \prod_{j=0}^{d-1} \left\{ \frac{d(n-j)}{(n-1-j)(d+1)}(zD - n + d + 1) + n - j \right\} \right] T_n(z, d) = 0, \quad (49)$$

which can be simplified to the desired result. ■

As particular cases of equation (38), we have:

- For $d = 1$ and $b = 1$, the differential equation satisfied by the Chebyshev polynomials of the first kind.
- For $d = 2$, the third-order differential equation satisfied by the Chebyshev 2-OPS of the first kind [12]:

$$4(z^3 - 1)P_n'''(z) + 18z^2P_n''(z) - (3n^2 + 3n - 10)zP_n'(z) - n^2(n + 3)P_n(z) = 0, \quad n \geq 0.$$

- For $d = m - 1$ and $b = (1/m - 1)$, we meet again the m -differential equation obtained by He and Saff [37].

3.3.2 Zeros of the polynomials $T_n(., d)$. From the d -symmetry property of $T_{p(d+1)+j}(., d)$, we easily state the following properties:

R1: If z_0 is a zero of $T_{p(d+1)+j}(., d)$, then $\omega_d^k z_0$, $k = 1, \dots, d$, $\omega_d = \exp(2i\pi/(d+1))$, are also zeros of $T_{p(d+1)+j}(., d)$, for $p \geq 1$ and $j = 0, 1, \dots, d$.

R2: 0 is a zero of $T_{p(d+1)+j}(., d)$ of multiplicity j .

On the other hand, by virtue of ([37] Theorem 1.1), we deduce

R3: For $b > 0$, $T_{p(d+1)+j}(., d)$ has p distinct positive real number zeros.

Then the zeros of $T_n(., d)$ lie on $(d+1)$ rays emanating from the origin.

For $b < 0$, let $S_n(z, d)$ be defined by

$$S_n(z, d) = \exp\left(\frac{-in\pi}{(d+1)}\right) T_n\left(\exp\left(\frac{i\pi}{(d+1)}\right) z, d\right).$$

If z_0 is a zero of S_n then $\exp(i\pi/(d+1))z_0$ is a zero of T_n .

By virtue of equation (25), $\{S_n\}_{n \geq 0}$ satisfies

$$\begin{cases} S_n(z, d) = z^n, & 0 \leq n \leq d, \\ S_{n+d+1}(z, d) = zS_{n+d}(z) - (-(d+1)^{\delta_{n,0}}b)S_n(z, d), & 0 \leq n. \end{cases} \quad (50)$$

Replacing, in **R3**, T_n and b , respectively, by S_n and $-(d+1)^{\delta_{n,0}}b$, we obtain that the zeros of S_n are localized on $(d+1)$ rays emanating from the origin. Thus, the zeros of $T_n(\cdot, d)$ lie on the $(d+1)$ rays:

$$\Gamma_k = \{r \exp(i\pi(2k+1)/(d+1)), \quad r \geq 0\}, \quad k = 0, 1, \dots, d.$$

4. Concluding remarks

The general problem **P1** was treated for some special choices of the integers r and d . In particular, Douak and Maroni [12] investigated the cases $(r, d) = (1, 1)$ and $(r, d) = (1, 2)$. The solutions are, exactly and respectively, the Faber OPS and the Faber 2-OPS. Their method was based on the solution of the nonlinear system obtained from the $(d+1)$ -order recurrence relations satisfied by $\{P_n\}_{n \geq 0}$ and $\{P_n^{(1)}\}_{n \geq 0}$. This method appears to be unable to solve the general case in problem **P1**.

In order to deduce more results about the solutions of **P1**, and since the obtained polynomials in the cases $d = 1, 2$ are, respectively, the Faber OPS and the Faber 2-OPS, it is natural to solve first problem **P2**. For this, Ben Cheikh and the author [35] used another approach based on generating functions and considered the following problem

P: Characterize all d -OPSs generated by functions of the type

$$\frac{A(t)}{1 - xC(t)} = \sum_{n=0}^{\infty} P_n(x)t^n,$$

where A and C are two formal power series satisfying $A(0) = C'(0) = 1$ and $C(0) = 0$.

The obtained result (Lemma 2.2) was used in this paper to determine the Faber d -OPSs (Theorem 2.1) which solves problem **P2**. Then, in Theorem 2.4, we proved that the Faber d -OPSs are solutions of **P1** for each non-negative integer r and general d . Therefore, with d -symmetry property as an additional condition, a partial solution of problem **P1**, was obtained in Theorem 3.5. The resulting polynomials are the d -symmetric Faber d -OPSs which are exactly the Chebyshev d -OPS of the first kind [35]. Some properties of these polynomials were established. It is then natural to state the following question:

Q: Are the Faber d -OPSs the unique solutions of **P1**?

This question (and then problem **P1**) remains unsolved.

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Generalized gamma-type functions involving Kummer's confluent hypergeometric function and associated probability distributions

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The object of this paper is to define and study a new generalization of the generalized gamma-type function in the form

$$S\left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \middle| p, k\right) := v^{-\lambda} \int_0^\infty t^{u-1} \Phi(\alpha, \beta; -pt) {}_2R_1\left(\lambda, a; b; k; -\frac{t}{v}\right) dt,$$

where $\Phi(\alpha, \beta; z)$ is the well known Kummer's confluent hypergeometric function and ${}_2R_1(\cdot)$ is a special case of Wright's generalized hypergeometric function. This generalization provides unification and extension of the various generalizations given earlier by Kobayashi, Al-Musallam and Kalla and Virchenko *et al.* Incomplete generalized gamma function corresponding to the above defined generalized gamma function are also defined and their basic properties are investigated, which extend the existing results in the field of generalized gamma-type functions. We study a class of probability density functions involving Kummer's confluent hypergeometric function and ${}_2R_1(\cdot)$.

Keywords: Gamma and incomplete gamma functions; Generalized hypergeometric function; Confluent hypergeometric function; Diffraction theory; Probability density function; Moment generating function

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1. Introduction and preliminaries

In the study of asymptotic expansion of a class of branch-cut integrals occurring in diffraction theory using Wiener–Hopf technique, Kobayashi [1, 2] encountered an integral of the following form

$$\Gamma_m(u, v) := \int_0^\infty \frac{t^{u-1} e^{-t}}{(t+v)^m} dt, \quad |\arg v| < \pi, \quad \operatorname{Re}(u) > 0, \quad (1)$$

where $m \in N_0$.

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For $m = 0$, equation (1) becomes the well-known gamma function

$$\Gamma(u) := \int_0^\infty t^{u-1} e^{-t} dt, \quad \operatorname{Re}(u) > 0. \quad (2)$$

Al-Musallam and Kalla [3, 4] considered a generalization of equation (1) associated with Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ in the form

$$D\left(\begin{matrix} \lambda, a; b; p \\ u, v \end{matrix}\right) := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} {}_2F_1\left(\lambda, a; b; -\frac{t}{v}\right) dt, \quad (3)$$

where $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

Saxena and Kalla [5] introduced a generalization of the gamma function by taking Clausenian hypergeometric function ${}_3F_2(\cdot)$ instead of the binomial function in the form

$$S\left(\begin{matrix} \lambda, a, b; c, d, p \\ u, v \end{matrix}\right) := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} {}_3F_2\left(\lambda, a, b; c, d; -\frac{t}{v}\right) dt, \quad (4)$$

where $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

If we take $b = d$ in equation (4), it reduces to equation (3).

Recently Virchenko *et al.* [6] introduced yet another extension of Kobayashi function [1, 2] in the following form

$$\Gamma\left(\begin{matrix} \lambda, a; b; \\ u, v \end{matrix} \quad p, k\right) := v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} {}_2R_1\left(\lambda, a; b; k; -\frac{t}{v}\right) dt, \quad (5)$$

where $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$, $|\arg v| < \pi$ and ${}_2R_1(\cdot)$, is a special case of Wright's generalized hypergeometric function ${}_p\Psi_q$ with p numerator and q denominator parameter pairs (see [7, 8]), defined by

$$\begin{aligned} {}_2R_1^k(z) &= {}_2R_1(\lambda, a; b; k; z) \\ &= \frac{\Gamma(b)}{\Gamma(\lambda)\Gamma(a)} \sum_{r=0}^{\infty} \frac{\Gamma(\lambda+r)\Gamma(a+kr)}{\Gamma(b+kr)} \cdot \frac{z^r}{r!}, \end{aligned} \quad (6)$$

where $|z| < 1$.

In the present paper the authors introduce a further generalization of the gamma-type function associated with Kummer's confluent hypergeometric function $\Phi(\alpha, \beta; z)$ in the form

$$S\left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k\right) := v^{-\lambda} \int_0^\infty t^{u-1} \Phi(\alpha, \beta; -pt) {}_2R_1\left(\lambda, a; b; k; -\frac{t}{v}\right) dt, \quad (7)$$

where $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$, $k > 0$ and $|\arg v| < \pi$, the confluent hypergeometric function is defined by

$$\Phi(\alpha, \beta; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{z^k}{k!}, \quad \beta \neq 0, -1, -2, \dots \quad (8)$$

The Mellin–Barnes type integral representation of ${}_2R_1^k(z)$ is obtained as

$$\begin{aligned} {}_2R_1^k(z) &= {}_2R_1(\lambda, a; b; k; z) \\ &= \frac{1}{2\pi i} \frac{\Gamma(b)}{\Gamma(\lambda)\Gamma(a)} \int_{-i\infty}^{i\infty} \frac{\Gamma(-s)\Gamma(\lambda+s)\Gamma(a+ks)}{\Gamma(b+ks)} (-z)^s ds, \end{aligned} \quad (9)$$

where $i = (-1)^{1/2}$, $|\arg(-z)| < \pi$, $k > 0$ and the poles of the gamma functions appearing in the integrand of equation (9) are assumed to be simple. Equation (6) can be established by

the application of calculus of residues, if we calculate the sum of the residues at the poles of $\Gamma(-s)$ at the points $s = v$ ($v = 0, 1, 2, \dots$).

If, however, we calculate the sum of the residues at the poles of the gamma functions $\Gamma(\lambda + s)$ and $\Gamma(a + ks)$ at the points $s = -\lambda - v_1$ and $s = -(a/k) - v_2$ ($v_1, v_2 = 0, 1, \dots$), respectively, in the integrand of equation (9), we find that

$$\begin{aligned} {}_2R_1^k(z) = & \frac{\Gamma(b)}{\Gamma(\lambda)\Gamma(a)} \left[(-z)^{-\lambda} \sum_{v=0}^{\infty} \frac{\Gamma(\lambda+v)\Gamma(a-k\lambda-kv)}{\Gamma(b-k\lambda-kv)} \left(\frac{1}{z}\right)^v \right. \\ & \left. + (-z)^{-a/k} \sum_{v=0}^{\infty} \frac{\Gamma(a/k+v)\Gamma(\lambda-v-a/k)}{\Gamma(b-a-kv)} \left(\frac{1}{z}\right)^v \right] \end{aligned} \quad (10)$$

where $|\arg(-z)| < \pi$, $|z| > 1$.

Equation (10) gives the analytic continuation formula for the function ${}_2R_1^k(z)$.

${}_2R_1^k(z)$ can be expressed in terms of the H -function (see [9]) in the form

$${}_2R_1(\lambda, a; b; k; -z) = \frac{\Gamma(b)}{\Gamma(\lambda)\Gamma(a)} H_{2,2}^{1,2} \left[z \left| \begin{matrix} (1-\lambda, 1), & (1-a, k) \\ (0, 1), & (1-b, k) \end{matrix} \right. \right]. \quad (11)$$

2. Generalized gamma-type function

Let us assume that $\Omega_v = \{v \in C; |\arg v| < \pi\}$ and

$$\Omega_u = \{u \in C; \operatorname{Re}(u) > 0\}.$$

DEFINITION Let $\lambda, \alpha, \beta, a, b, p \in C$ with $\beta, b \neq 0, -1, -2, \dots, k > 0$ and $\operatorname{Re}(p) > 0$, we define

$$S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \middle| p, k \right) := v^{-\lambda} \int_0^\infty t^{u-1} \Phi(\alpha, \beta; -pt) {}_2R_1 \left(\lambda, a; b; k; -\frac{t}{v} \right) dt, \quad (12)$$

where $u \in \Omega_u, v \in \Omega_v$ and ${}_2R_1(\lambda, a; b; k; -(t/v))$ is the generalized hypergeometric function.

When $\alpha = \beta$, the confluent hypergeometric function reduces to an exponential function e^{-pt} and consequently (12) reduces to the following result given earlier by Virchenko *et al.* [6]

$$S \left(\begin{matrix} \lambda, a, b; \alpha, \alpha; \\ u, v \end{matrix} \middle| p, k \right) = \Gamma \left(\begin{matrix} \lambda, a; b; \\ u, v \end{matrix} \middle| p, k \right) \quad (13)$$

$$= v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} {}_2R_1 \left(\lambda, a; b; k; -\frac{t}{v} \right) dt, \quad (14)$$

where $\operatorname{Re}(u) > 0, \operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

For $\alpha = \beta$ and $k = 1$, equation (12) yields the following generalized gamma function studied by Al-Mussallam and Kalla [4]

$$D \left(\begin{matrix} \lambda, a; b; p \\ u, v \end{matrix} \right) = v^{-\lambda} \int_0^\infty t^{u-1} e^{-pt} {}_2F_1 \left(\lambda, a; b; -\frac{t}{v} \right) dt, \quad (15)$$

where $\operatorname{Re}(u) > 0, \operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

If we set $\alpha = \beta$, $a = b$, $p = k = 1$ and $\lambda = m \in N_0$, equation (12) gives the generalized Gamma function defined by equation (1) studied by Kobayashi [1, 2].

THEOREM 1 S is analytic in the domain $\Omega_u \times \Omega_v$.

The proof is similar to the corresponding theorem for the generalized gamma function given by Saxena and Kalla [5, pp. 191–192], if we employ the asymptotic estimates [3]

$${}_2R_1(\lambda, a; b; k; z) = A_1 z^{-\lambda} + A_2 z^{-(a/k)} + O(z^{-\lambda-1}) + O(z^{-(a/k)-1}), \quad (16)$$

as $z \rightarrow \infty$ (A_1 and A_2 are numerical constants.), and

$$\Phi(a, c; z) \sim \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} e^z, \quad z \rightarrow \infty \quad (17)$$

LEMMA 1 Let $\lambda, \alpha, \beta, a, b, p \in C$ with $\beta, b \neq 0, -1, -2, \dots; k > 0$ and $\operatorname{Re}(vp) > 0$, then S has the representation

$$S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \middle| p, k \right) = v^{u-\lambda} \int_0^\infty t^{u-1} \Phi(\alpha, \beta; -pvt) {}_2R_1(\lambda, a; b; k; -t) dt, \quad (18)$$

where $u \in \Omega_u$, $v \in \Omega_v$.

Proof The proof is trivial.

When $\alpha = \beta$, equation (18) reduces to a result given by Virchenko *et al.* [6, p. 97] ■

LEMMA 2 The partial derivatives of S are

$$\frac{\partial^n}{\partial u^n} S = v^{-\lambda} \int_0^\infty t^{u-1} \Phi(\alpha, \beta; -pt) (\log t)^n {}_2R_1 \left(\lambda, a; b; k; -\frac{t}{v} \right) dt, \quad (19)$$

and

$$\frac{\partial^n}{\partial v^n} S = (-1)^n (\lambda)_n S \left(\begin{matrix} \lambda + n, a, b; \alpha, \beta; \\ u, v \end{matrix} \middle| p, k \right) \quad (20)$$

the proof of equations (19) and (20) is straight forward.

LEMMA 3 Let $\lambda, \alpha, \beta, a, b, p \in C$ with $\beta, b \neq 0, -1, -2, \dots; k > 0$ and $\operatorname{Re}(p) > 0$, then the following relation holds

$$\begin{aligned} S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \middle| p, k \right) &= \frac{p\alpha}{u\beta} S \left(\begin{matrix} \lambda, a, b; \alpha + 1, \beta + 1; \\ u + 1, v \end{matrix} \middle| p, k \right) \\ &\quad + \frac{\lambda \Gamma(b) \Gamma(a+k)}{u \Gamma(a) \Gamma(b+k)} S \left(\begin{matrix} \lambda + 1, a + k, b + k; \alpha, \beta; \\ u + 1, v \end{matrix} \middle| p, k \right) \end{aligned} \quad (21)$$

Proof If we use [6, equation (17)] for $d/dz [{}_2R_1^k(z)]$ and integrate by parts, then equation (18) leads to equation (21). ■

3. Incomplete S-functions

A generalized incomplete gamma function corresponding to the S -function (18) is defined in the form

$$S_0^x \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k \right) := v^{-\lambda} \int_0^x t^{u-1} \Phi(\alpha, \beta; -pt)_2 R_1 \left(\lambda, a; b; k; -\frac{t}{v} \right) dt, \quad (22)$$

where $x, k > 0$, $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

The generalized complementary incomplete Gamma function is defined analogously by

$$S_x^\infty \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k \right) := v^{-\lambda} \int_x^\infty t^{u-1} \Phi(\alpha, \beta; -pt)_2 R_1 \left(\lambda, a; b; k; -\frac{t}{v} \right) dt, \quad (23)$$

where $x, k > 0$, $\operatorname{Re}(u) > 0$, $\operatorname{Re}(p) > 0$ and $|\arg v| < \pi$.

Thus, the definitions (22) and (23) yield

$$S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k \right) = S_0^x \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k \right) + S_x^\infty \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \\ u, v \end{matrix} \quad p, k \right) \quad (24)$$

For $\alpha = \beta$, equations (22) and (23) reduce to the generalized incomplete gamma functions introduced by Virchenko *et al.* [6, p. 98]. Further, for $\alpha = \beta$ and $k = 1$, equations (22) and (23) yield the incomplete gamma functions introduced by Al-Musallam and Kalla [3].

Remark If we set $\alpha = \beta$, $a = b$, $p = k = 1$ in equations (22) and (23) and $\lambda \rightarrow 0$, then we find that

$$\lim_{\lambda \rightarrow 0} S_0^x \left(\begin{matrix} \lambda, a, a; \alpha, \alpha; \\ u, v \end{matrix} \quad 1, 1 \right) = \gamma(u, x) = \int_0^x t^{u-1} e^{-t} dt, \quad (25)$$

where $\gamma(u, x)$ is the well known incomplete gamma function and

$$\lim_{\lambda \rightarrow 0} S_x^\infty \left(\begin{matrix} \lambda, a, a; \alpha, \alpha; \\ u, v \end{matrix} \quad 1, 1 \right) = \Gamma(u, x) = \int_x^\infty t^{u-1} e^{-t} dt, \quad (26)$$

where $\Gamma(u, x)$ is the complimentary incomplete gamma function.

4. A class of probability density functions

Let m and δ represent the shape parameters. Further, let σ and γ denote the scale parameters. Then, by taking

$$t = \sigma x^\delta \quad \text{and} \quad dt = \sigma \delta x^{\delta-1} dx,$$

the definition (12) with

$$p = \frac{\gamma}{\sigma} (\gamma > 0; \sigma > 0), \quad u = \frac{m + \delta}{\delta} (m + \delta > 0), \quad \text{and} \quad v = n (n > 0)$$

yields

$$\begin{aligned} & \delta \sigma^{(m+\delta)/\delta} \int_0^\infty x^{m+\delta-1} \Phi(\alpha, \beta; -\gamma x^\delta)_2 R_1 \left(\lambda, a; b; k; -\frac{\sigma x^\delta}{n} \right) dx \\ &= n^\lambda S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; \gamma \\ 1 + m/\delta, n \end{matrix} \quad \frac{\gamma}{\sigma}, k \right) (\min\{\gamma, \sigma, m + \delta, n\} > 0). \end{aligned} \quad (27)$$

By virtue of the integral formula (27), a class of probability density functions associated with the S -function can be defined by

$$f(x) := \delta \sigma^{(m+\delta)/\delta} n^{-\lambda} x^{m+\delta-1} \Phi(\alpha, \beta; -\gamma x^\delta) {}_2R_1 \left(\lambda, a; b; k; -\frac{\sigma x^\delta}{n} \right) \\ \times \left\{ S \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} (x > 0), \\ = 0, \text{ elsewhere.} \quad (28)$$

provided that the various parameters and variable x occurring in equation (28) are so constrained that the density function is always non-negative. It is evident that

$$\int_{-\infty}^{\infty} f(x) dx = 1, \quad (29)$$

$$f(0) = \begin{cases} \delta \sigma^{1/\delta} n^{-\lambda} \left\{ S \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} & (m + \delta = 1) \\ 0 & (m + \delta > 1) \end{cases} \quad (30)$$

$$f(x) \longrightarrow \infty \text{ as } x \longrightarrow 0+ \text{ when } m + \delta < 1 \quad (31)$$

$$\lim_{x \rightarrow \infty} f(x) = 0 \quad (\delta > 0), \quad (32)$$

and

$$f'(x) = \left(\frac{m + \delta - 1}{x} - \gamma \delta x^{\delta-1} - \frac{\sigma \delta}{n} x^{\delta-1} \Psi \right) f(x), \quad (33)$$

where, for convenience,

$$\Psi := \frac{\lambda \Gamma(a+k) \Gamma(b)}{\Gamma(a) \Gamma(b+k)} \frac{{}_2R_1(\lambda+1, a+k; b+k; k; -(\sigma x^\delta/n))}{{}_2R_1(\lambda, a; b; k; -\sigma x^\delta/n)} \quad (34)$$

The formula (33) can be derived, if we differentiate both sides of equation (28) with respect to x logarithmically and apply the following formula

$$\frac{d}{dx} \left\{ {}_2R_1 \left(\lambda, a; b; k; -\frac{\sigma x^\delta}{n} \right) \right\} = -\frac{\sigma \delta \lambda \Gamma(a+k) \Gamma(b)}{n \Gamma(a) \Gamma(b+k)} x^{\delta-1} {}_2R_1 \left(\lambda+1, a+k; b+k; k; -\frac{\sigma x^\delta}{n} \right) \quad (35)$$

It is interesting to note that for $\alpha = \beta$, the results of this section reduce to the results due to Virchenko *et al.* [6].

5. Some statistical functions

In this section, several basic statistical functions associated with the probability density function $f(x)$ defined by equation (28) will be evaluated explicitly.

5.1 The r^{th} moment

The r^{th} moment μ'_r about the origin of a continuous real random variable X with the probability density function $f(x)$ is given by

$$\mu'_r := \int_{-\infty}^{\infty} x^r f(x) dx = : E[X^r] (r \in N), \quad (36)$$

which, on using equation (27) and the definition (28), gives

$$\mu'_r = \sigma^{-r/\delta} S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + (m+r)/\delta, n & \end{matrix} \right) \left\{ S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + m/\delta, n & \end{matrix} \right) \right\}^{-1} \quad (37)$$

In particular, for $r = 1$, the expected value of the random variable X (also referred to as the mean or the first moment of X) is obtained as

$$E[x] := \int_{-\infty}^{\infty} x f(x) dx = \sigma^{-1/\delta} S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + (m+1)/\delta, n & \end{matrix} \right) \left\{ S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + m/\delta, n & \end{matrix} \right) \right\}^{-1} \quad (38)$$

5.2 The moment generating function

The moment generating function $M(t; \delta)$ of a continuous random variable X having the probability density function $f(x)$ is defined by

$$\begin{aligned} M(t; \delta) &= E[e^{tX}] := \int_{-\infty}^{\infty} e^{tx} f(x) dx \\ &= \delta \sigma^{(m+\delta)/\delta} n^{-\lambda} \int_0^{\infty} x^{m+\delta-1} e^{tx} \Phi(\alpha, \beta; -\gamma x^\delta) {}_2R_1 \left(\lambda, a, b; k; -\frac{\sigma x^\delta}{n} \right) \\ &\quad \times \left\{ S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + m/\delta, n & \end{matrix} \right) \right\}^{-1}, \end{aligned} \quad (39)$$

which itself is a generalization of the result given by Kalla *et al.* [10] to which it reduces for $\alpha = \beta$ and $k = 1$.

5.3 The hazard rate function

For a continuous random variable X having the probability density function $f(x)$, the cumulative distribution function $F(t)$ is given by

$$F(t) := \int_{-\infty}^t f(x) dx =: \text{Pr ob}\{X \in (-\infty, t)\}, \quad (40)$$

that is, by

$$F(t) = S_0^{\sigma t^\delta} \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + m/\delta, n & \end{matrix} \right) \left\{ S \left(\begin{matrix} \lambda, a, b; \alpha, \beta; & \frac{\gamma}{\sigma}, k \\ 1 + m/\delta, n & \end{matrix} \right) \right\}^{-1}, \quad (41)$$

where we have used the definitions (28) and (22). Thus, by virtue of the relationship (24), the hazard (or failure) rate function $h(t)$ is given by

$$h(t) := \frac{f(t)}{1 - F(t)} = \delta \sigma^{(m+\delta)/\delta} n^{-\lambda} t^{m+\delta-1} \Phi(\alpha, \beta; -\gamma x^\delta)_2 R_1 \left(\lambda, a; b; k; -\frac{\sigma x^\delta}{n} \right) \\ \times \left\{ S_{\sigma t^\delta}^\infty \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} \quad (t > 0), \quad (42)$$

in terms of the complementary incomplete S -function defined by equation (23).

In passing, we remark that the above derivations would also give the survival (or reliability) function $S^*(t)$ in the form

$$S^*(t) := 1 - F(t) = \int_t^\infty f(x) dx \\ = S_{\sigma t^\delta}^\infty \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \left\{ S \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} \quad (t > 0), \quad (43)$$

5.4 The mean residual life (or remaining life expectancy) function

For a continuous random variable X , the mean residual life (or remaining life expectancy) function $K(t)$ is given by

$$K(t) := E[X - t | X \geq t] = \frac{1}{S^*(t)} \int_t^\infty S^*(x) dx = \frac{1}{S^*(t)} \int_t^\infty (x - t) f(x) dx, \quad (44)$$

that is, by

$$K(t) = \frac{1}{S^*(t)} \int_t^\infty x f(x) dx - t, \quad (45)$$

since

$$\frac{t}{S^*(t)} \int_t^\infty f(x) dx = \frac{t}{S^*(t)} \left(1 - \int_{-\infty}^t f(x) dx \right) = t, \quad (46)$$

where $S^*(t)$ denotes the survivor (or reliability) function denoted by equation (43).

By virtue of the definition (28), if we use the substitution

$$w = \sigma x^\delta \quad \text{and} \quad dw = \sigma \delta x^{\delta-1} dx,$$

the integral in equation (45) can be evaluated as

$$\int_t^\infty x f(x) dx = \sigma^{-1/\delta} S_{\sigma t^\delta}^\infty \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \left\{ S \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} \quad (47)$$

so that

$$K(t) = \sigma^{-1/\delta} S_{\sigma t^\delta}^\infty \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \left\{ S_{\sigma t^\delta}^\infty \left(\lambda, a, b; \alpha, \beta; \frac{\gamma}{\sigma}, k \right) \right\}^{-1} - t, \quad (48)$$

in terms of the complementary incomplete S -function defined by equation (23).

Finally, it is expected that the results of this paper may find some applications in physical problems due to occurrence of the confluent hypergeometric function in the density function (28), which includes various density functions that are useful in those problems arising in probability models.

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Commutative convolution of functions and distributions

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The commutative convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ is defined as the limit of the sequence $\{(f\tau_n) * (g\tau_n)\}$, provided the limit exists, where $\{\tau_n\}$ is a certain sequence of functions τ_n in $\mathcal{D}$ converging to 1. It is proved that

$$|x|^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1}) = \pi [\cot(\pi\lambda) - \operatorname{cosec}(\pi\lambda)] \operatorname{sgn} x |x|^0,$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$, where B denotes the Beta function.

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In the following, $\mathcal{D}$ denotes the space of infinitely differentiable functions with compact support and $\mathcal{D}'$ denotes the space of distributions defined on $\mathcal{D}$.

The classical definition of the convolution is as follows.

DEFINITION 1 *If f and g are locally summable functions, then the convolution $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) dt = \int_{-\infty}^{\infty} f(x-t)g(t) dt \quad (1)$$

for all x for which the integrals exist.

The convolution of certain pairs of distributions in $\mathcal{D}'$ is usually defined as follows, see, for example, Gel'fand and Shilov [1].

DEFINITION 2 *Let f and g be distributions in $\mathcal{D}'$ satisfying either of the following conditions:*

- (a) *either f or g has bounded support;*
- (b) *the supports of f and g are bounded on the same side.*

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Then the convolution $f * g$ is defined by the equation

$$\langle (f * g)(x), \varphi(x) \rangle = \langle g(x), \langle f(t), \varphi(x + t) \rangle \rangle \quad (2)$$

for arbitrary test function φ in $\mathcal{D}$.

Note that if f and g are locally summable functions satisfying either of the conditions (a) or (b) in Definition 2, then Definition 2 is in agreement with Definition 1.

It follows that if the convolution $f * g$ exists by Definitions 1 or 2, then the following equations hold:

$$f * g = g * f, \quad (3)$$

$$(f * g)' = f * g' = f' * g. \quad (4)$$

Definition 2 is rather restrictive, and we now give an extension of Definition 2 based on a definition of the convolution by Jones [5], see also [1, 2]. We first of all let τ be the function in $\mathcal{D}$ satisfying the following conditions:

- (i) $\tau(x) = \tau(-x)$;
- (ii) $0 \leq \tau(x) \leq 1$;
- (iii) $\tau(x) = 1, |x| \leq 1/2$;
- (iv) $\tau(x) = 0, |x| \geq 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n. \end{cases}$$

DEFINITION 3 Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ and $g_n = g\tau_n$ for $n = 1, 2, \dots$. Then, the convolution $f * g$ is defined to be the limit of the sequence $\{f_n * g_n\}$, provided the limit h exists in the sense that

$$\lim_{n \rightarrow \infty} \langle f_n * g_n, \varphi \rangle = \langle h, \varphi \rangle$$

for all φ in $\mathcal{D}$.

Note that the convolution $f_n * g_n$ in this definition is in the sense of Definition 2, the supports of f_n of g_n being bounded. It is clear that if the convolution $f * g$ exists by Definition 2 and is equal to h , then the convolution $f * g$ also exists by Definition 3 and is equal to h . It is also clear that equation (3) holds for Definition 3, but equations (4) do not necessarily hold. We do however have the following theorems.

THEOREM 1 Let f and g be distributions in $\mathcal{D}'$ and suppose that the convolution $f * g$ exists and $\lim_{n \rightarrow \infty} \langle (f\tau_n)' * g_n, \varphi \rangle$ exists and equals $\langle h, \varphi \rangle$ for all φ in $\mathcal{D}$. Then the convolution $f' * g$ exists and

$$(f * g)' = f' * g + h. \quad (5)$$

Proof Suppose that $f * g$ exists. As f_n has compact support, equation (4) holds and so

$$\langle (f_n * g_n)', \varphi \rangle = \langle (f_n)' * g_n, \varphi \rangle = \langle (f')_n * g_n + (f\tau_n)' * g, \varphi \rangle \quad (6)$$

for all φ in $\mathcal{D}$. Equation (5) follows on letting n tend to infinity in equation (6). ■

THEOREM 2 *Let f and g be distributions in $\mathcal{D}'$ and suppose that the convolution $f' * g$ exists. Suppose further that $\lim_{n \rightarrow \infty} \langle (f\tau'_n)' * g_n, \varphi \rangle$ exists and equals $\langle h_1, \varphi \rangle$ for all φ in $\mathcal{D}$ and $\lim_{n \rightarrow \infty} \langle (f_n * (g\tau'_n)', \varphi \rangle$ exists and equals $\langle h_2, \varphi \rangle$ for all φ in $\mathcal{D}$. Then the convolution $f * g'$ exists and*

$$f * g' + h_2 = f' * g + h_1. \quad (7)$$

Proof Suppose that $f' * g$ exists. As f_n has compact support, equation (4) holds and so

$$\begin{aligned} \langle f_n * (g_n)', \varphi \rangle &= \langle (f_n)' * g_n, \varphi \rangle = \langle f_n * (g')_n + f_n * (g\tau'_n)', \varphi \rangle \\ &= \langle (f')_n * g_n + (f\tau'_n)' * g_n, \varphi \rangle \end{aligned} \quad (8)$$

for all φ in $\mathcal{D}$. Equation (7) follows on letting n tend to infinity in equation (8).

Before proving our main theorem, we need the following lemma. ■

LEMMA 1 *If $-1 < \lambda < 0$ and $0 < \alpha < 1$, then*

$$\begin{aligned} \int_{\alpha}^1 u^{-1}(1-u)^{\lambda} du &= -(1-\alpha)^{\lambda+1}[\ln \alpha - \ln(1-\alpha)] + (\lambda+1) \\ &\quad \times \int_{\alpha}^1 (1-u)^{\lambda}[\ln u - \ln(1-u)] du. \end{aligned} \quad (9)$$

Proof We have

$$\begin{aligned} \int_{\alpha}^1 u^{-1}(1-u)^{\lambda} du &= \int_{\alpha}^1 \frac{(1-u)^{\lambda+1}}{u(1-u)} du \\ &= \int_{\alpha}^1 (1-u)^{\lambda+1} d[\ln u - \ln(1-u)] \\ &= -(1-\alpha)^{\lambda+1}[\ln \alpha - \ln(1-\alpha)] \\ &\quad + (\lambda+1) \int_{\alpha}^1 (1-u)^{\lambda}[\ln u - \ln(1-u)] du, \end{aligned}$$

proving equation (9). ■

Note that if we put

$$B_{p,q}(\lambda, \mu) = \frac{\partial^p \partial^q}{\partial^p \lambda \partial^q \mu} B(\lambda, \mu),$$

where B denotes the Beta function, then

$$\lim_{\alpha \rightarrow 0} \int_{\alpha}^1 (1-u)^{\lambda} \ln u du = B_{1,0}(1, \lambda+1), \quad (10)$$

$$\lim_{\alpha \rightarrow 0} \int_{\alpha}^1 (1-u)^{\lambda} \ln(1-u) du = B_{0,1}(1, \lambda+1). \quad (11)$$

LEMMA 2 *If $\lambda \neq 0, \pm 1, \pm 2, \dots$, then*

$$B_{1,0}(1, \lambda) = \frac{\Gamma'(1)}{\lambda} - \frac{1}{\lambda^2} - \frac{\Gamma'(\lambda)}{\lambda \Gamma(\lambda)}, \quad (12)$$

$$B_{0,1}(1, \lambda) = -\lambda^{-2}, \quad (13)$$

$$\frac{\Gamma'(\lambda+1)}{\Gamma(\lambda+1)} - \frac{\Gamma'(-\lambda)}{\Gamma(-\lambda)} = -\pi \cot(\pi \lambda). \quad (14)$$

Proof We have

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and so

$$B_{1,0}(x, y) = \frac{\Gamma'(x)\Gamma(y)}{\Gamma(x+y)} - \frac{\Gamma(x)\Gamma(y)\Gamma'(x+y)}{\Gamma^2(x+y)}.$$

Thus,

$$B_{1,0}(1, \lambda) = \frac{\Gamma'(1)}{\lambda} - \frac{\Gamma'(\lambda+1)}{\lambda\Gamma(\lambda+1)}, \quad (15)$$

$$B_{0,1}(1, \lambda) = \frac{\Gamma'(\lambda)}{\Gamma(\lambda+1)} - \frac{\Gamma'(\lambda+1)}{\lambda\Gamma(\lambda+1)}. \quad (16)$$

Differentiating the equation

$$\Gamma(\lambda+1) = \lambda\Gamma(\lambda), \quad (17)$$

we get

$$\Gamma'(\lambda+1) = \lambda\Gamma'(\lambda) + \Gamma(\lambda). \quad (18)$$

Equation (12) now follows from equations (15), (17) and (18) and equation (13) follows from equations (16), (17) and (18).

Next, differentiating the equation

$$\Gamma(\lambda+1)\Gamma(-\lambda) = -\pi \operatorname{cosec}(\pi\lambda), \quad (19)$$

we get

$$\Gamma'(\lambda+1)\Gamma(-\lambda) - \Gamma(\lambda+1)\Gamma'(-\lambda) = \pi^2 \cot(\pi\lambda) \operatorname{cosec}(\pi\lambda) \quad (20)$$

and equation (14) follows from equations (19) and (20).

From now on, we will use Definition 4 for the definition of the convolution.

We now prove our main theorem. ■

THEOREM 3 *The convolutions $|x|^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1})$ and $(\operatorname{sgn} x |x|^\lambda) * |x|^{-\lambda-1}$ exist and*

$$|x|^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1}) = \pi [\cot(\pi\lambda) - \operatorname{cosec}(\pi\lambda)] \operatorname{sgn} x |x|^0, \quad (21)$$

$$(\operatorname{sgn} x |x|^\lambda) * |x|^{-\lambda-1} = -\pi [\cot(\pi\lambda) + \operatorname{cosec}(\pi\lambda)] \operatorname{sgn} x |x|^0 \quad (22)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$

Proof We will first of all suppose that $-1 < \lambda < 0$ and put

$$|x|_n^\lambda = |x|^\lambda \tau_n(x), \quad (\operatorname{sgn} x |x|^\lambda)_n = \operatorname{sgn} x |x|^\lambda \tau_n(x), \quad (x_+^\lambda)_n = x_+^\lambda \tau_n(x), \quad (x_-^\lambda)_n = x_-^\lambda \tau_n(x).$$

Then,

$$\begin{aligned}
 |x|_n^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1})_n &= [(x_+^\lambda)_n + (x_-^\lambda)_n] * [(x_+^{-\lambda-1})_n - (x_-^{-\lambda-1})_n] \\
 &= (x_+^\lambda)_n * (x_+^{-\lambda-1})_n + (x_-^\lambda)_n * (x_+^{-\lambda-1})_n \\
 &\quad - (x_+^\lambda)_n * (x_-^{-\lambda-1})_n - (x_-^\lambda)_n * (x_-^{-\lambda-1})_n \\
 &= I_1 + I_2 - I_3 - I_4,
 \end{aligned} \tag{23}$$

the convolutions existing by Definition 2. It is clear that

$$\lim_{n \rightarrow \infty} I_1 = x_+^\lambda * x_+^{-\lambda-1} = B(\lambda + 1, -\lambda)x_+ = -\pi \operatorname{cosec}(\pi\lambda)x_+^0, \tag{24}$$

$$\lim_{n \rightarrow \infty} I_4 = x_-^\lambda * x_-^{-\lambda-1} = B(\lambda + 1, -\lambda)x_- = -\pi \operatorname{cosec}(\pi\lambda)x_-^0, \tag{25}$$

where B denotes the Beta function. Equations (24) and (25) in fact exist for all $\lambda > -1$ and $\lambda \neq 1, 2, \dots$ by Definition 1 and for all $\lambda \neq \pm 1, \pm 2, \dots$ by Definition 2.

Further,

$$\begin{aligned}
 (x_-^\lambda)_n * (x_+^{-\lambda-1})_n &= \int_{-n}^0 |t|^\lambda (x-t)_+^{-\lambda-1} \tau_n(x-t) dt \\
 &\quad + \int_{-n-n^{-n}}^{-n} |t|^\lambda (x-t)_+^{-\lambda-1} \tau_n(t) \tau_n(x-t) dt.
 \end{aligned} \tag{26}$$

If $-n < x < 0$, we have on making the substitution $t = xu^{-1}$

$$\begin{aligned}
 \int_{-n}^0 |t|^\lambda (x-t)_+^{-\lambda-1} \tau_n(x-t) dt &= \int_{-n}^x |t|^\lambda (x-t)^{-\lambda-1} dt \\
 &= \int_{-x/n}^1 u^{-1} (1-u)^{-\lambda-1} du \\
 &= -\left(\frac{1-|x|}{n}\right)^{-\lambda} \left[\ln\left(\frac{|x|}{n}\right) - \ln\left(\frac{1-|x|}{n}\right) \right] \\
 &\quad - \lambda \int_{-x/n}^1 (1-u)^{-\lambda-1} [\ln u - \ln(1-u)] du \\
 &= -\ln|x| + \ln n - \lambda[B_{1,0}(1, -\lambda) - B_{0,1}(1, -\lambda)] \\
 &\quad + O(n^{-1} \ln n)
 \end{aligned} \tag{27}$$

on using equations (9)–(11). This equation also shows that the convolution $x_-^\lambda * x_+^{-\lambda-1}$ does not exist.

If $n > x > 0$, we have

$$\begin{aligned}
 \int_{-n}^0 |t|^\lambda (x-t)_+^{-\lambda-1} \tau_n(x-t) dt &= \int_{x-n}^0 |t|^\lambda (x-t)^{-\lambda-1} dt \\
 &\quad + \int_{x-n-n^{-n}}^{x-n} |t|^\lambda (x-t)^{-\lambda-1} \tau_n(x-t) dt,
 \end{aligned} \tag{28}$$

where, on making the substitution $t = x(1 - u^{-1})$, we have

$$\begin{aligned}
 \int_{x-n}^0 |t|^\lambda (x-t)^{-\lambda-1} dt &= \int_{x/n}^1 u^{-1} (1-u)^\lambda du \\
 &= -\left(\frac{1-x}{n}\right)^{\lambda+1} \left[\ln\left(\frac{x}{n}\right) - \ln\left(\frac{1-x}{n}\right) \right] \\
 &\quad + (\lambda+1) \int_{x/n}^1 (1-u)^\lambda [\ln u - \ln(1-u)] du \\
 &= -\ln x + \ln n + (\lambda+1)[B_{1,0}(1, \lambda+1) - B_{0,1}(1, \lambda+1)] \\
 &\quad + O(n^{-1} \ln n), \tag{29}
 \end{aligned}$$

on using equations (9)–(11).

It is easily seen that

$$\lim_{n \rightarrow \infty} \int_{-n-n^{-n}}^{-n} |t|^\lambda (x-t)_+^{-\lambda-1} \tau_n(t) \tau_n(x-t) dt = 0, \tag{30}$$

$$\lim_{n \rightarrow \infty} \int_{x-n-n^{-n}}^{x-n} |t|^\lambda (x-t)^{-\lambda-1} \tau_n(x-t) dt = 0 \tag{31}$$

for all x .

Similarly,

$$(x_+^\lambda)_n * (x_-^{-\lambda-1})_n = \int_0^n t^\lambda (x-t)_-^{-\lambda-1} \tau_n(x-t) dt + \int_n^{n+n^{-n}} t^\lambda (x-t)_-^{-\lambda-1} \tau_n(t) \tau_n(x-t) dt. \tag{32}$$

If $n > x > 0$, we have on making the substitution $t = xu^{-1}$

$$\begin{aligned}
 \int_0^n t^\lambda (x-t)_-^{-\lambda-1} \tau_n(x-t) dt &= \int_x^n t^\lambda (t-x)^{-\lambda-1} dt \\
 &= \int_{x/n}^1 u^{-1} (1-u)^{-\lambda-1} du \\
 &= -\left(1 - \frac{x}{n}\right)^{-\lambda} \left[\ln\left(\frac{x}{n}\right) - \ln\left(\frac{1-x}{n}\right) \right] \\
 &\quad - \lambda \int_{x/n}^1 (1-u)^{-\lambda-1} [\ln u - \ln(1-u)] du \\
 &= -\ln x + \ln n - \lambda[B_{1,0}(1, -\lambda) - B_{0,1}(1, -\lambda)] \\
 &\quad + O(n^{-1} \ln n), \tag{33}
 \end{aligned}$$

on using equations (9)–(11).

If $-n < x < 0$, we have

$$\begin{aligned}
 \int_0^n t^\lambda (x-t)_-^{-\lambda-1} \tau_n(x-t) dt &= \int_0^{x+n} t^\lambda (t-x)^{-\lambda-1} dt \\
 &\quad + \int_{x+n}^{x+n+n^{-n}} t^\lambda (t-x)^{-\lambda-1} \tau_n(x-t) dt \tag{34}
 \end{aligned}$$

where, on making the substitution $t = x(1 - u^{-1})$, we have

$$\begin{aligned}
 \int_0^{x+n} t^\lambda (t-x)^{-\lambda-1} dt &= \int_{-x/n}^1 u^{-1} (1-u)^\lambda du \\
 &= -\left(1 + \frac{x}{n}\right)^{\lambda+1} \left[\ln\left(\frac{|x|}{n}\right) - \ln\left(\frac{1+x}{n}\right) \right] \\
 &\quad + (\lambda+1) \int_{-x/n}^1 u^{-1} (1-u)^\lambda [\ln u - \ln(1-u)] du \\
 &= -\ln|x| + \ln n + (\lambda+1)[B_{1,0}(1, \lambda+1) - B_{0,1}(1, \lambda+1)] \\
 &\quad + O(n^{-1} \ln n), \tag{35}
 \end{aligned}$$

on using equations (9)–(11).

It is easily seen that

$$\lim_{n \rightarrow \infty} \int_n^{n+n^{-n}} t^\lambda (x-t)^{-\lambda-1} \tau_n(t) \tau_n(x-t) dt = 0, \tag{36}$$

$$\lim_{n \rightarrow \infty} \int_{x+n}^{x+n+n^{-n}} t^\lambda (x-t)^{-\lambda-1} \tau_n(x-t) dt = 0 \tag{37}$$

for all x .

It now follows from equations (26) and (28)–(33) that if $x > 0$, then

$$\begin{aligned}
 \lim_{n \rightarrow \infty} [I_2 - I_3] &= (\lambda+1)[B_{1,0}(1, \lambda+1) - B_{0,1}(1, \lambda+1)] + \lambda[B_{1,0}(1, -\lambda) - B_{0,1}(1, -\lambda)] \\
 &= \pi \cot(\pi\lambda) x_+^0 \tag{38}
 \end{aligned}$$

on using equations (12)–(14). Similarly, it follows from equations (12)–(14), (26), (27), (32) and (34)–(37) that if $x < 0$, then

$$\lim_{n \rightarrow \infty} [I_2 - I_3] = -\pi \cot(\pi\lambda) x_-^0. \tag{39}$$

Thus, from equations (38) and (39), we have

$$\lim_{n \rightarrow \infty} [I_2 - I_3] = \pi \cot(\pi\lambda) \operatorname{sgn} x |x|^0 \tag{40}$$

and equation (21) now follows from equations (23)–(25) and (40).

Equation (22) follows from equation (21) on noting that the convolution is commutative and $-1 < -\lambda - 1 < 0$ when $-1 < \lambda < 0$.

Now suppose that equations (21) and (22) hold when $r - 1 < \lambda \leq r$ for some non-negative integer r . This is true when $r = 0$. Then with $|x| < n$, we have

$$\begin{aligned}
 &[|x|^\lambda \tau'_n(x)] * (\operatorname{sgn} x |x|^{-\lambda-1}) \\
 &= - \int_n^{n+n^{-n}} t^\lambda (t-x)^{-\lambda-1} d\tau_n(t) + \int_{-n-n^{-n}}^{-n} |t|^\lambda (x-t)^{-\lambda-1} d\tau_n(t) \\
 &= n^\lambda (n-x)^{-\lambda-1} + n^\lambda (x+n)^{-\lambda-1} + \int_n^{n+n^{-n}} [\lambda t^{\lambda-1} (t-x)^{-\lambda-1} - (\lambda+1) t^\lambda (t-x)^{-\lambda-2}] \\
 &\quad \times \tau_n(t) dt - \int_{-n-n^{-n}}^{-n} [-\lambda |t|^{\lambda-1} (x-t)^{-\lambda-1} + (\lambda+1) t^\lambda (x-t)^{-\lambda-2}] \tau_n(t) dt
 \end{aligned}$$

and it follows that

$$\lim_{n \rightarrow \infty} [|x|^\lambda \tau'_n(x)] * (\operatorname{sgn} x |x|^{-\lambda-1}) = 0. \quad (41)$$

It now follows from Theorem 2, our assumptions and equation (40) that

$$\begin{aligned} (|x|^{\lambda+1})' * |x|^{-\lambda-1} &= |x|^{\lambda+1} * (|x|^{-\lambda-1})' \\ &= -(\lambda + 1)|x|^{\lambda+1} * (\operatorname{sgn} x |x|^{-\lambda-2}) \\ &= (\lambda + 1)(\operatorname{sgn} x |x|^\lambda) * |x|^{-\lambda-1} \end{aligned}$$

and so the convolution $|x|^{\lambda+1} * (\operatorname{sgn} x |x|^{-\lambda-2})$ exists and

$$\begin{aligned} |x|^{\lambda+1} * (\operatorname{sgn} x |x|^{-\lambda-2}) &= -(\operatorname{sgn} x |x|^\lambda) * |x|^{-\lambda-1} \\ &= \pi [\cot(\pi\lambda) + \operatorname{cosec}(\pi\lambda)] \operatorname{sgn} x |x|^0 \\ &= \pi \{\cot[\pi(\lambda + 1)] - \pi \operatorname{cosec}[\pi(\lambda + 1)]\} \operatorname{sgn} x |x|^0. \end{aligned}$$

Equation (21) therefore holds for $r < \lambda < r + 1$ and so follows by induction for $\lambda > -1$ and $\lambda \neq 0, 1, 2, \dots$

Similarly,

$$\begin{aligned} (\operatorname{sgn} x |x|^{\lambda+1})' * (\operatorname{sgn} x |x|^{-\lambda-1}) &= (\operatorname{sgn} x |x|^{\lambda+1}) * (\operatorname{sgn} x |x|^{-\lambda-1})' \\ &= -(\lambda + 1)(\operatorname{sgn} x |x|^{\lambda+1}) * |x|^{-\lambda-2} \\ &= (\lambda + 1)|x|^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1}) \end{aligned}$$

and so the convolution $(\operatorname{sgn} x |x|^{\lambda+1}) * |x|^{-\lambda-2}$ exists and

$$\begin{aligned} (\operatorname{sgn} x |x|^{\lambda+1}) * |x|^{-\lambda-2} &= -|x|^\lambda * (\operatorname{sgn} x |x|^{-\lambda-1}) \\ &= \pi [\cot(\pi\lambda) - \operatorname{cosec}(\pi\lambda)] \operatorname{sgn} x |x|^0 \\ &= -\pi \{\cot[\pi(\lambda + 1)] + \operatorname{cosec}[\pi(\lambda + 1)]\} \operatorname{sgn} x |x|^0. \end{aligned}$$

Equation (22) therefore holds for $r < \lambda < r + 1$ and so follows by induction for $\lambda > -1$ and $\lambda \neq 0, 1, 2, \dots$

As the convolution is commutative, it follows that equations (21) and (22) hold for $\lambda < -1$ and $\lambda \neq -2, -3, \dots$. This completes the proof of the theorem. $\blacksquare$

THEOREM 4 *The convolutions $|x|^{\lambda-2r+1} * |x|^{-\lambda-1}$, $|x|^{\lambda-2r} * (\operatorname{sgn} x |x|^{-\lambda-1})$, $(\operatorname{sgn} x |x|^{\lambda-2r}) * |x|^{-\lambda-1}$ and $(\operatorname{sgn} x |x|^{\lambda-2r+1}) * (\operatorname{sgn} x |x|^{-\lambda-1})$ exist and*

$$|x|^{\lambda-2r+1} * |x|^{-\lambda-1} = -\frac{2\pi [\cot(\pi\lambda) + \operatorname{cosec}(\pi\lambda)]}{(\lambda)_{2r-1}} \delta^{(2r-2)}(x), \quad (42)$$

$$(\operatorname{sgn} x |x|^{\lambda-2r}) * |x|^{-\lambda-1} = -\frac{2\pi [\cot(\pi\lambda) + \operatorname{cosec}(\pi\lambda)]}{(\lambda)_{2r}} \delta^{(2r-1)}(x), \quad (43)$$

$$(\operatorname{sgn} x |x|^{\lambda-2r+1}) * (\operatorname{sgn} x |x|^{-\lambda-1}) = \frac{2\pi [\cot(\pi\lambda) - \operatorname{cosec}(\pi\lambda)]}{(\lambda)_{2r-1}} \delta^{(2r-2)}(x), \quad (44)$$

$$|x|^{\lambda-2r} * (\operatorname{sgn} x |x|^{-\lambda-1}) = \frac{2\pi [\cot(\pi\lambda) - \operatorname{cosec}(\pi\lambda)]}{(\lambda)_{2r}} \delta^{(2r-1)}(x) \quad (45)$$

for $\lambda \neq 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$, where

$$(\lambda)_r = \prod_{i=1}^r (\lambda - r + i),$$

for $r = 1, 2, \dots$.

Proof Using Theorem 1 and equation (41), equation (42) follows on differentiating equation (22) $2r - 1$ times and equation (43) follows on differentiating equation (22) $2r$ times. Similarly, equation (44) follows on differentiating equation (21) $2r - 1$ times and equation (45) follows on differentiating equation (21) $2r$ times. This completes the proof of the theorem. ■

A non-commutative convolution version of the above results with $-1 < \lambda + \mu < 0$, was given in [3]

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Kernel theorems for the spaces of tempered ultradistributions

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We give a simple proof of the Kernel theorem for the spaces of tempered ultradistributions of Beurling–Komatsu and of Roumieu–Komatsu types, by using the characterization of Fourier–Hermite coefficients of the elements of the spaces. As a consequence of the Kernel theorem, we have that the Weyl transform can be extended on spaces of tempered ultradistributions.

Keywords: Hermite coefficients; Tempered; Ultradistributions; Kernel theorem; Weyl transform

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1. Introduction

Tempered ultradistributions spaces, as good spaces for harmonic analysis, have appeared in papers of many authors. These spaces have been developed in the framework of several ultradistribution theories: Komatsu’s theory, Beurling–Björk’s theory, Cioranescu–Zsidó’s and Braun–Meise–Taylor’s theory.

In the paper, we study tempered ultradistributions, following Komatsu’s approach to ultradistribution theory, (see [1]). We study the space $\mathcal{S}^{(M_p)'}(\mathbb{R}^d)$ of tempered ultradistributions of Beurling–Komatsu type, and the space $\mathcal{S}^{\{M_p\}'}(\mathbb{R}^d)$ of tempered ultradistributions of Roumieu–Komatsu type, which were introduced and investigated in [2, 6–8, 16]. They are the generalizations of the space of tempered distributions $\mathcal{S}'$.

The examples of the spaces $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ are

- The Gelfand–Shilov space $\mathcal{S}_\alpha^\alpha$, ([3]) is the space $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$, where $M_p = p^{\alpha p}$, $\alpha > 1$.
- The Pilipović space Σ'_α , (see [4]), is an example of the space of tempered ultradistributions, where $\{M_p\}$ is a Gevrey sequence $p^{\alpha p}$, $p \in \mathbb{N}$, $\alpha > 1$, which appeared to be interesting for the development of pseudodifferential operator (Ψ DO) theory in the framework of Time–Frequency analysis (see the papers of Pilipović, and Teofanov, [5]).

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- Braun–Meise–Taylor space $\mathcal{S}_{\{\omega\}}$, $\omega \in \mathcal{W}$, introduced in [6] and studied in the series of papers by the same authors, is the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, where

$$M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.$$

The sequence satisfies the conditions (M.1), (M.2) and (M.3)', and it is, in general, different from a Gevrey sequence.

- Beurling–Björk space $\mathcal{S}_\omega$, $\omega \in \mathcal{M}_c$, introduced in [7], is equal to the space $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, where

$$M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.$$

The sequence satisfies the conditions (M.1) and (M.3)', and it is, in general, different from a Gevrey sequence. If we assume additionally that $\omega(\rho) \geq C(\log \rho)^2$ for some $C > 0$, then (M.2) is satisfied.

After introduction, in section 2. we present without proof the basic identification of the spaces $\mathcal{S}^{\{M_p\}}$, (respectively, $\mathcal{S}^{\{M_p\}'}$) and $\mathcal{S}^{\{M_p\}'}$ (respectively, $\mathcal{S}^{\{M_p\}}$) with the sequence spaces. In the sequence representation, the test space $\mathcal{S}^{\{M_p\}}$, (respectively, $\mathcal{S}^{\{M_p\}'}$), represent the space of sequences of ultrapolynomial falloff and its dual space $\mathcal{S}^{\{M_p\}'}$, (respectively, $\mathcal{S}^{\{M_p\}}$), represents the space of sequences of ultrapolynomial growth. We use these results to prove the kernel theorems for tempered ultradistributions. The proof is based on the idea from B. Simons paper [8]. As a consequence of the Kernel theorem we have that the Weyl transform can be extended on spaces of tempered ultradistributions and this gives the possibility to introduce the Ψ DO theory in the framework of tempered ultradistributions, where $\{M_p\}_{p \in \mathbb{N}_0}$ is not necessarily a Gevrey sequence, but can be, for example

$$M_p = (p^\nu (\log p)^\mu)^p, \quad p \in \mathbb{N},$$

where $\nu > 1$ and $\mu \in \mathbb{R}$, or $\nu = 1$ and $\mu > 1$.

In section 3. we give the detailed proofs of characterizations of Fourier–Hermite coefficients of elements of the spaces $\mathcal{S}^{\{M_p\}}$, (respectively, $\mathcal{S}^{\{M_p\}'}$) and $\mathcal{S}^{\{M_p\}'}$, (respectively, $\mathcal{S}^{\{M_p\}}$), which are only stated in section 2. This characterization appeared in the papers, [9], [10], but in different forms and without a proof. Here we give complete and detailed proofs, based on the ideas from [11].

1.1 Notations

Let $\{M_p\}_{p \in \mathbb{N}_0}$ be a sequence of positive numbers, where $M_0 = 1$. An infinitely differentiable function φ on $\mathbb{R}^d$ is an ultradifferentiable function of class (M_p) , (respectively, of class $\{M_p\}$), if on each compact set K in $\mathbb{R}^d$ its derivatives satisfy that for every $h > 0$ (respectively, for some $h > 0$), there exists $C > 0$ such

$$||\varphi^{(\alpha)}||_{C(K)} \leq Ch^{|\alpha|} M_{|\alpha|}, \quad |\alpha| = 0, 1, \dots,$$

where we use multi-index notation: for $d \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$, $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, we put $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$, $\alpha! = \alpha_1! \alpha_2! \dots \alpha_d!$, $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_d^{\alpha_d}$, $|x| = \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$.

For $x \in \mathbb{R}^d$, $\varphi(x) \in C^\infty(\mathbb{R}^d)$,

$$\varphi^{(\alpha)}(x) = \left(\frac{\partial}{\partial x} \right)^\alpha \varphi(x) = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial x_2} \right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_d} \right)^{\alpha_d} \varphi(x).$$

We impose the conditions on the sequence $\{M_p, p \in \mathbb{N}_0\}$, which are standard in ultradistribution theory (for their detailed analysis see, for example Komatsu's papers [1])

(M.1) $M_p^2 \leq M_{p-1}M_{p+1}$, $p = 1, 2, \dots$ (logarithmic convexity)

(M.2) There exist constants $A, H > 0$ such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \dots \text{ (stability under ultradifferential operators)}$$

(M.3)' $B := \sum_{p=1}^{\infty} M_{p-1}M_p < \infty$. (non-quasi-analyticity)

We will often use the inequality

$$M_p M_q \leq M_0 M_{p+q}, \quad p, q = 0, 1, 2, \dots, \quad (1)$$

which follows from the condition (M.1), (see [1, p. 45]).

The associated function for the sequence $\{M_p, p \in \mathbb{N}_0\}$ is

$$M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \quad \rho > 0.$$

The Gevrey sequence $\{p^{sp}\}_{p \in \mathbb{N}_0}$, for $s > 1$, satisfies all the above conditions and in that case $M(\rho) \sim \rho^{1/s}$. The examples of sequences, which satisfy the above conditions are also

$$M_p = (p^\nu (\log p)^\mu)^p, \quad p \in \mathbb{N}, \quad (2)$$

where $\nu > 1$ and $\mu \in \mathbb{R}$, or $\nu = 1$ and $\mu > 1$. In this special case $M(\rho) = \rho^{1/2s} (\log \rho)^{-(t/s)}$, $\rho \gg 0$.

Following Komatsu, we denote by $\mathcal{E}^{(M_p)}$, (respectively, $\mathcal{E}^{\{M_p\}}$), the spaces of all ultradifferentiable functions on $\mathbb{R}^d$ of class (M_p) , (respectively, of class $\{M_p\}$), and by $\mathcal{D}^{(M_p)}$, (respectively, $\mathcal{D}^{\{M_p\}}$), the subspace of $\mathcal{E}^{(M_p)}$, (respectively, $\mathcal{E}^{\{M_p\}}$), of all ultradifferentiable functions with compact support. We denote by $\mathcal{D}^{(M_p)'}$, the strong dual space of $\mathcal{D}^{(M_p)}$, and call its elements ultradistributions of Beurling–Komatsu type. We denote by $\mathcal{D}^{\{M_p\}'}$, the strong dual space of $\mathcal{D}^{\{M_p\}}$, and call its elements ultradistributions of Roumieu–Komatsu type.

The sign $*$ will stand for (M_p) or $\{M_p\}$.

1.2 Space of tempered ultradistributions of Beurling–Komatsu type

The space $\mathcal{S}^{(M_p)'}$, (respectively, $\mathcal{S}^{\{M_p\}'}$), of tempered ultradistributions of type (M_p) , (respectively, of type $\{M_p\}$) is a subspace of the space of Beurling–Komatsu ultradistributions (respectively, Roumieu–Komatsu ultradistributions). The set $\mathcal{D}^{(M_p)}$ is dense in $\mathcal{S}^{(M_p)}$ and the set $\mathcal{S}^{(M_p)} \setminus \mathcal{D}^{(M_p)}$ is non-empty. The set $\mathcal{D}^{\{M_p\}}$ is dense in $\mathcal{S}^{\{M_p\}}$ and the set $\mathcal{S}^{\{M_p\}} \setminus \mathcal{D}^{\{M_p\}}$ is

non-empty. Moreover, we have

$$\begin{array}{ccccc} \mathcal{D}' & \longleftarrow & \mathcal{S}' & \longleftarrow & \mathcal{E}' \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{D}^{*'} & \longleftarrow & \mathcal{S}^{*'} & \longleftarrow & \mathcal{E}^{*'} \\ & & \downarrow & & \\ & & \mathcal{G}' & & \end{array},$$

where the arrows denote continuous inclusions, $*$ denotes (M_p) or $\{M_p\}$, and $\mathcal{S}'$ is the Schwartz space tempered distributions and $\mathcal{G}'$ is the space of extended Fourier hyperfunctions. For the proof see [12].

Let us now give the precise definitions.

By $\mathcal{S}^{M_p, m}$, $m > 0$, we denote the Banach space of smooth functions φ on $\mathbb{R}^d$, such that for some constant $C > 0$

$$||x^\beta \varphi^{(\alpha)}||_2 \leq C \frac{M_{|\alpha|} M_{|\beta|}}{m^{|\alpha|+|\beta|}}, \quad \text{for every } \alpha, \beta \in \mathbb{N}_0^d, \quad (3)$$

where $|| \cdot ||_2$ is the standard norm in space $L^2(\mathbb{R}^d)$, equipped with the norm

$$s_m(\varphi) = \sup_{\alpha, \beta \in \mathbb{N}_0^d} \frac{m^{|\alpha|+|\beta|}}{M_{|\alpha|} M_{|\beta|}} ||x^\beta \varphi^{(\alpha)}(x)||_2. \quad (4)$$

The test spaces are defined by

$$\mathcal{S}^{(M_p)} = \text{proj} \lim_{m \rightarrow \infty} \mathcal{S}^{M_p, m}, \quad \mathcal{S}^{\{M_p\}} = \text{ind} \lim_{m \rightarrow 0} \mathcal{S}^{M_p, m}.$$

They are Frechet spaces.

The strong dual $\mathcal{S}^{(M_p)'} of the space $\mathcal{S}^{(M_p)}$ is the space of tempered ultradistributions of Beurling–Komatsu type, and the strong dual $\mathcal{S}^{\{M_p\}'}$ of the space $\mathcal{S}^{\{M_p\}}$ is the space of tempered ultradistributions of Roumieu–Komatsu type. The spaces of tempered ultradistributions are good space for Harmonic analysis, since the Fourier transform is an isomorphism of $\mathcal{S}^{*'} onto itself, (see [10]). It is defined on $\mathcal{S}^*$ by$$

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx, \quad \varphi \in \mathcal{S}^*,$$

and on $\mathcal{S}^{*'} by$

$$\langle \mathcal{F}f, \varphi \rangle = \langle f, \mathcal{F}\varphi \rangle,$$

for $f \in \mathcal{S}^{*'} and $\varphi \in \mathcal{S}^*$.$

2. Main results

Komatsu proved, in his papers [1] the Kernel theorem for ultradistributions and ultradistributions with compact support, which are an analogue of L. Schwartz Kernel theorem for distributions.

The main result of the paper is the Kernel theorem for tempered ultradistributions. Applying the theorem we show that the Weyl transform, which is a representation of the convolution algebra on L^1 functions over the reduced Heisenberg group H_d^{red} , extends uniquely to a bijection

from $\mathcal{S}'(\mathbb{R}^{2l})$ to the space of continuous linear maps from $\mathcal{S}^*(\mathbb{R}^l)$ to $\mathcal{S}'(\mathbb{R}^l)$, where $*$ stands for (M_p) or $\{M_p\}$.

The simple proof of the Kernel theorem for tempered ultradistributions, which will be given in this section, depend on the characterization of the Fourier–Hermite coefficients of elements of $\mathcal{S}^*$ and $\mathcal{S}'$. In this section, we will state the theorems, which give these characterizations and prove the Kernel theorem.

The elements of the space $\mathcal{S}^*$ are in L^2 , thus they have the Hermite expansion $\sum a_n(\varphi)h_n$, where h_n are Hermite functions:

$$h_n(x) = (-1)^n \pi^{-(1/4)} 2^{-(1/2)n} (n!)^{-(1/2)} e^{(1/2)x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}, \quad n \in \mathbb{N},$$

which are harmonic oscillator wave functions, since

$$\left(-\frac{d}{dx^2} + x^2 \right) h_n = (2n + 1)h_n.$$

Let $\varphi \in \mathcal{S}^*$, the Fourier–Hermite coefficients of φ are

$$a_n(\varphi) = \int_{\mathbb{R}} \varphi(x) h_n(x) dx, \quad n \in \mathbb{N}_0.$$

The sequence of the Fourier–Hermite coefficients $\{a_n\}_{n \in \mathbb{N}_0}$ of φ , we call as the *Hermite representation* of φ .

The first main result of the paper is the Hermite representation of the space $\mathcal{S}^*(\mathbb{R})$:

THEOREM 2.1 (a) *If $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R})$, (respectively, $\mathcal{S}^{\{M_p\}}(\mathbb{R})$), then for every $\theta > 0$, (respectively, for some $\theta > 0$),*

$$\|\varphi\|_{\theta} = \sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty. \quad (5)$$

(b) *Conversely, if a sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfies that for every $\theta > 0$, (respectively, for some $\theta > 0$),*

$$\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty,$$

then the series $\sum a_n h_n$ converges in the space $\mathcal{S}^{(M_p)}(\mathbb{R})$, (respectively, $\mathcal{S}^{\{M_p\}}(\mathbb{R})$), to some φ , and the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ is the Hermite representation of φ .

From the proof of Theorem 2.1, which will be given in the next section, it follows that the test space of tempered ultradistributions of Beurling–Komatsu type $\mathcal{S}^{(M_p)}(\mathbb{R})$ can be identified with the space $s^{(M_p)}$ of sequences of ultrapolynomial falloff, and that the test space of tempered ultradistributions of Beurling–Komatsu type $\mathcal{S}^{\{M_p\}}(\mathbb{R})$ can be identified with the space $s^{\{M_p\}}$

of sequences of ultrapolynomial falloff, where we define:

$$s^{(M_p)} = \left\{ \{a_n\}_{n \in \mathbb{N}_0} \mid \forall \theta > 0, \sum_{n=0}^{\infty} |a_n|^2 \exp [2M(\theta \sqrt{2n+1})] < \infty \right\},$$

and equip it with the projective limit topology, which is generated by the family of semimorms

$$\left\{ ||\{a_n\}||_{\theta} = \sum_{n=0}^{\infty} |a_n|^2 \exp [2M(\theta \sqrt{2n+1})], \quad \theta > 0 \right\}, \quad (6)$$

and

$$s^{\{M_p\}} = \left\{ \{a_n\}_{n \in \mathbb{N}_0} \mid \exists \theta > 0, \sum_{n=0}^{\infty} |a_n|^2 \exp [2M(\theta \sqrt{2n+1})] < \infty \right\},$$

which are equipped with the inductive limit topology, generated by the family of semimorms (6).

More precisely, the map

$$\mathcal{S}^*(\mathbb{R}) \longrightarrow s^*, \quad \varphi \longmapsto \{a_n(\varphi)\}_{n \in \mathbb{N}_0}, \quad a_n(\varphi) = \langle \varphi, h_n \rangle$$

is a topological isomorphism. It is easy to prove that the space s^* is nuclear and, therefore, space $\mathcal{S}^*(\mathbb{R})$ is nuclear.

If $f \in \mathcal{S}^{*'}(\mathbb{R})$, then $a_n(f) = \langle f, h_n \rangle$, $n = 0, 1, 2, \dots$, are the Fourier–Hermite coefficients of f , and the sequence $\{a_n(f)\}_{n \in \mathbb{N}_0}$, is the *Hermite representation* of tempered ultradistribution f .

Hermite representation of the spaces $\mathcal{S}^{(M_p)'}(\mathbb{R})$ and $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$ are given by

THEOREM 2.2 (a) *Let $f \in \mathcal{S}^{(M_p)'}(\mathbb{R})$, (respectively, $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$). Then for some $\theta > 0$, (respectively, for all $\theta > 0$),*

$$\sum_{n=0}^{\infty} |a_n(f)|^2 \exp [-2M(\theta \sqrt{2n+1})] < \infty \quad (7)$$

and $\langle f, \varphi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\varphi)$, where the sequence $\{a_n(\varphi)\}_{n \in \mathbb{N}_0}$ is the Hermite representative of $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R})$, (respectively, $\mathcal{S}^{\{M_p\}}(\mathbb{R})$).

(b) *Conversely, if a sequence $\{b_n\}_{n \in \mathbb{N}_0}$ satisfies that for some $\theta > 0$, (respectively, for all $\theta > 0$),*

$$\sum_{n=0}^{\infty} |b_n|^2 \exp [-2M(\theta \sqrt{2n+1})] < \infty,$$

then the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ is the Hermite representation of some $f \in \mathcal{S}^{(M_p)'}(\mathbb{R})$, (respectively, $\mathcal{S}^{\{M_p\}'}(\mathbb{R})$), and the Parseval equation holds $\langle f, \varphi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\varphi)$.

Although we have stated the results for $\mathcal{S}^*(\mathbb{R})$ and $\mathcal{S}^{*'}(\mathbb{R})$ in one dimensional case, analogous results hold in the multidimensional case, for $\mathcal{S}^*(\mathbb{R}^l)$ and $\mathcal{S}^{*'}(\mathbb{R}^l)$. The Hermite functions in a multidimensional case are defined by

$$h_n(x) = h_{n_1}(x_1) h_{n_2}(x_2) \dots h_{n_d}(x_d), \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d,$$

where $n = (n_1, n_2, \dots, n_d) \in \mathbb{N}^d$. By $h_{(n,k)}$ we denote

$$h_{(n,k)} = h_{n_1}(x_1) h_{n_2}(x_2) \dots h_{n_l}(x_l) h_{k_1}(x_{l+1}) h_{k_2}(x_{l+2}) \dots h_{k_s}(x_{l+s}), \quad (n, k) \in \mathbb{N}_0^l \times \mathbb{N}_0^s.$$

2.1 The Kernel theorem

The Kernel theorem for tempered ultradistributions states essentially that every continuous linear map $\mathcal{K}$ of the space $(\mathcal{S}^*(\mathbb{R}^l))_x$ of test functions in some variable x , into the space $(\mathcal{S}'^*(\mathbb{R}^s))_y$ of tempered ultradistributions in a second variable y , is given by a unique tempered ultradistributions K in both variables.

THEOREM 2.3 Kernel theorem *Every jointly continuous bilinear functional K on $\mathcal{S}^*(\mathbb{R}^l) \times \mathcal{S}^*(\mathbb{R}^s)$ defines a linear map $\mathcal{K}: \mathcal{S}^*(\mathbb{R}^s) \rightarrow \mathcal{S}'^*(\mathbb{R}^l)$ by*

$$\langle \mathcal{K}\varphi, \psi \rangle = K(\psi \otimes \varphi), \quad \varphi \in \mathcal{S}^*(\mathbb{R}^s), \quad \psi \in \mathcal{S}^*(\mathbb{R}^l). \quad (8)$$

and $(\varphi \otimes \psi)(x, y) = \varphi(x)\psi(y)$, which is continuous, in the sense, that $\mathcal{K}\varphi_j \rightarrow 0$ in $\mathcal{S}'^*(\mathbb{R}^l)$ if $\varphi_j \rightarrow 0$ in $\mathcal{S}^*(\mathbb{R}^s)$.

Conversely, for linear map $\mathcal{K}: \mathcal{S}^*(\mathbb{R}^s) \rightarrow \mathcal{S}'^*(\mathbb{R}^l)$ there is unique tempered ultradistribution $K \in \mathcal{S}'^*(\mathbb{R}^{l+s})$ such that equation (8) is valid. The tempered ultradistribution K is called the kernel of $\mathcal{K}$.

Proof Let us prove the theorem in Beurling–Komatsu’s case. If K is a jointly continuous bilinear functional on $\mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)}(\mathbb{R}^s)$, then equation (8) defines a tempered ultradistribution $(\mathcal{K}\varphi) \in \mathcal{S}^{(M_p)'}(\mathbb{R}^l)$ since $\psi \mapsto K(\psi \otimes \varphi)$ is continuous. The mapping $\mathcal{K}: \mathcal{S}^{(M_p)}(\mathbb{R}^s) \rightarrow \mathcal{S}^{(M_p)'}(\mathbb{R}^l)$ is continuous since the mapping $\varphi \mapsto K(\psi \otimes \varphi)$ is continuous.

Let us prove the converse. To prove the existence we define a bilinear form B on $\mathcal{S}^{(M_p)'}(\mathbb{R}^l) \otimes \mathcal{S}^{(M_p)'}(\mathbb{R}^s)$ by

$$B(\varphi, \psi) = \langle \mathcal{K}\psi, \phi \rangle, \quad \psi \in \mathcal{S}^{(M_p)}(\mathbb{R}^l), \quad \varphi \in \mathcal{S}^{(M_p)}(\mathbb{R}^s).$$

The form B is a separately continuous bilinear form on the product $\mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)}(\mathbb{R}^s)$ of Frechet spaces and, therefore, it is jointly continuous, see [13].

Let $C > 0$, $\theta \in \mathbb{R}_+^l$, $\nu \in \mathbb{R}_+^s$ be chosen so that

$$|B(\varphi, \psi)| \leq C \|\varphi\|_\theta \|\psi\|_\nu, \quad (9)$$

and let

$$t_{(n,k)} = B(h_n, h_k), \quad n \in \mathbb{N}^l, \quad k \in \mathbb{N}^s. \quad (10)$$

Since B is jointly continuous on $\mathcal{S}^{(M_p)}(\mathbb{R}^l) \times \mathcal{S}^{(M_p)}(\mathbb{R}^s)$, for $\varphi = \sum a_n h_n$ and $\psi = \sum b_k h_k$, we have that

$$B(\varphi, \psi) = \sum t_{(n,k)} a_n b_k. \quad (11)$$

On the other hand, for $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$ and $(\theta, \nu) \in \mathbb{R}^l \times \mathbb{R}^s$, by equation (9) we have

$$\begin{aligned} |t_{(n,k)}| &\leq C \|h_n\|_\theta \|h_k\|_\nu \\ &= \|h_{n_1}\|_{\theta_1} \|h_{n_2}\|_{\theta_2} \dots \|h_{n_l}\|_{\theta_l} \|h_{k_1}\|_{\nu_1} \|h_{k_2}\|_{\nu_2} \dots \|h_{k_s}\|_{\nu_s} \\ &= \exp \left[2 \sum_{i=1}^l M(\theta_i \sqrt{n_i}) \right] \exp \left[2 \sum_{j=1}^s M(\nu_j \sqrt{k_j}) \right]. \end{aligned}$$

Thus, from Theorem 2.2 it follows that the sequence $\{t_{(n,k)}\}_{(n,k)}$ is a Hermite representation of a tempered ultradistribution $K \in \mathcal{S}^{(M_p)'}(\mathbb{R}^l \times \mathbb{R}^s)$. Thus,

$$\langle K, \varphi \rangle = \sum t_{(n,k)} c_{(n,k)}, \quad (12)$$

for $\varphi = \sum c_{(n,k)} h_{n,k} \in \mathcal{S}^{(M_p)}(\mathbb{R}^{l+s})$.

If $\varphi = \sum a_n h_n \in \mathcal{S}^{(M_p)}(\mathbb{R}^l)$ and $\psi = \sum b_k h_k \in \mathcal{S}^{(M_p)}(\mathbb{R}^s)$ then $\varphi \otimes \psi$ has the Hermite representation $\{a_n b_k\}_{(n,k)}$ and we have that for tempered ultradistribution K defined by equation (12)

$$K(\varphi \otimes \psi) = \sum_{(n,k)} t_{(n,k)} a_n b_k = B(\varphi, \psi),$$

so $K = B$. This proves the existence.

The uniqueness follows from the fact that K is completely determined by its Hermite representation $\{\langle K, h_{(n,k)} \rangle\}_{(n,k)}$ and the fact that for every $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$

$$\langle K, h_{(n,k)} \rangle = \langle K, h_n \otimes h_k \rangle = B(h_n, h_k) = t_{(n,k)}. \quad \blacksquare$$

The unitary representation ρ of the reduced Heisenberg group H_n^{red} might be considered as a representation of $L^1(\mathbb{R}^{2l})$, with a non-standard convolution structure (see for example [14]. Accordingly, for $F \in L^1(\mathbb{R}^{2l})$ let us define (see [14]) a bounded operator $\rho(F)$ on $L^2(\mathbb{R}^l)$ (it is sometimes called the Weyl transform of F) as

$$\rho(F)\varphi(x) = \iint F(y-x, q) e^{\pi i q(x+y)} \varphi(y) dy dq. \quad (13)$$

In other words, $\rho(F)$ is an integral operator with kernel

$$K_F(x, y) = \mathcal{F}_2^{-1} F\left(y-x, \frac{y+x}{2}\right),$$

where $\mathcal{F}_2$ denotes Fourier transform in the second variable and $\mathcal{F}_2^{-1}$ its inverse transform. Terming $\rho(F)$ as the Weyl transform is historically inaccurate. In fact, the Weyl transform of F should be denoted as $\rho(\mathcal{F}(F))$.

COROLLARY 2.4 *The map ρ from $L^1(\mathbb{R}^{2l})$ to the space of bounded operators on $L^2(\mathbb{R}^l)$, defined by equation (13), extends uniquely to a bijection from $\mathcal{S}^{*'}(\mathbb{R}^{2l})$ to the space of continuous linear maps from $\mathcal{S}^*(\mathbb{R}^l)$ to $\mathcal{S}^{*'}(\mathbb{R}^l)$.*

Proof The kernel K_F is well defined when F is an arbitrary ultradistribution and belongs to the space $\mathcal{S}^{*'}$. So the Weyl transform extends to a bijection from $\mathcal{S}^{*' }(\mathbb{R}^{2l})$ to the space of continuous linear maps from $\mathcal{S}^*(\mathbb{R}^l)$ to $\mathcal{S}^{*'}(\mathbb{R}^l)$.

The uniqueness of the extension follows from the kernel theorem, since every continuous linear map from $\mathcal{S}^*(\mathbb{R}^l)$ to $\mathcal{S}^{*'}(\mathbb{R}^l)$ is of the form (8). ■

3. Proofs of Theorems 2.1 and 2.2

In the proof of Theorem 2.1, we follow the ideas of Zemanian [11] and we will use the following facts.

- (i) Since the sequence $m_n = M_n/M_{n-1}$, $n = 1, 2, \dots$ is increasing (which is equivalent to the condition (M.1), see [1, p. 50]), we have

$$\frac{k}{m_k} \leq \sum_{k=0}^{\infty} \frac{1}{m_k} =: B < \infty, \quad (14)$$

and

$$\frac{k! M_n}{n! M_k} = \frac{k}{m_k} \frac{k-1}{m_{k-1}} \dots \frac{n+1}{m_{n+1}} \leq B^{k-n}, \quad n, k \in \mathbb{N}, \quad n \leq k. \quad (15)$$

From the above and Stirling formula

$$n^n = n! \frac{1}{\sqrt{2\pi n}} e^n e^{-g(n)} \leq C e^n n!, \quad |g(n)| \leq \frac{1}{12n}, \quad (16)$$

it follows that

$$\frac{k^k M_n}{n^n M_k} \leq C e^k \sqrt{n} B^{k-n}, \quad k, n \in \mathbb{N}, \quad n \leq k. \quad (17)$$

- (ii) For $n, m \in \mathbb{N}$

$$x^m h_n(x) = 2^{-m/2} \sum_{k=0}^m \alpha_{k,m}^{(n)} h_{n-m+2k}(x), \quad x \in \mathbb{R}, \quad (18)$$

where

$$0 \leq |\alpha_{k,m}^{(n)}| \leq \binom{m}{k} ((2n+1)^{m/2} + m^{m/2}) \quad (19)$$

- (iii) If we denote $\mathcal{R} = (-d/dx^2 + x^2)$, then for $\varphi \in \mathcal{S}$ and $k \in \mathbb{N}$ it holds

$$\mathcal{R}^k \varphi = \sum_{n=0}^k \sum_{p=0}^{2n} C_{p,2n-p}^{(k)} x^p \varphi^{(2n-p)}, \quad |C_{p,2n-p}^{(k)}| \leq 10^k k^{k-n}. \quad (20)$$

The (ii) and (iii) were proved by mathematical induction in [15].

Proof (Theorem 2.1) We give a detailed proof for Beurling–Komatsu case in one dimensional case.

- a) Let us first prove that if $\varphi \in \mathcal{S}^{(M_p)}$, then for every $\theta > 0$

$$\sum_{n=0}^{\infty} |a_n|^2 \exp [2M(\theta \sqrt{2n+1})] < \infty. \quad (21)$$

Let $\varphi \in \mathcal{S}^{(M_p)}$. From equations (20), (3) (M.1) and (1), Stirling formula, (M.2), (15) and (17) it follows that there exists C such that for each $m > 0$ and each $k \in \mathbb{N}_0$,

$$\begin{aligned} & \left(\sum_{n \in \mathbb{N}_0} |a_n|^2 (2n+1)^{2k} \right)^{1/2} \\ &= \|\mathcal{R}^k \varphi\|_2 \leq \sum_{n=0}^k \sum_{p=0}^{2n} C_{p,2n-p}^{(k)} \|x^p \varphi^{(2n-p)}\|_2 \\ &\leq C \sum_{n=0}^k \sum_{p=0}^{2n} 10^k k^{k-n} m^{-2n} M_p M_{2n-p} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{n=0}^k \sum_{p=0}^{2n} 10^k \frac{k^k}{n^n} m^{-2n} M_{2n} \\
&\leq C \sum_{n=0}^k \sum_{p=0}^{2n} 10^k \frac{k!}{n!} e^{k-n+(1/12n)} \sqrt{n} m^{-2n} M_{2n} \\
&\leq C 20^k e^k (1+H^2)^k m^{2k} M_{2k} \sum_{n=0}^k (2n+1) \frac{k!}{n!} \frac{M_n}{M_k} \frac{M_n}{M_k} \frac{1}{m^{2(k-n)}} \\
&\leq C 80^k e^k (1+H^2)^k m^{2k} M_{2k} \sum_{n=0}^k \frac{1}{2^n} B^{k-n} \frac{M_n}{M_k} \left(\frac{1}{m^2} \right)^{k-n} \\
&\leq C 80^k e^k (1+H^2)^k B^k \left(1 + \frac{1}{B} \right)^k m^{2k} M_{2k} \\
&\quad \times \left(\sum_{n=0}^{\lfloor 1/m^2 \rfloor} \frac{H^{k+1}}{2^n} \frac{M_n}{M_{k+1}} \frac{(k+1)^{k+1}}{n^n} + \sum_{n=\lfloor 1/m^2 \rfloor+1}^k \frac{H^{n+k+1}}{2^n} \frac{M_{n-1}}{M_{k+1}} \frac{(k+1)^{k+1}}{(n-1)^{n-1}} \right) \\
&\leq C 80^k e^k (1+H^2)^k B^k \left(1 + \frac{1}{B} \right)^k m^{2k} M_{2k} \\
&\quad \times \left(\sum_{n=0}^{\lfloor 1/m^2 \rfloor} \frac{H^{k+1}}{2^k} e^{k+1} \sqrt{n} B^{k+1-n} + \sum_{n=\lfloor 1/m^2 \rfloor+1}^k \frac{H^{n+k+1}}{2^n} e^{k+1} \sqrt{n-1} B^{k+1-n} \right) \\
&\leq C 160^k e^{2k} (1+H^2)^k (1+H)^{2k} B^{2k} \left(1 + \frac{1}{B} \right)^{2k} m^{2k} M_{2k} \\
&\leq C m_1^{2k} M_{2k},
\end{aligned}$$

where $m_1 = \sqrt{160(1+H^2)} e(1+H)B(1+(1/B))m$.

It follows that, for $2k = \alpha + 2$

$$|a_n|^2 (2n+1)^{\alpha+2} \leq C m_1^{\alpha+2} M_{\alpha+2}^2.$$

Applying (M.2), we obtain

$$|a_n|^2 (2n+1)^{\alpha+2} \leq C m_1^{\alpha+2} A^2 H^{2(\alpha+2)} M_\alpha^2,$$

which implies that for each $\alpha, n \in \mathbb{N}_0$ and $\theta = m_1 H$, there exists C , such that

$$\frac{|a_n|^2 \theta^{2\alpha} (2n+1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n+1)^2},$$

which implies

$$|a_n|^2 \exp[2M(\theta\sqrt{2n+1})] = |a_n|^2 \sup_{\alpha \in \mathbb{N}_0} \frac{\theta^{2\alpha} (2n+1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n+1)^2}.$$

Therefore,

$$\sum_{n \in \mathbb{N}_0} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty,$$

for every $\theta > 0$.

b) Let us now prove that if

$$\sum_{n \in \mathbb{N}_0} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty, \quad (22)$$

for every $\theta > 0$, then the series $\sum a_n h_n$ converges in the space $\mathcal{S}^{(M_p)}$.

Suppose that the inequality (22) holds for every $\theta > 0$. Applying equations (18) and (19), Cauchy–Schwartz inequality and

$$\exp[-M(\theta\sqrt{2n+1})] \leq \frac{M_4}{\theta^4(2n+1)^2}, \quad (23)$$

which follows from the definition of the associated function. We get that for every $\theta > 0$, there exists a constant C such that for each $p \in \mathbb{N}_0$

$$\begin{aligned} \|x^p \varphi\|_2 &\leq 2^{-p/2} \left\| \sum_{n \in \mathbb{N}_0} a_n \left(\sum_{k \leq p} \alpha_{k,p}^{(n)} h_{n-p+2k} \right) \right\|_2 \\ &\leq 2^{-p/2} \sum_{n \in \mathbb{N}_0} a_n ((2n+1)^{p/2} + p^{p/2}) \left(\sum_{k \leq p} \binom{p}{k} \right) \|h_{n-p+2k}\|_2 \\ &\leq 2^{p/2} \sum_{n \in \mathbb{N}_0} a_n \exp[M(\theta\sqrt{2n+1}) - M(\theta\sqrt{2n+1})] ((2n+1)^{p/2} + p^{p/2}) \\ &\leq 2^{p/2} \left(\sum_{n \in \mathbb{N}_0} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] \right)^{1/2} \\ &\quad \times \left(\sum_{n \in \mathbb{N}_0} \exp[-2M(\theta\sqrt{2n+1})] ((2n+1)^{p/2} + p^{p/2})^2 \right)^{1/2} \\ &\leq C 2^{p/2} \left(\sum_{n \in \mathbb{N}_0} \exp[-2M(\theta\sqrt{2n+1})] ((2n+1)^{p/2} + p^{p/2})^2 \right)^{1/2} \\ &\leq C 2^{p/2} \sup_{n \in \mathbb{N}_0} \left(((2n+1)^{p/2} + p^{p/2}) \exp\left[-\frac{1}{2}M(\theta\sqrt{2n+1})\right] \right) \\ &\quad \times \left(\sum_{n \in \mathbb{N}_0} \exp[-M(\theta\sqrt{2n+1})] \right)^{1/2} \\ &\leq C \left(\left(\frac{2}{\theta} \right)^{p/2} M_p \left(\theta^{p/2} \sup_n \frac{(2n+1)^{p/2} \exp[-(1/2)M(\theta\sqrt{2n+1})]}{M_p} \right) + \frac{\theta^{p/2} p^{p/2}}{M_p} \right) \\ &\quad \times \frac{M_4}{\theta^4} \sum_{n \in \mathbb{N}_0} \frac{1}{(2n+1)^2} \end{aligned}$$

Since,

$$\frac{\theta^{p/2} p^{p/2}}{M_p} \leq \frac{p! e^p \theta^{p/2}}{M_p} \rightarrow 0, \text{ as } p \rightarrow \infty,$$

and

$$\begin{aligned} \sup_{n \in \mathbb{N}_0} \frac{\theta^{p/2} (2n+1)^{p/2} \exp[-(1/2)M(\theta\sqrt{2n+1})]}{M_p} &= \frac{1}{M_p} \sup_{n \in \mathbb{N}_0} \left(\frac{\theta^{p/2} (2n+1)^p}{\exp[M(\theta\sqrt{2n+1})]} \right)^{1/2} \\ &\leq \frac{\sqrt{M_p}}{M_p} \longrightarrow 0, \text{ as } p \longrightarrow \infty, \end{aligned}$$

which follows from [1, (3.3)], we have that for each $\theta > 0$ there exists $C > 0$ such that

$$\|x^p \varphi\|_2 \leq C \left(\sqrt{\frac{2}{\theta}} \right)^p M_p, \quad p \in \mathbb{N}_0. \quad (24)$$

From the above and from the properties of the Fourier transform we obtain that for each $\theta > 0$ there exists $C > 0$ such that

$$\|\varphi^{(q)}\|_2 = \frac{1}{\sqrt{2\pi}} \|x^q \mathcal{F}(\varphi)\|_2 = \frac{1}{\sqrt{2\pi}} \left\| x^q \sum_{k \in \mathbb{N}_0} a_k h_k \right\|_2 = \frac{1}{\sqrt{2\pi}} \|x^q \varphi\|_2 \leq C \left(\sqrt{\frac{2}{\theta}} \right)^q M_q. \quad (25)$$

If $p, q \in \mathbb{N}_0$ and $\gamma = \min(q, 2p)$, by using

$$(\|x^p \varphi^{(q)}\|_2)^2 = (x^p \varphi^{(q)}, x^p \varphi^{(q)})_{L^2} = |(x^{2p} \varphi^{(q)})^{(q)}, \varphi)_{L^2}| \quad (26)$$

$$\leq \left| \sum_{k=0}^q \binom{q}{k} \frac{(2p)!}{(2p-k)!} (x^{2p-k} \varphi^{(2q-k)}, \varphi)_{L^2} \right| \quad (27)$$

$$\leq \sum_{k=0}^q \binom{q}{k} \binom{2p}{k} k! \|x^{2p-k} \varphi\|_2 \|\varphi^{(2q-k)}\|_2 \quad (28)$$

$$\leq C \sum_{k=0}^q \binom{q}{k} \binom{2p}{k} k! \left(\frac{2}{\theta} \right)^{(q+p-k)} \frac{M_k^2}{M_k^2} M_{2q-k} M_{2p-k} \quad (29)$$

$$\leq C \sum_{k=0}^q \binom{q}{k} \binom{2p}{k} \left(\frac{2}{\theta} \right)^{(q+p-k)} \frac{k!}{M_k^2} M_{2q} M_{2p} \quad (30)$$

$$\leq C H^{2(q+p)} M_q^2 M_p^2 \sum_{k=0}^q \binom{q}{k} \binom{2p}{k} (2/\theta)^{(q+p-k)} \quad (31)$$

$$\leq C m^{q+p} M_q^2 M_p^2, \quad (32)$$

■

Let us now prove Theorem 2.2 in Beurling–Komatsu case.

Proof (Theorem 2.2) (b) Let us suppose that for some $\theta > 0$

$$\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{2n+1})] < \infty. \quad (33)$$

We will prove the convergence of the series $\sum_{n=1}^{\infty} \langle b_n h_n, \varphi \rangle$ for every $\varphi \in \mathcal{S}^{(M_p)}$. From that it follows that the mapping $f \mapsto \sum_{n=1}^{\infty} a_n b_n$ is an element of $\mathcal{S}'^{(M_p)}$. Using Schwartz inequality,

we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} |\langle b_n h_n, \varphi \rangle| &= \sum_{n=0}^{\infty} |b_n \langle h_n, \varphi \rangle| \\
 &= \sum_{n=0}^{\infty} (|b_n| \exp[-M(\theta\sqrt{2n+1})] \cdot |\langle \varphi, h_n \rangle| \exp[M(\theta\sqrt{2n+1})]) \\
 &\leq \left(\sum_{n=1}^{\infty} |b_n|^2 \exp[-2M(\theta\sqrt{2n+1})] \right)^{1/2} \\
 &\quad \times \left(\sum_{n=0}^{\infty} |\langle \varphi, h_n \rangle|^2 \exp[2M(\theta\sqrt{2n+1})] \right)^{1/2}.
 \end{aligned}$$

The first sum on the right-hand side converges by supposition, and the convergence of the second sum follows from Theorem 2.1, since $\varphi \in \mathcal{S}^{(M_p)}$.

(a) Suppose that $f \in \mathcal{S}^{(M_p)'}$ and let $b_n = \langle f, h_n \rangle$. Then, for every $\varphi = \sum a_n h_n \in \mathcal{S}^{(M_p)}$, we have that the series $\sum a_n b_n$ converges. We will prove that for some $\theta > 0$

$$\sum_{n \in \mathbb{N}_0} |b_n|^2 \exp[-2M(\theta\sqrt{2n+1})] < \infty. \quad (34)$$

Let us first prove that the sequence

$$\{|b_n| \exp[-M(\theta\sqrt{2n+1})]\}_{n \in \mathbb{N}} \quad (35)$$

is bounded for some θ , say θ_0 . If it is not so, then for every $q \in \mathbb{N}$, there exists $n_q \geq q$ such that

$$|b_{n_q}| \exp[-M(q\sqrt{2n_q+1})] \geq 1.$$

Let $\varphi = \sum a_m b_m$, where

$$|a_m| = \exp[-M(q\sqrt{2n_q+1})] q^{-1}$$

for $a_m = n_q$, and $a_m = 0$ for $m \neq n_q$, and a_m satisfies that $a_m b_m = |a_m b_m|$. We will prove that $\varphi \in \mathcal{S}^{(M_p)}$. Using [1, Proposition 3.4, p.50],

$$M(k\rho) - M(\rho) \geq \frac{\log(\rho/A) \log k}{\log H} \geq \frac{\log(\rho/A) \log k}{\log(1+H)} \quad (36)$$

for $\rho = \mu\sqrt{2n_q+1}$, $k = q/\mu$, we have that

$$\begin{aligned}
 \sum_{m=1}^{\infty} |a_m|^2 \exp[2M\mu\sqrt{2n+1}] &= \sum_q q^{-2} \exp[-2M(q\sqrt{2n_q+1}) + 2M(\mu\sqrt{2n_q+1})] \\
 &\leq \sum_q
 \end{aligned}$$

The above sum converges, since for large enough q

$$\exp \left[-2 \frac{\log((\mu/A)\sqrt{2n_q+1}) \log((q/\mu))}{\log(1+H)} \right]$$

is bounded. From Theorem 2.1 it follows that $\varphi \in \mathcal{S}^{(M_p)}$, and, therefore,

$$\sum_{n=0}^{\infty} a_n b_n = \langle f, \varphi \rangle < \infty,$$

which is in contradiction with

$$\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} |a_n b_n| \geq \sum_{q=1}^{\infty} q^{-1} = \infty.$$

Therefore, sequence (35) is bounded for some θ_0 .

Using inequality (36) for $\rho = \theta_0 \sqrt{2n+1}$ and $k = \theta/\theta_0$, we have that

$$\exp[-M(\theta\sqrt{2n+1})] \leq \exp[-M(\theta_0\sqrt{2n+1})] \exp\left[-\frac{\log(\theta_0\sqrt{2n+1}/A) \log(\theta/\theta_0)}{\log(1+H)}\right],$$

which implies that the sequence

$$b_n \exp[-M(\theta\sqrt{2n+1})] \longrightarrow 0, \quad n \longrightarrow \infty.$$

for some $\theta > \theta_0$.

And finally, we will prove that the series (7) converges for some $\theta > \theta_0$. If we suppose that the sequence diverges for every integer $\theta = q > \theta_0$, then there exists an increasing sequence of integers k_q such that

$$1 \leq \sum_{n=k_{q-1}}^{k_q-1} |b_n| \exp[-M(q\sqrt{2n+1})] < 2, \quad q = \theta_0 + 1, \theta_0 + 2, \dots$$

Put

$$|a_n| = |b_n| \exp[-2M(q\sqrt{2n+1})] q^{-1}$$

for $k_{q-1} \leq n, k_q, q) \theta_0$. Then, for every fixed $\theta > 0$, we have

$$\begin{aligned} & \sum_{n=k_{q-1}}^{k_q-1} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] \\ &= \sum_{n=k_{q-1}}^{k_q-1} |b_n| \exp[-2M(q\sqrt{2n+1})] \exp[2M(\theta\sqrt{2m+1}) - 2M(q\sqrt{2n+1})]. \end{aligned}$$

Using once again the inequality (36), we see that

$$\exp[-2M(q\sqrt{2n+1}) + 2M(\theta\sqrt{2n+1})] \leq 1,$$

and therefore, for every $\theta > 0$,

$$\sum_{n=k_{q-1}}^{k_q-1} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] \leq 2q^{-2}.$$

It implies that the sum

$$\sum_{n=0}^{\infty} |a_n| \exp [2M(\theta\sqrt{2n+1})]$$

converges for every $\theta > 0$. From Theorem 2.1 it follows that $\varphi = \sum a_n h_n$ belongs to $\mathcal{S}^{(M_p)}$, which contradict the fact that the sum $\sum a_n b_n$ diverges, since

$$\sum_{n=k_{q-1}}^{k_q-1} |a_n b_n| = \sum_{n=k_{q-1}}^{k_q-1} |b_n|^2 \exp [-2M(q\sqrt{2n+1})] q^{-1} \geq q^{-1}. \quad \blacksquare$$

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Generalized Jacobi orthogonal polynomials

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In this paper, by evaluating a new Hankel determinant of order n , we give explicitly the recurrence coefficients of a symmetric semi-classical sequence of polynomials of class $s \leq 2$. Then, by using the quadratic decomposition of such sequence, we give a nonsymmetric semi-classical sequence of polynomials of class $\tilde{s} \leq 1$ generalizing the Jacobi polynomial sequence.

Keywords: Orthogonal polynomials; Recurrence coefficients; Semi-classical polynomials

Mathematics Subject Classification: 33C45; 42C05

1. Introduction

In recent years, several papers have been published in combinatorics, quantum mechanics, computations and so forth, dealing with orthogonal polynomials which are not classical. It happens that they belong to a greater family called semi-classical orthogonal polynomials. The classical polynomials (Hermite, Jacobi, Bessel and Laguerre) are called semi-classical of class $s = 0$ and are completely determined.

In order to determine semi-classical polynomials of class $s = 1$ many authors gave systems fulfilled by the recurrence coefficients [1, 2]. Such systems are not linear and have not been sorted before except in the symmetric case [3, 4]. For semi-classical polynomials of class $s = 2$ the system is more complicated and remains unsolved in the symmetric and the nonsymmetric case [5]. This urges many authors to use the technique of doing some transformations of the classical polynomials or their corresponding linear functionals in order to find some examples of semi-classical polynomials of class s greater or equal to one [6–11, 18, 19].

In ref. [12] I and J. Alaya present a method with which we give all symmetric semi-classical sequences of polynomials of class $s \leq 2$ orthogonal with respect to a linear functional u fulfilling $(\Phi u)' + \Psi u = 0$ with $\Phi(x) = x^2$ and Ψ is an arbitrary polynomial of degree less or equal to 3. The method used consists in evaluating the Hankel determinants of order n , $n \geq 0$ of u by doing some elementary operations under the columns. Then, by using the quadratic decomposition of such sequences, we find new examples of nonsymmetric semi-classical sequences of polynomials of class $\tilde{s} = 1$.

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In this paper, we should use this method further with a symmetric linear functional u fulfilling the functional equation above but with $\Phi(x) = x^4 - x^2$ and Ψ is an arbitrary polynomial of degree less or equal to 3. We show that u depends on three parameters: τ , θ and γ_1 .

Section 2 is devoted to the preliminary results and notations used in the sequel. In section 3, the coefficients in the three term recurrence relation, satisfied by the polynomial sequence orthogonal with respect to u , are given and the regularity conditions are discussed. In section 4, we show that the even part of u is semi-classical of class $\tilde{s} \leq 1$ and generalizes the Jacobi linear functional for which we give the functional equation and the coefficients in the three term recurrence relation that the corresponding monic orthogonal polynomial sequence satisfies.

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathcal{C}$ and $\mathcal{P}'$ be its dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}$ on $f \in \mathcal{P}'$ in particular we denote by $(u)_n = \langle u, x^n \rangle$, $n \geq 0$ the moments of u . Let us define the following operations on $\mathcal{P}'$.

The left multiplication of a linear functional by a polynomial

$$\langle gu, f \rangle = \langle u, gf \rangle, \quad g \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (1)$$

The derivative of a linear functional

$$\langle u', f \rangle = -\langle u, f' \rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (2)$$

The homothetic of a linear functional

$$\langle h_a u, f \rangle = \langle u, h_a f \rangle, \quad a \in \mathcal{C} - \{0\}, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \quad (3)$$

where

$$(h_a f)(x) = f(ax). \quad (4)$$

The translation of a linear functional

$$\langle \tau_b u, f \rangle = \langle u, \tau_{-b} f \rangle, \quad b \in \mathcal{C}, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \quad (5)$$

where

$$(\tau_{-b} f)(x) = f(x + b). \quad (6)$$

The division of a linear functional by a polynomial of first degree

$$\langle (x - c)^{-1} u, f \rangle = \langle u, \theta_c f \rangle, \quad (7)$$

where

$$\theta_c f(x) = \frac{f(x) - f(c)}{x - c}, \quad c \in \mathcal{C}, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (8)$$

The even part of a linear functional

$$\langle \sigma u, f \rangle = \langle u, \sigma f \rangle, \quad (9)$$

where

$$\sigma f(x) = f(x^2). \quad (10)$$

DEFINITION 2.1 [13] A sequence of polynomials $\{B_n\}_{n \geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to a linear functional u if

- (i) $\deg B_n = n$ and the leading coefficient of $B_n(x)$ is equal to 1.
- (ii) $\langle u, B_n B_m \rangle = r_n \delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$.

A sequence of monic orthogonal polynomials satisfies a second-order recurrence relation [13]

$$\begin{aligned} B_0(x) &= 1, \quad B_1(x) = x - \beta_0, \\ B_{n+2}(x) &= (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0, \end{aligned} \quad (11)$$

with $(\beta_n, \gamma_n) \in \mathcal{C} \times \mathcal{C} \setminus \{0\}$, $n \geq 0$.

In such conditions we say that u is regular. In what follows we take all regular linear functionals normalized that is $(u)_0 = 1$.

Let us denote by

$$\Delta_n = \begin{vmatrix} (u)_0 & (u)_1 & \cdots & (u)_n \\ (u)_1 & (u)_2 & \cdots & (u)_{n+1} \\ \vdots & \vdots & & \vdots \\ (u)_n & (u)_{n+1} & \cdots & (u)_{2n} \end{vmatrix}, \quad n \geq 0, \quad \Delta_{-1} = 1.$$

the Hankel determinant of order $n + 1$ of u . Then, it is well known that the existence of a MOPS for u is equivalent to the following condition [13–15]:

$$\Delta_n \neq 0, \quad n \geq 0.$$

In this conditions the sequence γ_n in (11) is given by [14]

$$\gamma_{n+1} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad n \geq 0; \quad \gamma_0 = 1, \quad (12)$$

which is equivalent to

$$\Delta_{n+1} = \prod_{k=1}^{n+1} \tau_k, \quad n \geq 0, \quad (13)$$

where

$$\tau_k = \prod_{i=1}^k \gamma_i, \quad k \geq 1. \quad (14)$$

The orthogonality is kept by shifting. In fact the sequence $\{\tilde{B}_n\}_{n \geq 0}$, defined by $\tilde{B}_n(x) = a^{-n}B_n(ax + b)$, $n \geq 0$, is orthogonal with respect to $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ and satisfies the recurrence relation [14, 15]

$$\begin{aligned} \tilde{B}_0(x) &= 1, \quad \tilde{B}_1(x) = x - \frac{\beta_0 - b}{a}, \\ \tilde{B}_{n+2}(x) &= \left(x - \frac{\beta_{n+1} - b}{a}\right) \tilde{B}_{n+1}(x) - \frac{\gamma_{n+1}}{a^2} \tilde{B}_n(x), \quad n \geq 0. \end{aligned} \quad (15)$$

A linear functional u is said to be symmetric if and only if $(u)_{2n+1} = 0$, $n \geq 0$. Or equivalently $\sigma(xu) = 0$. Or in (11) $\beta_n = 0$, $n \geq 0$.

DEFINITION 2.2 (see [13, 14, 16]) *A linear functional u is semi-classical of class s , if it is regular and the following statement holds: There exist two polynomials Ψ of degree $p \geq 1$ and Φ of degree $t \geq 0$, such that*

$$(\Phi u)' + \Psi u = 0, \quad (16)$$

$$\prod_{c \in Z_\Phi} (|\Psi(c) + \Phi'(c)| + |\langle u, \theta_c \Psi + \theta_c^2(\Phi) \rangle|) > 0, \quad (17)$$

where Z_Φ is the set of zeros of Φ .

The class of u is given by $s = \max(p - 1, t - 2)$. The sequence $\{B_n\}_{n \geq 0}$ corresponding to u is said semi-classical of class s .

If u is a semi-classical functional of class s fulfilling (16) and (17) then $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ is also semi-classical of the same class and satisfies the functional equation

$$(\tilde{\Phi} \tilde{u})' + \tilde{\Psi} \tilde{u} = 0, \quad (18)$$

where

$$\tilde{\Phi}(x) = a^{-t} \Phi(ax + b), \quad \tilde{\Psi}(x) = a^{1-t} \Psi(ax + b). \quad (19)$$

PROPOSITION 2.3 *Let u be a symmetric semi classical linear functional of class s fulfilling (16) and (17) then if s is even Φ is even and Ψ is odd; if s is odd Φ is odd and Ψ is even.*

Proof See [4] with $B(x) = 0$. ■

3. A symmetric semi-classical form of class $s \leq 2$

In the sequel we suppose u symmetric, semi-classical linear functional of class $s \leq 2$ fulfilling (16) with

$$\Phi(x) = x^4 - x^2. \quad (20)$$

We denote by $\{B_n\}_{n \geq 0}$ the (MOPS) corresponding to u and we suppose that it satisfies (11).

According to the proposition 2.3 we have $\Psi(x) = a_3 x^3 + a_1 x$, where $a_3, a_1 \in \mathcal{C}$, $a_3 - n \neq 0, n \geq 0$.

Putting $a_3 = -(2\tau + 1)$ and $a_1 = 2\theta + 1$ we have

$$\Psi(x) = -(2\tau + 1)x^3 + (2\theta + 1)x, \quad (21)$$

therefore, (16) becomes

$$((x^4 - x^2)u)' + (-(2\tau + 1)x^3 + (2\theta + 1)x)u = 0. \quad (22)$$

Let's see when the functional equation (22) can be simplified by x . This will play an important role in the calculus of γ_{n+1} , $n \geq 0$ in what follows.

Using the definition of the operator θ_c in (8) we have

$$\begin{aligned}\langle u, \theta_0 \Psi + \theta_0^2 \Phi \rangle &= \langle u, (-2\tau - 1)x^2 + (2\theta + 1)x + x^2 - 1 \rangle \\ &= \langle u, -2\tau x^2 + 2\theta \rangle \\ &= -2\tau(u)_2 + 2\theta \\ &= -2(\tau\gamma_1 - \theta).\end{aligned}$$

Therefore, according to (17), if $\tau \neq 0$ and $\gamma_1 = (\theta/\tau)$ then $s \leq 1$ and (22) becomes after simplification by x

$$((x^3 - x)u)' + 2(-\tau x^2 + \theta)u = 0. \quad (23)$$

3.1 Expression of the moments

From (22) we have

$$\langle ((x^4 - x^2)u)' + ((-2\tau - 1)x^3 + (2\theta + 1)x)u, x^n \rangle = 0, \quad n \geq 0,$$

which implies

$$-n\langle u, x^{n+3} - x^{n+1} \rangle - (2\tau + 1)\langle u, x^{n+3} \rangle + (2\theta + 1)\langle u, x^{n+1} \rangle = 0, \quad n \geq 0.$$

Therefore,

$$(u)_{n+3} = \frac{n + 2\theta + 1}{n + 2\tau + 1}(u)_{n+1}, \quad n \geq 0.$$

But, with $(u)_0 = 1$, $(u)_1 = 0$ and $(u)_2 = \gamma_1$ we obtain by induction

$$(u)_{2n+1} = 0, \quad n \geq 0, \quad (24)$$

$$(u)_{2n} = \gamma_1 \prod_{k=1}^{n-1} \frac{\theta + k}{\tau + k}, \quad n \geq 2. \quad (25)$$

Note that u depends on γ_1, θ and τ . Then, in order to distinguish between cases, we will denote by

$$u = u(\theta, \tau, \gamma_1). \quad (26)$$

Remark We must take $\theta + n \neq 0$, $n \geq 1$ and $\tau \neq \theta$. Indeed, if $\theta = -n_0$, where n_0 is a positive integer (respectively $\tau = \theta$), then, according to equation (25), $(u)_{2n} = 0$ for every $n \geq n_0 + 1$ (respectively $(u)_{2n} = \gamma_1$, $n \geq 1$), therefore $\Delta_{2n} = 0$, $n \geq n_0 + 1$, (respectively $\Delta_n = 0$, $n \geq 3$), which is contradictory with the regularity of u .

In what follows, we take $\theta + n \neq 0$, $n \geq 1$, $\tau \neq \theta$ and we denote by $\{w_n\}_{n \geq 0}$ the sequence defined by

$$w_{2n} = \frac{\Gamma(\tau + 1) \Gamma(\theta + n)}{\Gamma(\theta + 1) \Gamma(\tau + n)}, \quad n \geq 1, \quad w_0 = 1, \quad (27)$$

$$w_{2n+1} = 0, \quad n \geq 0. \quad (28)$$

where Γ is the gamma function. Then, (24) and (25) become

$$(u)_n = \gamma_1 w_n, \quad n \geq 1. \quad (29)$$

3.2 Expression of the recurrence coefficients

The sequence $\{B_n\}_{n \geq 0}$ is symmetric then $\beta_n = 0, n \geq 0$. To explicitly determine the recurrence coefficients $\gamma_{n+1}, n \geq 0$, we will calculate $\Delta_n = \Delta_n(\theta, \tau, \gamma_1)$ the Hankel determinants of $u(\theta, \tau, \gamma_1)$ and use (12).

We have by (29)

$$\begin{aligned} \Delta_n(\theta, \tau, \gamma_1) &= \begin{vmatrix} (u)_0 & (u)_1 & \dots & (u)_n \\ (u)_1 & (u)_2 & \dots & (u)_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ (u)_n & (u)_{n+1} & \dots & (u)_{2n} \end{vmatrix} \\ &= \begin{vmatrix} 1 & \gamma_1 w_1 & \dots & \gamma_1 w_n \\ \gamma_1 w_1 & \gamma_1 w_2 & \dots & \gamma_1 w_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 w_n & \gamma_1 w_{n+1} & \dots & \gamma_1 w_{2n} \end{vmatrix} \\ &= \gamma_1^n \begin{vmatrix} 1 & w_1 & \dots & w_n \\ \gamma_1 w_1 & w_2 & \dots & w_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 w_n & w_{n+1} & \dots & w_{2n} \end{vmatrix}. \end{aligned}$$

Expansion of this determinant according to the first column gives

$$\Delta_n(\theta, \tau, \gamma_1) = \gamma_1^n (b_n(\theta, \tau) + \gamma_1 a_n(\theta, \tau)), \quad n \geq 1 \quad (30)$$

where $b_n(\theta, \tau)$ and $a_n(\theta, \tau)$ do not depend on γ_1 since w_n does not depend on γ_1 .

It is clear that

$$b_n(\theta, \tau) = \det(w_{i+j})_{1 \leq i, j \leq n} = \begin{vmatrix} w_2 & \dots & w_{n+1} \\ \vdots & & \vdots \\ w_{n+1} & \dots & w_{2n} \end{vmatrix}, \quad n \geq 1.$$

3.2.1 Expression of $b_n(\theta, \tau)$. To calculate $b_n(\theta, \tau), n \geq 1$ we need the two following lemmas

LEMMA 3.1 For $n \geq 1$ and $1 \leq i \leq n$ we have

$$\begin{aligned} &\sum_{j=1}^n (-1)^j \binom{n-1}{j-1} \frac{\Gamma(\theta + i + j)}{\Gamma(\theta + j + 1)} \frac{\Gamma(\tau + n + j - 1)}{\Gamma(\tau + i + j)} \\ &= (n-1)! (-1)^n (\tau - \theta)_{n-1} \frac{\Gamma(\tau + n)}{\Gamma(\tau + 2n)} \delta_{n,i}, \end{aligned} \quad (31)$$

$$\begin{aligned} &\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\Gamma(\theta + i + j + 1)}{\Gamma(\theta + j + 1)} \frac{\Gamma(\tau + n + j)}{\Gamma(\tau + i + j + 1)} \\ &= n! (\tau - \theta)_n \frac{(-1)^n \Gamma(\tau + n)}{\Gamma(\tau + 2n + 1)} \delta_{n,i}, \end{aligned} \quad (32)$$

where

$$\delta_{n,i} = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}.$$

and $(a)_k$ is the Pochhammer symbol or rising factorial

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1) \cdots (a+k-1).$$

Proof Let us recall that the generalized hypergeometric function ${}_3F_2\left[\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; x\right]$ is given by

$${}_3F_2\left[\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; x\right] = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{x^k}{k!}.$$

We have then,

$${}_3F_2\left[\begin{smallmatrix} n+\nu, i+\theta+1, 1-n \\ 2+\theta, i+\nu+1 \end{smallmatrix}; 1\right] = \sum_{k=0}^{+\infty} \frac{(n+\nu)_k (i+\theta+1)_k (1-n)_k}{(2+\theta)_k (i+\nu+1)_k} \frac{1}{k!}.$$

Taking into account that $(1-n)_k = 0$ for $k \geq n$ we get

$$\begin{aligned} & {}_3F_2\left[\begin{smallmatrix} n+\tau, i+\theta+1, 1-n \\ 2+\theta, i+\tau+1 \end{smallmatrix}; 1\right] \\ &= \frac{\Gamma(\theta+2)}{\Gamma(\theta+i+1)} \frac{\Gamma(\tau+i+1)}{\Gamma(\tau+n)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\Gamma(\theta+i+k+1)}{\Gamma(\theta+k+2)} \frac{\Gamma(\tau+n+k)}{\Gamma(\tau+i+k+1)}. \end{aligned}$$

But, from the Saalschutz theorem:

$${}_3F_2\left[\begin{smallmatrix} a, b, c \\ d, e \end{smallmatrix}; x\right] = \frac{(d-a)_{|c|} (d-b)_{|c|}}{(d)_{|c|} (d-a-b)_{|c|}},$$

where $d+e = a+b+c+1$ and c is a negative integer, we have also for $1 \leq i \leq n$

$$\begin{aligned} & {}_3F_2\left[\begin{smallmatrix} n+\tau, i+\theta+1, 1-n \\ 2+\theta, i+\tau+1 \end{smallmatrix}; 1\right] = \frac{(\theta-\tau-n+2)_{n-1} (1-i)_{n-1}}{(\theta+2)_{n-1} (1-i-n-\tau)_{n-1}} \\ &= (-1)^{n-1} (n-1)! (\tau-\theta)_{n-1} \frac{\Gamma(\theta+2) \Gamma(\tau+n+1)}{\Gamma(\theta+n+1) \Gamma(\tau+2n)} \delta_{n,i}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\Gamma(\theta+i+k+1)}{\Gamma(\theta+k+2)} \frac{\Gamma(\tau+n+k)}{\Gamma(\tau+i+k+1)} \\ &= (-1)^{n-1} (n-1)! (\tau-\theta)_{n-1} \frac{\Gamma(\tau+n)}{\Gamma(\tau+2n)} \delta_{n,i} \end{aligned}$$

The change of indices $j = k+1$ in the sum gives (31).

Similarly, we have for $1 \leq i \leq n$

$${}_3F_2 \left[\begin{matrix} n+\tau, i+\theta+1, -n \\ 1+\theta, i+\tau+1 \end{matrix}; 1 \right] \\ = \frac{\Gamma(\theta+1)}{\Gamma(\theta+i+1)} \frac{\Gamma(\tau+i+1)}{\Gamma(\tau+n)} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma(\theta+i+k+1)}{\Gamma(\theta+k+1)} \frac{\Gamma(\tau+n+k)}{\Gamma(\tau+i+k+1)}$$

and

$${}_3F_2 \left[\begin{matrix} n+\tau, i+\theta+1, -n \\ 1+\theta, i+\tau+1 \end{matrix}; 1 \right] = \frac{(\theta-\tau-n+1)_n (-i)_n}{(\theta+1)_n (-i-n-\tau)_n} \\ = n!(\tau-\theta)_n \frac{(-1)^n \Gamma(\theta+1) \Gamma(\tau+n+1)}{\Gamma(\theta+n+1) \Gamma(\tau+2n+1)} \delta_{n,i}$$

hence,

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{\Gamma(\theta+i+j+1)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j)}{\Gamma(\tau+i+j+1)} = n!(\tau-\theta)_n \frac{(-1)^n \Gamma(\tau+n)}{\Gamma(\tau+2n+1)} \delta_{n,i},$$

which gives (32) and completes the proof. ■

LEMMA 3.2 *We have the following relations*

$$b_{2n}(\theta, \tau) = \frac{\Gamma(\tau+2n) \Gamma(\theta+n+1) \Gamma(\tau+n)}{((\tau-\theta)_n)^2 \Gamma^2(\tau) \Gamma(\theta+1) (n!)^2} \\ \times \prod_{k=1}^n \left(\frac{k! (\tau-\theta)_k \Gamma(\tau+1) \Gamma(\theta+k) \Gamma(\tau+k-1)}{\Gamma(\theta+1) \Gamma(\tau+2k) \Gamma(\tau+2k-1)} \right)^2, \quad n \geq 1, \quad (33)$$

$$b_{2n+1}(\theta, \tau) = \frac{\tau \Gamma^2(\tau+n) \Gamma^2(\theta+n+1)}{(\tau-\theta)_n \Gamma(\tau) \Gamma^2(\theta+1) \Gamma(\tau+2n+1) n!} \\ \times \prod_{k=1}^n \left(\frac{k! (\tau-\theta)_k \Gamma(\tau+1) \Gamma(\theta+k) \Gamma(\tau+k-1)}{\Gamma(\theta+1) \Gamma(\tau+2k) \Gamma(\tau+2k-1)} \right)^2, \quad n \geq 1, \\ b_1(\theta, \tau) = 1. \quad (34)$$

Proof Let us denote by C_j the j -th column of $b_n(\theta, \tau) = \det(w_{i+j})_{1 \leq i, j \leq n}$.

On $b_{2n}(\theta, \tau)$ we do the following elementary operation

$$C_{2n} \longleftarrow C_{2n} + \sum_{j=1}^{n-1} \lambda_{2j} C_{2j}$$

where

$$\lambda_{2j} = (-1)^{n-j} \binom{n-1}{j-1} \frac{\Gamma(\theta+n+1)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j-1)}{\Gamma(\tau+2n-1)},$$

we obtain

$$b_{2n}(\theta, \tau) = \begin{vmatrix} w_2 & \dots & w_{2n} & w_{2n+1} + \sum_{j=1}^{n-1} \lambda_{2j} w_{2j+1} \\ w_3 & \dots & w_{2n+1} & \vdots \\ \vdots & \dots & \vdots & \vdots \\ w_{2n+1} & \dots & \dots & w_{4n} + \sum_{j=1}^{n-1} \lambda_{2j} w_{2n+2j} \end{vmatrix}.$$

Now let us calculate the $(2i+1)$ -th and the $2i$ -th coefficients of C_{2n} .

We have:

For $0 \leq i \leq n-1$, (by (28))

$$w_{2n+2i+1} + \sum_{j=1}^{n-1} \lambda_{2j} w_{2i+2j+1} = 0.$$

For $1 \leq i \leq n-1$, (by (27) and (31))

$$\begin{aligned} w_{2n+2i} + \sum_{j=1}^{n-1} \lambda_{2j} w_{2i+2j} \\ &= \sum_{j=1}^n (-1)^{n-j} \binom{n-1}{j-1} \frac{\Gamma(\theta+n+1)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j-1)}{\Gamma(\tau+2n-1)} \frac{\Gamma(\theta+i+j)}{\Gamma(\theta+1)} \frac{\Gamma(\tau+1)}{\Gamma(\tau+i+j)} \\ &= \frac{(-1)^n \Gamma(\theta+n+1) \Gamma(\tau+1)}{\Gamma(\tau+2n-1) \Gamma(\theta+1)} \sum_{j=1}^n (-1)^j \binom{n-1}{j-1} \frac{\Gamma(\theta+i+j)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j-1)}{\Gamma(\tau+i+j)} \\ &= 0. \end{aligned}$$

For $i = n$ (by (31))

$$\begin{aligned} w_{4n} + \sum_{j=1}^{n-1} \lambda_{2j} w_{2n+2j} \\ &= (-1)^n \frac{\Gamma(\theta+n+1) \Gamma(\tau+1)}{\Gamma(\tau+2n-1) \Gamma(\theta+1)} \sum_{j=1}^n (-1)^j \binom{n-1}{j-1} \frac{\Gamma(\theta+n+j)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j-1)}{\Gamma(\tau+n+j)} \\ &= (n-1)! (\tau-\theta)_{n-1} \frac{\Gamma(\theta+n+1) \Gamma(\tau+1)}{\Gamma(\tau+2n-1) \Gamma(\theta+1)} \frac{\Gamma(\tau+n)}{\Gamma(\tau+2n)}. \end{aligned}$$

Then, we deduce that

$$b_{2n}(\theta, \tau) = \begin{vmatrix} w_2 & \cdots & w_{2n} & 0 \\ \vdots & \cdots & \vdots & \vdots \\ w_{2n} & \cdots & w_{4n-2} & 0 \\ \cdots & \cdots & \cdots & (n-1)! (\tau-\theta)_{n-1} \frac{\Gamma(\theta+n+1) \Gamma(\tau+1)}{\Gamma(\tau+2n-1) \Gamma(\theta+1)} \frac{\Gamma(\tau+n)}{\Gamma(\tau+2n)} \end{vmatrix},$$

or,

$$b_{2n}(\theta, \tau) = (n-1)! (\tau-\theta)_{n-1} \frac{\Gamma(\theta+n+1) \Gamma(\tau+1)}{\Gamma(\tau+2n-1) \Gamma(\theta+1)} \frac{\Gamma(\tau+n)}{\Gamma(\tau+2n)} b_{2n-1}(\theta, \tau), \quad n \geq 1. \quad (35)$$

Similarly on $b_{2n+1}(\theta, \tau)$, $n \geq 1$ we do the following elementary operation

$$C_{2n+1} \leftarrow C_{2n+1} + \sum_{j=0}^{n-1} \lambda_{2j+1} C_{2j+1},$$

where

$$\lambda_{2j+1} = (-1)^{n-j} \binom{n}{j} \frac{\Gamma(\theta+n+1)}{\Gamma(\theta+j+1)} \frac{\Gamma(\tau+n+j)}{\Gamma(\tau+2n)}.$$

and we take into account (32) we get

$$b_{2n+1}(\theta, \tau) = n!(\tau - \theta)_n \frac{\Gamma(\theta + n + 1)\Gamma(\tau + 1)}{\Gamma(\tau + 2n)\Gamma(\theta + 1)} \frac{\Gamma(\tau + n)}{\Gamma(\tau + 2n + 1)} b_{2n}(\theta, \tau), \quad n \geq 1, \quad (36)$$

$$b_1(\theta, \tau) = 1.$$

Finally, using (35) and (36) we prove by induction (33) and (34). ■

3.2.2 Expression of $a_n(\theta, \tau)$ ($\tau \neq 0$ and $\theta \neq 0$). Let $\tau \neq 0$ and $\theta \neq 0$.

In (30) we take $\gamma_1 = (\theta/\tau)$, we get

$$\Delta_n \left(\theta, \tau, \frac{\theta}{\tau} \right) = \left(\frac{\theta}{\tau} \right)^n \left(b_n(\theta, \tau) + \frac{\theta}{\tau} a_n(\theta, \tau) \right), \quad n \geq 1,$$

which gives

$$a_n(\theta, \tau) = \frac{\tau^{n+1}}{\theta^{n+1}} \Delta_n \left(\theta, \tau, \frac{\theta}{\tau} \right) - \frac{\tau}{\theta} b_n(\theta, \tau), \quad n \geq 1. \quad (37)$$

Since the sequence $(b_n(\theta, \tau))_{n \geq 0}$ is given in equations (33) and (34), to calculate $a_n(\theta, \tau)$ in (37), we will evaluate $\Delta_n(\theta, \tau, (\theta/\tau))$ -the Hankel determinant of $u(\theta, \tau, (\theta/\tau))$.

The linear functional $u(\theta, \tau, (\theta/\tau))$ is of class $s \leq 1$ and the recurrence coefficients of its corresponding (MOPS) are given by [3]

$$\begin{cases} \gamma_{2n+2} = \frac{(n+1)(n+\tau-\theta)}{(2n+\tau+1)(2n+\tau)}, & n \geq 0, \\ \gamma_{2n+1} = \frac{(n+\theta)(n+\tau-1)}{(2n+\tau-1)(2n+\tau)}, & n \geq 0. \end{cases}$$

with $n + \tau \neq 0$, $n \geq -1$, $n + \theta \neq 0$, $n \geq 0$, and $\tau - \theta + n \neq 0$, $n \geq 0$.

Therefore, using (13) with (14) we obtain

$$\left\{ \begin{array}{l} \Delta_{2n} \left(\theta, \tau, \frac{\theta}{\tau} \right) = \frac{\theta^{2n} \Gamma(\tau - \theta) \Gamma(\tau + 2n)}{\tau^{2n} n! \Gamma(\tau - \theta + n) \Gamma(\tau)} \\ \prod_{k=1}^n \left(\frac{k!(\tau - \theta)_k \Gamma(\tau + 1) \Gamma(\tau + k - 1) \Gamma(\theta + k)}{\Gamma(\theta + 1) \Gamma(\tau + 2k) \Gamma(\tau + 2k - 1)} \right)^2, \quad n \geq 1, \\ \Delta_{2n+1} \left(\theta, \tau, \frac{\theta}{\tau} \right) = \frac{\theta^{2n} \Gamma(\tau + n) \Gamma(\theta + n + 1)}{\tau^{2n} \Gamma(\theta) \Gamma(\tau + 2n + 1)} \\ \prod_{k=1}^n \left(\frac{k!(\tau - \theta)_k \Gamma(\tau + 1) \Gamma(\tau + k - 1) \Gamma(\theta + k)}{\Gamma(\theta + 1) \Gamma(\tau + 2k) \Gamma(\tau + 2k - 1)} \right)^2, \quad n \geq 1. \end{array} \right. \quad (38)$$

3.2.3 Expression of γ_n . Let $\tau \neq 0$, $\theta \neq 0$. Substitution of $a_n(\theta, \nu)$ from (38) into (30) gives:

$$\Delta_n(\theta, \tau, \gamma_1) = \gamma_1^n \left(\left(1 - \gamma_1 \frac{\tau}{\theta} \right) b_n(\theta, \tau) + \gamma_1 \frac{\tau^{n+1}}{\theta^{n+1}} \Delta_n \left(\theta, \tau, \frac{\theta}{\tau} \right) \right), \quad n \geq 1.$$

Therefore, on account of equations (33), (34) and (38), we obtain

$$\begin{aligned} \Delta_{2n}(\theta, \tau, \gamma_1) &= \frac{\gamma_1^{2n} \Gamma(\tau + 2n)}{\theta \Gamma^2(\tau) \Gamma(\theta + 1) (n!)^2 ((\tau - \theta)_n)^2} \\ &\quad \times \prod_{k=1}^n \left(\frac{k! (\tau - \theta)_k \Gamma(\tau + 1) \Gamma(\theta + k) \Gamma(\tau + k - 1)}{\Gamma(\theta + 1) \Gamma(\tau + 2k) \Gamma(\tau + 2k - 1)} \right)^2 \\ &\quad \times (\gamma_1 \tau \Gamma(\tau) \Gamma(\theta + 1) n! (\tau - \theta)_n \\ &\quad + (\theta - \gamma_1 \tau) \Gamma(\theta + n + 1) \Gamma(\tau + n)), \quad n \geq 0, \\ \Delta_{2n+1}(\theta, \tau, \gamma_1) &= \frac{\gamma_1^{2n+1} \tau \Gamma(\theta + n + 1) \Gamma(\tau + n)}{\theta \Gamma^2(\theta + 1) \Gamma(\tau) \Gamma(\tau + 2n + 1) n! (\tau - \theta)_n} \\ &\quad \times \prod_{k=1}^n \left(\frac{k! (\tau - \theta)_k \Gamma(\tau + 1) \Gamma(\theta + k) \Gamma(\tau + k - 1)}{\Gamma(\theta + 1) \Gamma(\tau + 2k) \Gamma(\tau + 2k - 1)} \right)^2 \\ &\quad \times (\gamma_1 \tau \Gamma(\tau) \Gamma(\theta + 1) n! (\tau - \theta)_n \\ &\quad + (\theta - \gamma_1 \tau) \Gamma(\theta + n + 1) \Gamma(\tau + n)), \quad n \geq 0, \end{aligned} \quad (39)$$

(with the convention that empty product equals 1).

Using (12), we deduce the following proposition

PROPOSITION 3.3 For $\tau \neq 0$ and $\theta \neq 0$ one has

$$\gamma_{2n+1} = \frac{n(\tau + n - 1)(\tau - \theta + n - 1)(\theta + n)}{(\tau + 2n - 1)(\tau + 2n)} \frac{\alpha_{n-1}}{\alpha_n}, \quad n \geq 1, \quad (40)$$

$$\gamma_{2n+2} = \frac{1}{(\tau + 2n + 1)(\tau + 2n)} \frac{\alpha_{n+1}}{\alpha_n}, \quad n \geq 0, \quad (41)$$

where

$$\alpha_n = \gamma_1 \Gamma(\tau + 1) \Gamma(\theta + 1) n! (\tau - \theta)_n + (\theta - \gamma_1 \tau) \Gamma(\theta + n + 1) \Gamma(\tau + n), \quad n \geq 0. \quad (42)$$

3.2.4 Regularity of u . Let $\tau \neq 0$ and $\theta \neq 0$. According to the Proposition 3.3 one has:

The linear functional u is regular if and only if $\gamma_1 \neq 0$, $\theta + n \neq 0$, $\tau + n \neq 0$, $\tau - \theta + n - 1 \neq 0$, $n \geq 1$ and $\alpha_n \neq 0$, $n \geq 0$.

The coefficient α_n given in (42) is a polynomial of first degree on γ_1 then we distinguish two cases:

If $\Gamma(\tau + 1) \Gamma(\theta + 1) n! (\tau - \theta)_n - \tau \Gamma(\theta + n + 1) \Gamma(\tau + n) = 0$ then

$$\alpha_n = \theta \Gamma(\theta + n + 1) \Gamma(\tau + n) \neq 0.$$

If $\Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \tau\Gamma(\theta + n + 1)\Gamma(\tau + n) \neq 0$, $n \geq 0$, then

$$\left(\alpha_n \neq 0 \text{ if and only if } \gamma_1 \neq \frac{-\theta\Gamma(\theta + n + 1)\Gamma(\tau + n)}{\Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \tau\Gamma(\theta + n + 1)\Gamma(\tau + n)} \right).$$

Consequently, form u is regular if and only if $\theta + n \neq 0$, $\tau + n \neq 0$, $\tau - \theta + n \neq 0$, $n \geq 0$ and $\gamma_1 \neq (-\theta\Gamma(\theta + n + 1)\Gamma(\tau + n) / \Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \tau\Gamma(\theta + n + 1)\Gamma(\tau + n))$ for any $n \geq 0$ such that $\Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \nu\Gamma(\theta + n + 1)\Gamma(\nu + n) \neq 0$.

COROLLARY 3.4 For $\tau = 0$ and $\theta \neq 0$ one has

$$\begin{cases} \gamma_{2n+1} = \frac{(-\theta + n - 1)(\theta + n)}{2(2n - 1)} \frac{\mu_{n-1}}{\mu_n}, & n \geq 1, \\ \gamma_{2n+2} = \frac{1}{2(2n + 1)} \frac{\mu_{n+1}}{\mu_n}, & n \geq 1 \\ \gamma_2 = \frac{\mu_1}{\mu_0} \end{cases} \quad (43)$$

where

$$\mu_n = \gamma_1 n \Gamma(\theta + 1)(-\theta)_n + \theta \Gamma(\theta + n + 1), \quad n \geq 0 \quad (44)$$

with the regularity conditions: $-\theta + n - 1 \neq 0$, $\theta + n \neq 0$, $n \geq 1$ and $\mu_n \neq 0$, $n \geq 0$.

Proof Let $\theta \neq 0$.

The continuity of the function

$$\Delta_n : \tau \longrightarrow \Delta_n(\theta, \tau, \gamma_1) = \begin{vmatrix} (u)_0 & (u)_1 & \dots & (u)_n \\ (u)_1 & (u)_2 & \dots & (u)_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ (u)_n & (u)_{n+1} & \dots & (u)_{2n} \end{vmatrix}$$

implies : $\Delta_n(\theta, 0, \gamma_1) = \lim_{\tau \rightarrow 0} \Delta_n(\theta, \tau, \gamma_1)$.

So, by taking the limit of γ_{n+1} , $n \geq 0$ in equations (40), (41) and (42) as τ goes to zero, which is equivalent to take the limit of $\Delta_n(\theta, \tau, \gamma_1)$ in equation (39) as τ goes to zero and use (12), we find (43) with (44). ■

COROLLARY 3.5 For $\theta = 0$ and $\tau \neq 0$ one has

$$\begin{cases} \gamma_{2n+1} = \frac{n(\tau + n - 1)}{(\tau + 2n - 1)(\tau + 2n)} \frac{v_{n-1}}{v_n}, & n \geq 1, \\ \gamma_{2n+2} = \frac{(n + 1)(n + \tau)}{(\tau + 2n + 1)(\tau + 2n)} \frac{v_{n+1}}{v_n}, & n \geq 0. \end{cases} \quad (45)$$

where

$$v_n = -\gamma_1 \tau \sum_{k=1}^n \left(\frac{1}{\tau + k - 1} + \frac{1}{k} \right) + 1, \quad n \geq 0 \quad (46)$$

with the convention that empty sum equals zero.

The regularity conditions is: $\tau + n \neq 0$ and $v_n \neq 0$, $n \geq 0$.

Proof The proof is similar to the previous one. The only difference between them is that in this one, precisely in the passage to the limit in (40) and (41) as θ goes to 0 we have an indeterminate form because $\lim_{\theta \rightarrow 0} \alpha_n = 0$. Thus, we wrote

$$\alpha_n = \gamma_1 \Gamma(\tau + 1) \Gamma(\theta + 1) [(-1)^n n! (\theta - \tau)(\theta - (\tau + 1)) \cdot \cdot (\theta - (\tau + n - 1) - (\tau)_n (\theta + 1) \cdot (\theta + n)] + \theta \Gamma(\theta + n + 1) \Gamma(\tau + n)$$

and we applied the Hôpital rule. ■

4. A nonsymmetric semi-classical sequence of class $\tilde{s} \leq 1$ generalizing Jacobi sequence

The (MOPS) $\{B_n\}_{n \geq 0}$ studied in the previous section is symmetric then it is characterized by the following quadratic decomposition [17]

$$\begin{cases} B_{2n}(x) = P_n(x^2), & n \geq 0, \\ B_{2n+1}(x) = xR_n(x^2), & n \geq 0. \end{cases} \quad (47)$$

where $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ are two (MOPS) called components of the polynomial $\{B_n\}_{n \geq 0}$. The sequences $\{P_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ are orthogonal with respect to σu and $x\sigma u$ respectively.

The aim of this section is to study the sequence $\{P_n\}_{n \geq 0}$.

4.1 Expression of the recurrence coefficients

The sequence $\{P_n\}_{n \geq 0}$ satisfies the following recurrence relation [17]

$$\begin{cases} P_0(x) = 1, & P_1(x) = x - \beta_0^P \\ P_{n+2}(x) = (x - \beta_{n+1}^P)P_{n+1}(x) - \gamma_{n+1}^P P_n(x), & n \geq 0. \end{cases} \quad (48)$$

where

$$\beta_0^P = \gamma_1, \quad \beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3}, \quad \gamma_{n+1}^P = \gamma_{2n+1}\gamma_{2n+2}, \quad n \geq 0. \quad (49)$$

We distinguish three cases ($\tau \neq 0, \theta \neq 0$), ($\tau = 0, \theta \neq 0$), and ($\tau \neq 0, \theta = 0$)

Case $\tau \neq 0, \theta \neq 0$

From (40), (41) and (49) we have

$$\begin{aligned} \beta_0^P &= \gamma_1, \\ \beta_{n+1}^P &= \frac{1}{(\tau + 2n + 1)(\tau + 2n)} \frac{\alpha_{n+1}}{\alpha_n} \\ &\quad + \frac{(n + 1)(\tau + n)(\tau - \theta + n)(\theta + n + 1)}{(\tau + 2n + 1)(\tau + 2n + 2)} \frac{\alpha_n}{\alpha_{n+1}}, \quad n \geq 0, \\ \gamma_1^P &= \gamma_1 \frac{1}{\tau(\tau + 1)} \frac{\alpha_1}{\alpha_0}, \\ \gamma_{n+1}^P &= \frac{n(\tau + n - 1)(\tau - \theta + n - 1)(\theta + n)}{(\tau + 2n + 1)(\tau + 2n - 1)(\tau + 2n)^2} \frac{\alpha_{n+1}\alpha_{n-1}}{\alpha_n^2}, \quad n \geq 1, \end{aligned} \quad (50)$$

where $\theta + n \neq 0$, $\tau + n \neq 0$, $\tau - \theta + n \neq 0$, $n \geq 0$ and $\gamma_1 \neq (-\theta\Gamma(\theta + n + 1)\Gamma(\tau + 2n)/\Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \tau\Gamma(\theta + n + 1)\Gamma(\tau + n))$ for any $n \geq 0$ such that $\Gamma(\tau + 1)\Gamma(\theta + 1)n!(\tau - \theta)_n - \nu\Gamma(\theta + n + 1)\Gamma(\nu + n) \neq 0$.

Case $\tau = 0$ and $\theta \neq 0$

From (43) and (49) we have

$$\left\{ \begin{array}{l} \beta_0^P = \gamma_1, \quad \beta_1^P = \frac{\mu_1}{\mu_0} - \frac{\theta(\theta+1)}{2} \frac{\mu_0}{\mu_1}, \\ \beta_{n+1}^P = \frac{1}{2(2n+1)} \frac{\mu_{n+1}}{\mu_n} + \frac{(-\theta+n)(\theta+n+1)}{2(2n+1)} \frac{\mu_n}{\mu_{n+1}}, \quad n \geq 1, \\ \gamma_1^P = \gamma_1 \frac{\mu_1}{\mu_0}, \\ \gamma_{n+1}^P = \frac{(-\theta+n-1)(\theta+n)}{4(2n-1)(2n+1)} \frac{\mu_{n-1}\mu_{n+1}}{\mu_n^2}, \quad n \geq 1 \end{array} \right.$$

where $-\theta+n-1 \neq 0$, $\theta+n \neq 0$ and $\mu_n \neq 0$, $n \geq 0$.

Case $\theta = 0$ and $\tau \neq 0$

From (45) and (49) we have

$$\left\{ \begin{array}{l} \beta_0^P = \gamma_1, \\ \beta_{n+1}^P = \frac{(n+1)(n+\tau)}{(\tau+2n+1)(\tau+2n)} \frac{v_{n+1}}{v_n} + \frac{(n+1)(\tau+n)}{(\tau+2n+1)(\tau+2n+2)} \frac{v_n}{v_{n+1}}, \quad n \geq 0, \\ \gamma_1^P = \gamma_1 \frac{1}{(\tau+1)} \frac{v_1}{v_0}, \\ \gamma_{n+1}^P = \frac{n(n+1)(n+\tau)(\tau+n-1)}{(\tau+2n+1)(\tau+2n-1)(\tau+2n)^2} \frac{v_{n-1}v_{n+1}}{v_n^2}, \quad n \geq 1. \end{array} \right.$$

Where $\tau+n \neq 0$ and $v_n \neq 0$, $n \geq 0$.

Remark Note that in all the previous cases $\{P_n\}_{n \geq 0}$ is a nonsymmetric sequence since we have $\beta_0^P \neq 0$.

4.2 Functional equation

The quadratic decomposition of the sequence $\{B_n\}_{n \geq 0}$ allows us to find a new semi-classical sequence of polynomials.

PROPOSITION 4.1 *The functional σu is semi-classical and satisfies*

$$(\Phi^P \sigma u)' + \Psi^P \sigma u = 0 \quad (51)$$

where

$$\Phi^P(x) = x^3 - x^2, \quad \Psi^P(x) = -(\tau+1)x^2 + (\theta+1)x. \quad (52)$$

Proof Multiplying (22) by x , we get

$$((x^5 - x^3)u)' + (-2(\tau + 1)x^4 + 2(\theta + 1)x^2)u = 0.$$

Applying the operator σ , we obtain

$$\langle \sigma[(x^5 - x^3)u]' + (-2(\tau + 1)x^4 + 2(\theta + 1)x^2)u, x^n \rangle = 0, \quad n \geq 0.$$

Hence

$$-2n \langle u, x^{2n+4} \rangle + 2n \langle u, x^{2n+2} \rangle - 2(\tau + 1) \langle u, x^{2n+4} \rangle + 2(\theta + 1) \langle u, x^{2n+2} \rangle = 0, \quad n \geq 0.$$

Therefore,

$$\langle ((x^3 - x^2)\sigma u)' + (-2(\tau + 1)x^2 + 2(\theta + 1)x)\sigma u, x^n \rangle = 0, \quad n \geq 0.$$

Which gives (51) with (52) and completes the proof. ■

Let's denote by

$$\tilde{u} = (h_2 \circ \tau_{-(1/2)})\sigma u \quad (53)$$

and $\{\tilde{P}_n\}_{n \geq 0}$ the (MOPS) corresponding to $\tilde{u}$. The recurrence coefficients of such sequence can be obtained by application of (15).

According to (18) and (19), the form $\tilde{u}$ satisfies:

$$(\tilde{\Phi}\tilde{u})' + \tilde{\Psi}\tilde{u} = 0 \quad (54)$$

where

$$\tilde{\Phi} = (x + 1)^2(x - 1), \quad \tilde{\Psi} = (x + 1)(-(\tau + 1)(x + 1) + 2(\theta + 1)). \quad (55)$$

Therefore the class of $\tilde{u}$ is $\tilde{s} \leq 1$. Let's see when functional equation (54) can be simplified by $x + 1$.

We have

$$\begin{aligned} \langle \tilde{u}, \theta_{-1}\tilde{\Psi} + \theta_{-1}^2\tilde{\Phi} \rangle &= \langle (h_2 \circ \tau_{-1/2})\sigma u, -\tau x + 2\theta - \tau \rangle \quad (\text{by equations (53) and (55)}) \\ &= \langle u, -2\tau x^2 + 2\theta \rangle \quad (\text{by equations (3)–(6), (9) and (10)}) \\ &= -2\tau(u)_2 + 2\theta \\ &= -2\tau\gamma_1 + 2\theta. \end{aligned}$$

Consequently, if $\tau \neq 0$ and $\gamma_1 = \theta/\tau$ then $\tilde{s} = 0$ and (54) becomes after simplification by $x + 1$

$$((x^2 - 1)\tilde{u})' + (-\tau x + 2\theta)\tilde{u} = 0.$$

In this case, we have

$$\tilde{u} = J\left(\frac{1}{2}\tau - \theta - 1, \frac{1}{2}\tau + \theta - 1\right)$$

where $J(\alpha, \beta)$ is the Jacobi linear functional characterized by [14–16]

$$((x^2 - 1)J(\alpha, \beta))' + (-(\alpha + \beta + 2)x + \beta - \alpha)J(\alpha, \beta) = 0.$$

5. Conclusion

The sequence $\{\tilde{P}_n\}_{n \geq 0}$ is a nonsymmetric semi-classical sequence of polynomials of class $\tilde{s} \leq 1$. This sequence generalizes the Jacobi polynomial sequence, that is it depends on three parameters γ_1 , τ and θ in such a way that for $\tau \neq 0$ and $\gamma_1 = \theta/\tau$ one has the well-known Jacobi polynomials.

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Explicit representations of some orthogonal polynomials

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In this paper, polynomials orthogonal on $[-1, 1]$ with respect to either $x^{2q+1}(1-x)^k(1-x^2)^\alpha$, q integer, $k = 1, 2$, or $|x|^\mu(1-x)^k(1-x^2)^\alpha$, $k = 1, 2, 3$, are written as simple summations. In these cases, even with the use of the Christoffel's formula, it is very difficult to give simple summations because the elements in this Christoffel determinant demand to compute the value of generalized Gegenbauer polynomial at $x = 0$ and the values of $(k-1)$ derivatives of generalized Gegenbauer polynomial at $x = +1$, and does not directly give explicit representation of polynomials function of n . For the applications, the second weight has interest in Birth–Death process, and, in general, these summations can be applied in many fields, specially, in the following areas - Generating functions - Localization of zeros - Theory of approximations.

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1. Introduction

The generalized Gegenbauer (Ultraspherical) polynomials [1], $p_n^{(\alpha, \mu)}(x)$, orthogonal on $[-1, 1]$ with respect to $|x|^\mu(1-x^2)^\alpha$ can be given in terms of Jacobi polynomials as in Szego's book, and these polynomials have an explicit representation as a ${}_2F_1$ hypergeometric series

$$p_{2n}^{(\alpha, \mu)}(x) = x^{2n} {}_2F_1 \left(\frac{-2n - \mu + 1}{2}, -n; \frac{-\mu - 4n + 1 - 2\alpha}{2}; \frac{1}{x^2} \right), \quad (1)$$

which we can write as

$$p_{2n}^{(\alpha, \mu)}(x) = \frac{2^{-2n} \Gamma((-2n+1)/2)(2n)!}{\Gamma(-n - (\mu/2) + 1/2) \Gamma(\alpha + 1/2 + 2n + \mu/2)} \times \sum_{k=0}^n \frac{(-1)^k \Gamma((4n-2k+2\alpha+1+\mu)/2) \Gamma((2k+1-2n-\mu)/2) (2x)^{2n-2k}}{\Gamma((2k+1-2n)/2) (2n-2k)! k!} \quad (2)$$

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and

$$p_{2n+1}^{(\alpha, \mu)}(x) = x^{2n+1} {}_2F_1 \left(\frac{-2n - \mu - 1}{2}, -n; \frac{-\mu - 4n - 1 - 2\alpha}{2}; \frac{1}{x^2} \right), \quad (3)$$

which can be written as

$$p_{2n+1}^{(\alpha, \mu)}(x) = \frac{2^{-2n-1} \Gamma((-2n-1)/2)(2n+1)!}{\Gamma(-n - \mu/2 - 1/2) \Gamma(\alpha + 3/2 + 2n + \mu/2)} \times \sum_{k=0}^n \frac{(-1)^k \Gamma((4n-2k+2\alpha+3+\mu)/2) \Gamma((2k-1-2n-\mu)/2)(2x)^{2n+1-2k}}{\Gamma(k-1/2-n)(2n+1-2k)!k!}. \quad (4)$$

Thus, for polynomials orthogonal with respect to $(1-x)^k |x|^\mu (1-x^2)^\alpha$, $k = 1, 2, 3$, [2] the Christoffel's formula can be used, but it is very difficult to give a simple summation using the following determinants

$$(1-x)q_n(x) = \begin{vmatrix} p_n^{(\alpha, \mu)}(x) & p_{n+1}^{(\alpha, \mu)}(x) \\ p_n^{(\alpha, \mu)}(1) & p_{n+1}^{(\alpha, \mu)}(1) \end{vmatrix},$$

$$(1-x)^2 q_n(x) = \begin{vmatrix} p_n^{(\alpha, \mu)}(x) & p_{n+1}^{(\alpha, \mu)}(x) & p_{n+2}^{(\alpha, \mu)}(x) \\ p_n^{(\alpha, \mu)}(1) & p_{n+1}^{(\alpha, \mu)}(1) & p_{n+2}^{(\alpha, \mu)}(1) \\ (p_n^{(\alpha, \mu)})'(1) & (p_{n+1}^{(\alpha, \mu)})'(1) & (p_{n+2}^{(\alpha, \mu)})'(1) \end{vmatrix},$$

and

$$(1-x)^3 q_n(x) = \begin{vmatrix} p_n^{(\alpha, \mu)}(x) & p_{n+1}^{(\alpha, \mu)}(x) & p_{n+2}^{(\alpha, \mu)}(x) & p_{n+3}^{(\alpha, \mu)}(x) \\ p_n^{(\alpha, \mu)}(1) & p_{n+1}^{(\alpha, \mu)}(1) & p_{n+2}^{(\alpha, \mu)}(1) & p_{n+3}^{(\alpha, \mu)}(1) \\ (p_n^{(\alpha, \mu)})'(1) & (p_{n+1}^{(\alpha, \mu)})'(1) & (p_{n+2}^{(\alpha, \mu)})'(1) & (p_{n+3}^{(\alpha, \mu)})'(1) \\ (p_n^{(\alpha, \mu)})''(1) & (p_{n+1}^{(\alpha, \mu)})''(1) & (p_{n+2}^{(\alpha, \mu)})''(1) & (p_{n+3}^{(\alpha, \mu)})''(1) \end{vmatrix},$$

where $p_n^{(\alpha, \mu)}(x)$ are given by equations (2) and (4), because the elements in this Christoffel determinant pose to compute the values of the $(k-1)$ derivatives of generalized Gegenbauer polynomial at $x = +1$, and does not directly give explicit representation of polynomials function of n . This is possible here, using the induction that we are able to give very simple summations.

On the other hand, for polynomials orthogonal on $[-1, 1]$ with respect to $x^{2q+1}(1-x)^k(1-x^2)^\alpha$, q integer, $k = 1, 2$ [3], which we write as $x(1-x)^k x^{2q}(1-x^2)^\alpha$ it is also very difficult to give a simple summation because the elements in Christoffel determinants pose to compute not only the values of the $(k-1)$ derivatives of generalized Gegenbauer polynomial at $x = +1$ but also the value of generalized Gegenbauer polynomial at $x = 0$. In all these cases, simple summations will be given.

2. General features

The *Pochhammer* symbol or shifted factorial is defined by

$$(a)_0 = 1, \quad (a)_n = a(a+1)(a+2) \dots (a+n-1), \quad n \geq 0, \quad (5)$$

which can be written as $(a)_n = \Gamma(a+n)/\Gamma(a)$, $n \geq 0$. This is very useful in the notation for hypergeometric functions

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \dots \quad |z| < 1. \quad (6)$$

When a or b are non-positive integers the series (6) terminates, and F reduces to a polynomial. In fact, taking into account

$$(-m)_n = \begin{cases} 0, & \text{if } n > m \\ (-1)^n \frac{m!}{(m-n)!}, & \text{if } n \leq m, \end{cases}$$

we have

$$F(-m, b; c; z) = \sum_{n=0}^m \frac{(-m)_n(b)_n}{(c)_n n!} z^n = \sum_{n=0}^m \frac{(-1)^n \binom{m}{n} (b)_n}{(c)_n} z^n. \quad (7)$$

We are looking for properties of some class of monic polynomials, which are orthogonal on $[-1, +1]$ with respect to the positive weight

$$|x|^\mu (1-x^2)^\alpha (1-x)^k, \quad \mu > -1, \quad \alpha > -1, \quad k = 1, 2, 3,$$

and monic polynomials, which are orthogonal on $[-1, +1]$ with respect to the non-positive weight

$$x^{2q+1} (1-x^2)^\alpha (1-x)^k, \quad q \text{ integer}, \quad \alpha > -1, \quad k = 1, 2.$$

We are mainly interested to use the explicit recurrence coefficients given in [2–4] β_n, γ_{n+1} , $n \geq 0$ in the recurrence relation

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x - \beta_0, \\ p_{n+2}(x) &= (x - \beta_{n+1})p_{n+1}(x) - \gamma_{n+1}p_n(x), \quad n \geq 0, \end{aligned} \quad (8)$$

to give simple summation of these polynomials.

3. Definite positive case

In this section, polynomials orthogonal with respect $(1-x)^k |x|^\mu (1-x^2)^\alpha$, $k = 1, 2, 3$ are given as summation.

3.1 Case $k = 1$

In [1] and [2], we already gave explicitly the recurrence coefficients in the three term recurrence relation (8) fulfilled by polynomials orthogonal on $[-1, 1]$ with respect to $(1-x)|x|^\mu (1-x^2)^\alpha$. The results obtained were ($n \geq 0$)

$$\begin{aligned} \beta_0 &= \frac{\mu + 1}{\mu + 2\alpha + 3}, \quad \beta_{n+1} = (-1)^{n+1} \frac{\mu(\mu + 2n + 2\alpha + 4) + (-1)^{n+1}(2\alpha + 1)}{(2n + 2\alpha + 3 + \mu)(2n + 2\alpha + 5 + \mu)} \\ \gamma_{2n+1} &= 2 \frac{(n + \alpha + 1)(2n + 1 + \mu)}{(4n + 2\alpha + 3 + \mu)^2}, \quad \gamma_{2n+2} = \frac{(2n + 2)(2n + 2\alpha + 3 + \mu)}{(4n + 2\alpha + 5 + \mu)^2}, \end{aligned}$$

with $-\mu \neq 2n + 1$, and $-\mu \neq 2n + 2\alpha + 3$.

With these expressions, we immediately have the explicit representation for their corresponding polynomials

PROPOSITION 1

$$p_{2n+1}(x) = x^{2n+1} + \sum_{k=0}^{n-1} \frac{(-1)^{k+1} \Gamma((2n+3+\mu)/2) \Gamma(((4n+3+2\alpha+\mu)/2) - k)}{\Gamma((2n+1+\mu)/2 - k) \Gamma((4n+5+2\alpha+\mu)/2)} \\ \times \left(\binom{n}{k} x^{2n-2k} + \binom{n}{k+1} x^{2n-2k-1} \right) \\ + \frac{(-1)^{n+1} \Gamma((2n+3+\mu)/2) \Gamma((2n+3+2\alpha+\mu)/2)}{\Gamma(1+\mu/2) \Gamma(4n+5+2\alpha+\mu/2)}, \quad n \geq 0.$$

Which we write

$$p_{2n+1}(x) = x^{2n+1} {}_2F_1 \left(-\frac{2n+\mu+1}{2}, -n; -\frac{2\alpha+4n+\mu+3}{2}; \frac{1}{x^2} \right) \\ - \frac{(2n+\mu+1)x^{2n}}{(4n+2\alpha+\mu+3)} {}_2F_1 \left(-\frac{2n+\mu-1}{2}, -n; -\frac{2\alpha+4n+\mu+1}{2}; \frac{1}{x^2} \right) \\ p_0(x) = 1, \quad p_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k \Gamma((2n+1+\mu)/2) \Gamma(((4n+1+2\alpha+\mu)/2) - k)}{\Gamma((2n+1+\mu)/2 - k) \Gamma((4n+3+2\alpha+\mu)/2)} \\ \times \left(\left(\frac{4n+1+2\alpha+\mu}{2} - k \right) \binom{n}{k} x^{2n-2k} \right. \\ \left. - (k+1) \binom{n}{k+1} x^{2n-2k-1} \right), \quad n \geq 1.$$

which we write

$$p_{2n}(x) = x^{2n} {}_2F_1 \left(-\frac{2n+\mu-1}{2}, -n; -\frac{2\alpha+4n+\mu+1}{2}; \frac{1}{x^2} \right) \\ - \frac{2nx^{2n-1}}{4n+2\alpha+\mu+1} {}_2F_1 \left(-\frac{2n+\mu-1}{2}, -n+1; -\frac{2\alpha+4n+\mu-1}{2}; \frac{1}{x^2} \right)$$

Proof

Step 1 Computing by iterations from the three term recurrence relation (8): $p_1(x)$, $p_2(x)$, $p_3(x)$, $\dots$, and writing $p_1(x)$, $p_3(x)$, $p_5(x)$, $\dots$, together,

$$p_1(x) = x - \frac{(\mu+1)}{(3+2\alpha+\mu)}, \\ p_3(x) = x^3 - \frac{(3+\mu)}{(7+2\alpha+\mu)} x^2 - \frac{(3+\mu)}{(7+2\alpha+\mu)} x + \frac{(1+\mu)(3+\mu)}{(5+2\alpha+\mu)(7+2\alpha+\mu)}, \\ p_5(x) = x^5 - \frac{(5+\mu)}{(11+2\alpha+\mu)} x^4 - \frac{2(5+\mu)}{(11+2\alpha+\mu)} x^3 + \frac{2(3+\mu)(5+\mu)}{(9+2\alpha+\mu)(11+2\alpha+\mu)} x^2 \\ + \frac{(3+\mu)(5+\mu)}{(9+2\alpha+\mu)(11+2\alpha+\mu)} x - \frac{(1+\mu)(3+\mu)(5+\mu)}{(7+2\alpha+\mu)(9+2\alpha+\mu)(11+2\alpha+\mu)},$$

and writing $p_2(x)$, $p_4(x)$, $p_6(x) \dots$ together,

$$\begin{aligned}
 p_2(x) &= x^2 - \frac{2}{(5+2\alpha+\mu)}x - \frac{(1+\mu)}{(5+2\alpha+\mu)}, \\
 p_4(x) &= x^4 - \frac{4}{(9+2\alpha+\mu)}x^3 - \frac{2(3+\mu)}{(9+2\alpha+\mu)}x^2 \\
 &\quad + \frac{4(3+\mu)}{(7+2\alpha+\mu)(9+2\alpha+\mu)}x + \frac{(1+\mu)(3+\mu)}{(7+2\alpha+\mu)(9+2\alpha+\mu)}, \\
 p_6(x) &= x^6 - \frac{6}{(13+2\alpha+\mu)}x^5 - \frac{3(5+\mu)}{(13+2\alpha+\mu)}x^4 \\
 &\quad + \frac{12(5+\mu)}{(11+2\alpha+\mu)(13+2\alpha+\mu)}x^3 + \frac{3(3+\mu)(5+\mu)}{(11+2\alpha+\mu)(13+2\alpha+\mu)}x^2 \\
 &\quad - \frac{6(3+\mu)(5+\mu)}{(9+2\alpha+\mu)(11+2\alpha+\mu)(13+2\alpha+\mu)}x \\
 &\quad - \frac{(1+\mu)(3+\mu)(5+\mu)}{(9+2\alpha+\mu)(11+2\alpha+\mu)(13+2\alpha+\mu)}
 \end{aligned}$$

we get, by induction, the expression of sums written in Proposition 1.

Step 2 We, also, use the three term recurrence relation (8) to prove by recurrence the result, found by induction, in the step.

We prove both, $p_{2n+2}(x) = (x - \beta_{2n+1})p_{2n+1}(x) - \gamma_{2n+1}p_{2n}(x)$ and $p_{2n+1}(x) = (x - \beta_{2n})p_{2n}(x) - \gamma_{2n}p_{2n-1}(x)$ by analysing, in each case, two terms

Odd term The term $x^{2n-2k+1}$ in these previous sums $(x - \beta_{2n+1})p_{2n+1}(x) - \gamma_{2n+1}p_{2n}(x)$ is found only in

$$\begin{aligned}
 &(x - \beta_{2n+1}) \frac{(-1)^{k+1} \Gamma((2n+3+\mu)/2) \Gamma(((4n+3+2\alpha+\mu)/2) - k)}{\Gamma(((2n+1+\mu)/2) - k) \Gamma((4n+5+2\alpha+\mu)/2)} \\
 &\quad \times \left(\binom{n}{k} x^{2n-2k} + \binom{n}{k+1} x^{2n-2k-1} \right) + (x - \beta_{2n+1}) \\
 &\quad \times \frac{(-1)^{(k-1)+1} \Gamma((2n+3+\mu)/2) \Gamma(((4n+3+2\alpha+\mu)/2) - (k-1))}{\Gamma(((2n+1+\mu)/2) - (k-1)) \Gamma((4n+5+2\alpha+\mu)/2)} \\
 &\quad \times \left(\binom{n}{k+1} x^{2n-2(k-1)} + \binom{n}{(k-1)+1} x^{2n-2(k-1)-1} \right) \\
 &\quad - \gamma_{2n+1} \frac{(-1)^k \Gamma((2n+1+\mu)/2) \Gamma(((4n+1+2\alpha+\mu)/2) - k)}{\Gamma(((2n+1+\mu)/2) - k) \Gamma((4n+3+2\alpha+\mu)/2)} \\
 &\quad \times \left(\left(\frac{4n+1+2\alpha+\mu}{2} - k \right) \binom{n}{k} x^{2n-2k} - (k+1) \binom{n}{k+1} x^{2n-2k-1} \right).
 \end{aligned}$$

The term $x^{2n-2k+1}$ in $(x - \beta_{2n+1})p_{2n+1}(x) - \gamma_{2n+1}p_{2n}(x)$ is given, exactly, by

$$\begin{aligned} & \frac{(-1)^{k+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+3+2\alpha+\mu)/2)-k)}{\Gamma((2n+1+\mu)/2-k)\Gamma((4n+5+2\alpha+\mu)/2)} \binom{n}{k} x^{2n-2k+1} \\ & - \beta_{2n+1} \frac{(-1)^{(k-1)+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+3+2\alpha+\mu)/2)-(k-1))}{\Gamma(((2n+1+\mu)/2)-(k-1))\Gamma((4n+5+2\alpha+\mu)/2)} \\ & \times \binom{n}{k} x^{2n-2k+1} \\ & - \gamma_{2n+1} \frac{(-1)^{k-1}\Gamma((2n+1+\mu)/2)\Gamma(((4n+1+2\alpha+\mu)/2)-(k-1))}{\Gamma(((2n+1+\mu)/2)-(k-1))\Gamma((4n+3+2\alpha+\mu)/2)} \\ & \times \left(\left(\frac{4n+1+2\alpha+\mu}{2} - (k-1) \right) \binom{n}{k-1} x^{2n-2(k-1)} \right. \\ & \left. - ((k-1)+1) \binom{n}{k-1+1} x^{2n-2(k-1)-1} \right) \end{aligned}$$

Replacing β_{2n+1} and γ_{2n+1} by their values and using, not only $\Gamma(x+1) = x\Gamma(x)$, $x > 0$ but also $\binom{n}{k} = (k+1)/(n+1) \binom{n+1}{k+1}$, we obtain

$$= - \frac{(-1)^k \Gamma((2n+3+\mu)/2) \Gamma(((4n+5+2\alpha+\mu)/2)-k)}{\Gamma(((2n+3+\mu)/2)-k) \Gamma(((4n+7+2\alpha+\mu)/2))} (k+1) \binom{n+1}{k+1} x^{2n-2k+1}$$

and this coincides with the coefficient of the term $x^{2n-2k+1}$ in $p_{2n+2}(x)$, in fact

$$\begin{aligned} p_{2n+2}(x) &= \sum_{k=0}^{n+1} \frac{(-1)^k \Gamma((2(n+1)+1+\mu)/2) \Gamma(((4(n+1)+1+2\alpha+\mu)/2)-k)}{\Gamma(((2(n+1)+1+\mu)/2)-k) \Gamma((4(n+1)+3+2\alpha+\mu)/2)} \\ & \times \left(\left(\frac{4(n+1)+1+2\alpha+\mu}{2} - k \right) \binom{n+1}{k} x^{2(n+1)-2k} \right. \\ & \left. - (k+1) \binom{n+1}{k+1} x^{2(n+1)-2k-1} \right) \end{aligned}$$

and the coefficient of the term $x^{2n-2k+1}$ in $p_{2n+2}(x)$ is

$$\begin{aligned} & - \frac{(-1)^k \Gamma((2(n+1)+1+\mu)/2) \Gamma(((4(n+1)+1+2\alpha+\mu)/2)-k)}{\Gamma(((2(n+1)+1+\mu)/2)-k) \Gamma((4(n+1)+3+2\alpha+\mu)/2)} \\ & \times (k+1) \binom{n+1}{k+1} x^{2(n+1)-2k-1} \end{aligned}$$

Even term The term x^{2n-2k} in these previous sums

$(x - \beta_{2n+1})p_{2n+1}(x) - \gamma_{2n+1}p_{2n}(x)$ is found only in

$$\begin{aligned} & (x - \beta_{2n+1}) \frac{(-1)^{k+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+3+2\alpha+\mu)/2)-k)}{\Gamma(((2n+1+\mu)/2)-k)\Gamma((4n+5+2\alpha+\mu)/2)} \binom{n}{k} x^{2n-2k} \\ & + \binom{n}{k+1} x^{2n-2k-1} - \gamma_{2n+1} \frac{(-1)^k \Gamma((2n+1+\mu)/2) \Gamma(((4n+1+2\alpha+\mu)/2)-k)}{\Gamma(((2n+1+\mu)/2)-k) \Gamma((4n+3+2\alpha+\mu)/2)} \\ & \times \left(\left(\frac{4n+1+2\alpha+\mu}{2} - k \right) \binom{n}{k} x^{2n-2k} - (k+1) \binom{n}{k+1} x^{2n-2k-1} \right). \end{aligned}$$

The term x^{2n-2k} in $(x - \beta_{2n+1})p_{2n+1}(x) - \gamma_{2n+1}p_{2n}(x)$ is

$$\begin{aligned} & \frac{(-1)^{k+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+3+2\alpha+\mu)/2)-k)}{\Gamma(((2n+1+\mu)/2)-k)\Gamma((4n+5+2\alpha+\mu)/2)} \binom{n}{k+1} x^{2n-2k} \\ & - \beta_{2n+1} \frac{(-1)^{k+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+3+2\alpha+\mu)/2)-k)}{\Gamma(((2n+1+\mu)/2)-k)\Gamma((4n+5+2\alpha+\mu)/2)} \binom{n}{k} x^{2n-2k} \\ & - \gamma_{2n+1} \frac{(-1)^k\Gamma((2n+1+\mu)/2)\Gamma(((4n+1+2\alpha+\mu)/2)-k)}{\Gamma(((2n+1+\mu)/2)-k)\Gamma((4n+3+2\alpha+\mu)/2)} \\ & \times \left(\frac{4n+1+2\alpha+\mu}{2} - k \right) \binom{n}{k} x^{2n-2k}. \end{aligned}$$

Replacing β_{2n+1} and γ_{2n+1} by their values and using then, we obtain

$$\begin{aligned} & \frac{(-1)^{k+1}\Gamma((2n+3+\mu)/2)\Gamma(((4n+5+2\alpha+\mu)/2)-(k+1))}{\Gamma(((2n+3+\mu)/2)-(k+1))\Gamma((4n+7+2\alpha+\mu)/2)} \\ & \times \left(\frac{4n+5+2\alpha+\mu}{2} - (k+1) \right) \binom{n+1}{k+1} x^{2(n+1)-2(k+1)}, \end{aligned}$$

which is the coefficient of The term x^{2n-2k} in $p_{2n+2}(x)$ (by shifting in $p_{2n}(x)$, $n \rightarrow n+1$ and $k \rightarrow k+1$).

By the same way we prove $p_{2n+1}(x) = (x - \beta_{2n})p_{2n}(x) - \gamma_{2n}p_{2n-1}(x)$ by analysing the two terms. ■

3.2 Case $k = 2$

In [2], we already gave explicitly the recurrence coefficients in the three term recurrence relation fulfilled by polynomials orthogonal on $[-1, 1]$ with respect to $(1-x)^2|x|^\mu(1-x^2)^\alpha$. The results obtained were :

$$\begin{aligned} \beta_0 &= \frac{\mu+1}{\alpha+2+\mu}, \quad \beta_{2n} = \frac{(\mu+1)(\alpha+1)}{(2n+\alpha+1)(2n+\alpha+2+\mu)}, \quad n \geq 1, \\ \beta_{2n+1} &= \frac{(-\mu+1)(\alpha+1)}{(2n+\alpha+2+\mu)(2n+\alpha+3)}, \quad n \geq 0, \\ \gamma_{2n+1} &= \frac{(\mu+2n+1)(\mu+2\alpha+2n+3)(2n+\alpha+1)(2n+\alpha+3)}{(2n+\alpha+2+\mu)^2(\mu+4n+2\alpha+3)(\mu+4n+2\alpha+5)}, \quad n \geq 0, \\ \gamma_{2n+2} &= 4(n+1) \frac{(\mu+\alpha+2n+2)(\mu+\alpha+2n+4)(n+\alpha+2)}{(2n+\alpha+3)^2(\mu+4n+2\alpha+5)(\mu+4n+2\alpha+7)}, \quad n \geq 0. \end{aligned}$$

we must assume that $\mu \neq 2n+1$, $\mu \neq 2\alpha+2n+3$, $\mu \neq \alpha+2n+2$, $\alpha \neq -n-1$ and $n \geq 0$, in order to have regularity ($\gamma_n \neq 0$, $n \geq 1$).

With these expressions, we immediately have the explicit representation for their corresponding polynomials

PROPOSITION 2

$$\begin{aligned}
 p_1(x) &= x - \frac{(\mu + 1)}{(2 + \alpha + \mu)}, \quad p_{2n+1}(x) = \frac{\Gamma((2n + 3 + \mu)/2)}{(2n + 2 + \alpha + \mu)\Gamma((4n + 5 + 2\alpha + \mu)/2)} \\
 &\times \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} \Gamma(((4n + 5 + 2\alpha + \mu)/2) - k)}{\Gamma(((2n + 3 + \mu)/2) - k)} \\
 &\times ((2n + \alpha + \mu - 2k + 2)x^{2n+1-2k} - (2n + 1 + \mu - 2k)x^{2n-2k}), \quad n \geq 1
 \end{aligned}$$

which we write

$$\begin{aligned}
 p_{2n+1}(x) &= x^{2n+1} {}_2F_1 \left(-n, -\frac{2n + \mu + 1}{2}; -\frac{2\alpha + \mu + 3 + 4n}{2}; \frac{1}{x^2} \right) \\
 &+ x \frac{2n - 1_{2n(2n+\mu+1)}}{(2\alpha + 3 + \mu + 4n)(2n + 2 + \alpha + \mu) {}_2F_1(-n + 1, (-(2n + \mu - 1)/2); \\
 &\quad (-(2\alpha + \mu + 1 + 4n)/2); (1/x^2))} \\
 &- \frac{(2n + 1 + \mu)x^{2n}}{(2n + 2 + \alpha + \mu)} {}_2F_1 \left(-n, -\frac{2n + \mu - 1}{2}; -\frac{2\alpha + \mu + 3 + 4n}{2}; \frac{1}{x^2} \right)
 \end{aligned}$$

and the even part is given by

$$\begin{aligned}
 p_0(x) &= 1, \quad p_{2n}(x) = \frac{\Gamma((2n + 1 + \mu)/2)}{(2n + \alpha + 1)\Gamma((4n + 3 + 2\alpha + \mu)/2)} \\
 &\times \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} \Gamma(((4n + 3 + 2\alpha + \mu)/2) - k)}{\Gamma(((2n + 3 + \mu)/2) - k)} \\
 &\times ((2n + \alpha + \mu - 2k + 1)x^{2n-2k} - 2(n - k)x^{2n-2k-1}), \quad n \geq 1
 \end{aligned}$$

Which we write

$$\begin{aligned}
 p_{2n}(x) &= x^{2n} {}_2F_1 \left(-n, -\frac{2n + \mu - 1}{2}; -\frac{2\alpha + \mu + 1 + 4n}{2}; \frac{1}{x^2} \right) \\
 &+ 2n \left(\frac{(\mu - 1 + 2n)x^{2n-2}}{(2n + \alpha + 1)(2\alpha + 1 + \mu + 4n) {}_2F_1(-n + 1, (-(2n + \mu - 3)/2); \\
 &\quad (-(2\alpha + \mu - 1 + 4n)/2); (1/x^2))} \right. \\
 &\left. - \frac{2nx^{2n-1}}{(2n + 1 + \alpha)} {}_2F_1 \left(-n + 1, -\frac{2n + \mu - 1}{2}; -\frac{2\alpha + \mu + 1 + 4n}{2}; \frac{1}{x^2} \right) \right)
 \end{aligned}$$

Proof Exactly the same proof as done for proposition 1. ■

3.3 Case $k = 3$

In [2], we already gave explicitly the recurrence coefficients in the three term recurrence relation fulfilled by polynomials orthogonal on $[-1, 1]$ with respect to $(1 - x)^3|x|^\mu(1 - x^2)^\alpha$.

Using the notation $E_n = \sum_{k=0}^n \beta_k$, the results obtained were

$$\begin{aligned} E_{2n+1} &= -2(n+1) \frac{2\mu^2 + (4\alpha + 10n + 16)\mu + 3(2n + \alpha + 3)(2n + \alpha + 4)}{(\mu + 4n + 2\alpha + 7)(2(n+1)\mu + (2n + \alpha + 3)(2n + \alpha + 4))}, \\ E_{2n} &= -\frac{(2n+1+\mu)(3(2n+\alpha+2)(2n+\alpha+3) + 2\mu(n+\alpha+2))}{(4n+2\alpha+5+\mu)((2n+\alpha+2)(2n+\alpha+3) + 2\mu(n+\alpha+2))}, \\ \gamma_{2n+1} &= (2n+2\alpha+4)(2n+1+\mu) \frac{(n+2\alpha+3+\mu-\tau_-)}{(4n+2\alpha+5+\mu)^2} \\ &\quad \times \frac{(n+2\alpha+3+\mu+\tau_-)(n+2\alpha+7+\mu-\tau_-)(n+2\alpha+7+\mu+\tau_-)}{(n+2\alpha+5+\mu-\tau_+)^2(n+2\alpha+5+\mu+\tau_+)^2}, \end{aligned}$$

and for $n \geq 1$,

$$\begin{aligned} \gamma_{2n} &= 2n(2n+2\alpha+3+\mu) \frac{(n+2\alpha+1+\mu-\tau_+)}{(4n+2\alpha+3+\mu)^2} \\ &\quad \times \frac{(n+2\alpha+1+\mu+\tau_+)(n+2\alpha+5+\mu-\tau_+)(n+2\alpha+5+\mu+\tau_+)}{(n+2\alpha+3+\mu-\tau_-)^2(n+2\alpha+3+\mu+\tau_-)^2}, \end{aligned}$$

where we factorize the numerator and denominator of γ_n as first degree polynomial in n introducing:

$$\tau_+ = \sqrt{\mu^2 - (4\alpha + 6)\mu + 1}, \quad \tau_- = \sqrt{\mu^2 + (4\alpha + 6)\mu + 1}$$

in order to control the regularity of the γ_n , which happened when $\gamma_n \neq 0$.

With these expressions we immediately have the explicit representation for their corresponding polynomials

PROPOSITION 3

$$p_0(x) = 1, \quad p_1(x) = x + \frac{(2\mu + 9 + 3\alpha)(\mu + 1)}{(2\mu + 3 + \alpha)(5 + 2\alpha + \mu)},$$

and for $n \geq 1$, we have

$$\begin{aligned} p_{2n}(x) &= \sum_{k=0}^n \frac{\binom{n}{k} (-1)^k \Gamma((\mu/2) + 2n + (3/2) - k + \alpha) \Gamma((\mu/2) + n + (1/2))}{(2n\mu + (2n+1+\alpha)(2n+2+\alpha)) \Gamma((\mu/2) + 2n + (5/2) + \alpha)} \\ &\quad \times \Gamma((\mu/2) + n + (1/2) - k) \\ &\quad \times \left(\left(\frac{\mu}{2} + 2n + \frac{3}{2} - k + \alpha \right) ((2n+2+\alpha)(\alpha+2n-3)) \right. \\ &\quad + (2n-4)\mu - 4(k-1)(\mu+2n+\alpha+2)) x^{2n-2k} \\ &\quad + (n-k)(2\mu^2 + (4\alpha+6+10n-4k)\mu \\ &\quad \left. + (2n+2+\alpha)(3\alpha+6n+3-4k)) x^{2n-2k-1} \right), \end{aligned}$$

$$\begin{aligned}
p_{2n+1}(x) &= \frac{\Gamma((\mu/2) + n + (3/2))}{\Gamma((\mu/2) + 2n + (7/2) + \alpha)} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\Gamma((\mu/2) + 2n + (5/2) - k + \alpha)}{\Gamma((\mu/2) + n + (1/2) - k)} \\
&\quad \times \left(\frac{((\mu/2) + 2n + (5/2) - k + \alpha)((2n + \alpha + 3))}{((\mu/2) + n + (1/2) - k)(2\mu(n + \alpha + 2))} x^{2n-2k+1} \right. \\
&\quad \left. + \frac{(3(2n + \alpha + 2)(2n + \alpha + 3) - 4k\alpha - 4(2n + 3)k + 2\mu(n + \alpha + 2))}{(2\mu(n + \alpha + 2) + (2n + \alpha + 2)(2n + \alpha + 3))} x^{2n-2k} \right).
\end{aligned}$$

Proof Exactly the same proof as done for proposition 1. ■

4. Non-Definite positive case

In this section, polynomials orthogonal with respect to $(1-x)^k x^{2q+1} (1-x^2)^\alpha$, $k = 1, 2$ are given as summation.

4.1 Case $k = 1$

In [3], we already gave explicitly the recurrence coefficients in the three term recurrence relation fulfilled by polynomials orthogonal on $[-1, 1]$ with respect to $(1-x)x^{2q+1}(1-x^2)^\alpha$. The results obtained were

$$\begin{aligned}
\beta_n &= (-1)^{n+1}, \quad n \geq 0 \\
\gamma_{2n+1} &= -2 \frac{(n + \alpha + 1)(2n + 2\alpha + 2q + 3)}{(4n + 2\alpha + 2q + 3)(4n + 2\alpha + 2q + 5)}, \quad n \geq 0, \\
\gamma_{2n+2} &= - \frac{(2n + 2)(2n + 2q + 3)}{(4n + 2\alpha + 2q + 5)(4n + 2\alpha + 2q + 7)}, \quad n \geq 0
\end{aligned}$$

With these expressions we immediately have the explicit representation for their corresponding polynomials

PROPOSITION 4

$$p_0(x) = 1, \quad p_1(x) = x + 1,$$

and for $n \geq 1$, we have

$$\begin{aligned}
p_{2n+1}(x) &= \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} \Gamma((2q + 2n + 3)/2) \Gamma(((2\alpha + 2q + 4n + 5)/2) - k)}{\Gamma(((2q + 2n + 3)/2) - k) \Gamma(((2\alpha + 2q + 4n + 5)/2))} \\
&\quad \times (x^{2n+1-2k} + x^{2n-2k}), \quad n \geq 1,
\end{aligned}$$

which we can write as

$$p_{2n+1}(x) = (x^{2n+1} + x^{2n}) {}_2F_1 \left(-n, -\frac{2q + 2n + 1}{2}; -\frac{2\alpha + 3 + 4n + 2q}{2}; \frac{1}{x^2} \right), \quad n \geq 0$$

$$p_{2n}(x) = \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} \Gamma((2q+2n+3)/2) \Gamma(((2\alpha+2q+4n+3)/2) - k)}{\Gamma(((2q+2n+3)/2) - k) \Gamma((2\alpha+2q+4n+3)/2)} x^{2n-2k},$$

$$n \geq 1,$$

which we can write as

$$p_{2n}(x) = x^{2n} {}_2F_1 \left(-n, -\frac{2q+2n+1}{2}; -\frac{2\alpha+1+4n+2q}{2}; \frac{1}{x^2} \right), \quad n \geq 0$$

4.2 Case $k = 2$

In [3], we already gave explicitly the recurrence coefficients in the three term recurrence relation fulfilled by polynomials orthogonal on $[-1, 1]$ with respect to $(1-x)^2 x^{2q+1} (1-x^2)^\alpha$. The results obtained were

$$\begin{aligned} \beta_{2n} &= -\frac{4n+\alpha+2q+4}{4n+2\alpha+2q+5} - \frac{2n(2n+2q+1)}{(\alpha+1)(4n+2\alpha+2q+3)}, \quad n \geq 0. \\ \beta_{2n+1} &= \frac{4n+\alpha+2q+4}{4n+2\alpha+2q+5} + \frac{(2n+2)(2n+2q+3)}{(\alpha+1)(4n+2\alpha+2q+7)}, \quad n \geq 0. \\ \gamma_{2n} &= -\frac{4n(2n+2q+1)(n+\alpha+1)(2n+2\alpha+2q+3)}{(\alpha+1)^2(4n+2\alpha+2q+3)^2}, \quad n \geq 1. \\ \gamma_{2n+1} &= -\frac{(\alpha+1)^2}{(4n+2\alpha+2q+5)^2}, \quad n \geq 0 \end{aligned}$$

With these expressions, we immediately have the explicit representation

PROPOSITION 5

$$p_0(x) = 1, \quad p_1(x) = x + \frac{(2q+4+\alpha)}{(2q+5+2\alpha)},$$

and for $n \geq 1$, we have

$$\begin{aligned} p_{2n+1}(x) &= \sum_{k=0}^n \frac{(-1)^k \binom{n}{k} \Gamma((2q+2n+3)/2) \Gamma(((2\alpha+2q+4n+5)/2) - k)}{\Gamma(((2q+2n+3)/2) - k) \Gamma((2\alpha+2q+4n+7)/2)} \\ &\quad \times \left(\left(\frac{2\alpha+2q+4n+5}{2} - k \right) x^{2n+1-2k} + \left(\frac{\alpha+2q+4n+4}{2} - k \right) x^{2n-2k} \right), \quad n \geq 1. \end{aligned}$$

which we write

$$\begin{aligned} p_{2n+1}(x) &= x^{2n+1} {}_2F_1 \left(-n, -n-q-1/2; -\alpha-q-2n-5/2; \frac{1}{x^2} \right) \\ &\quad + \frac{(2\alpha+2q+4n+3)x^{2n}}{2(\alpha+q+2n+5/2)} {}_2F_1 \left(-n, -n-q-1/2; -\alpha-q-2n-3/2; \frac{1}{x^2} \right) \\ &\quad + \frac{n(2q+2n+1)x^{2n-2}}{(\alpha+q+2n+5/2)(2\alpha+2q+4n+3)} {}_2F_1 \left(-n-q+1/2, -n+1; -\alpha-q-2n-1/2; \frac{1}{x^2} \right) \end{aligned}$$

$$p_{2n}(x) = x^{2n} + \sum_{k=1}^n \frac{(-1)^k \binom{n}{k} \Gamma((2q+2n+3)/2) \Gamma((2\alpha+2q+4n+5)/2) - k}{(\alpha+1) \Gamma((2q+2n+3)/2) - k \Gamma((2\alpha+2q+4n+5)/2)} \\ \times \left(2kx^{2n+1-2k} + (\alpha+2k+1)x^{2n-2k} \right), \quad n \geq 0.$$

Which we write

$$p_{2n}(x) = x^{2n} {}_2F_1 \left(-n, -n-q-1/2; -\alpha-q-2n-3/2; \frac{1}{x^2} \right) \\ - \frac{2n(2q+2n+1)x^{2n-2}}{(\alpha+1)(2\alpha+2q+4n+3)} {}_2F_1 \left(-n-q+1/2, -n+1; -\alpha-q-2n-1/2; \frac{1}{x^2} \right) \\ - \frac{2n(q+n+1/2)x^{2n-1}}{(\alpha+1)(\alpha+q+2n+5/2)} {}_2F_1 \left(-n-q+1/2, -n+1; -\alpha-q-2n-1/2; \frac{1}{x^2} \right).$$

Application These summations can be applied in many fields and, specially, in the following areas

- Generating functions.
- Localization of zeros.
- Theory of approximations.

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Chebyshev functions of half-integer order

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The Chebyshev functions of half-integer order are defined. We find that this definition will reduce to that of Chebyshev functions of integer order for $m = 1$.

Keywords: Chebyshev polynomials; Ladder operators

AMS classification: 05E35; 33C47; 33D45

1. Introduction

It is known that the Chebyshev polynomials (CHPs) have wide applications in various fields of mathematics and physics. The CHPs combine useful features of the Fourier series (to which they are related) and the analytic simplicity of polynomials. Since the CHPs are the most efficient polynomials to approximate arbitrary functions, they are widely applied in numerical analysis. There are several possible approaches to the problem of fitting a curve. The minimax approach leads us to the use of the CHPs. The Chebyshev polynomials Type I (CHP-I) has also been used in a quadrature scheme to find the integral of a nonsingular function defined on a finite range [1–4]. From the discrete orthogonality relationship of the CHPs, it is found that various methods to solve linear and nonlinear ordinary differential equations [5, 6] as well as integral differential equations [7] have considerable advantage over finite-difference method. An alternative approach to these problems is given in ref. [8] and most recently in ref. [9], in which the old spectral method was extended, a Chebyshev expansion can be applied to numerically solve complicated nonlinear integral equations encountered in quantum mechanics, field

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theory and fluid dynamics. With respect to CHP-II, one of its applications is in the development of four-dimensional spherical harmonics [10]. In the ‘standard formalism’ to discuss quantum phase problems [11], the Hermitian cosine and sine operators are related to the CHP-II and this relationship can be extended to arbitrary sets of orthogonal polynomials on the interval $[-1, 1]$ as shown in ref. [12]. We should point out that almost all contributions on the CHPs are of integer order. Motivated by our recent work about the ladder operators for the trigonometric functions [13], we attempt to define Chebyshev functions of half-integer order in this work.

The paper is organized as follows. In section 2 we define the Chebyshev functions of half-integer order based on the operators obtained in ref. [13]. Some properties are studied in section 3. Section 4 is devoted to establishing the relationship between the polynomials $\phi_{n+1/2}(x)$ and CHP-II and to evaluating integrals with orthogonal polynomials. Finally, we summarize our conclusions in section 5.

2. Chebyshev functions of half-integer order

As shown in our recent work [13], the ν, μ -dependent differential operators ($\nu, \mu \in \mathbb{R}$)

$$A(\nu, \mu)_\theta \equiv \left(f(\theta) \frac{d}{d\theta} - W(\theta) \right) = \frac{1}{\nu} \sin(\mu\theta) \frac{d}{d\theta} + \cos(\mu\theta); \quad \nu \neq 0 \quad (1)$$

are ladder operators for the trigonometric functions, *i.e.*

$$f_{\nu \pm \mu}(\theta) = A(\nu, \pm \mu) f_\nu(\theta), \quad (2)$$

where

$$f_\nu(\theta) \equiv \begin{cases} \psi_\nu(\theta) = K \sin(\nu\theta), \\ \tilde{\psi}_\nu(\theta) = \tilde{K} \cos(\nu\theta). \end{cases} \quad (3)$$

The functions $\psi_\nu(\theta)$ and $\tilde{\psi}_\nu(\theta)$ are linear independent solutions of differential equation

$$\frac{d^2}{d\theta^2} f_\nu(\theta) + \nu^2 f_\nu(\theta) = 0. \quad (4)$$

In particular, when $\nu \equiv n = \pm 1, \pm 2, \dots$, the functions $f_n(\theta)$ in the interval $[\theta_0, \theta_0 + 2\pi]$ are linear independent solutions of corresponding Sturm–Liouville problem.

In this paper, based on the general ladder operators defined in equation (1), we attempt to study the particular case, where $\nu \equiv n$ and $\mu \equiv m$ are natural numbers. If so, we have

$$A(n, \pm m)_\theta = \pm \frac{1}{n} \sin(m\theta) \frac{d}{d\theta} + \cos(m\theta). \quad (5)$$

If taking

$$x = \cos(m\theta); \quad x \in [-1, 1], \quad (6)$$

then equations (4) and (5) become

$$\left[(1-x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} + \left(\frac{n}{m} \right)^2 \right] f_n(\theta(x)) = 0, \quad (7)$$

$$A(n, \pm m)_x \equiv A(n, \pm m)_{\theta(x)} = \mp \frac{m}{n} (1-x^2) \frac{d}{dx} + x, \quad n \neq 0, \quad (8)$$

which reduce to the Chebyshev equation Type I (CHE-I) and the ladder operators for the CHP-I, respectively for $m = 1$. In this case, it is known that a special solution of equation (7)

is given by [14]

$$T_n(x) \equiv \frac{1}{\tilde{K}} \tilde{\psi}_n(\theta(x)) = \cos(n \cos^{-1}(x)), \quad (9)$$

where equation (6) was used.

We make a remark about $m \neq 1$. If so, the solution $\tilde{\psi}_n(\theta(x))$ of equation (7) must correspond to

$$\tilde{\psi}_n(\theta(x)) \rightarrow \tilde{K} \cos \left[\frac{n}{m} \cos^{-1}(x) \right], \quad (10)$$

from which we may define $T_{n/m}(x)$ as

$$T_{n/m}(x) \equiv \cos \left[\frac{n}{m} \cos^{-1}(x) \right]. \quad (11)$$

It is obvious to show from equations (6) and (11) that

$$T_{1/2}(x) = \left(\frac{x+1}{2} \right)^{1/2}. \quad (12)$$

On the other hand, it is known from equation (2) that

$$A(n, \pm m)_\theta \tilde{\psi}_n(\theta) = \tilde{\psi}_{n \pm m}(\theta), \quad (13)$$

which, in terms of variable x , can be further written as

$$A(n, \pm m)_x T_{n/m}(x) \equiv T_{(n/m) \pm 1}(x), \quad (14)$$

where equations (8) and (11) were used. As a result, the function $T_{n/m}(x)$, which is solution of CHE-I of order n/m , generates the solution of CHE-I of order $(n/m) \pm 1$.

Moreover, from equation (8) we have

$$2x = A(n, m)_x + A(n, -m)_x, \quad (15)$$

from which, together with equation (14), one obtains

$$2x T_{n/m}(x) = T_{(n/m)+1}(x) + T_{(n/m)-1}(x). \quad (16)$$

Obviously, this will reduce to the recurrence relation for the CHP-I ($m = 1$).

3. Properties of the functions $T_{n+(1/2)}(x)$

We proceed to investigate the properties of $T_{n/m}(x)$ with $m = 2$. If $n/2$ in equations (7) and (14) is replaced by $n + 1/2$, then we have

$$\left[(1-x^2) \frac{d^2}{dx^2} - x \frac{d}{dx} + \left(n + \frac{1}{2} \right)^2 \right] T_{n+(1/2)}(x) = 0, \quad (17)$$

$$T_{n+(1/2) \pm 1}(x) = A(2n+1, \pm 2)_x T_{n+(1/2)}(x), \quad (18)$$

where n is, from now on, taken as $n = 0, 1, 2, 3, \dots$

For $n = 0$, equation (17) has a general solution

$$T_{1/2}(x) = C_1(x-1)^{1/2} + C_2(x+1)^{1/2}. \quad (19)$$

Since we know that the CHP-I are defined in the interval $[-1, 1]$ we simply choose $C_1 = 0$ and $C_2 = 1/\sqrt{2}$ which is consistent with equation (12). Moreover, because of the differential operator in equation (17) is even, the function

$$T_{1/2}(-x) \equiv \left(\frac{1-x}{2}\right)^{1/2}, \quad (20)$$

is the other independent solution of equation (17) and our approach could be also used for this function. We note that our definition for $T_{1/2}(x)$ is such that

$$T_{-1/2}(x) = A(1, -2)_x T_{1/2}(x) \equiv T_{1/2}(x). \quad (21)$$

Repeated application of the up ladder operators given in equation (18) on equation (12) leads to the Chebyshev functions of half-integer order

$$T_{n+(1/2)}(x) = \left(\frac{x+1}{2}\right)^{1/2} \phi_{n+(1/2)}(x), \quad (22)$$

where $\phi_{n+1/2}(x)$ are the polynomials with a few special cases

$$\phi_{1/2}(x) \equiv 1, \quad \phi_{3/2}(x) = 2x - 1, \quad \phi_{5/2}(x) = 4x^2 - 2x - 1, \dots \quad (23)$$

When ordered from smallest to largest powers, the triangle of nonzero coefficients is $1; -1, 2; -1, -2, 4; 1, -4, -4, 8; 1, 4, -12, -8, 16; \dots$

It should be noted that the polynomials $\phi_{n+1/2}(x)$ are identified with the CHP-III [15] and can be defined as the determinant

$$\phi_{n+(1/2)}(x) = \begin{vmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 2x & 1 & \ddots & 0 \\ 0 & 1 & 2x & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 1 & 2x \end{vmatrix}_{(n+1) \times (n+1)}. \quad (24)$$

The zeros of $T_{n+(1/2)}(x)$ are given by

$$\xi_j^{(n+(1/2))} = \cos \left[2\theta_j^{(n+(1/2))} \right]; \quad n = 0, 1, 2, 3, \dots, \quad (25)$$

where

$$\theta_j^{(n+(1/2))} = \frac{(2j-1)\pi}{2n+1}; \quad j = 1, 2, \dots, n+1. \quad (26)$$

Also, we have special cases

$$\phi_{n+(1/2)}(x) = \begin{cases} (-1)^n(2n+1), & x = -1, \\ 1, & x = 1, \end{cases} \quad (27)$$

and

$$\phi_{n+(1/2)}(0) = \begin{cases} 1; & n = 0, 3, 4, \dots, \\ -1; & n = 1, 2, 5, 6, \dots \end{cases} \quad (28)$$

The leading term in $\phi_{n+(1/2)}(x)$ is $2^n x^n$. By substituting equation (22) into equation (7), the function $\phi_{n+(1/2)}(x)$ is the solution of the nonself-adjoint second-order differential equation

$$(1-x^2) \frac{d^2}{dx^2} \phi_{n+(1/2)}(x) + (1-2x) \frac{d}{dx} \phi_{n+(1/2)}(x) - \left[\frac{1}{4} - (n+(1/2))^2 \right] \phi_{n+(1/2)}(x) = 0, \quad (29)$$

which is nothing but the Jacobi differential equation with the following solution [16, 17]

$$P_n^{(-1/2, 1/2)}(x) \sim {}_2F_1\left(-n, n+1, \frac{1}{2}; \frac{1-x}{2}\right) \equiv \phi_{n+(1/2)}(x). \quad (30)$$

It is shown that the polynomials $\phi_{n+(1/2)}(x)$ can be expressed by hypergeometric functions.

As a consequence of equation (18) we have

$$\phi_{n+(1/2)\pm 1}(x) = B_{\pm}(n) \phi_{n+(1/2)}(x), \quad (31)$$

where

$$B_{\pm}(n) \equiv \pm \frac{2}{2n+1} \left[(x^2-1) \frac{d}{dx} + x \left(\pm n \pm \frac{1}{2} + \frac{1}{2} \right) - \frac{1}{2} \right], \quad (32)$$

which are the ladder operators for polynomials $\{\phi_{n+(1/2)}(x)\}$ and for Jacobi polynomials, $\{P_n^{(-1/2, 1/2)}(x)\}$ given in equation (30).

One self-adjoint differential equation can be obtained by multiplying equation (29) by $\sqrt{(1+x)/(1-x)}$. Once it is obtained, we identify the weight function and eigenvalue as

$$w(x) = \left(\frac{1+x}{1-x} \right)^{1/2}; \quad x \neq 1, \quad \lambda_n = n(n+1), \quad (33)$$

respectively. Therefore, the orthogonality of the polynomials $\{\phi_{n+(1/2)}(x)\}$ is written as

$$\int_{-1}^1 \phi_{n+\frac{1}{2}}(x) \phi_{n'+\frac{1}{2}}(x) w(x) dx = \pi \delta_{nn'}, \quad (34)$$

or equivalently using equation (22)

$$\int_{-1}^1 T_{n+(1/2)}(x) T_{n'+(1/2)}(x) (1-x^2)^{-1/2} dx = \frac{\pi}{2} \delta_{nn'}. \quad (35)$$

The set $\{\phi_{n+(1/2)}(x)\}$ is a basis in $[-1, 1]$. It can be written with the CHP-I as

$$\phi_{n+(1/2)}(x) = \sum_{i=0}^n (-1)^{n+i} C_i T_i(x); \quad C_i = \begin{cases} 1; & i = 0, \\ 2; & i \neq 0, \end{cases} \quad (36)$$

which from orthogonality of the CHP-I, we get

$$\int_{-1}^1 \phi_{n+(1/2)}(x) T_{n'}(x) (1-x^2)^{-1/2} dx = \pi \begin{cases} 0, & n' > n, \\ (-1)^{n+n'}, & n' \leq n. \end{cases} \quad (37)$$

4. Polynomials $\phi_{n+(1/2)}(x)$ and CHP-II

Analogously, we obtain that

$$\phi_{n+(1/2)}(x) = U_n(x) - U_{n-1}(x); (U_{-1}(x) \equiv 0), \quad (38)$$

where the set $\{U_n(x)\}$ is CHP-II [10].

By using

$$\frac{1}{n+1} \left[(1-x^2) \frac{d}{dx} + nx \right] U_n(x) = U_{n-1}(x), \quad (39)$$

we can rewrite equation (38) as

$$\phi_{n+(1/2)}(x) = - \left[\frac{(1-x^2)}{n+1} \frac{d}{dx} + \frac{n}{n+1}x - 1 \right] U_n(x), \quad (40)$$

which is another definition for polynomials $\phi_{n+(1/2)}(x)$ in terms of the CHP-II.

Since the CHP-II satisfies

$$\int_{-1}^1 U_m(x) U_n(x) (1-x^2)^{1/2} dx = \frac{\pi}{2} \delta_{mn}, \quad (41)$$

the equation (38) implies

$$\int_{-1}^1 \phi_{n+(1/2)}(x) U_{n'}(x) (1-x^2)^{1/2} dx = \frac{\pi}{2} \begin{cases} 0, & n' \neq n, n-1, \\ (-1)^{n+n'}, & n' = n, n-1. \end{cases} \quad (42)$$

From equations (36) and (38) we find the relationship between CHP-I and II, *i.e.*

$$\begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & 0 & \cdots \\ 1 & 0 & 2 & 0 & \cdots \\ 0 & 2 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ \vdots \end{pmatrix}, \quad (43)$$

and again by orthogonality of the CHP-I, we deduce that

$$\int_{-1}^1 U_n(x) T_{n'}(x) (1-x^2)^{1/2} dx = \pi \begin{cases} 1; & 0 \leq n' = n, n-2, \dots, \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

5. Conclusions

We have defined Chebyshev functions of half-integer order. They are basically obtained by repeated application of ladder operators for trigonometric functions. These functions are the polynomials $\phi_{n+(1/2)}(x)$ except for a common factor $(x+1/2)^{1/2}$. The properties of these polynomials as well as its relationship with the CHP-I and II have been studied.

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Series involving the product of a trigonometric integral and a trigonometric function

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This paper is concerned with the summation of series (1). To find the sum of the series (1) we first derive formulas for the summation of series whose general term contains a product of two trigonometric functions. These series are expressed in terms of Riemann's zeta, Catalan's beta function or Dirichlet functions eta and lambda, and in certain cases, thoroughly investigated here, they can be brought in closed form, meaning that the infinite series are represented by finite sums.

Keywords: Riemann's zeta; Catalan's beta function; Dirichlet; Bessel functions

Mathematics Subject Classification 1991: Primary: 33C10; Secondary: 11M06, 65B10

1. Introduction

In this paper we shall find the sum of the series

$$I_{\alpha}^{T,f} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} T((an-b)x)}{(an-b)^{\alpha}} f((an-b)z), \quad \alpha \in \mathbb{R}^+, \quad (1)$$

where $a = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, $b = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, $s = 1$ or -1 , f is sin or cos, and T denotes trigonometric integral S_{ϕ} or C_{ϕ} defined by

$$S_{\phi}(x) = \int_0^1 \phi(y) \sin xy \, dy, \quad C_{\phi}(x) = \int_0^1 \phi(y) \cos xy \, dy. \quad (2)$$

To obtain the sum of the series (1) we do not have to calculate integrals $T((an-b)x)$ previously. At first, we assume that ϕ is integrable. Yet, in order to extend the class of

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Table 1.

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		-1	0	η	$-\pi < x < \pi$
2	1	1	1/2	λ	$0 < x < \pi$
		-1	0	β	$-(\pi/2) < x < (\pi/2)$

summable series, we admit that ϕ is differentiable on $(0, 1)$, but not necessarily bounded in the neighborhood of 0 or 1, or not integrable on $(0, 1)$, however, such that there exists at least one of the integrals (2). We further require that the functions $y^k \phi(y)$ ($k \in \mathbb{N}$) are integrable on $(0, 1)$ as well.

Obtaining sums of the series (1) relies on the summation of some trigonometric series. Making use of the method for finding the formula (6) from our article [1] (see Theorem 1, p. 396), we can derive the other particular cases as well, writing all of them, in terms of Riemann's ζ or Catalan's β function or Dirichlet functions η and λ , in the form of a single formula, *i.e.*,

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)x)}{(an-b)^\alpha} = \frac{c\pi}{2\Gamma(\alpha)f(\pi\alpha/2)} x^{\alpha-1} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha-2i-\delta)}{(2i+\delta)!} x^{2i+\delta}, \quad (3)$$

where $\alpha > 0$, $a = \left\{\frac{1}{2}\right\}$, $b = \left\{\frac{0}{1}\right\}$, $s = 1$ or -1 , and $f = \left\{\frac{\sin}{\cos}\right\}$, $\delta = \left\{\frac{1}{0}\right\}$. The values for F and c are in the table 1, where ζ is Riemann's zeta function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, η and λ are Dirichlet functions $\eta(z) = \sum_{k=1}^{\infty} (-1)^{k-1} k^{-z} = (1-2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty} (2k+1)^{-z} = (1-2^{-z})\zeta(z)$, and β is Catalan's function $\beta(z) = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-z}$.

Remark 1 We note that the functions ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole. The integral representation $\beta(z) = (1/\Gamma(z)) \int_0^\infty (x^{z-1} e^x / (e^{2x} + 1)) dx$ of Catalan's function defines an analytical function for $\operatorname{Re} z \geq 1$, but also it satisfies the functional equation $\beta(z) = (\pi/2)^{z-1} \Gamma(1-z) \cos(\pi z/2) \beta(1-z)$ extending beta to the left side of the complex plane $\operatorname{Re} z < 1$.

Remark 2 After multiplying by $(2(x/2)^\nu)/(\Gamma(1/2)\Gamma(\nu + (1/2)))$ the integrals (2) with $\phi(y) = (1-y^2)^{\nu-1/2}$, and substituting $y = \cos \theta$, we obtain

$$\varphi_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu + (1/2))} \int_0^{\pi/2} \sin^{2\nu} \theta g(z \cos \theta) d\theta, \quad (4)$$

where $\operatorname{Re} \nu > -(1/2)$, $\varphi_\nu = \left\{\frac{J_\nu}{\mathbf{H}_\nu}\right\}$, $g = \left\{\frac{\cos}{\sin}\right\}$, J_ν and $\mathbf{H}_\nu$ are the Bessel and Struve functions respectively of the first kind and order ν . The integrals $S_\phi(x)$ and $C_\phi(x)$ in equation (2) can be considered as generalizations of Bessel and Struve functions, respectively, since their integral representations (4) are obtained when the particular function $\phi(y) = (1-y^2)^{\nu-1/2}$ is chosen. Just because of this, in this paper we generalize some of our results from [2].

2. Preliminaries

In order to find a sum of the series (1) we set one of the integrals (2) instead of $T(x)$, so that in the sequel we use denotations $T(x) = \left\{\frac{S_\phi}{C_\phi}\right\}$, $g = \left\{\frac{\sin}{\cos}\right\}$, meaning that g is one of the

trigonometric functions in equation (2), which depends on the choice of S_ϕ or C_ϕ . Afterwards, we shall prove that we are allowed to interchange summation and integration, i.e.,

$$\begin{aligned} I_\alpha^{T,f} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^\alpha} \int_0^1 \phi(y) g((an-b)xy) dy \\ &= \int_0^1 \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z) g((an-b)xy)}{(an-b)^\alpha} \right) \phi(y) dy \quad (\alpha \in \mathbb{R}^+), \end{aligned} \quad (5)$$

relying on uniform convergence of the right-hand series with respect to y by virtue of Dirichlet's test, which says that (see [3]) the series $\sum_{n=0}^{\infty} a_n(y)b_n(y)$ is uniformly convergent in D , if the partial sums of $\sum_{n=0}^{\infty} a_n(y)$ are uniformly bounded in D , and the sequence $b_n(y)$, being monotonic for every fixed y , uniformly converges to 0. We emphasize that we treat the above right-hand series as a function of $y \in (0, 1)$, regarding x and z as variable parameters.

LEMMA 1 *Let $\phi(y)$ be integrable. Then the right-hand series converges uniformly and there holds equation (5).*

Proof We make use of an elementary trigonometric identity, whereby representing the product of f and g , as the sum of two trigonometric functions $\sin$ or $\cos$. Now this series is split up into two series of the type

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1} \tau((an-b)(z \pm xy))}{(an-b)^\alpha} \quad (\alpha \in \mathbb{R}^+), \quad (6)$$

where $\tau = \sin$ or $\tau = \cos$. Let us suppose first that $a = 1, b = 0$. If $s = 1$, then there holds

$$\left| \sum_{k=1}^n \tau(k(z \pm xy)) \right| \leq \frac{1}{\sin((z \pm xy)/2)} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < 2\pi$, because $\sin((z \pm xy)/2) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq ((z \pm xy)/2) \leq \pi - \varepsilon$.

Similarly, if $s = -1$, we would have

$$\left| \sum_{k=1}^n (-1)^{k-1} \tau(k(z \pm xy)) \right| \leq \frac{1}{\cos((z \pm xy)/2)} \leq \frac{1}{\sin \varepsilon}$$

for $-\pi < z \pm xy < \pi$, because $\cos((z \pm xy)/2) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \leq ((z \pm xy)/2) \leq (\pi/2) - \varepsilon$.

We now suppose $a = 2, b = 1$. If $s = 1$, then

$$\left| \sum_{k=1}^n \tau((2k-1)(z \pm xy)) \right| \leq \frac{1}{\sin(z \pm xy)} \leq \frac{1}{\sin \varepsilon}$$

for $0 < z \pm xy < \pi$, because $\sin(z \pm xy) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $\varepsilon \leq z \pm xy \leq \pi - \varepsilon$. Finally, if $s = -1$, then

$$\left| \sum_{k=1}^n (-1)^{k-1} \tau((2k-1)(z \pm xy)) \right| \leq \frac{1}{\cos(z \pm xy)} \leq \frac{1}{\sin \varepsilon}$$

for $-(\pi/2) < z \pm xy < (\pi/2)$, because $\cos(z \pm xy) \geq \sin \varepsilon > 0$ for each $\varepsilon > 0$ satisfying $-(\pi/2) + \varepsilon \leq z \pm xy \leq (\pi/2) - \varepsilon$.

So in all these cases, partial sums are uniformly bounded with respect to $y \in (0, 1)$, and on this basis, we determine values of x and z giving rise to this, thus finding boundaries for convergence regions of the series (1) in each of four considered cases. They are in table 2. First, we have immediately $-|x| \leq \pm xy \leq |x|$, and to obtain the condition $0 < z \pm xy < 2\pi$, it is necessary to take $|x| < z < 2\pi - |x|$, from where there follows $|x| < \pi$. In the second case, to satisfy $-\pi < z \pm xy < \pi$, it is necessary to take $|x| - \pi < z < \pi - |x|$, and we have again $|x| < \pi$. In the third, $0 < z \pm xy < \pi$, it is necessary to take $|x| < z < \pi - |x|$, so that we easily find $|x| < (\pi/2)$. Finally, for $-(\pi/2) < z \pm xy < (\pi/2)$, it is necessary to take $|x| - (\pi/2) < z < (\pi/2) - |x|$ giving again $|x| < (\pi/2)$.

On the other hand, the sequence $1/((an - b)^\alpha)$ obviously monotonically tends to 0 for $\alpha > 0$, and uniformly converges to 0 with respect to y . Together with the above, this proves an uniform convergence of the right-hand series in equation (5) with respect to $y \in (0, 1)$. So the interchange of integration and summation in equation (5) is permitted, and accordingly, there holds equation (5). ■

LEMMA 2 Suppose that, for a differentiable on $(0, 1)$ and unbounded in the neighborhood of 0 or 1 function ϕ , the integral $\int_0^1 \phi(y) dy$ does not converge. Let there exist at least one of the integrals $T((an - b)x)$ (T is S_ϕ or C_ϕ defined by equation (2)), so that $|T((an - b)x)| \leq M_n(x)$, and for each corresponding x from table 2, the sequence $(M_n(x))/((an - b)^\alpha)$, $\alpha > 0$, monotonically tends to 0. Then there holds (5).

Proof Because of the assumption that ϕ is differentiable function, we know that it is continuous on each closed interval within $(0, 1)$, and, as it is not bounded in the neighborhood of 0 or 1, without loss of generality, we can consider $[\delta, 1 - \delta]$, $0 < \delta < 1$. Continuous function on a closed interval is bounded, and referring again to Dirichlet's test, we prove uniform convergence of the right-hand series in equation (5) with respect to y , but this time on $[\delta, 1 - \delta]$, so that there holds

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z)}{(an - b)^\alpha} \int_{\delta}^{1-\delta} \phi(y) g((an - b)xy) dy \\ &= \int_{\delta}^{1-\delta} \left(\sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an - b)z) g((an - b)xy)}{(an - b)^\alpha} \right) \phi(y) dy \quad (\alpha \in \mathbb{R}^+). \end{aligned} \quad (7)$$

We regard the left-hand series as a function of δ with variable parameters z and x . In view of the conditions, by virtue of Dirichlet's test, the left-hand series in equation (7) converges

Table 2.

a	b	s	c	F	Convergence regions
1	0	1	1	ζ	$\{(x, z) : -\pi < x < \pi, x < z < 2\pi - x \}$
1	0	-1	0	η	$\{(x, z) : -\pi < x < \pi, x - \pi < z < \pi - x \}$
2	1	1	1/2	λ	$\{(x, z) : -(\pi/2) < x < (\pi/2), x < z < \pi - x \}$
2	1	-1	0	β	$\{(x, z) : -(\pi/2) < x < (\pi/2), x - (\pi/2) < z < (\pi/2) - x \}$

uniformly with respect to δ on $(0, 1)$. Hence, we have

$$\begin{aligned} \lim_{\delta \rightarrow 0+} \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha}} \int_{\delta}^{1-\delta} \phi(y) g((an-b)xy) dy \\ = \sum_{n=1}^{\infty} \frac{(s)^{n-1} f((an-b)z)}{(an-b)^{\alpha}} \int_0^1 \phi(y) g((an-b)xy) dy, \end{aligned}$$

meaning that the right-hand integral in equation (7) converges, so there holds equation (5). ■

3. Series over the product of trigonometric functions

We have seen that in finding the summation formula for equation (1), the key role plays the series including the product of two trigonometric functions. That is why we are going to investigate them thoroughly. Now, by applying equation (3) to both series in each of the above particular cases, we obtain the following general formula

$$\begin{aligned} S_{\alpha}^{f,g} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an-b)xy) f((an-b)z)}{(an-b)^{\alpha}} \\ &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)} \left((z+xy)^{\alpha-1} + (-1)^{\delta}(z-xy)^{\alpha-1} \right) \\ &\quad + \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha-2i-d)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta}, \quad (8) \end{aligned}$$

where $g = \begin{cases} \sin \\ \cos \end{cases}$, $\delta = \begin{cases} 1 \\ 0 \end{cases}$, and $d = \begin{cases} 0 & f=g \\ 1 & f \neq g \end{cases}$, and also there holds $h = \begin{cases} \cos & f=g \\ \sin & f \neq g \end{cases}$. All the other relevant parameters are in table 2.

3.1 Limiting values of equation (8)

When on the right-hand side of equation (8) appears $h = \sin$ and $\alpha = 2m$ or $h = \cos$ and $\alpha = 2m - 1$, where $m \in \mathbb{N}$, one should take limit. Let us consider a particular case of the formula (8), taking $a = 1$, $b = 0$, $s = 1$, which implies $c = 1$, $F = \zeta$. If $g = \cos$ and $f = \sin$ then $\delta = 0$, $h = \sin$, $d = 1$, so we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin nz \cos nxy}{n^{\alpha}} &= \pi \frac{(z+xy)^{\alpha-1} + (z-xy)^{\alpha-1}}{4\Gamma(\alpha) \sin(\pi\alpha/2)} \\ &\quad + \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j} z^{2i-2j+1} (xy)^{2j}. \end{aligned}$$

As we have said, we take limit

$$\begin{aligned} \lim_{\alpha \rightarrow 2m} \left[\pi \frac{(z+xy)^{\alpha-1} + (z-xy)^{\alpha-1}}{4\Gamma(\alpha) \sin(\pi\alpha/2)} \right. \\ \left. + \sum_{i=0}^{m-1} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j} z^{2i-2j+1} (xy)^{2j} \right] = G_{2m}(x, y, z), \end{aligned}$$

where we have found

$$\begin{aligned} G_{2m}(x, y, z) &= \frac{(-1)^m}{2(2m-1)!} \left((z+xy)^{2m-1} (\log(z+xy) - \psi(2m) - \gamma) \right. \\ &\quad \left. + (z-xy)^{2m-1} (\log(z-xy) - \psi(2m) - \gamma) \right) \\ &\quad + \sum_{k=1}^{m-1} \frac{(-1)^{k-1} \zeta(2m-2k+1)}{(2k-1)!} \sum_{j=0}^{k-1} \binom{2k-1}{2j} z^{2k-2j-1} (xy)^{2j}, \end{aligned}$$

where γ is Euler's constant and ψ is the digamma function, $\psi(s) = (\Gamma'(s)/\Gamma(s))$, whose relation to the harmonic numbers $H_n = \sum_{j=1}^n (1/j)$ is $\psi(n) = H_{n-1} - \gamma$, with $\psi(1) = -\gamma = \Gamma'(1)$. Finally,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin nz \cos nxy}{n^{2m}} &= G_{2m}(x, y, z) + \sum_{i=m}^{\infty} \frac{(-1)^i \zeta(2m-2i-1)}{(2i+1)!} \\ &\quad \times \sum_{j=0}^i \binom{2i+1}{2j} z^{2i-2j+1} (xy)^{2j}. \end{aligned}$$

Example 1 If we additionally choose $m = 1$, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\sin nz \cos nxy}{n^2} &= z - \frac{1}{2} \left(z \log(z^2 - (xy)^2) + xy \log \frac{z+xy}{z-xy} \right) \\ &\quad + \sum_{i=1}^{\infty} \frac{(-1)^i \zeta(1-2i)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j} z^{2i-2j+1} (xy)^{2j}. \end{aligned}$$

In the same way, we can come to a similar result for any of particular cases if $h = \sin$ and $\alpha = 2m$. Also, we repeat the same procedure, if $h = \cos$ and $\alpha = 2m - 1$.

Example 2 If we choose the same parameters as above, but take $f = \cos$ instead of $f = \sin$ and again $m = 1$, then $h = \cos$ and $\alpha = 1$, so we first have to find the limiting value

$$\lim_{\alpha \rightarrow 1} \left(\pi \frac{(z+xy)^{\alpha-1} + (z-xy)^{\alpha-1}}{4\Gamma(\alpha) \sin(\pi\alpha/2)} + z\zeta(\alpha) \right) = -\frac{1}{2} \log(z^2 - (xy)^2),$$

so that we have

$$\sum_{n=1}^{\infty} \frac{\cos nxy \cos nz}{n} = -\frac{1}{2} \log(z^2 - (xy)^2) + \sum_{i=1}^{\infty} \frac{(-1)^i \zeta(1-2i)}{(2i)!} \sum_{j=0}^i \binom{2i}{2j} z^{2i-2j} (xy)^{2j}.$$

However, based on the Fourier expansion of the function $-(1/2) \log(2 \sin(t/2))$ for $0 < t < 2\pi$, there holds $-(1/2) \log(2 \sin(t/2)) = \sum_{n=1}^{\infty} (1/n) \cos nt$, and we can easily find

$$\sum_{n=1}^{\infty} \frac{\cos nxy \cos nz}{n} = -\frac{1}{2} \log \left(4 \sin \frac{z-xy}{2} \sin \frac{z+xy}{2} \right),$$

where $0 < z \pm xy < 2\pi$. Relying on this result, we can conclude

$$\sum_{i=1}^{\infty} \frac{(-1)^i \zeta(1-2i)}{(2i)!} \sum_{j=0}^i \binom{2i}{2j} z^{2i-2j} (xy)^{2j} = \log \sqrt{\frac{(z-xy)(z+xy)}{4 \sin((z-xy)/2) \sin((z+xy)/2)}},$$

which is a new summation formula for the series involving Riemann's zeta function. Further, if we let $xy \rightarrow z$, and make use of the combinatorial identity $\sum_{j=0}^i \binom{2i}{2j} = 2^{2i-1}$, the above summation formula will be reduced to

$$\sum_{i=1}^{\infty} \frac{(-4z^2)^i \zeta(1-2i)}{(2i)!} = \log \left(\frac{z}{\sin z} \right).$$

3.2 Closed form cases and some applications of equation (8)

If $\alpha - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - d = 2m - 1$ and $F = \beta$ ($m \in \mathbb{N}$), the sum of the series on the right-hand side of equation (8) consists of a finite number of terms because the functions ζ, η, λ vanish at even negative integers, and the function β vanishes at odd negative integers. We shall denote $\alpha = 2m + d - \varepsilon$, where $\varepsilon = \begin{cases} 0, & F = \zeta, \eta, \lambda \\ 1, & F = \beta \end{cases}$. So, for this choice of parameters, the formula (8) is brought into a so-called closed form

$$\begin{aligned} S_{2m+d-\varepsilon}^{f,g} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} g((an-b)xy) f((an-b)z)}{(an-b)^{2m+d-\varepsilon}} \\ &= \frac{c\pi}{2} \sum_{j=0}^m \binom{2m+d-\varepsilon-1}{2j+\delta} \frac{(-1)^{\delta(\delta-d)} z^{2m-2j+d-\delta-\varepsilon-1} (xy)^{2j+\delta}}{(2m+d-\varepsilon-1)! h(m\pi + ((d-\varepsilon)\pi)/2)} \\ &\quad + \sum_{i=0}^m \frac{(-1)^{\delta(\delta-d)+i} F(2m-2i-\varepsilon)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} (xy)^{2j+\delta}, \quad (9) \end{aligned}$$

where $g = \begin{cases} \sin \\ \cos \end{cases} \delta = \begin{cases} 1 \\ 0 \end{cases}$, and independently of that $d = \begin{cases} 0 & f=g \\ 1 & f \neq g \end{cases}$, $h = \begin{cases} \cos & f=g \\ \sin & f \neq g \end{cases}$. The other relevant parameters are in table 2. The formula (9) comprises some particular results from [4], but it is more suitable for immediately obtaining sums of some infinite series as well. For example, in [2], §5.4.15, entry 8, we find

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nxy \cos nz = \frac{xy}{12} (\pi^2 - (xy)^2 - 3z^2).$$

It holds if $|xy \pm z| \leq \pi$, which is the condition for uniform convergence of the left-hand side series in equation (8) with $a = 1, b = 0$ and $s = -1$. After referring to table 2 and the conditions following equation (9), we obtain the above formula, placing in equation (9): $g = \sin, f = \cos, m = 1, d = 1, \varepsilon = 0, c = 0, F = \eta, \delta = 1, h = \sin$.

3.3 Some applications

In [5] the solution of the boundary value problem

$$U''_{tt} = a^2 U''_{xx}, \quad U(x, 0) = \frac{4hx(L-x)}{L^2}, \quad U'_t(x, 0) = 0, \quad 0 \leq x \leq L, \quad t \geq 0,$$

is given by the infinite series

$$U(x, t) = \frac{32h}{\pi^3} \sum_{n=1}^{\infty} \frac{\cos((\pi(2n-1)at)/L) \sin(((2n-1)\pi x)/L)}{(2n-1)^3}.$$

However, using equation (9), where we take $m = 1$, $a = 2$, $b = 1$, $s = 1$, $c = (1/2)$ and $F = \lambda$, then $\delta = 0$, $d = 1$ and $\varepsilon = 0$, and denoting $(at\pi/L) = xy$, $(x\pi/L) = z$, we obtain its solution in closed form

$$U(x, t) = \frac{4h}{L^2}(xL - x^2 - a^2t^2), \quad 0 \leq \frac{at}{L} \leq \frac{1}{2}, \quad \left| \frac{at}{L} \right| \leq \frac{x}{L} \leq 1 - \left| \frac{at}{L} \right|.$$

Also (see [5]), the solution of the boundary value problem

$$\begin{aligned} U''_{tt} &= U''_{xx} + x(x - L), & U(x, 0) &= U'_t(x, 0) \\ &= U(0, t) = U(L, t) = 0, & 0 \leq x \leq L, t \geq 0, \end{aligned}$$

is

$$U(x, t) = \frac{8L^4}{\pi^5} \sum_{n=1}^{\infty} \frac{\cos(\pi t(2n-1)/L) \sin(\pi x(2n-1)/L)}{(2n-1)^5} - \frac{x}{12}(x^3 - 2x^2L + L^3).$$

Applying equation (9) with $m = 2$, using the same parameters as above, but denoting $(t\pi/L) = xy$ and $(x\pi/L) = z$, this solution in closed form becomes

$$U(x, t) = \frac{1}{2}x^2t^2 - \frac{1}{2}Lxt^2 + \frac{1}{12}t^4, \quad 0 \leq t \leq \frac{L}{2}, \quad t \leq x \leq L - t.$$

4. Sum of the series (1)

Now we make use of equation (8), which is, as we have shown, the formula for finding sum of the right-hand side series of equation (5). We develop in equation (8) the binomials $(z \pm xy)^{\alpha-1}$ into binomial series, and after a rearrangement we actually obtain the summation formula of the left-hand side series in equation (5), which is actually the summation formula for the series (1)

$$\begin{aligned} I_{\alpha}^{T,f} &= \frac{c\pi(-1)^{\delta(\delta-d)}}{4\Gamma(\alpha)h(\pi\alpha/2)} \sum_{k=1}^{\infty} \binom{\alpha-1}{2k-\delta} z^{\alpha-1-2k+\delta} x^{2k-\delta} \int_0^1 y^{2k-\delta} \phi(y) dy \\ &+ \sum_{i=0}^{\infty} \frac{(-1)^{\delta(\delta-d)+i} F(\alpha-2i-d)}{(2i+d)!} \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} x^{2j+\delta} \int_0^1 y^{2j+\delta} \phi(y) dy, \end{aligned} \quad (10)$$

where $T(x) = \left\{ \begin{smallmatrix} S_{\phi}(x) \\ C_{\phi}(x) \end{smallmatrix} \right\}$, $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$, $\delta = \{0, 1\}$, $h = \left\{ \begin{smallmatrix} \cos, & f=g \\ \sin, & f \neq g \end{smallmatrix} \right\}$ and $d = \left\{ \begin{smallmatrix} 0, & f=g \\ 1, & f \neq g \end{smallmatrix} \right\}$. The other relevant parameters are in table 2.

4.1 Limiting values

If $h = \sin$ and $\alpha = 2m$ or $h = \cos$ and $\alpha = 2m - 1$ ($m \in \mathbb{N}$), we encounter a singularity in the first term of equation (10). That is why the limit should be taken. We shall demonstrate it for

the following choice of parameters: $a = 1$, $b = 0$, $s = 1$, $c = 1$, $F = \zeta$, $g = \sin$, $f = \cos$, $\delta = 1$, $h = \sin$ and $d = 1$, and, apart from this, $\phi(y) = y^{-1}$, so that equation (10) becomes

$$I_{\alpha}^{S_{\phi}, \cos} = \sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^{\alpha}} \int_0^1 \frac{\sin nxy}{y} dy = \frac{\pi}{2\Gamma(\alpha) \sin(\pi\alpha/2)} \sum_{k=1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k} x^{2k-1}}{2k-1} \\ + \sum_{i=0}^{\infty} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{z^{2i-2j} x^{2j+1}}{2j+1}.$$

Having chosen $\phi(y) = 1/y$, the integral $\int_0^1 (dy/y)$ does not converge, but there exists the integral $\int_0^1 (\sin nxy/y) dy = \text{Si}(nx)$, where $\text{Si}(z) = \int_0^z (\sin t/t) dt$ is the integral sine, which could not be elementarily calculated, but we know it is bounded by $\pi/2$, so Lemma 2 holds. Irrespective of this, without even trying to find its value, and simply by applying equation (10), we obtain the above sum. Yet, because of $h = \sin$, there still remains to take the limit for $\alpha = 2m$, $m \in \mathbb{N}$, i.e.,

$$\lim_{\alpha \rightarrow 2m} \left[\frac{\pi}{2\Gamma(\alpha) \sin(\pi\alpha/2)} \sum_{k=1}^m \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k} x^{2k-1}}{2k-1} \right. \\ \left. + \sum_{i=0}^{m-1} \frac{(-1)^i \zeta(\alpha-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{z^{2i-2j} x^{2j+1}}{2j+1} \right] = G_{2m}(x, z),$$

and we find

$$G_{2m}(x, z) = \sum_{k=1}^{m-1} \frac{(-1)^{k-1} \zeta(2m-2k+1)}{(2k-1)!} \sum_{j=0}^{k-1} \binom{2k-1}{2j+1} \frac{z^{2k-2j-2} x^{2j+1}}{2j+1} + \frac{(-1)^m}{(2m-1)!} \\ \times \left(\sum_{k=1}^{m-1} \frac{\log z - \psi(2m-2k+1) - \gamma}{2k-1} \binom{2m-1}{2k-1} x^{2k-1} z^{2m-2k} + \frac{x^{2m-1} \log z}{2m-1} \right)$$

as well as

$$\lim_{\alpha \rightarrow 2m} \frac{\pi}{2\Gamma(\alpha) \sin(\pi\alpha/2)} \sum_{k=m+1}^{\infty} \binom{\alpha-1}{2k-1} \frac{z^{\alpha-2k} x^{2k-1}}{2k-1} \\ = \frac{(-1)^{m-1} z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m} (2k-1)},$$

so that we get

$$\sum_{n=1}^{\infty} \frac{\cos n\alpha}{n^{2m}} \int_0^1 \frac{\sin nxy}{y} dy = G_{2m}(x, z) + \frac{(-1)^{m-1} z^{2m-1}}{(2m)!} \sum_{k=m+1}^{\infty} \frac{(x/z)^{2k-1}}{\binom{2k-1}{2m} (2k-1)} \\ + \sum_{i=m}^{\infty} \frac{(-1)^i \zeta(2m-2i-1)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{z^{2i-2j} x^{2j+1}}{2j+1}.$$

For instance,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\cos nz}{n^4} \int_0^1 \frac{\sin nxy}{y} dy = & -\frac{11x^3}{108} - \frac{19xz^2}{18} + \left(\frac{x^2z}{4} + \frac{11z^3}{36} \right) \operatorname{arctgh} \frac{x}{z} \\ & + \frac{x^5}{24z^2} \Phi\left(\frac{x^2}{z^2}, 2, \frac{5}{2}\right) + \left(\frac{x^3}{36} + \frac{xz^2}{4} \right) \log\left(1 - \frac{x^2}{z^2}\right) \\ & + \left(\frac{x^3}{18} + \frac{xz^2}{2} \right) \log z + x\zeta(3) \\ & + \sum_{i=2}^{\infty} \frac{(-1)^i \zeta(3-2i)}{(2i+1)!} \sum_{j=0}^i \binom{2i+1}{2j+1} \frac{z^{2i-2j} x^{2j+1}}{2j+1}, \end{aligned}$$

where Φ is the Lerch transcendent function defined by the series $\Phi(z, s, \alpha) = \sum_{n=0}^{\infty} (z^n / (n + \alpha)^s)$, and $|x| < \pi$, $|x| < z < 2\pi - |x|$ (see table 2).

In the same way, we can come to a similar result for any of particular cases.

4.2 Closed form cases

They ensue if $\alpha - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - d = 2m - 1$ and $F = \beta$ ($m \in \mathbb{N}$). The other relevant parameters and convergence regions are in table 2. When we choose the function ϕ in equation (10), we additionally have to calculate both integrals having in fact a similar structure.

We have already seen that the formula (9) contains all closed form cases for the product of trigonometric functions. So, using the formula (9) we bring the sum of the series (1) into closed form:

$$\begin{aligned} I_{2m+d-\varepsilon}^{T,f} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1} T((an-b)x)}{(an-b)^{2m+d-\varepsilon}} f((an-b)z) \\ &= \frac{c\pi}{2} \sum_{j=0}^m \binom{2m+d-\varepsilon-1}{2j+\delta} \frac{(-1)^{\delta(\delta-d)} z^{2m-2j+d-\delta-\varepsilon-1} x^{2j+\delta}}{(2m+d-\varepsilon-1)! h(m\pi + ((d-\varepsilon)\pi)/2)} \\ &\quad \times \int_0^1 y^{2j+\delta} \phi(y) dy + \sum_{i=0}^m \frac{(-1)^{\delta(\delta-d)+i} F(2m-2i-\varepsilon)}{(2i+d)!} \\ &\quad \times \sum_{j=0}^i \binom{2i+d}{2j+\delta} z^{2i-2j+d-\delta} x^{2j+\delta} \int_0^1 y^{2j+\delta} \phi(y) dy, \end{aligned} \quad (11)$$

where $T(x) = \left\{ \begin{smallmatrix} S_{\phi}(x) \\ C_{\phi}(x) \end{smallmatrix} \right\}$, $g = \left\{ \begin{smallmatrix} \sin \\ \cos \end{smallmatrix} \right\}$, $\delta = \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\}$, and independently of that $d = \left\{ \begin{smallmatrix} 0 & f=g \\ 1 & f \neq g \end{smallmatrix} \right\}$, $h = \left\{ \begin{smallmatrix} \cos & f=g \\ \sin & f \neq g \end{smallmatrix} \right\}$. For $F = \zeta, \eta, \lambda$ there holds $\varepsilon = 0$, but for $F = \beta$ it is $\varepsilon = 1$. The other relevant parameters are in table 2.

Example 3 First we take $a = 1$, $b = 0$ and $s = 1$ in equation (11), there follows $c = 1$ and $F = \zeta$ (see table 2), then $\varepsilon = 0$. We choose $\phi(y) = \operatorname{ctgy}$. For $g = \cos$ the integral $\int_0^1 \operatorname{ctgy} \cos nxy dy$ does not exist, so there must be $g = \sin$, which means $\delta = 1$. If we take

$f = \cos$, then $h = \sin$ and $d = 1$ because $f \neq g$. Let $m = 1$. For $0 < |x| < \pi$, we have

$$\begin{aligned} \left| x \int_0^1 \operatorname{ctgy} \sin nxy \, dy \right| &= \left| \int_0^x \operatorname{ctg}(t/x) \sin nt \, dt \right| \leq \int_0^x \left| \operatorname{ctg}(t/x) \sin nt \right| dt \\ &\leq \int_0^\pi \left| \operatorname{ctg}(t/\pi) \sin nt \right| dt. \end{aligned}$$

Also, $|\operatorname{tg}(t/\pi)| > (|t|/\pi)$ implies $|\operatorname{ctg}(t/\pi)| < (\pi/|t|)$, for $|t| < \pi$. Additionally, $\lim_{t \rightarrow 0+} \operatorname{ctg}(t/\pi) \sin nt = n\pi$. The function $|\operatorname{ctg}(t/\pi) \sin nt|$ is non-negative, fast oscillatory, and its maximal value between two consecutive zeros constantly decreases. There follows

$$\begin{aligned} \int_0^\pi \left| \operatorname{ctg} \frac{t}{\pi} \sin nt \right| dt &= \int_0^{1/n} \left| \operatorname{ctg} \frac{t}{\pi} \sin nt \right| dt + \int_{1/n}^\pi \left| \operatorname{ctg} \frac{t}{\pi} \right| \cdot \left| \sin nt \right| dt \\ &< \int_0^{1/n} n\pi \, dt + \pi \int_{1/n}^\pi \frac{dt}{t} = \pi(1 + \log n\pi), \end{aligned}$$

which means that $\left| \int_0^1 \operatorname{ctgy} \sin nxy \, dy \right| < (\pi(1 + \log n\pi)/|x|) = M_n(x)$, and we can easily see that for $\alpha > 0$ and each x , $0 < |x| < \pi$, the sequence $(M_n(x)/n^\alpha) \rightarrow 0$ monotonically, so that we may apply Lemma 2. Also, we find

$$\int_0^1 y^{2j+1} \operatorname{ctgy} \, dy = \begin{cases} \log(2 \sin 1) + (1/2)\operatorname{Cl}_2(2), & j = 0 \\ \log(2 \sin 1) + (3/2)\operatorname{Cl}_2(2) + (3/2)\operatorname{Cl}_3(2) - (3/4)\operatorname{Cl}_4(2), & j = 1 \end{cases}$$

where on the right-hand side are Clausen functions defined by (see [1])

$$\operatorname{Cl}_{2v}(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{2v}}, \quad \operatorname{Cl}_{2v-1}(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^{2v-1}}, \quad v \in \mathbb{N}.$$

Making use of equation (9), we obtain

$$\begin{aligned} I_1^{S_\phi, \cos} &= \sum_{n=1}^{\infty} \frac{\cos nz}{n^3} \int_0^1 \operatorname{ctgy} \sin nxy \, dy \\ &= \frac{x}{12} (2\pi^2 - 6\pi z + 3z^2) \left(\log(2 \sin 1) + \frac{1}{2}\operatorname{Cl}_2(2) \right) \\ &\quad + \frac{x^3}{12} \left(\log(2 \sin 1) + \frac{3}{2}\operatorname{Cl}_2(2) + \frac{3}{2}\operatorname{Cl}_3(2) - \frac{3}{4}\operatorname{Cl}_4(2) \right), \end{aligned}$$

where $|x| < \pi$ and $|x| < z < 2\pi - |x|$ (see table 2).

Example 4 Let $a = 1, b = 0$ and $s = -1$ in equation (11), implying $c = 0, F = \eta$ and $\varepsilon = 0$. For $g = \cos$ there must be $\delta = 0$. Further, we take $f = \cos$, there follows $h = \cos, d = 0$ because $f = g$. Let $m = 2$. We choose $\phi(y) = (1 - y^2)^{-1/2}$. It is unbounded about 1, but integrable on $(0, 1)$, thus satisfying conditions of Lemma 1. So applying equation (11), we find

$$\begin{aligned} I_4^{C_\phi, \cos} &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos nz}{n^4} \int_0^1 \frac{\cos nxy}{\sqrt{1-y^2}} \, dy \\ &= \frac{7\pi^5}{1440} - \frac{\pi^3}{96} (x^2 + 2z^2) + \frac{\pi}{32} \left(\frac{x^4}{8} + x^2 z^2 + \frac{z^4}{3} \right), \end{aligned}$$

where $|x| < \pi$ and $|x| - \pi < z < \pi - |x|$ (see table 2).

Remark 3 We note that this particular result can be obtained in a different way. Namely, substituting $y = \cos \theta$, then multiplying with $2/\pi$, we deal with Bessel functions (see equation (4)), and the above series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} J_0(nx)}{n^4} \cos nz,$$

which is the series over the product of Bessel and trigonometric function, and its sum can be obtained by means of the formula on the page 289 in [2].

Example 5 Further, we take $a = 2$, $b = 1$ and $s = 1$ in equation (11). In table 2, we read $c = (1/2)$ and $F = \lambda$. So $\varepsilon = 0$. If we choose $g = \cos$, $f = \sin$, then we have $\delta = 0$, $d = 1$ and $h = \sin$. Let $m = 1$ and $\phi(y) = \log(\sin y)$. It is unbounded in the neighborhood of 0, however $\int_0^1 \log(\sin y) dy = -\log 2 - (1/2)\text{Cl}_2(2)$, and Lemma 1 holds. Applying equation (11), we obtain

$$\begin{aligned} I_1^{S_\phi, \sin} &= \sum_{n=1}^{\infty} \frac{\sin((2n-1)z)}{(2n-1)^3} \int_0^1 \log(\sin y) \cos((2n-1)xy) dy \\ &= \frac{\pi}{96} \left(6z(z-\pi)(\text{Cl}_2(2) + \log 4) + x^2(6\text{Cl}_2(2) + 6\text{Cl}_3(2) - 3\text{Cl}_4(2) + \log 16) \right), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

Example 6 Finally, we take in equation (11): $a = 2$, $b = 1$, $s = -1$, $c = 0$, $F = \beta$, $g = \sin$, $f = \sin$, $d = 0$, $\delta = 1$, $h = \cos$, $\varepsilon = 1$ and $m = 2$, and choose $\phi(y) = \log y$. The integral $\int_0^1 \log y dy$ converges, ϕ meets requirements of Lemma 1, and making use of equation (11), the infinite series is brought in closed form

$$\begin{aligned} I_5^{S_\phi, \sin} &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin((2n-1)z)}{(2n-1)^5} \int_0^1 \log y \sin((2n-1)xy) dy \\ &= \frac{\pi x z}{32} \left(\frac{\pi^2}{4} - \frac{x^2}{12} - \frac{z^2}{3} \right), \end{aligned}$$

where $|x| < (\pi/2)$ and $|x| - (\pi/2) < z < (\pi/2) - |x|$ (see table 2).

This series can take another form. At first, by integrating by parts, we come to the relation $\int_0^1 \log y \sin((2n-1)xy) dy = (1/((2n-1)x)) \int_0^1 ((\cos((2n-1)xy) - 1/y) dy$. Afterwards, we prove $\int_0^p ((1 - e^{-t})/t) dt - \int_p^\infty (e^{-t}/t) dt = \log p + \gamma$. On this basis, we find $\text{Ci}(p) + \text{Cin}(p) = \log p + \gamma$, where $\text{Ci}(p) = -\int_p^\infty (\cos t/t) dt$ is the integral cosine and $\text{Cin}(p) = \int_0^p ((1 - \cos t)/t) dt$ its related function. Hence, there follows

$$\begin{aligned} &\int_0^1 \log y \sin((2n-1)xy) dy \\ &= \frac{1}{(2n-1)x} (\text{Ci}((2n-1)x) - \gamma - \log((2n-1)x)) \quad (x > 0), \end{aligned}$$

and we obtain the sum of a new series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin((2n-1)z)}{(2n-1)^6 x} (\text{Ci}((2n-1)x) - \gamma - \log((2n-1)x)) \\ = \frac{\pi x z}{32} \left(\frac{\pi^2}{4} - \frac{x^2}{12} - \frac{z^2}{3} \right).$$

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Half inverse problem for operators with singular potential

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A half-inverse problem for Sturm-Liouville operators consists of reconstruction of this operator by its spectrum and half of the potential. In this study, we show that if $q(x)$ is prescribed on $(\pi/2, \pi)$, then, only one spectrum is sufficient to determine $q(x)$ on the interval $(0, \pi/2)$ for the Sturm-Liouville equation having singularity type $1/\sin^2 x$ on $(0, \pi)$.

Keywords: Singular Sturm-Liouville problem; Eigenfunction; Spectrum

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1. Introduction

In spectral theory, the inverse problem is the usual name for any problem in which it is required to ascertain the spectral data that will determine a differential operator uniquely and a method of construction this operator from the data. A problem of this kind was first formulated and investigated by Ambartsumyan in 1929 [1]. Since 1946, various forms of the inverse problem have been considered by numerous authors – G. Borg [2], N. Levinson [3], B.M. Levitan [4], etc. – and now there exists an extensive literature on the question [5–9]. Later, the inverse problems having specified singularities were considered by a number of authors [10–14].

In this work, merely one of the variants of the inverse problem will be treated and solved. In particular, we shall give the solution of the inverse problem for the equation with specified singularities. Some questions of the spectral problems having analogous singularities were considered in [7, 11, 13, 15, 16].

The first result on the half-inverse problem is due to Hochstadt and Lieberman [17], who proved that if $\{\lambda_n\} = \{\tilde{\lambda}_n\}$, $q = \tilde{q}$, $h = \tilde{h}$, $H = \tilde{H}$ on $(1/2, 1)$, then $q = \tilde{q}$ on $(0, 1)$. Later, Hald [15] showed that if the potential is known over half the interval and if one boundary condition is given, then the potential and the other boundary condition are uniquely determined by the eigenvalues. The position of the discontinuity and the jump in the eigenfunction are also uniquely determined. This is a generalized theorem by Hochstadt and Lieberman [17]. In [18], Sakhnovich studied the existence of solution to the half-inverse problem; that is, he presented sufficient conditions on a function $\tilde{q}$ on $(0, 1/2)$ and a sequence (λ_n^2) of pairwise

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distinct real numbers tending to $+\infty$ in order that there existed a Sturm-Liouville operator on $(0, 1)$ with given spectrum $\{\lambda_n\}$ and potential q coinciding with $\tilde{q}$ on $(0, 1/2)$. In [18], a half-inverse problem for a Sturm-Liouville operator with singular potential is considered by using a transformation operator. Necessary and sufficient conditions for solvability of the half-inverse spectral problem for the class of Sturm-Liouville operators with singular potentials from the space $W_2^{-1}(0, 1)$ are given and a reconstruction algorithm is provided.

In this paper, by using the Hochstadt and Lieberman's method [17], we solve the half inverse problem for differential operator having singularity type $1/4 \sin^2 x$ at the points 0 and π .

Let us consider

$$Lu = -u'' + \left[q(x) - \frac{1}{4 \sin^2 x} \right] u = \lambda u \quad (1)$$

with the boundary conditions

$$u(0) = 0, \quad (2)$$

$$u'(\pi, \lambda) + Hu(\pi, \lambda) = 0. \quad (3)$$

Suppose that the function $q(x)$ is summable, ε any positive number. The operator L is self adjoint on the $L_2(0, \pi)$ and with (2), (3) boundary conditions has a discrete spectrum $\{\lambda_n\}$.

2. Main results

Before giving the main results of this paper, we mention the properties of the classical Legendre equation. Legendre functions are special cases of the hypergeometric function. They arise in problems involving spherical symmetry.

The Legendre equation is

$$(1 - t^2)y'' - 2ty' + n(n + 1)y = 0.$$

First, we let

$$t = \cos x, \quad z = y(t)$$

then z satisfies

$$z'' + \cot x z' + n(n + 1)z = 0.$$

Later, when we consider transformation $u = z\sqrt{\sin x}$, it is obtained that [12]

$$-u'' - \frac{1}{4 \sin^2 x} u = \lambda u,$$

where $\lambda = (n + 1/2)^2$ and $0 < \varepsilon < x \leq \pi - \varepsilon < \pi$.

Solution of the equations (1)–(3) is

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} \int_0^x \sin \sqrt{\lambda}(x-t) \left[q(t) - \frac{1}{4 \sin^2 t} \right] y(t, \lambda) dt.$$

Eigenvalues of the problem (1)–(3) are the roots of the (3). These spectral characteristics and eigenfunctions satisfy the following asymptotic expression, respectively

$$\begin{aligned} \sqrt{\lambda_n} &= n + \frac{1}{2} + \frac{c}{\pi(n+1/2)} + O\left(\frac{1}{n^2}\right), \\ \varphi_n(x) &= \frac{1}{(n+1/2)} \sin\left(n + \frac{1}{2}\right)x + O\left(\frac{1}{n^2}\right) \end{aligned}$$

where c depends on H and q .

THEOREM We consider equation (1) with the boundary conditions (2), (3). Let $\{\lambda_n\}$ be the spectrum of the L with equations (2), (3). Considering a second operator

$$Lu = -u'' + \left[\tilde{q}(x) - \frac{1}{4 \sin^2 x} \right] u = \lambda u, \quad (4)$$

where $\tilde{q}(x)$ is summable on $(0, \pi)$ and $q(x) = \tilde{q}(x)$ on $(\pi/2, \pi)$. Suppose that the spectrum of $\tilde{L}$ with the (2), (3) is also $\{\lambda_n\}$. Then $q(x) = \tilde{q}(x)$ almost everywhere on $(0, \pi)$.

Proof Before proving the theorem, we will mention some results, which will be needed later. We get the initial value problems

$$-u'' + \left[q(x) - \frac{1}{4 \sin^2 x} \right] u = \lambda u, \quad (5)$$

$$u(0) = 0 \quad (6)$$

and

$$-\tilde{u}'' + \left[\tilde{q}(x) - \frac{1}{4 \sin^2 x} \right] \tilde{u} = \lambda \tilde{u}, \quad (7)$$

$$\tilde{u}(0) = 0. \quad (8)$$

It can be shown that there exists a kernel $K(x, t)$ ($\tilde{K}(x, t)$) continuous on $(0, \pi) \times (0, \pi)$ such that by using the transformation operator every solution of equations (5), (6) and (7), (8)

can be expressed in the form

$$u(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x K(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt, \quad (9)$$

$$\tilde{u}(x, \lambda) = \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \int_0^x \tilde{K}(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt, \quad (10)$$

respectively, where the kernel $K(x, t)$ ($\tilde{K}(x, t)$) is the solution of the equation

$$\frac{\partial^2 K(x, t)}{\partial x^2} + \left[\frac{1}{4 \sin^2 x} - q(x) \right] K(x, t) = \frac{\partial^2 K(x, t)}{\partial t^2} + \frac{1}{4 \sin^2 t} K(x, t)$$

subject to the boundary conditions

$$\begin{aligned} 2 \frac{dK(x, x)}{dx} &= q(x), \\ K(x, 0) &= 0. \end{aligned}$$

Using equations (9) and (10), we find that

$$\begin{aligned} u\tilde{u} &= \frac{\sin^2 \sqrt{\lambda} x}{\lambda} + \int_0^x \left[K(x, t) + \tilde{K}(x, t) \right] \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt \\ &\quad + \int_0^x K(x, t) \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} dt \times \int_0^x \tilde{K}(x, s) \frac{\sin \sqrt{\lambda} s}{\sqrt{\lambda}} ds. \end{aligned} \quad (11)$$

If the range of $K(x, t)$ and $\tilde{K}(x, t)$ is extended with respect to the second argument, by using the trigonometric addition formulas and change of variables, we find that

$$u\tilde{u} = \left\{ \left(1 - \cos 2\sqrt{\lambda} x \right) \times \int_0^x \tilde{K}(x, \tau) \left(1 - \cos 2\sqrt{\lambda} \tau \right) d\tau \right\}, \quad (12)$$

where

$$\begin{aligned} \tilde{K}(x, t) &= 2 \left[K(x, x - 2\tau) + \tilde{K}(x, x - 2\tau) \right] + 2 \left[\int_{-x+2\tau}^x K(x, s) \tilde{K}(x, s - 2\tau) ds \right. \\ &\quad \left. + \int_{-x}^{x-2\tau} K(x, s) \tilde{K}(x, s + 2\tau) ds \right]. \end{aligned} \quad (13)$$

Next, we define a function

$$\Omega(\lambda) = u'(\pi, \lambda) + Hu(\pi, \lambda) \quad (14)$$

Zeros of $\Omega(\lambda)$ are the eigenvalues of L or $\tilde{L}$ subject to equations (5), (6) and if the asymptotic results of u and $\tilde{u}$ are considered, $\Omega(\lambda)$ is an entire function of order $1/2$ of λ . If we multiply

equation (5) by $\tilde{u}$, equation (7) by u and subtract, we obtain, after integration,

$$(\tilde{u}u' - u\tilde{u}')|_0^{\pi-\varepsilon} + \int_0^\pi (\tilde{q} - q)u\tilde{u} \, dx = 0. \quad (15)$$

Using equations (6) and (8), we obtain for

$$[\tilde{u}(\pi, \lambda)u'(\pi, \lambda) - u(\pi, \lambda)\tilde{u}'(\pi, \lambda)] + \int_0^{\pi/2} (\tilde{q} - q)u\tilde{u} \, dx = 0. \quad (16)$$

When we consider equation (3) for u and $\tilde{u}$, the first term of equation (16) is zero. Let

$$Q = \tilde{q} - q \quad (17)$$

and

$$K(\lambda) = \int_0^{\pi/2} Q(x)u\tilde{u} \, dx. \quad (18)$$

If the properties of u and $\tilde{u}$ are considered, the function $K(\lambda)$ is an entire function for $\lambda = \lambda_n$, since the first term of equation (16) is zero, then

$$K(\lambda_n) = 0. \quad (19)$$

In addition, using equations (9) and (18) for $0 < x < \pi$

$$|K(\lambda)| \leq M \frac{\pi}{2} |b|, \quad b = \operatorname{Im} \left| \sqrt{\lambda} \right|, \quad (20)$$

where M is constant. Now, we define a rational function

$$\Psi(\lambda) = \frac{K(\lambda)}{\Omega(\lambda)}. \quad (21)$$

$\Psi(\lambda)$ is an entire function. Asymptotic form of $\Omega(\lambda)$ and with equation (21), we get

$$|\Psi(\lambda)| = O\left(\frac{1}{\sqrt{\lambda}}\right).$$

So, for all λ , from the Liouville theorem

$$\Psi(\lambda) = 0 \quad (22a)$$

or

$$K(\lambda) = 0. \quad (22b)$$

From now on, substituting equation (13) into equation (19)

$$\int_0^{\pi/2} Q(x) [1 - \cos 2\sqrt{\lambda}x] \, dx + \int_0^{\pi/2} Q(x) \int_0^x \tilde{K}(x, \tau) \cos 2\sqrt{\lambda}\tau \, d\tau \, dx = 0. \quad (23)$$

Using the change of variables, after some reordering, it can be written as

$$\int_0^{\pi/2} Q(x) \, dx - \int_0^{\pi/2} \cos 2\sqrt{\lambda}\tau \left[Q(\tau) + \int_\tau^{\pi/2} Q(x) \tilde{K}(x, \tau) \, dx \right] d\tau = 0.$$

Letting $\lambda \rightarrow \infty$ for real λ , we see from Riemann–Lebesgue Lemma that we must have

$$\int_0^{\pi/2} Q(x) dx = 0 \quad (24)$$

and

$$\int_0^{\pi/2} \cos 2\sqrt{\lambda}\tau \left[Q(\tau) + \int_{\tau}^{\pi/2} Q(x) \tilde{K}(x, \tau) dx \right] d\tau = 0. \quad (25)$$

On the other hand, from the completeness of the function $\cos$ we see that

$$Q(\tau) + \int_{\tau}^{\pi/2} Q(x) \tilde{K}(x, \tau) dx = 0, \quad 0 < \tau < \frac{\pi}{2}. \quad (26)$$

Since equation (26) is a Volterra integral equation, it has only trivial solution. Hence,

$$\tilde{q} = q, \quad \text{almost everywhere on } (0, \pi).$$

This completes the proof. ■

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Abel's method on summation by parts and hypergeometric contiguous relations

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The modified Abel lemma on summation by parts is employed to investigate contiguous relations of classical hypergeometric ${}_3F_2$ -series. Several known and new two- and three-term relations are established. Inspired by the recent work due to Krattenthaler and Rivoal [Krattenthaler, C. and Rivoal, T., 2006, How can we escape Thomae's relations. *Journal of Mathematical Society of Japan*, **58**, 183–210.], we shall construct further counter-examples to the conjecture made by Rhin and Viola [Rhin, G. and Viola, C. 2001, The group structure for $\zeta 3$. *Acta Arithmetica*, **97**(3), 269–293.], which claims that two linearly dependent ${}_3F_2(1)$ -series of irrational values over $\mathbb{Q}$ satisfy one of Thomae's relations.

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Generally, the classical hypergeometric series is defined as a series $\sum C_n$ with term ratio C_{n+1}/C_n , a rational function of n . Following Bailey [1, §2.1] and Slater [2, §2.1], it can be explicitly written as

$${}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!},$$

where we suppose that none of the denominator parameters is a non-positive integer so that the series is well defined. Here, we have used the notation of the rising shifted factorial given by

$$(x)_0 \equiv 1 \quad \text{and} \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1) \cdots (x+n-1), \quad n = 1, 2, \dots,$$

with the Γ -function being defined through Euler integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du \quad \text{with} \quad \Re(x) > 0.$$

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The fractional forms of the shifted factorials and the Γ -function are abbreviated, respectively, as

$$\begin{aligned} \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, D \end{matrix} \right]_n &= \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (D)_n}, \\ \Gamma \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, D \end{matrix} \right] &= \frac{\Gamma(\alpha) \Gamma(\beta) \cdots \Gamma(\gamma)}{\Gamma(A) \Gamma(B) \cdots \Gamma(D)}. \end{aligned}$$

Throughout the paper, we shall also use the *shifted* hypergeometric series

$${}_pF_q^* \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix} \middle| z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{n z^n}{n!}$$

whose summation index begins essentially with $n = 1$, instead of $n = 0$.

Recently, Abel's lemma on summation by parts has successfully been used by Chu [3] to review several fundamental summation theorems of classical hypergeometric series. For the subsequent application, we reproduce this lemma as follows. For an arbitrary complex sequence τ_k , define the backward and forward difference operators ∇ and Δ , respectively, by

$$\nabla \tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta \tau_k = \tau_k - \tau_{k+1}.$$

It should be pointed out that Δ is adopted for convenience in the present paper, which differs from the usual operator Δ only by the minus sign. Then *Abel's lemma* on summation by parts for non-terminating series may be reformulated as

$$\sum_{k=0}^{\infty} U_k \Delta V_k = U_{-1} V_0 + \sum_{k=0}^{\infty} V_k \nabla U_k,$$

provided that one of the series on both sides converges and $U_k V_{k+1} \rightarrow 0$ as $k \rightarrow \infty$.

For all the series treated in this paper, the extra term $U_{-1} V_0$ will vanish because of the presence of $k! = \Gamma(k+1)$ in the denominator of U_k , which forces $U_{-1} = 0$.

Proof Let n be a natural number. According to the definition of the forward difference, we have

$$\sum_{k=0}^n U_k \Delta V_k = \sum_{k=0}^n U_k \{V_k - V_{k+1}\} = \sum_{k=0}^n U_k V_k - \sum_{k=0}^n U_k V_{k+1}.$$

Replacing k by $k-1$ for the last sum, we get the following expression:

$$\begin{aligned} \sum_{k=0}^n U_k \Delta V_k &= U_{-1} V_0 - U_n V_{n+1} + \sum_{k=0}^n V_k \{U_k - U_{k-1}\} \\ &= U_{-1} V_0 - U_n V_{n+1} + \sum_{k=0}^n V_k \nabla U_k. \end{aligned}$$

Letting $n \rightarrow \infty$ leads us to the identity stated in the lemma. ■

In this paper, we shall employ this lemma further to investigate systematically contiguous relations of hypergeometric series and their applications. In section 1, the modified Abel lemma on summation by parts will be utilized to establish four very general three-term contiguous relations of non-terminating ${}_3F_2(1)$ -series, denominated by patterns 'A', 'B', 'C' and 'D'. Then

they will be applied in section 2 to derive 10 two-term relations of the ${}_3F_2(1)$ -series, including the four main theorems discovered recently by Krattenthaler and Rivoal [4]. Section 3 will be devoted to the proof of 17 exotic relations as well as seven two-term exotic relations from §3.1 to §3.15. Finally in the last section, we shall construct, by means of these two-term contiguous and exotic relations, further counter-examples to the conjecture due to Rhin–Viola, who claim that two linearly dependent ${}_3F_2(1)$ -series of irrational values with integer parameters over $\mathbb{Q}$ satisfy one of Thomae's relations.

1. Four patterns of three-term relations

In this section, we show four patterns A, B, C and D of contiguous relations satisfied by three generic hypergeometric ${}_3F_2(1)$ -series through the modified Abel lemma on summation by parts. In order to justify the limiting process, we have to invoke the asymptotic relation

$$\Gamma(x+n) \simeq n^x(n-1)! \quad \text{as } n \longrightarrow +\infty$$

which follows easily from the Stirling asymptotic formula. Throughout this section, we assume that the parameters of all the ${}_3F_2\left[\begin{smallmatrix} a, c, e \\ b, d \end{smallmatrix} \middle| 1 \right]$ series are subject to the condition $\Re(b+d-a-c-e) > 1$ in order that the modified Abel lemma on summation by parts can be applied smoothly.

1.1 Pattern A

For the two sequences defined by

$$U_k = \begin{bmatrix} 1+a, & d-a \\ 1, & d \end{bmatrix}_k \quad \text{and} \quad V_k = \begin{bmatrix} c, & e \\ b, & d-a-1 \end{bmatrix}_k,$$

we can easily check the following limiting relation

$$U_{-1}V_0 = \lim_{n \rightarrow +\infty} U_n V_{n+1} = 0$$

as well as the finite differences

$$\begin{aligned} \nabla U_k &= \begin{bmatrix} a, & d-a-1 \\ 1, & d \end{bmatrix}_k, \\ \Delta V_k &= \begin{bmatrix} c, & e \\ 1+b, & d-a \end{bmatrix}_k \left\{ \frac{b(1+a-d)+ce}{b(1+a-d)} + \frac{k(1+a+c+e-b-d)}{b(1+a-d)} \right\}. \end{aligned}$$

According to the modified Abel lemma on summation by parts, we can manipulate the following ${}_3F_2$ -series:

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] &= \sum_{k \geq 0} \begin{bmatrix} a, d-a-1 \\ 1, d \end{bmatrix}_k \begin{bmatrix} c, e \\ b, d-a-1 \end{bmatrix}_k = \sum_{k \geq 0} V_k \nabla U_k = \sum_{k \geq 0} U_k \Delta V_k \\ &= \frac{b(1+a-d)+ce}{b(1+a-d)} \sum_{k \geq 0} \begin{bmatrix} 1+a, d-a \\ 1, d \end{bmatrix}_k \begin{bmatrix} c, e \\ 1+b, d-a \end{bmatrix}_k \\ &\quad + \frac{(1+a+c+e-b-d)}{b(1+a-d)} \sum_{k \geq 0} k \begin{bmatrix} 1+a, d-a \\ 1, d \end{bmatrix}_k \begin{bmatrix} c, e \\ 1+b, d-a \end{bmatrix}_k. \end{aligned}$$

Shifting the summation index by $k \rightarrow k + 1$ for the last sum, we state the result as the following contiguous relation.

THEOREM 1 (Pattern A) *For five complex numbers a, b, c, d, e subject to the condition $\Re(b + d - a - c - e) > 1$, there holds the three-term relation:*

$${}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = \mathcal{A} {}_3F_2 \left[\begin{matrix} a+1, c, e \\ b+1, d \end{matrix} \middle| 1 \right] + \mathbb{A} {}_3F_2 \left[\begin{matrix} a+2, c+1, e+1 \\ b+2, d+1 \end{matrix} \middle| 1 \right], \quad (1a)$$

$${}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = \mathcal{A} {}_3F_2 \left[\begin{matrix} a+1, c, e \\ b+1, d \end{matrix} \middle| 1 \right] + \mathfrak{A} {}_3F_2^* \left[\begin{matrix} a+1, c, e \\ b+1, d \end{matrix} \middle| 1 \right], \quad (1b)$$

where the coefficients $\mathcal{A}$, $\mathbb{A}$ and $\mathfrak{A}$ are defined by

$$\mathcal{A} := \mathcal{A}(a, c, e; b, d) = \frac{(1+a-d)b+ce}{(1+a-d)b}, \quad (1c)$$

$$\mathbb{A} := \mathbb{A}(a, c, e; b, d) = \frac{(1+a+c+e-b-d)(1+a)ce}{(1+a-d)(1+b)bd}, \quad (1d)$$

$$\mathfrak{A} := \mathfrak{A}(a, c, e; b, d) = \frac{1+a+c+e-b-d}{(1+a-d)b}. \quad (1e)$$

1.2 Pattern B

For the two sequences defined by

$$U_k = \left[\begin{matrix} c, & e \\ d-1, & 1+c+e-d \end{matrix} \right]_k \quad \text{and} \quad V_k = \left[\begin{matrix} a, & 2+c+e-d \\ 1, & b \end{matrix} \right]_k$$

we can verify without difficulty the following limiting relation

$$U_0 V_{-1} = \lim_{n \rightarrow +\infty} U_{n+1} V_n = 0$$

as well as the finite differences

$$\begin{aligned} \Delta U_k &= \left[\begin{matrix} c, & e \\ d, & 2+c+e-d \end{matrix} \right]_k \frac{(1+c-d)(1+e-d)}{(1+c+e-d)(1-d)}, \\ \nabla V_k &= \left[\begin{matrix} a-1, & 1+c+e-d \\ 1, & b \end{matrix} \right]_k \left\{ 1 + k \frac{1+a+c+e-b-d}{(a-1)(1+c+e-d)} \right\}. \end{aligned}$$

By means of the modified Abel lemma on summation by parts, we can reformulate the following ${}_3F_2$ -series:

$$\begin{aligned} \frac{(1+c-d)(1+e-d)}{(1+c+e-d)(1-d)} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= \sum_{k \geq 0} V_k \Delta U_k = \sum_{k \geq 0} U_k \nabla V_k \\ &= \sum_{k \geq 0} \left[\begin{matrix} a-1, & c, & e \\ 1, & b, & d-1 \end{matrix} \right]_k + \frac{(1+a+c+e-b-d)}{(a-1)(1+c+e-d)} \sum_{k \geq 0} k \left[\begin{matrix} a-1, & c, & e \\ 1, & b, & d-1 \end{matrix} \right]_k, \end{aligned}$$

which leads to the following contiguous relation.

THEOREM 2 (Pattern B) For five complex numbers a, b, c, d, e subject to $\Re(b + d - a - c - e) > 1$, there holds the three-term relation:

$${}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = \mathcal{B} {}_3F_2 \left[\begin{matrix} a-1, c, e \\ b, d-1 \end{matrix} \middle| 1 \right] + \mathbb{B} {}_3F_2 \left[\begin{matrix} a, c+1, e+1 \\ b+1, d \end{matrix} \middle| 1 \right] \quad (2a)$$

$${}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = \mathcal{B} {}_3F_2 \left[\begin{matrix} a-1, c, e \\ b, d-1 \end{matrix} \middle| 1 \right] + \mathfrak{B} {}_3F_2^* \left[\begin{matrix} a-1, c, e \\ b, d-1 \end{matrix} \middle| 1 \right], \quad (2b)$$

where the coefficients $\mathcal{B}$, $\mathbb{B}$ and $\mathfrak{B}$ are defined by

$$\mathcal{B} := \mathcal{B}(a, c, e; b, d) = \frac{(1+c+e-d)(1-d)}{(1+c-d)(1+e-d)}, \quad (2c)$$

$$\mathbb{B} := \mathbb{B}(a, c, e; b, d) = \frac{(1+a+c+e-b-d)ce}{(1+c-d)(d-e-1)b}, \quad (2d)$$

$$\mathfrak{B} := \mathfrak{B}(a, c, e; b, d) = \frac{(1+a+c+e-b-d)(1-d)}{(1-a)(1+c-d)(d-e-1)}. \quad (2e)$$

1.3 Pattern C

Define the two sequences by

$$U_k = \begin{bmatrix} 1+c, & 1+e \\ 1, & 1+c+e \end{bmatrix}_k \quad \text{and} \quad V_k = \begin{bmatrix} a, & 1+c+e \\ b, & d \end{bmatrix}_k.$$

Then it is not hard to check the following limiting relation

$$U_{-1}V_0 = \lim_{n \rightarrow +\infty} U_n V_{n+1} = 0$$

as well as the finite differences

$$\begin{aligned} \nabla U_k &= \begin{bmatrix} c, & e \\ 1, & 1+c+e \end{bmatrix}_k, \\ \Delta V_k &= \begin{bmatrix} a, & 1+c+e \\ 1+b, & 1+d \end{bmatrix}_k \left\{ \frac{bd-a(1+c+e)}{b \times d} + k \frac{b+d-a-c-e-1}{b \times d} \right\}. \end{aligned}$$

In view of the modified Abel lemma on summation by parts, we can proceed with the ${}_3F_2$ -series as follows:

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= \sum_{k \geq 0} V_k \nabla U_k = \sum_{k \geq 0} U_k \Delta V_k \\ &= \frac{bd-a(1+c+e)}{b \times d} \sum_{k \geq 0} \begin{bmatrix} a, & 1+c, & 1+e \\ 1, & 1+b, & 1+d \end{bmatrix}_k \\ &\quad + \frac{b+d-a-c-e-1}{b \times d} \sum_{k \geq 0} k \begin{bmatrix} a, & 1+c, & 1+e \\ 1, & 1+b, & 1+d \end{bmatrix}_k. \end{aligned}$$

We therefore establish the following contiguous relation.

THEOREM 3 (Pattern C) For five complex numbers a, b, c, d, e subject to $\Re(b + d - a - c - e) > 1$, there holds the three-term relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \mathcal{C} {}_3F_2 \left[\begin{matrix} a, & 1+c, & 1+e \\ & 1+b, & 1+d \end{matrix} \middle| 1 \right] + \mathbb{C} {}_3F_2 \left[\begin{matrix} 1+a, & 2+c, & 2+e \\ & 2+b, & 2+d \end{matrix} \middle| 1 \right], \quad (3a)$$

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \mathcal{C} {}_3F_2 \left[\begin{matrix} a, & 1+c, & 1+e \\ & 1+b, & 1+d \end{matrix} \middle| 1 \right] + \mathfrak{C} {}_3F_2^* \left[\begin{matrix} a, & 1+c, & 1+e \\ & 1+b, & 1+d \end{matrix} \middle| 1 \right], \quad (3b)$$

where the coefficients $\mathcal{C}$, $\mathbb{C}$ and $\mathfrak{C}$ are defined by

$$\mathcal{C} := \mathcal{C}(a, c, e; b, d) = \frac{bd - a(1 + c + e)}{b \times d}, \quad (3c)$$

$$\mathbb{C} := \mathbb{C}(a, c, e; b, d) = \frac{(b + d - a - c - e - 1)(1 + c)(1 + e)a}{(1 + b)(1 + d)bd}, \quad (3d)$$

$$\mathfrak{C} := \mathfrak{C}(a, c, e; b, d) = \frac{b + d - a - c - e - 1}{bd}. \quad (3e)$$

1.4 Pattern D

Define the two sequences by

$$U_k = \left[\begin{matrix} a, & b + d - a - 2 \\ b - 1, & d - 1 \end{matrix} \right]_k \quad \text{and} \quad V_k = \left[\begin{matrix} c, & e \\ 1, & b + d - a - 2 \end{matrix} \right]_k.$$

Then it is not difficult to verify the following limiting relation

$$U_0 V_{-1} = \lim_{n \rightarrow +\infty} U_{n+1} V_n = 0$$

as well as the finite differences

$$\Delta U_k = \left[\begin{matrix} a, & b + d - a - 2 \\ b, & d \end{matrix} \right]_k \frac{(1 + a - b)(1 + a - d)}{(b - 1)(d - 1)},$$

$$\nabla V_k = \left[\begin{matrix} c - 1, & e - 1 \\ 1, & b + d - a - 2 \end{matrix} \right]_k \left\{ 1 + k \frac{1 + a + c + e - b - d}{(c - 1)(e - 1)} \right\}.$$

Applying the modified Abel lemma on summation by parts, we can deal with the ${}_3F_2$ -series as follows:

$$\begin{aligned} & \frac{(1 + a - b)(1 + a - d)}{(b - 1)(d - 1)} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \sum_{k \geq 0} V_k \Delta U_k = \sum_{k \geq 0} U_k \nabla V_k \\ & = \sum_{k \geq 0} \left[\begin{matrix} a, & c - 1, e - 1 \\ 1, b - 1, d - 1 \end{matrix} \right]_k + \frac{1 + a + c + e - b - d}{(c - 1)(e - 1)} \\ & \quad \times \sum_{k \geq 0} k \left[\begin{matrix} a, & c - 1, & e - 1 \\ 1, & b - 1, & d - 1 \end{matrix} \right]_k. \end{aligned}$$

This may be expressed in terms of the following contiguous relation.

THEOREM 4 (Pattern D) For five complex numbers a, b, c, d, e subject to $\Re(b + d - a - c - e) > 1$, there holds the three-term relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \mathcal{D} {}_3F_2 \left[\begin{matrix} a, & c-1, & e-1 \\ & b-1, & d-1 \end{matrix} \middle| 1 \right] + \mathbb{D} {}_3F_2 \left[\begin{matrix} 1+a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right], \quad (4a)$$

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \mathcal{D} {}_3F_2 \left[\begin{matrix} a, & c-1, & e-1 \\ & b-1, & d-1 \end{matrix} \middle| 1 \right] + \mathfrak{D} {}_3F_2^* \left[\begin{matrix} a, & c-1, & e-1 \\ & b-1, & d-1 \end{matrix} \middle| 1 \right], \quad (4b)$$

where the coefficients $\mathcal{D}$, $\mathbb{D}$ and $\mathfrak{D}$ are defined by

$$\mathcal{D} := \mathcal{D}(a, c, e; b, d) = \frac{(1-b)(1-d)}{(1+a-b)(1+a-d)}, \quad (4c)$$

$$\mathbb{D} := \mathbb{D}(a, c, e; b, d) = \frac{a(1+a+c+e-b-d)}{(1+a-b)(1+a-d)}, \quad (4d)$$

$$\mathfrak{D} := \mathfrak{D}(a, c, e; b, d) = \frac{(1+a+c+e-b-d)(1-b)(1-d)}{(1+a-b)(1+a-d)(1-c)(1-e)}. \quad (4e)$$

2. Two-term contiguous relations for ${}_3F_2$ -series

By comparing the four patterns A, B, C and D, we obtain several two-term contiguous relations for ${}_3F_2(1)$ -series. This shows that there exist many more two-term contiguous relations hidden behind the theorems due to Krattenthaler and Rivoal [4]. We shall label each section by the two letters corresponding to the combination of two patterns chosen from the last section.

2.1 A&A

Let $\text{Eq}'(1a)$ stand for $\text{Eq}(1a)$ under the parameter replacements

$$a \longrightarrow c-1, \quad c \longrightarrow a+1, \quad b \longrightarrow d-1, \quad d \longrightarrow b+1.$$

Then consider the linear combination of two equations

$$[\text{AA}]: \quad \text{Eq}(1a) - \text{Eq}'(1a) \frac{(1+b-c)(1-d)}{(1+a-d)b} \quad \text{with } d = 1 + e - \frac{a(b-e)}{(1-c)}.$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AA] leads us to the following identity.

THEOREM 5 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & 1+e-\frac{a(b-e)}{1-c} \end{matrix} \middle| 1 \right] &= {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b+1, & e-\frac{a(b-e)}{1-c} \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(1+b-c)(ab+ce-ae-e)}{b(a+ab+ce-ac-ae-e)}. \end{aligned}$$

Specifying the parameters by

$$a \longrightarrow \alpha, \quad b \longrightarrow 1 + \alpha, \quad c \longrightarrow \beta, \quad e \longrightarrow \gamma,$$

we recover the following relation with one free parameter less.

PROPOSITION 6 [4, Theorem 2] *For three complex numbers α, β, γ subject to $\Re(2 - \beta - \alpha(\alpha - \gamma + 1)/(\beta - 1)) > 0$, there holds the relation:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} \alpha, \beta, \gamma \\ \alpha + 1, \gamma + \frac{\alpha(\alpha - \gamma + 1)}{\beta - 1} + 1 \end{matrix} \middle| 1 \right] &= {}_3F_2 \left[\begin{matrix} \alpha + 1, \beta - 1, \gamma \\ \alpha + 2, \gamma + \frac{\alpha(\alpha - \gamma + 1)}{\beta - 1} \end{matrix} \middle| 1 \right] \\ &\times \frac{(\alpha + \beta + 2)(\alpha + \alpha^2 - \gamma - \alpha\gamma + \beta\gamma)}{(1 + \alpha)(2\alpha + \alpha^2 - \alpha\beta - \gamma - \alpha\gamma + \beta\gamma)}. \end{aligned}$$

2.2 A&B

Let Eq'(2a) stand for Eq(2a) under the parameter replacements

$$a \longrightarrow a + 2, \quad b \longrightarrow d, \quad d \longrightarrow b + 2.$$

Then consider the linear combination of two equations

$$\begin{aligned} [\text{AB}] : \quad & \text{Eq}(1a) - \text{Eq}'(2a) \frac{(1 + a)(1 + b - c)(1 + b - e)}{b(b + 1)(d - a - 1)} \quad \text{with} \\ & d = 2 + 2a + \frac{(1 + a)(1 - c - e) + ce}{b}. \end{aligned}$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AB] leads us to the following identity.

THEOREM 7 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a, c, e \\ b, 2 + 2a + \frac{(a + 1)(1 - c - e) + ce}{b} \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} 2 + a, c, e \\ 2 + b, 2 + 2a + \frac{(a + 1)(1 - c - e) + ce}{b} \end{matrix} \middle| 1 \right] \\ &\times \frac{(1 + a)(1 + b - e)(1 + b - c)}{(1 + b)(a + b + ab + ce - c - e - ac - ae + 1)}. \end{aligned}$$

Specifying the parameters by

$$a \longrightarrow \alpha, \quad c \longrightarrow \beta, \quad e \longrightarrow \gamma, \quad b \longrightarrow 1 + \alpha,$$

we recover the following relation with one free parameter less.

PROPOSITION 8 [4, Theorem 10] *For three complex numbers α, β, γ such that $\Re(4 + 2\alpha - 2\beta - 2\gamma + \beta/(\alpha + 1)) > 0$, we have the identity*

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} \alpha, \beta, \gamma \\ \alpha + 1, 2\alpha - \beta - \gamma + \frac{\beta\gamma}{\alpha + 1} + 3 \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} \alpha + 2, \beta, \gamma \\ \alpha + 3, 2\alpha - \beta - \gamma + \frac{\beta\gamma}{\alpha + 1} + 3 \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(1 + \alpha)(\alpha - \beta + 2)(\alpha - \gamma + 2)}{(2 + \alpha)(3\alpha + \alpha^2 - \beta - \alpha\beta - \gamma - \alpha\gamma + \beta\gamma + 2)}. \end{aligned}$$

2.3 A&B

Let $\text{Eq}'(2a)$ stand for $\text{Eq}(2a)$ under the parameter replacements

$$a \longrightarrow a + 2, \quad b \longrightarrow b + 1, \quad d \longrightarrow d + 1.$$

Then consider the linear combination of two equations

$$\begin{aligned} [\text{AB}] : \quad & \text{Eq}(1a) - \text{Eq}'(2a) \frac{(a + 1)(d - c)(d - e)}{bd(d - a - 1)} \quad \text{with} \\ & d = 1 + a - \frac{(1 + a - c)(1 + a - e)}{1 + a - b}. \end{aligned}$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AB] leads us to the following identity.

THEOREM 9 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a, \quad c, \quad e \\ b, 1 + a - \frac{(1 + a - c)(1 + a - e)}{1 + a - b} \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} a + 2, \quad c, \quad e \\ 1 + b, 2 + a - \frac{(1 + a - c)(1 + a - e)}{1 + a - b} \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(1 + a)(b - e)(c - b)}{b(c + e - b + ac + ae - ab - ce)}. \end{aligned}$$

2.4 A&B

Let $\text{Eq}'(2a)$ stand for $\text{Eq}(2a)$ under the parameter replacements

$$a \longrightarrow c + 1, \quad c \longrightarrow a + 1, \quad b \longrightarrow d, \quad d \longrightarrow b + 2.$$

Then consider the linear combination of two equations

$$[\text{AB}] : \quad \text{Eq}(1a) - \text{Eq}'(2a) \frac{c(b - a)(1 + b - e)}{b(b + 1)(d - a - 1)} \quad \text{with } d = 1 + a + c - \frac{ac}{b}.$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AB] leads us to the following identity.

THEOREM 10 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & 1+a+c-\frac{ac}{b} \end{matrix} \middle| 1 \right] = \frac{1+b-e}{1+b} {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e \\ & b+2, & 1+a+c-\frac{ac}{b} \end{matrix} \middle| 1 \right].$$

Specifying the parameters by

$$a \longrightarrow \beta, \quad c \longrightarrow \gamma, \quad e \longrightarrow \beta + \gamma - \delta, \quad b \longrightarrow \frac{\beta\gamma}{\delta},$$

we recover the following relation with one free parameter less.

PROPOSITION 11 [4, Theorem 9] *For three complex numbers α, β, γ such that $\Re(\beta\gamma/\delta - \beta - \gamma + 1) > 0$, we have the identity:*

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} \beta + \gamma - \delta, \beta, \gamma \\ \beta + \gamma - \delta + 1, \frac{\beta\gamma}{\delta} \end{matrix} \middle| 1 \right] \\ &= \frac{\beta\gamma + \delta - \beta\delta - \gamma\delta + \delta^2}{\beta\gamma + \delta} {}_3F_2 \left[\begin{matrix} \beta + \gamma - \delta, \beta + 1, \gamma + 1 \\ \beta + \gamma - \delta + 1, \frac{\beta\gamma}{\delta} + 2 \end{matrix} \middle| 1 \right]. \end{aligned}$$

2.5 A&C

Let $\text{Eq}'(3a)$ stand for $\text{Eq}(3a)$ under the parameter replacements

$$a \longrightarrow a + 1, \quad c \longrightarrow c - 1, \quad e \longrightarrow e - 1, \quad d \longrightarrow d - 1.$$

Then consider the linear combination of two equations

$$[\text{AC}]: \quad \text{Eq}(1a) - \text{Eq}'(3a) \frac{d-1}{d-a-1} \quad \text{with } b = c + e - 1 - \frac{(1-c)(1-e)}{a}.$$

With this specific value b , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AC] leads us to the following identity.

THEOREM 12 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a, & c, & e \\ d, & c+e-1-\frac{(1-c)(1-e)}{a} \end{matrix} \middle| 1 \right] \\ &= \frac{1-d}{1+a-d} {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e-1 \\ d-1, & c+e-1-\frac{(1-c)(1-e)}{a} \end{matrix} \middle| 1 \right]. \end{aligned}$$

2.6 A&D

Let $\text{Eq}'(4a)$ stand for $\text{Eq}(4a)$ under the parameter replacements

$$a \longrightarrow c, \quad c \longrightarrow a + 2, \quad e \longrightarrow e + 1, \quad b \longrightarrow b + 2, \quad d \longrightarrow d + 1.$$

Then consider the linear combination of two equations

$$\begin{aligned} [\text{AD}]: \quad & \text{Eq}(1a) - \text{Eq}'(4a) \frac{(1+b-c)(a+1)(d-c)e}{(1+a-d)(b+1)bd} \quad \text{with} \\ & d = 1 + a - \frac{(1+a-c)e}{b}. \end{aligned}$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [AD] leads us to the following identity.

THEOREM 13 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, \\ b, & a + \frac{e(c-a-1)}{b} + 1 \end{matrix} \middle| 1 \right] &= {}_3F_2 \left[\begin{matrix} a+2, & c, \\ b+2, & a + \frac{e(c-a-1)}{b} + 2 \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(1+a)(b-e)(1+b-c)}{(1+b)(b+ab+ce-ae-e)}. \end{aligned}$$

2.7 B&C

Let $\text{Eq}'(3a)$ stand for $\text{Eq}(3a)$ under the parameter replacements

$$a \longrightarrow c, \quad c \longrightarrow a - 2, \quad e \longrightarrow e - 1, \quad b \longrightarrow b - 1, \quad d \longrightarrow d - 2.$$

Then consider the linear combination of two equations

$$\begin{aligned} [\text{BC}]: \quad & \text{Eq}(2a) - \text{Eq}'(3a) \frac{(1-b)(1-d)(2-d)}{(1-a)(1+c-d)(1+e-d)} \quad \text{with} \\ & d = 2 + \frac{(1-e)(1+c-a)}{a-b}. \end{aligned}$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [BC] leads us to the following identity.

THEOREM 14 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & 2 + \frac{(e-1)(1+c-a)}{b-a} \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} a-2, & c, & e-1 \\ b-1, & \frac{(1-e)(1+c-a)}{a-b} \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(b-1)(1+c-a)(b+e-c+ce-ae-1)}{(1-a)(1+c-b)(bc-ac-b-e-ce+ae+c+1)}. \end{aligned}$$

2.8 C&D

Let $\text{Eq}'(4a)$ stand for $\text{Eq}(4a)$ under the parameter replacements

$$c \longrightarrow c+2, \quad e \longrightarrow e+2, \quad b \longrightarrow b+2, \quad d \longrightarrow d+2.$$

Then consider the linear combination of two equations

$$\begin{aligned} [\text{CD}] : \quad & \text{Eq}(3a) - \text{Eq}'(4a) \frac{(c+1)(e+1)(a-b-1)(d-a+1)}{bd(b+1)(d+1)} \quad \text{with} \\ & d = \frac{a(c+e+1) - (c+1)(e+1)}{b}. \end{aligned}$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [CD] leads us to the following identity.

THEOREM 15 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & \frac{(a-1)(c+e+1)-ce}{b} \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} a, & c+2, & e+2 \\ b+2, & 2 + \frac{(a-1)(c+e+1)-ce}{b} \end{matrix} \middle| 1 \right] \\ &\quad \times \frac{(1+c)(1+e)(a-b-1)\{(c+e-b+1)(a-1)-ce\}}{(b+1)\{(a-1)(c+e+1)-ce\}\{b-ce+(a-1)(c+e+1)\}}. \end{aligned}$$

2.9 C&D

Let $\text{Eq}'(4a)$ stand for $\text{Eq}(4a)$ under the parameter replacements

$$a \longrightarrow c+1, \quad c \longrightarrow a+1, \quad e \longrightarrow e+2, \quad b \longrightarrow b+2, \quad d \longrightarrow d+2.$$

Then consider the linear combination of two equations

$$[\text{CD}] : \quad \text{Eq}(3a) - \text{Eq}'(4a) \frac{a(e+1)(c-b)(d-c)}{bd(b+1)(d+1)} \quad \text{with } d = \frac{ac}{b}.$$

With this specific value d , we can check that the right member of the last equation vanishes. After some simplification, the corresponding left-hand side of [CD] leads us to the following identity.

THEOREM 16 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, c, & e \\ b, & \frac{ac}{b} \end{matrix} \middle| 1 \right] = \frac{(c-b)(1+e)(a-b)}{(b+1)(b+ac)} {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e+2 \\ b+2, & 2+\frac{ac}{b} \end{matrix} \middle| 1 \right].$$

2.10 Contiguous relation for nearly poised series

In addition, we are now going to prove a very general contiguous relation for the nearly poised series.

For this purpose, let us first define two sequences by

$$\mathcal{A}_k = \begin{bmatrix} 1+2a, & 1+c, & 1+a-c \\ 1, & 1+2a-c, & 1+a+c \end{bmatrix}_k \quad \text{and} \quad \mathcal{B}_k = \begin{bmatrix} b+\lambda, & d-\lambda \\ b, & d \end{bmatrix}_k.$$

Then it is almost trivial to verify the following differences

$$\begin{aligned} \nabla \mathcal{A}_k &= \begin{bmatrix} 2a, & c, & a-c \\ 1, & 1+2a-c, & 1+a+c \end{bmatrix}_k, \\ \Delta \mathcal{B}_k &= \begin{bmatrix} b+\lambda, & d-\lambda \\ 1+b, & 1+d \end{bmatrix}_k \frac{\lambda(\lambda+b-d)}{b \times d}, \end{aligned}$$

as well as the following limiting relation

$$A_{-1}B_0 = \lim_{n \rightarrow +\infty} A_n B_{n+1} = 0.$$

Applying the modified Abel lemma on summation by parts, we can manipulate the following ${}_5F_4$ -series:

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} 2a, & c, & a-c, & b+\lambda, & d-\lambda \\ 1+2a-c, & 1+a+c, & b, & d \end{matrix} \middle| 1 \right] \\ &= \sum_{k \geq 0} \begin{bmatrix} 2a, & c, & a-c \\ 1, & 1+2a-c, & 1+a+c \end{bmatrix}_k \begin{bmatrix} b+\lambda, & d-\lambda \\ b, & d \end{bmatrix}_k \\ &= \sum_{k \geq 0} \mathcal{B}_k \nabla \mathcal{A}_k = \sum_{k \geq 0} \mathcal{A}_k \Delta \mathcal{B}_k \\ &= \frac{\lambda(\lambda+b-d)}{b \times d} \sum_{k \geq 0} \begin{bmatrix} 1+2a, & 1+c, & 1+a-c \\ 1, & 1+2a-c, & 1+a+c \end{bmatrix}_k \begin{bmatrix} b+\lambda, & d-\lambda \\ 1+b, & 1+d \end{bmatrix}_k. \end{aligned}$$

This leads us to the following transformation formula.

THEOREM 17 (Contiguous relation on nearly poised series)

$$\begin{aligned} & {}_5F_4 \left[\begin{matrix} 2a, & c, & a-c, & b+\lambda, & d-\lambda \\ 1+2a-c, & 1+a+c, & b, & d \end{matrix} \middle| 1 \right] \\ &= \frac{\lambda(\lambda+b-d)}{b \times d} {}_5F_4 \left[\begin{matrix} 1+2a, & 1+c, & 1+a-c, & b+\lambda, & d-\lambda \\ 1+2a-c, & 1+a+c, & 1+b, & 1+d \end{matrix} \middle| 1 \right]. \end{aligned}$$

When the parameters are specified with

$$a \longrightarrow \alpha + \beta, \quad b \longrightarrow 2\alpha + 2\beta, \quad c \longrightarrow \alpha, \quad d \longrightarrow \infty, \quad \lambda \longrightarrow \gamma - 2\alpha - 2\beta,$$

the last theorem reduces immediately to the result recently discovered by Krattenthaler and Rivoal, which has been fundamental in their construction of counter-examples to the Rhin–Viola conjecture.

PROPOSITION 18 [4, Theorem 1] *Let α, β, γ be complex numbers such that $\Re(2\alpha + 2\beta - \gamma) > 0$. There holds the identity:*

$${}_3F_2 \left[\begin{matrix} \alpha + 1, \beta + 1, \gamma \\ 2\alpha + \beta + 1, 2\beta + \alpha + 1 \end{matrix} \middle| 1 \right] = \frac{2(\alpha + \beta)}{2(\alpha + \beta) - \gamma} {}_3F_2 \left[\begin{matrix} \alpha, \beta, \gamma \\ 2\alpha + \beta + 1, 2\beta + \alpha + 1 \end{matrix} \middle| 1 \right].$$

Krattenthaler and Rivoal derived it from the limiting case $q \rightarrow 1$ of the transformation due to Verma and Jain (see [5, Eq 3.5.10]). However, our proof for the even more general result is more accessible and elegant.

3. More exotic contiguous relations

By comparing the four patterns A, B, C and D , we derive several exotic contiguous relations, including all the propositions established by Krattenthaler and Rivoal [4] as particular cases. Some of them may be specialized to further two-term relations.

3.1 A&A

Denote by $\text{Eq}'(1b)$ the resulting equation of Eq (1b) under the parameter replacements

$$a \longrightarrow c - 1, \quad c \longrightarrow a + 1, \quad b \longrightarrow d - 1, \quad d \longrightarrow b + 1.$$

Then for an arbitrary variable w , the difference $\text{Eq} (1b) - w \times \text{Eq}'(1b)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a + 1, & c - 1, & e \\ & b + 1, & d - 1 \end{matrix} \middle| 1 \right] \quad (5a)$$

$$= (\mathcal{A} - \mathcal{A}'w) \times {}_4F_3 \left[\begin{matrix} a + 1, c, e, & 1 + \frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} \\ & b + 1, d, & \frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} \end{matrix} \middle| 1 \right], \quad (5b)$$

where the following notations have been used for the coefficients

$$\mathcal{A}' = \mathcal{A}(c - 1, a + 1, e; d - 1, b + 1),$$

$$\mathfrak{A}' = \mathfrak{A}(c - 1, a + 1, e; d - 1, b + 1),$$

with $\mathcal{A}$ and $\mathfrak{A}$ being defined by equations (1c) and (1e), respectively. Resolving the equation

$$\frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} = b \Longleftrightarrow w = \frac{(1 + b - c)(b - c)(b - e)(1 - d)}{b(1 + a - d)(1 - ab + b^2 - c - bc - d + cd + e + ae - be)},$$

we get from equations (5a) and (5b) the following contiguous relation.

THEOREM 19 With w being given above, there holds the three-term relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = w \times {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b+1, & d-1 \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ \times \frac{(1+a+c+e-b-d)(1+ae+cd+e-ab-ce-c-d)}{(1+a-d)(1-ab+b^2-c-bc-d+cd+e+ae-be)}.$$

For $b = 1 + a$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (cf. [1, §1.3]):

$${}_2F_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| 1 \right] = \Gamma \left[\begin{matrix} c, c-a-b \\ c-a, c-b \end{matrix} \right], \quad \text{where } \Re(c-a-b) > 0. \quad (6)$$

We therefore recover the following exotic relation.

PROPOSITION 20 [4, Proposition 1] For four complex numbers a, c, d, e subject to $\Re(1-c+d-e) > 0$, there holds the identity:

$${}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, d \end{matrix} \middle| 1 \right] = \frac{(1+a-c)(2+a-c)(1+a-e)(d-1)}{(1+a)(c-1)(1+a-d)(2+a-d)} {}_3F_2 \left[\begin{matrix} a+1, c-1, e \\ a+2, d-1 \end{matrix} \middle| 1 \right] \\ + \frac{(1-a-a^2-c+e+ae-ce-d+cd)}{(c-1)(1+a-d)(2+a-d)} \times \Gamma \left[\begin{matrix} d, d-c-e+1 \\ d-c, d-e \end{matrix} \right].$$

3.2 A&A

Denote by Eq'(1a) the resulting equation of Eq (1a) under the parameter replacements

$$a \longrightarrow c-1 \quad \text{and} \quad c \longrightarrow a+1.$$

Then the difference Eq(1a) – $w \times$ Eq'(1a) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \quad (7a)$$

$$= (\mathcal{A} - \mathcal{A}'w) \times {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b+1, & d \end{matrix} \middle| 1 \right] \quad (7b)$$

$$+ (\mathbb{A} - \mathbb{A}'w) \times {}_3F_2 \left[\begin{matrix} a+2, & c+1, & e+1 \\ & b+2, & d+1 \end{matrix} \middle| 1 \right] \quad (7c)$$

where the following notations have been used for the coefficients

$$\mathcal{A}' = \mathcal{A}(c-1, a+1, e; b, d),$$

$$\mathbb{A}' = \mathbb{A}(c-1, a+1, e; b, d),$$

with $\mathcal{A}$ and $\mathbb{A}$ being defined by equations (1c) and (1d), respectively. Resolving the equation

$$\mathbb{A} - \mathbb{A}'w = 0 \Rightarrow w = \frac{d-c}{d-a-1},$$

we get from equation (7a)–(7c) the following contiguous relation.

THEOREM 21 (Three-term relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \frac{c-d}{1+a-d} {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ + \frac{(b-e)(1+a-c)}{b(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b+1, & d \end{matrix} \middle| 1 \right].$$

For $b = 1 + a$, the second ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore recover the following exotic relation after having relabeled parameters.

PROPOSITION 22 [4, Proposition 3] *For any complex numbers a, b, c, d subject to $\Re(d - b - c + 1) > 0$, we have the identity*

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & a+1, & d \end{matrix} \middle| 1 \right] = \frac{(1+a-b)(1+a-c)}{(1+a)(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & b, & c \\ & a+2, & d \end{matrix} \middle| 1 \right] \\ - \frac{1}{1+a-d} \Gamma \left[\begin{matrix} d, & d-b-c+1 \\ d-b, & d-c \end{matrix} \right].$$

Instead, resolving the equation

$$\mathcal{A} - \mathcal{A}'w = 0 \Rightarrow w = \frac{(c-d)(b+ab-bd+ce)}{(1+a-d)(bc+ae+e-bd)},$$

we get from equations (7a)–(7c) the following contiguous relation.

THEOREM 23 (Three-term relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \frac{(d-c)(b+ab-bd+ce)}{(1+a-d)(bc+ae+e-bd)} {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ + \frac{ce(1+a)(b-e)(1+a-c)(1+a+c+e-b-d)}{bd(1+b)(1+a-d)(bd-bc-ae-e)} {}_3F_2 \left[\begin{matrix} a+2, & c+1, & e+1 \\ & b+2, & d+1 \end{matrix} \middle| 1 \right].$$

For $b = 1 + a$, the second ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore obtain the exotic relation.

PROPOSITION 24 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & a+1, & d \end{matrix} \middle| 1 \right] = \frac{ce(1+a-c)(1+a-e)}{d(1+a)(2+a)(d-a-1)} {}_3F_2 \left[\begin{matrix} a+2, & c+1, & e+1 \\ & a+3, & d+1 \end{matrix} \middle| 1 \right] \\ + \frac{1+a^2+2a+ce-d-ad}{(1+a)(1+a-d)} \Gamma \left[\begin{matrix} d-c-e, & d \\ d-c, & d-e \end{matrix} \right].$$

When $d = 1 + a + ce/(1 + a)$, we can check that the last fraction in front of the Gamma function vanishes. After some simplification, we get the following identity with one independent parameter less.

COROLLARY 25 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & a+1, & 1+a+\frac{ce}{1+a} \end{matrix} \middle| 1 \right] = {}_3F_2 \left[\begin{matrix} a+2, & c+1, & e+1 \\ & a+3, & 2+a+\frac{ce}{1+a} \end{matrix} \middle| 1 \right] \\ \times \frac{(1+a)(1+a-e)(1+a-c)}{(2+a)(1+2a+a^2+ce)}.$$

3.3 A&A

Denote by Eq'(1b) the resulting equation of Eq(1b) under the parameter replacements

$$b \longrightarrow d-1 \quad \text{and} \quad d \longrightarrow b+1.$$

Then for an arbitrary variable w , the difference Eq(1b) $- w \times$ Eq'(1b) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b+1, & d-1 \end{matrix} \middle| 1 \right] \quad (8a)$$

$$= (\mathcal{A} - \mathcal{A}'w) \times {}_4F_3 \left[\begin{matrix} a+1, c, e, & 1 + \frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} \\ & b+1, d, & \frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} \end{matrix} \middle| 1 \right], \quad (8b)$$

where the following notations have been used for the coefficients

$$\mathcal{A}' = \mathcal{A}(a, c, e; d-1, b+1),$$

$$\mathfrak{A}' = \mathfrak{A}(a, c, e; d-1, b+1),$$

with $\mathcal{A}$ and $\mathfrak{A}$ being defined by equations (1c) and (1e), respectively. Resolving the equation

$$\frac{\mathcal{A} - \mathcal{A}'w}{\mathfrak{A} - \mathfrak{A}'w} = b \Leftrightarrow w = \frac{(d-1)(b-a)(b-c)(b-e)}{b(1+a-d)(a+ab+bc+be-b^2-ce-ad)},$$

we get from equations (8a) and (8b) the following contiguous relation.

THEOREM 26 *With w being given above, there holds the three-term relation:*

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = w \times {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b+1, & d-1 \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ \times \frac{a(1+b-d)(1+a-b+c-d+e)}{(1+a-d)(a+ab+bc+be-b^2-ce-ad)}.$$

For $b = 1 + a$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 27 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, d \end{matrix} \middle| 1 \right] &= \frac{a(2+a-d)(c+e-d)}{(1+a-d)(ac+c+ae+e-ce-ad-1)} \Gamma \left[\begin{matrix} d, d-c-e \\ d-e, d-c \end{matrix} \right] \\ &+ \frac{(d-1)(1+a-c)(1+a-e)}{(a+1)(1+a-d)(ac+c+ae+e-ce-ad-1)} \\ &\times {}_3F_2 \left[\begin{matrix} a, c, e \\ a+2, d-1 \end{matrix} \middle| 1 \right]. \end{aligned}$$

3.4 A&A

Denote by Eq'(1a) the resulting equation of Eq(1a) under the parameter replacements

$$b \longrightarrow d-1 \quad \text{and} \quad d \longrightarrow b+1.$$

Then for an arbitrary variable w , the difference Eq(1a) $- w \times$ Eq'(1a) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b+1, & d-1 \end{matrix} \middle| 1 \right] \quad (9a)$$

$$= (\mathcal{A} - \mathcal{A}'w) \times {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ b+1, & d \end{matrix} \middle| 1 \right] \quad (9b)$$

$$+ (\mathbb{A} - \mathbb{A}'w) \times {}_3F_2 \left[\begin{matrix} a+2, & c+1, & e+1 \\ b+2, & d+1 \end{matrix} \middle| 1 \right], \quad (9c)$$

where the following notations have been used for the coefficients

$$\mathcal{A}' = \mathcal{A}(a, c, e; d-1, b+1),$$

$$\mathbb{A}' = \mathbb{A}(a, c, e; d-1, b+1),$$

with $\mathcal{A}$ and $\mathbb{A}$ being defined by equations (1c) and (1d), respectively. Resolving the equation

$$\mathbb{A} - \mathbb{A}'w = 0 \Leftrightarrow w = \frac{(d-1)(b-a)}{b(d-a-1)},$$

we get from equations (9a)–(9c) the following contiguous relation.

THEOREM 28 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= \frac{(a-b)(d-1)}{b(1+a-d)} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b+1, & d-1 \end{matrix} \middle| 1 \right] \\ &+ \frac{a(1+b-d)}{b(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ b+1, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $d = c+1$, the second ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). Exchanging a and c , we therefore get the following exotic relation.

PROPOSITION 29 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, b \end{matrix} \middle| 1 \right] = \frac{c(a-b)}{b(a-c)} {}_3F_2 \left[\begin{matrix} a, c+1, e \\ a+1, b+1 \end{matrix} \middle| 1 \right] + \frac{a}{a-c} \Gamma \left[\begin{matrix} b, 1+b-c-e \\ b-c, 1+b-e \end{matrix} \right].$$

Instead, resolving the equation

$$\mathcal{A} - \mathcal{A}'w = 0 \Rightarrow w = \frac{(d-1)(b-a)(b+ab-bd+ce)}{b(1+a-d)(a+bd-ad-b-ce)},$$

we get from equations (9a)–(9c) the following contiguous relation.

THEOREM 30 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] &= \frac{(a-b)(1-d)(b+ab-bd+ce)}{b(1+a-d)(a+bd-ad-b-ce)} {}_3F_2 \left[\begin{matrix} a, c, e \\ b+1, d-1 \end{matrix} \middle| 1 \right] \\ &+ \frac{ace(1+a)(1+b-d)(1+a+c+e-b-d)}{bd(1+b)(1+a-d)(a+bd-ad-b-ce)} {}_3F_2 \left[\begin{matrix} a+2, c+1, e+1 \\ b+2, d+1 \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $b = c$, the first ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation under the replacements $a \rightarrow c, c \rightarrow a$ and $d \rightarrow d+1$.

PROPOSITION 31 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & d \end{matrix} \middle| 1 \right] &= \frac{ad-cd-ae}{a-c} \Gamma \left[\begin{matrix} d, & d-c-e \\ d-c, & 1+d-e \end{matrix} \right] \\ &+ \frac{c(1+c)e(a-d)}{d(1+a)(1+d)(c-a)} {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+1 \\ a+2, & d+2 \end{matrix} \middle| 1 \right]. \end{aligned}$$

When $d = ae/(a-c)$, the fraction in front of the Gamma function vanishes. After some simplification, we get the following identity with one free parameter less.

COROLLARY 32 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & \frac{ae}{a-c} \end{matrix} \middle| 1 \right] = \frac{c(1+c)(c+e-a)}{(1+a)(a-c+ae)} {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+1 \\ a+2, & 2+\frac{ae}{a-c} \end{matrix} \middle| 1 \right].$$

3.5 A&B

Denote by Eq'(2b) the resulting equation of Eq(2b) under the parameter replacements

$$a \longrightarrow c + 1, \quad c \longrightarrow a + 1, \quad b \longrightarrow d, \quad d \longrightarrow b + 2.$$

Then for an arbitrary variable w , the difference $\text{Eq}(1b) - w \times \text{Eq}'(2b)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e \\ & b+2, & d \end{matrix} \middle| 1 \right] \quad (10a)$$

$$= (\mathcal{A} - \mathcal{B}'w) \times {}_4F_3 \left[\begin{matrix} a+1, c, e, & 1 + \frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} \\ & b+1, d, & \frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} \end{matrix} \middle| 1 \right], \quad (10b)$$

where the following notations have been used for the coefficients

$$\mathcal{B}' = \mathcal{B}(c+1, a+1, e; d, b+2),$$

$$\mathfrak{B}' = \mathfrak{B}(c+1, a+1, e; d, b+2),$$

with $\mathcal{B}$ and $\mathfrak{B}$ being defined by equations (2c) and (2e), respectively. Resolving the equation

$$\frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} = e \Rightarrow w = \frac{c(a-b)(b-e)(1+b-e)(1+a+e-d)}{b(1+b)(1+a-d)(ac-bc-e-ae+be+de-e^2)},$$

we get from equations (10a) and (10b) the following contiguous relation.

THEOREM 33 *With w being given above, there holds the three-term relation:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= w \times {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e \\ & b+2, & d \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a+1, c, e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right] \\ &\times \frac{e(b+d-a-c-e-1)(b+ab-bd-ac+bc)}{b(1+a-d)(ac-bc-e-ae+be+de-e^2)}. \end{aligned}$$

For $d = e + 1$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore recover the following exotic relation after some simplification.

PROPOSITION 34 [4, Proposition 2] *For four complex numbers a, b, c, d subject to $\Re(d - b - c + 1) > 0$, there holds the identity:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & b, & c \\ & a+1, & d \end{matrix} \middle| 1 \right] &= \frac{bc(a-d-1)(a-d)}{(a-b)(a-c)(d+1)d} {}_3F_2 \left[\begin{matrix} a, & b+1, & c+1 \\ & a+1, & d+2 \end{matrix} \middle| 1 \right] \\ &+ \frac{a(bc+ad-bd-cd)}{(a-b)(a-c)} \Gamma \left[\begin{matrix} d, d-b-c+1 \\ d-b+1, d-c+1 \end{matrix} \right]. \end{aligned}$$

Instead, resolving the equation

$$\frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} = b \Leftrightarrow w = \frac{(1+b-e)(b-e)(b-c)(a-b)c}{b(b+1)(1+a-d)(b^2+ac-b-ab-2bc+bd-be+ce)},$$

we get from equations (10a) and (10b) the following contiguous relation.

THEOREM 35 *With w being given above, there holds the three-terms relation:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= w \times {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e \\ & b+2, & d \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a+1, & c, & e+1 \\ & b, & d \end{matrix} \middle| 1 \right] \\ &\times \frac{(b+d-a-c-e-1)(b+ab-bd-ac+bc)}{(1+a-d)(b^2-b-ab+ac-2bc+bd-be+ce)}. \end{aligned}$$

For $b = 1 + a$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation after some simplification.

PROPOSITION 36 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, d \end{matrix} \middle| 1 \right] &= \frac{(d-c-e)(1+2a+a^2+c-d-ad)}{(1+a-d)(d-2c-e+ce+ad-ac-ae)} \Gamma \left[\begin{matrix} d, d-c-e-1 \\ d-c, d-e-1 \end{matrix} \right] \\ &+ \frac{c(1+a-c)(1+a-e)(2+a-e)}{(a+1)(a+2)(1+a-d)(d-2c-e+ce+ad-ac-ae)} {}_3F_2 \left[\begin{matrix} a+1, c+1, e \\ a+3, d \end{matrix} \middle| 1 \right]. \end{aligned}$$

When $d = 1 + a - c/(1+a)$, the above identity reduces to the following relation.

COROLLARY 37 (Contiguous relation of ${}_3F_2(1)$ -series)

$$\begin{aligned} &{}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & 1+a-c/(1+a) \end{matrix} \middle| 1 \right] \\ &= {}_3F_2 \left[\begin{matrix} a+1, & c+1, & e \\ a+3, & 1+a-c/(1+a) \end{matrix} \middle| 1 \right] \\ &\times \frac{(c-a-1)(1+a-e)(2+a-e)}{(2+a)(1+2a+a^2-3c-e-ac-ae+ce)}. \end{aligned}$$

3.6 A&B

Denote by Eq'(2b) the resulting equation of Eq(2b) under the parameter replacements

$$a \longrightarrow c + 1, \quad c \longrightarrow a + 1, \quad b \longrightarrow b + 1, \quad d \longrightarrow d + 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(1b) - w \times \text{Eq}'(2b)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a + 1, & c + 1, & e \\ & b + 1, & d + 1 \end{matrix} \middle| 1 \right] \quad (11a)$$

$$= (\mathcal{A} - \mathcal{B}'w) \times {}_4F_3 \left[\begin{matrix} a + 1, c, e, & 1 + \frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} \\ & b + 1, d, & \frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} \end{matrix} \middle| 1 \right], \quad (11b)$$

where the following notations have been used for the coefficients

$$\begin{aligned} \mathcal{B}' &= \mathcal{B}(c + 1, a + 1, e; b + 1, d + 1), \\ \mathfrak{B}' &= \mathfrak{B}(c + 1, a + 1, e; b + 1, d + 1), \end{aligned}$$

with $\mathcal{B}$ and $\mathfrak{B}$ being defined by equations (2c) and (2e), respectively. Resolving the equation

$$\frac{\mathcal{A} - \mathcal{B}'w}{\mathfrak{A} - \mathfrak{B}'w} = b \Rightarrow w = \frac{c(d - e)(b - e)}{bd(b + d - a - e - 1)},$$

we get from equations (11a) and (11b) the following contiguous relation.

THEOREM 38 *With w being given above, there holds the three-term relation:*

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= w \times {}_3F_2 \left[\begin{matrix} a + 1, & c + 1, & e \\ & b + 1, & d + 1 \end{matrix} \middle| 1 \right] \\ &+ \frac{1 + a + c + e - b - d}{1 + a + e - b - d} {}_3F_2 \left[\begin{matrix} a + 1, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $b = a + 1$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 39 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, c, e \\ a + 1, d \end{matrix} \middle| 1 \right] = \frac{c(1 + a - e)}{d(1 + a)} {}_3F_2 \left[\begin{matrix} a + 1, c + 1, e \\ a + 2, d + 1 \end{matrix} \middle| 1 \right] + \Gamma \left[\begin{matrix} d, 1 + d - c - e \\ d - c, 1 + d - e \end{matrix} \middle| 1 \right].$$

3.7 A&D

Denote by $\text{Eq}'(4a)$ the resulting equation of $\text{Eq}(4a)$ under the parameter replacements

$$a \longrightarrow a + 1, \quad c \longrightarrow c + 1, \quad e \longrightarrow e + 1, \quad b \longrightarrow b + 2, \quad d \longrightarrow d + 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(1a) - w \times \text{Eq}'(4a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a + 1, & c + 1, & e + 1 \\ & b + 2, & d + 1 \end{matrix} \middle| 1 \right] \quad (12a)$$

$$= (\mathcal{A} - \mathcal{D}'w) \times {}_3F_2 \left[\begin{matrix} a + 1, & c, & e \\ & b + 1, & d \end{matrix} \middle| 1 \right] \quad (12b)$$

$$+ (\mathbb{A} - \mathbb{D}'w) \times {}_3F_2 \left[\begin{matrix} a + 2, & c + 1, & e + 1 \\ & b + 2, & d + 1 \end{matrix} \middle| 1 \right], \quad (12c)$$

where the following notations have been used for the coefficients

$$\mathcal{D}' = \mathcal{D}(a + 1, c + 1, e + 1; b + 2, d + 1),$$

$$\mathbb{D}' = \mathbb{D}(a + 1, c + 1, e + 1; b + 2, d + 1)$$

with $\mathcal{D}$ and $\mathbb{D}$ being defined by equations (4c) and (4d), respectively. Resolving the equation

$$\mathbb{A} - \mathbb{D}'w = 0 \Leftrightarrow w = \frac{ce(a - b)}{bd(b + 1)},$$

we get from equation (12a–12c) the following contiguous relation.

THEOREM 40 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= \frac{ce(a - b)}{bd(1 + b)} {}_3F_2 \left[\begin{matrix} a + 1, & c + 1, & e + 1 \\ & b + 2, & d + 1 \end{matrix} \middle| 1 \right] \\ &+ {}_3F_2 \left[\begin{matrix} a + 1, & c, & e \\ & b + 1, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $d = 1 + a$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation after some simplification.

PROPOSITION 41 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & 1 + a, & b \end{matrix} \middle| 1 \right] &= \frac{ce(a - b)}{b(1 + a)(1 + b)} {}_3F_2 \left[\begin{matrix} a + 1, & c + 1, & e + 1 \\ & a + 2, & b + 2 \end{matrix} \middle| 1 \right] \\ &+ \Gamma \left[\begin{matrix} b + 1, & 1 + b - c - e \\ 1 + b - c, & 1 + b - e \end{matrix} \right]. \end{aligned}$$

3.8 B&B

Denote by $\text{Eq}'(2a)$ the resulting equation of $\text{Eq}(2a)$ under the parameter replacements

$$a \rightarrow c + 1 \quad \text{and} \quad c \rightarrow a - 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(2a) - w \times \text{Eq}'(2a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \quad (13a)$$

$$= (\mathcal{B} - \mathcal{B}'w) \times {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ & b, & d-1 \end{matrix} \middle| 1 \right] \quad (13b)$$

$$+ (\mathbb{B} - \mathbb{B}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right], \quad (13c)$$

where the following notations have been used for the coefficients

$$\mathcal{B}' = \mathcal{B}(c+1, a-1, e; b, d),$$

$$\mathbb{B}' = \mathbb{B}(c+1, a-1, e; b, d),$$

with $\mathcal{B}$ and $\mathbb{B}$ being defined by equations (2c) and (2d), respectively. Resolving the equation

$$\mathcal{B} - \mathcal{B}'w = 0 \Leftrightarrow w = \frac{(a-d)(1+c+e-d)}{(a+e-d)(1+c-d)},$$

we get from equation (13a)–(13c) the following contiguous relation.

THEOREM 42 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= \frac{(a-d)(1+c+e-d)}{(a+e-d)(1+c-d)} {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ &+ \frac{e(1+c-a)(b+d-a-c-e-1)}{b(1+c-d)(a+e-d)} {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $b = a$, the first ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 43 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & a+1, & d \end{matrix} \middle| 1 \right] &= \frac{e(c-a-1)}{(a+1)(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c, & e+1 \\ & a+2, & d \end{matrix} \middle| 1 \right] \\ &+ \frac{1+a+e-d}{1+a-d} \Gamma \left[\begin{matrix} d-c-e, & d \\ d-c, & d-e \end{matrix} \right]. \end{aligned}$$

When $d = 1 + a + e$, this proposition reduces to the following two-term relation.

COROLLARY 44 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & a+1, & 1+a+e \end{matrix} \middle| 1 \right] = \frac{1+a-c}{1+a} {}_3F_2 \left[\begin{matrix} a+1, & c, & e+1 \\ & a+2, & 1+a+e \end{matrix} \middle| 1 \right].$$

3.9 B&B

Denote by $\text{Eq}'(2a)$ the resulting equation of $\text{Eq}(2a)$ under the parameter replacements

$$b \rightarrow d - 1 \quad \text{and} \quad d \rightarrow b + 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(2a) - w \times \text{Eq}'(2a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b+1, & d-1 \end{matrix} \middle| 1 \right] \quad (14a)$$

$$= (\mathcal{B} - \mathcal{B}'w) \times {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ b, & d-1 \end{matrix} \middle| 1 \right] \quad (14b)$$

$$+ (\mathbb{B} - \mathbb{B}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right], \quad (14c)$$

where the following notations have been used for the coefficients

$$\mathcal{B}' = \mathcal{B}(a, c, e; d - 1, b + 1),$$

$$\mathbb{B}' = \mathbb{B}(a, c, e; d - 1, b + 1),$$

with $\mathcal{B}$ and $\mathbb{B}$ being defined by equations (2c) and (2d), respectively. Resolving the equation

$$\mathcal{B} - \mathcal{B}'w = 0 \Leftrightarrow w = \frac{(b-c)(b-e)(d-1)(1+c+e-d)}{b(1+c-d)(1+e-d)(c+e-b)},$$

we get from equation (14a)–(14c) the following contiguous relation.

THEOREM 45 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= \frac{(b-c)(b-e)(d-1)(1+c+e-d)}{b(1+c-d)(1+e-d)(c+e-b)} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b+1, & d-1 \end{matrix} \middle| 1 \right] \\ &+ \frac{ce(1+b-d)(1+a+c+e-b-d)}{b(1+c-d)(1+e-d)(c+e-b)} {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $d = 1 + a$, the second ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 46 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & b \end{matrix} \middle| 1 \right] &= \frac{ce(b-a)}{b(c-a)(e-a)} {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ a+1, & b+1 \end{matrix} \middle| 1 \right] \\ &+ \frac{a(a-c-e)}{(e-a)(c-a)} \Gamma \left[\begin{matrix} b-c-e, & b \\ b-c, & b-e \end{matrix} \middle| 1 \right]. \end{aligned}$$

When $a = c + e$, we obtain the following two-term relation.

COROLLARY 47 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} c+e, & c, & e \\ 1+c+e, & b \end{matrix} \middle| 1 \right] = \frac{b-c-e}{b} {}_3F_2 \left[\begin{matrix} c+e, & c+1, & e+1 \\ c+e+1, & b+1 \end{matrix} \middle| 1 \right].$$

3.10 B&C

Denote by Eq'(3b) the resulting equation of Eq(3b) under the parameter replacements

$$a \rightarrow c, \quad c \rightarrow a - 2, \quad e \rightarrow e - 1, \quad b \rightarrow b - 1, \quad d \rightarrow d - 2.$$

Then for an arbitrary variable w , the difference Eq(2b) $- w \times$ Eq'(3b) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a-2, & c, & e-1 \\ & b-1, & d-2 \end{matrix} \middle| 1 \right] \quad (15a)$$

$$= (\mathcal{B} - \mathcal{C}'w) \times {}_4F_3 \left[\begin{matrix} a-1, c, e, & 1 + \frac{\mathcal{B} - \mathcal{C}'w}{\mathfrak{B} - \mathfrak{C}'w} \\ & b, d-1, & \frac{\mathfrak{B} - \mathfrak{C}'w}{\mathfrak{B} - \mathfrak{C}'w} \end{matrix} \middle| 1 \right], \quad (15b)$$

where the following notations have been used for the coefficients

$$\mathcal{C}' = \mathcal{C}(c, a - 2, e - 1; b - 1, d - 2),$$

$$\mathfrak{C}' = \mathfrak{C}(c, a - 2, e - 1; b - 1, d - 2),$$

with $\mathcal{C}$ and $\mathfrak{C}$ being defined by equations (3c) and (3e), respectively. Resolving the equation

$$\frac{\mathcal{B} - \mathcal{C}'w}{\mathfrak{B} - \mathfrak{C}'w} = d - 2 \Rightarrow w = \frac{(b-1)(d-1)(d-2)}{(a-1)(1+c-d)(2+c-d)(d-e-1)(a+e-d)} \\ \times (2b-3a-1-c-ac+2d+2ad-bd+cd-d^2-e-ae+de)$$

we get from equation (15a)–(15b) the following contiguous relation.

THEOREM 48 *With w being given above, there holds the three-term relation:*

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = w \times {}_3F_2 \left[\begin{matrix} a-2, & c, & e-1 \\ & b-1, & d-2 \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ & b, & d-2 \end{matrix} \middle| 1 \right] \\ \times \frac{(d-1)(d-2)(1+a+c+e-b-d)}{(a-1)(1+c-d)(2+c-d)} \\ \times \frac{1+a-2b+c-ad+bd-e+ae-ce}{(d-e-1)(d-a-e)}.$$

For $d = a + 1$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 49 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, b \end{matrix} \middle| 1 \right] = \frac{a(a-1)(b-1)(a-b)}{(a-c)(a-c-1)(a-e)(e-1)} {}_3F_2 \left[\begin{matrix} a-2, c, e-1 \\ a-1, b-1 \end{matrix} \middle| 1 \right] \\ + \frac{a(c+e-b)(a^2+b-c+e+ce-ab-ae-1)}{(a-c)(a-c-1)(a-e)(e-1)} \Gamma \left[\begin{matrix} b, b-c-e \\ b-c, b-e \end{matrix} \right].$$

When $b = 1 + a - e + c(e-1)/(a-1)$, we can check that the fraction in front of the Gamma function vanishes. After some simplification, we get the following identity with one free parameter less.

COROLLARY 50 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & 1+a-e & + \frac{c(e-1)}{a+1} \end{matrix} \middle| 1 \right] = {}_3F_2 \left[\begin{matrix} a+2, & c, & e+1 \\ a+3, & 2+a-e & + \frac{c(e-1)}{a+1} \end{matrix} \middle| 1 \right] \\ \times \frac{(1+a)(2+a-c)(1+a-e)}{(2+a)(a^2+4a-ae-e+ce+1)}.$$

3.11 B&D

Denote by Eq'(4b) the resulting equation of Eq(4b) under the parameter replacements

$$a \longrightarrow a-1, \quad c \longrightarrow c+1, \quad e \longrightarrow e+1, \quad b \longrightarrow b+1.$$

Then for an arbitrary variable w , the difference Eq(2b) $- w \times$ Eq'(4b) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right] \quad (16a)$$

$$= (\mathcal{B} - \mathcal{D}'w) \times {}_4F_3 \left[\begin{matrix} a-1, c, e, & 1 + \frac{\mathcal{B} - \mathcal{D}'w}{\mathfrak{B} - \mathfrak{D}'w} \\ b, d-1, & \frac{\mathcal{B} - \mathcal{D}'w}{\mathfrak{B} - \mathfrak{D}'w} \end{matrix} \middle| 1 \right], \quad (16b)$$

where the following notations have been used for the coefficients

$$\mathcal{D}' = \mathcal{D}(a-1, c+1, e+1; b+1, d),$$

$$\mathfrak{D}' = \mathfrak{D}(a-1, c+1, e+1; b+1, d),$$

with $\mathcal{D}$ and $\mathfrak{D}$ being defined by equation (4c) and (4e), respectively. Resolving the equation

$$\frac{\mathcal{B} - \mathcal{D}'w}{\mathfrak{B} - \mathfrak{D}'w} = d-2 \Leftrightarrow w = \frac{(a-b-1)(a-d)ce}{(a-1)(1+c-d)b} \\ \times \frac{(a-1)(1+c+e-d) + (d-2)(b+d-a-c-e-1)}{(d-e-1)\{ce + (d-2)(b+d-a-c-e-1)\}},$$

we get from equations (16a) and (16b) the following contiguous relation.

THEOREM 51 With w being given above, there holds the three-term relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] = w \times {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right] + {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ b, & d-2 \end{matrix} \middle| 1 \right] \\ \times \frac{(d-1)(d-2)(1+a+c+e-b-d)}{(1-a)(1+c-d)(1+e-d)} \\ \times \frac{(1-a+c-ac+ce-d+ad+e-ae)}{ce + (d-2)(b+d-a-c-e-1)}.$$

For $d = a+1$, the last ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation.

PROPOSITION 52 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, c, e \\ a+1, b \end{matrix} \middle| 1 \right] = \frac{a(b-c-e)(a^2+c+ce+e-a-ac-ae)}{(a-c)(a-e)\{ce+(a-1)(b-c-e)\}} \Gamma \left[\begin{matrix} b, b-c-e \\ b-c, b-e \end{matrix} \middle| 1 \right] \\ + \frac{ce(1+b-a)(b-a)}{b(c-a)(a-e)\{ce+(a-1)(b-c-e)\}} {}_3F_2 \left[\begin{matrix} a-1, c+1, e+1 \\ a+1, b+1 \end{matrix} \middle| 1 \right].$$

3.12 B&D

Denote by Eq'(4a) the resulting equation of Eq(4a) under the parameter replacements

$$a \rightarrow a-1, \quad c \rightarrow c+1, \quad e \rightarrow e+1, \quad b \rightarrow b+1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(2a) - w \times \text{Eq}'(4a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right] \quad (17a)$$

$$= (\mathcal{B} - \mathcal{D}'w) \times {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ b, & d-1 \end{matrix} \middle| 1 \right] \quad (17b)$$

$$+ (\mathbb{B} - \mathbb{D}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right], \quad (17c)$$

where the following notations have been used for the coefficients

$$\mathcal{D}' = \mathcal{D}(a-1, c+1, e+1; b+1, d),$$

$$\mathbb{D}' = \mathbb{D}(a-1, c+1, e+1; b+1, d),$$

with $\mathcal{D}$ and $\mathbb{D}$ being defined by equations (4c) and (4d), respectively. Resolving the equation

$$\mathbb{B} - \mathbb{D}'w = 0 \Leftrightarrow w = \frac{ce(a-d)(a-b-1)}{b(a-1)(1+c-d)(d-e-1)},$$

we get from equations (17a)–(17c) the following contiguous relation.

THEOREM 53 (Three-term relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] = \frac{ce(a-d)(a-b-1)}{b(1-a)(1+c-d)(1+e-d)} {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e+1 \\ b+1, & d \end{matrix} \middle| 1 \right] \\ + \frac{(d-1)\{(a-1)(1+c+e-d)-ce\}}{(1-a)(1+c-d)(1+e-d)} {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ b, & d-1 \end{matrix} \middle| 1 \right].$$

For $b = a$, the first ${}_3F_2$ -series reduces to a ${}_2F_1$ -series, which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation after some simplification.

PROPOSITION 54 (Exotic contiguous relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ a+1, & d \end{matrix} \middle| 1 \right] = \frac{a}{ce-ac-ae+ad} \Gamma \left[\begin{matrix} d, & 1+d-c-e \\ d-c, & d-e \end{matrix} \right] \\ + \frac{ce(a-d)}{d(1+a)(ac+ae-ad-ce)} {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ a+2, & d+1 \end{matrix} \middle| 1 \right].$$

3.13 B&D

Denote by $\text{Eq}'(4a)$ the resulting equation of $\text{Eq}(4a)$ under the parameter replacements

$$a \rightarrow c, \quad c \rightarrow a, \quad e \rightarrow e + 1, \quad b \rightarrow b + 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(2a) - w \times \text{Eq}'(4a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a, & c, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right] \quad (18a)$$

$$= (\mathcal{B} - \mathcal{D}'w) \times {}_3F_2 \left[\begin{matrix} a-1, & c, & e \\ & b, & d-1 \end{matrix} \middle| 1 \right] \quad (18b)$$

$$+ (\mathbb{B} - \mathbb{D}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right], \quad (18c)$$

where the following notations have been used for the coefficients

$$\mathcal{D}' = \mathcal{D}(c, a, e+1; b+1, d),$$

$$\mathbb{D}' = \mathbb{D}(c, a, e+1; b+1, d),$$

with $\mathcal{D}$ and $\mathbb{D}$ being defined by equations (4c) and (4d) respectively. Resolving the equation

$$\mathcal{B} - \mathcal{D}'w = 0 \Leftrightarrow w = \frac{(b-c)(1+c+e-d)}{b(1+e-d)},$$

we get from equation (18a)–(18c) the following contiguous relation.

THEOREM 55 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] &= \frac{(b-c)(1+c+e-d)}{b(1+e-d)} {}_3F_2 \left[\begin{matrix} a, & c, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right] \\ &+ \frac{c(1+a+c+e-b-d)}{b(1+e-d)} {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $b = a$, the first ${}_3F_2$ -series, reduces to a ${}_2F_1$ -series which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation after some simplification.

PROPOSITION 56 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ & a+1, & d \end{matrix} \middle| 1 \right] &= \frac{c}{c-a} {}_3F_2 \left[\begin{matrix} a, & c+1, & e \\ & a+1, & d \end{matrix} \middle| 1 \right] \\ &+ \frac{a}{a-c} \Gamma \left[\begin{matrix} d, & d-c-e \\ d-c, & d-e \end{matrix} \right]. \end{aligned}$$

3.14 C&C

Denote by $\text{Eq}'(3a)$ the resulting equation of $\text{Eq}(3a)$ under the parameter replacements

$$a \rightarrow c + 1 \quad \text{and} \quad c \rightarrow a - 1.$$

Then for an arbitrary variable w , the difference $\text{Eq}(3a) - w \times \text{Eq}'(3a)$ leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e \\ b, & d \end{matrix} \middle| 1 \right] \quad (19a)$$

$$= (\mathcal{C} - \mathcal{C}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ b+1, & d+1 \end{matrix} \middle| 1 \right] \quad (19b)$$

$$+ (\mathbb{C} - \mathbb{C}'w) \times {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+2 \\ b+2, & d+2 \end{matrix} \middle| 1 \right], \quad (19c)$$

where the following notations have been used for the coefficients

$$\mathcal{C}' = \mathcal{C}(c+1, a-1, e; b, d),$$

$$\mathbb{C}' = \mathbb{C}(c+1, a-1, e; b, d);$$

with $\mathcal{C}$ and $\mathbb{C}$ being defined by equation (3c) and (3d), respectively. Resolving the equation

$$\mathcal{C} - \mathcal{C}'w = 0 \Rightarrow w = \frac{bd - a(1+c+e)}{bd - (c+1)(a+e)},$$

we get from equations (19a)–(19c) the following contiguous relation.

THEOREM 57 (Three-term relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ b, & d \end{matrix} \middle| 1 \right] &= \frac{bd - a(1+c+e)}{bd - (1+c)(a+e)} {}_3F_2 \left[\begin{matrix} a-1, & c+1, & e \\ b, & d \end{matrix} \middle| 1 \right] \\ &+ \frac{ae(1+c)(1+e)(1+c-a)(b+d-a-c-e-1)}{bd(1+b)(1+d)(a+e+ac+ce-bd)} {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+2 \\ b+2, & d+2 \end{matrix} \middle| 1 \right]. \end{aligned}$$

For $d = 1 + c$, the second ${}_3F_2$ -series reduces to a ${}_2F_1$ -series which can be evaluated by the Gauss summation theorem (6). We therefore get the following exotic relation after some simplification.

PROPOSITION 58 (Exotic contiguous relation)

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} a, & c, & e \\ c+1, & b \end{matrix} \middle| 1 \right] &= \frac{ae(1+e)(a-c-1)}{b(1+b)(1+c)(2+c)} {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+2 \\ c+3, & b+2 \end{matrix} \middle| 1 \right] \\ &+ \frac{a(1+c+e) - b(c+1)}{(a-b)(c+1)} \Gamma \left[\begin{matrix} b, & b-a-e \\ b-a, & b-e \end{matrix} \middle| 1 \right]. \end{aligned}$$

When $b = a + ae/(c+1)$, we can check that the fraction in front of the Gamma function vanishes. After some simplification, we get the following identity with one free parameter less.

COROLLARY 59 (Contiguous relation of ${}_3F_2(1)$ -series)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & c+1, & a+ae/(c+1) \end{matrix} \middle| 1 \right] = {}_3F_2 \left[\begin{matrix} a+1, & c+2, & e+2 \\ & c+3, & 2+a+ae/(c+1) \end{matrix} \middle| 1 \right] \\ \times \frac{(a-c-1)(1+c)(1+e)e}{(2+c)(1+c+e)(1+c+a+ac+ae)}.$$

3.15 D&D

Denote by Eq'(4a) the resulting equation of Eq(4a) under the parameter replacements

$$a \rightarrow c-1 \quad \text{and} \quad c \rightarrow a+1.$$

Then for an arbitrary variable w , the difference Eq(4a) $- w \times$ Eq'(4a) leads us to the following relation:

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] - w \times {}_3F_2 \left[\begin{matrix} a, & c-1, & e-1 \\ & b-1, & d-1 \end{matrix} \middle| 1 \right] \quad (20a)$$

$$= (\mathcal{D} - \mathcal{D}'w) \times {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] \quad (20b)$$

$$+ (\mathbb{D} - \mathbb{D}'w) \times {}_3F_2 \left[\begin{matrix} a, & c+1, & e+1 \\ & b+1, & d \end{matrix} \middle| 1 \right], \quad (20c)$$

where the following notations have been used for the coefficients

$$\mathcal{D}' = \mathcal{D}(c-1, a+1, e; b, d),$$

$$\mathbb{D}' = \mathbb{D}(c-1, a+1, e; b, d),$$

with $\mathcal{D}$ and $\mathbb{D}$ being defined by equations (4c) and (4d), respectively. Resolving the equation

$$\mathbb{D} - \mathbb{D}'w = 0 \Rightarrow w = \frac{a(c-b)(c-d)}{(c-1)(1+a-b)(1+a-d)},$$

we get from equation (20a)–(20c) the following contiguous relation.

THEOREM 60 (Three-term relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \frac{a(b-c)(c-d)}{(1-c)(1+a-b)(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ + \frac{(1-b)(1-d)(1+a-c)}{(1-c)(1+a-b)(1+a-d)} {}_3F_2 \left[\begin{matrix} a, & c-1, & e-1 \\ & b-1, & d-1 \end{matrix} \middle| 1 \right].$$

Instead, resolving the equation

$$\mathcal{D} - \mathcal{D}'w = 0 \Rightarrow w = \frac{(c-b)(c-d)}{(1+a-b)(1+a-d)},$$

we get from equations (20a)–(20c) the following contiguous relation.

THEOREM 61 (Three-term relation)

$${}_3F_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right] = \frac{(c-b)(c-d)}{(1+a-b)(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c-1, & e \\ & b, & d \end{matrix} \middle| 1 \right] \\ + \frac{(1+a-c)(1+a+c+e-b-d)}{(1+a-b)(1+a-d)} {}_3F_2 \left[\begin{matrix} a+1, & c, & e \\ & b, & d \end{matrix} \middle| 1 \right].$$

4. Further Counter-Examples to the Conjecture of Rhin–Viola

4.1 Conjecture of Rhin–Viola

In section 2 and 3 of this paper, we derived numerous relations between two generic hypergeometric ${}_3F_2$ -series with argument 1. In fact, there are 120 such relations found by Thomae [6, Art. 4] in 1879, which are obtainable by iterating the following identity

$${}_3F_2 \left[\begin{matrix} a, & b, & c \\ & d, & e \end{matrix} \middle| 1 \right] = \Gamma \left[\begin{matrix} e, d+e-a-b-c \\ e-a, d+e-b-c \end{matrix} \right] {}_3F_2 \left[\begin{matrix} a, & d-b, & d-c \\ & d, & d+e-b-c \end{matrix} \middle| 1 \right].$$

With the trivial symmetries being excluded, there remain only 10 independent relations. Whipple [7] put these 120 relations in more coherent order, which can be found in Bailey [1, §3.5]. The following theorem given by Hardy [8, p. 499] (see also [9]) characterizes clearly the 120 relations through the symmetric group $\mathfrak{S}_5$.

THEOREM 62 [8, Hardy] *Let $s = s(a, b, c, d, e) = a + b + c - d - e$. Then the function*

$$\frac{1}{\Gamma(s)\Gamma(2d)\Gamma(2e)} {}_3F_2 \left[\begin{matrix} 2a-s, & 2b-s, & 2c-s \\ & 2d, & 2e \end{matrix} \middle| 1 \right]$$

is a symmetric function of the five variables a, b, c, d and e .

For five nonnegative integers a, b, c, d, e , Rhin and Viola [10, 11] introduced recently the integral

$$I(a, b, c, d, e) = \int_0^1 \int_0^1 \frac{x^a(1-x)^b(1-y)^c y^d}{(1-xy)^{b+c-e+1}} dx dy \quad (21)$$

motivated by $I(a, b, c, d, e) \in \mathbb{Q} + \mathbb{Q}\zeta(2)$ to find a good irrationality measure for

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Writing the integral in terms of hypergeometric series

$$I(a, b, c, d, e) = \frac{a!b!c!d!}{(a+b+1)!(c+d+1)!} {}_3F_2 \left[\begin{matrix} a+1, d+1, b+c-e+1 \\ a+b+2, c+d+2 \end{matrix} \middle| 1 \right], \quad (22)$$

it is routine to check that Theorem 62 is equivalent to the following theorem.

THEOREM 63 [12, Dixon] *Assuming $x! = \Gamma(1+x)$, then the following integral*

$$\frac{I(a, b, c, d, e)}{a!b!c!d!e!} = \frac{1}{a!b!c!d!e!} \int_0^1 \int_0^1 \frac{x^a(1-x)^b(1-y)^c y^d}{(1-xy)^{b+c-e+1}} dx dy$$

is a symmetric function of the five parameters $a+b, b+c, c+d, d+e$ and $e+a$.

Then the conjecture made by Rhin and Viola [11] on the integrals $I(a, b, c, d, e)$ has been translated by Krattenthaler and Rivoal [4] in terms of the corresponding hypergeometric series: two linearly dependent ${}_3F_2$ -series over $\mathbb{Q}$ of non-negative integer parameters and argument 1 convergent to irrational values satisfy one of Thomae's 120 relations. The irrationality of the ${}_3F_2$ -series just described has been determined by Krattenthaler and Rivoal [4] as follows.

THEOREM 64 [4, Krattenthaler-Rivoal] *For five non-negative integers a, b, c, d, e , the integral $I(a, b, c, d, e)$ is an irrational number if $a + b - d$, $b + c - e$, $c + d - a$, $d + e - b$ and $e + a - c$ are all non-negative.*

However, the Rhin–Viola conjecture has been proved to be false by Susumn Sato [13] in his Master thesis in 2001 through constructing the following six ‘exotic’ counter-examples:

$$I(1, 1, 1, 1, 1) = 5 - 3\zeta(2) = I(3, 1, 1, 2, 0), \quad (23a)$$

$$I(3, 1, 2, 2, 1) = 79/4 - 12\zeta(2) = I(4, 2, 2, 3, 0), \quad (23b)$$

$$I(3, 1, 2, 1, 1) = 3\zeta(2) - \frac{59}{12} = I(3, 3, 1, 3, 0), \quad (23c)$$

$$I(3, 2, 2, 2, 1) = 10\zeta(2) - \frac{148}{9} = I(5, 1, 3, 2, 1), \quad (23d)$$

$$I(3, 0, 3, 1, 1) = 9\zeta(2) - \frac{59}{4} = 9I(3, 3, 1, 2, 1), \quad (23e)$$

$$I(3, 1, 3, 1, 0) = \zeta(2) - \frac{29}{18} = I(3, 2, 1, 2, 0). \quad (23f)$$

Krattenthaler and Rivoal [4] gave hypergeometric explanation to these six relations based on their main Theorems 1 and 2 (corresponding, respectively, to Propositions 6 and 18 of the present paper). They also pointed out that Sato’s counter-examples are not just rare exceptions, which are among the infinite families of counter-examples found successfully by them. Based on Theorem 62, Krattenthaler and Rivoal [4] wrote the following Maple commands to generate the parameter arrays $[a', b', c', d', e']$ for ${}_3F_2$ -series corresponding to $I(a, b, c, d, e)$ under Thomae’s relations.

```
> with(combinat):
> p := (u, v, w, x, y) -> permute([u, v, w, x, y]):
> s := (u, v, w, x, y) -> u + v + w - x - y:
> A := (u, v, w, x, y) -> [2 * u - s(u, v, w, x, y),
    2 * v - s(u, v, w, x, y), 2 * w - s(u, v, w, x, y), 2 * x, 2 * y]:
> T := (u, v, w, x, y) -> seq(A(op(1, op(j, p(u, v, w, x, y))),
    op(2, op(j, p(u, v, w, x, y))), op(3, op(j, p(u, v, w, x, y))),
    op(4, op(j, p(u, v, w, x, y))), op(5, op(j, p(u, v, w, x, y)))),
    j = 1..nops(p(u, v, w, x, y))):
> F := (a, b, c, d, e) -> T((d + e - b - c)/2,
    (d + e - c - a)/2, (d + e - a - b)/2, d/2, e/2):
> I := (h, i, j, k, l) -> T((h + l + 2)/2, (k + l + 2)/2,
    (i + j + 2)/2, (h + i + 2)/2, (j + k + 2)/2):
```

We shall utilize these maple commands to detect the new counter-examples to the Rhin–Viola conjecture from the two-term relations established in the last two section.

4.2 Counter-examples from two-term relations

Based on the contiguous relations established in section 2, we shall construct counter-examples to the Rhin–Viola conjecture.

THEOREM 65 *The following two relations are counter-examples to the Rhin–Viola conjecture:*

$$I(2, 1, 5, 1, 5) = \frac{1}{2}I(4, 0, 6, 1, 5), \quad (24a)$$

$$I(3, 2, 4, 2, 5) = I(5, 1, 5, 2, 5). \quad (24b)$$

Proof From the relation between ${}_3F_2(1)$ -series and $I(a, b, c, d, e)$, we can easily verify that the two equations (24a) and (24b) are the special cases of Theorem 9 specified, respectively, by $(a, b, c, e) = (3, 5, 2, 2)$ and $(a, b, c, e) = (4, 7, 3, 2)$.

Applying the Maple commands devised by Krattenthaler and Rivoal [4], we get 20 arrays generated from $I(2, 1, 5, 1, 5)$. Among them, there are only four free arrays from the trivial symmetries:

$$[3, 2, 2, 5, 8], \quad [6, 5, 2, 8, 8], \quad [6, 6, 3, 9, 8], \quad [3, 3, 3, 9, 5].$$

Similarly with the inputting entry $I(4, 0, 6, 1, 5)$, Maple outputs

$$[5, 2, 2, 6, 9], \quad [7, 4, 2, 8, 9], \quad [4, 1, 2, 8, 6], \quad [6, 1, 4, 8, 8], \quad [7, 7, 5, 11, 9], \\ [4, 4, 5, 11, 6], \quad [6, 4, 7, 11, 8],$$

which are free from the trivial symmetries. It is obvious that the intersection of the two sets of arrays corresponding to $I(2, 1, 5, 1, 5)$ and $I(4, 0, 6, 1, 5)$ is empty. Therefore, the relation (24a) is beyond Thomae's 120 relations and constitutes one counter-example to the Rhin–Viola conjecture.

By the same Maple commands, we get the arrays free from the trivial symmetries:

$$I[3, 2, 4, 2, 5] : [4, 3, 2, 7, 8], [5, 4, 2, 8, 8], [6, 4, 3, 9, 8], \\ [5, 3, 3, 9, 7], [6, 5, 4, 10, 8], [5, 4, 4, 10, 9], [6, 5, 6, 10, 9]; \\ I[5, 1, 5, 2, 5] : [6, 3, 2, 8, 9], [5, 2, 2, 8, 8], \\ [7, 3, 3, 9, 9], [7, 6, 6, 12, 9], [6, 5, 6, 12, 8].$$

Hence, equation (24b) gives another counter-example to the Rhin–Viola conjecture. ■

As done by Krattenthaler and Rivoal [4], we can even get infinitely many counter-examples to the Rhin–Viola conjecture by applying Hardy's theorem.

For $a = \alpha + 1$, $c = e = 2$ and $b = 2 + 2\alpha$ with α being a natural number, the relation displayed in Theorem 9 reads as

$${}_3F_2 \left[\begin{matrix} \alpha + 1, & 2, & 2 \\ & 2 + 2\alpha, & 2 + 2\alpha \end{matrix} \middle| 1 \right] = \frac{\alpha(\alpha + 2)}{(\alpha + 1)^2} {}_3F_2 \left[\begin{matrix} \alpha + 3, & 2, & 2 \\ & 3 + 2\alpha, & 3 + 2\alpha \end{matrix} \middle| 1 \right]. \quad (25)$$

According to equation (22), the last equation is equivalent to the following relation:

$$I[\alpha, \alpha, 2\alpha - 1, 1, 3\alpha - 2] = \frac{2\alpha}{\alpha + 1} I[\alpha + 2, \alpha - 1, 2\alpha, 1, 3\alpha - 2]. \quad (26)$$

Expressing the left side of equation (25) in the symmetric form stated in Theorem 62, we can figure out the parameters $2a = 4\alpha$, $2b = 2c = 3\alpha + 1$, $2d = 2e = 2\alpha + 2$ and $s = 3\alpha - 1$.

Under the action of the permutations group $\mathfrak{S}_5$ of a, b, c, d, e , we get four ${}_3F_2$ -series from that on the left-hand side of equation (25) free from the trivial symmetries:

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} \alpha + 1, 2\alpha, 2\alpha \\ 2\alpha + 2, 4\alpha \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} \alpha + 1, \alpha + 1, 3\alpha - 1 \\ 3\alpha + 1, 3\alpha + 1 \end{matrix} \middle| 1 \right]; \\ & {}_3F_2 \left[\begin{matrix} 2, \alpha + 1, 2\alpha \\ 2\alpha + 2, 3\alpha + 1 \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} 2\alpha, 2\alpha, 3\alpha - 1 \\ 3\alpha + 1, 4\alpha \end{matrix} \middle| 1 \right]. \end{aligned}$$

Similarly for the ${}_3F_2$ -series on the right-hand side of equation (25), the corresponding parameters in Theorem 62 read as $2a = 4\alpha + 2$, $2b = 2c = 3\alpha + 1$, $2d = 2e = 2\alpha + 3$ and $s = 3\alpha - 1$. Under the action of the same $\mathfrak{S}_5$, we have also four ${}_3F_2$ -series free from the trivial symmetries:

$$\begin{aligned} & {}_3F_2 \left[\begin{matrix} \alpha + 3, 2\alpha + 1, 2\alpha + 1 \\ 2\alpha + 3, 4\alpha + 2 \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} \alpha, \alpha, 3\alpha - 1 \\ 3\alpha + 1, 3\alpha + 1 \end{matrix} \middle| 1 \right]; \\ & {}_3F_2 \left[\begin{matrix} 2, \alpha, 2\alpha + 1 \\ 2\alpha + 3, 3\alpha + 1 \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} 2\alpha + 1, 2\alpha + 1, 3\alpha - 1 \\ 3\alpha + 1, 4\alpha + 2 \end{matrix} \middle| 1 \right]. \end{aligned}$$

Clearly, both ‘orbits’ generated, respectively, by the two ${}_3F_2$ -series displayed in equation (25) do not have any ${}_3F_2$ -series in common. Therefore, for any natural number α , the relation (25) does not lie among Thomae’s 120 relations. In addition, it is not hard to check, according to the definition of $I[a, b, c, d, e]$, the following asymptotic relation

$$\lim_{\alpha \rightarrow +\infty} I[\alpha, \alpha, 2\alpha - 1, 1, 3\alpha - 2]^{1/\alpha} = \max_{(x, y) \in [0, 1]^2} x(1-x)(1-y)^2 = \frac{1}{4} < 1,$$

which means that

$$\lim_{\alpha \rightarrow +\infty} I[\alpha, \alpha, 2\alpha - 1, 1, 3\alpha - 2] = 0.$$

Consequently, the relation displayed in equation (26) provides infinitely many counter-examples to the Rhin–Viola conjecture.

Further counter-examples with more free parameters can also be constructed. However, the computation involved in ${}_3F_2$ -series is too complicated to reproduce here.

4.3 Counter-examples from the exotic contiguous relations

Based on the exotic contiguous relations in section 3, we shall further construct counter-examples to the Rhin–Viola conjecture.

THEOREM 66 *The following two relations are counter-examples to the Rhin–Viola conjecture:*

$$I[4, 0, 4, 1, 2] = \frac{6}{5} I[6, 0, 4, 2, 1]; \quad (27a)$$

$$I[3, 0, 8, 2, 6] = 6 I[4, 0, 8, 4, 5]. \quad (27b)$$

From the relation between $I(a, b, c, d, e)$ and ${}_3F_2(1)$ -series displayed in equation (22), we can show that the two equations in the last theorem correspond to Corollaries 25 and 32 specified, respectively, with $(a, c, e) = (5, 2, 3)$ and $(a, c, e) = (4, 3, 3)$.

Proof Applying the Maple commands, we get 53 arrays from $I(4, 0, 4, 1, 2)$ with only the following seven arrays free from the trivial symmetries:

$$\begin{aligned} &[5, 2, 3, 6, 7], \quad [4, 1, 3, 6, 6], \quad [4, 2, 2, 5, 7], \quad [3, 1, 2, 5, 6], \\ &[4, 5, 5, 8, 7], \quad [3, 4, 5, 8, 6], \quad [3, 3, 4, 8, 5]. \end{aligned}$$

Similarly, $I(6, 0, 4, 2, 1)$ generates also seven arrays free from the trivial symmetries:

$$\begin{aligned} &[7, 3, 4, 8, 8], \quad [5, 1, 4, 6, 8], \quad [4, 1, 3, 8, 5], \quad [2, 1, 1, 5, 6], \\ &[4, 7, 5, 8, 9], \quad [2, 5, 5, 9, 6], \quad [2, 4, 4, 9, 5]. \end{aligned}$$

Observing that the last two *orbits* have no arrays in common, we conclude that equation (27a) results in a counter-example to the Rhin–Viola conjecture.

Analogously for the two integrals displayed in equation (27b), the Maple commands produce the following arrays free from the trivial symmetries:

$$\begin{aligned} I(3, 0, 8, 2, 6): & [4, 3, 3, 5, 12], [9, 8, 3, 10, 12], [2, 1, 3, 10, 5], [7, 1, 8, 10, 10], \\ & [9, 9, 4, 11, 12], [2, 2, 4, 11, 5], [7, 2, 9, 11, 10], \\ I(4, 0, 8, 4, 5): & [5, 5, 4, 6, 14], [9, 9, 4, 10, 14], [1, 1, 4, 10, 6], [10, 9, 5, 11, 14], \\ & [2, 1, 5, 11, 6], [6, 1, 9, 11, 10], [6, 2, 10, 11, 11]. \end{aligned}$$

Therefore, the equation shown in (27b) results again in counter-example to the Rhin–Viola conjecture. ■

By means of Hardy's theorem, we can further construct infinitely many counter-examples to the Rhin–Viola conjecture from Corollary 25.

For $a = 8\alpha - 1$ and $c = e = 4\alpha$ with α being a natural number, Corollary 25 reads as the following equation:

$${}_3F_2 \left[\begin{matrix} 8\alpha - 1, & 4\alpha, & 4\alpha \\ & 8\alpha, & 10\alpha \end{matrix} \middle| 1 \right] = \frac{8\alpha}{5(8\alpha + 1)} {}_3F_2 \left[\begin{matrix} 8\alpha + 1, & 4\alpha + 1, & 4\alpha + 1 \\ & 8\alpha + 2, & 10\alpha + 1 \end{matrix} \middle| 1 \right], \quad (28)$$

which is equivalent via equation (22) to the following relation:

$$I[8\alpha - 2, 0, 6\alpha - 1, 4\alpha - 1, 2\alpha] = \frac{4\alpha}{8\alpha - 1} I[8\alpha, 0, 6\alpha - 1, 4\alpha, 2\alpha - 1]. \quad (29)$$

Rewriting the left side of equation (28) in the symmetric form explicited in Theorem 62, we get the parameters $2a = 10\alpha$, $2b = 2c = 6\alpha + 1$, $2d = 8\alpha$, $2e = 10\alpha$ and $s = 2\alpha + 1$. Under the action of the permutations group $\mathfrak{S}_5$ of a, b, c, d, e , there are four ${}_3F_2$ -series generated by that on the left-hand side of equation (28) free from the trivial symmetries:

$$\begin{aligned} &{}_3F_2 \left[\begin{matrix} 8\alpha - 1, 6\alpha, 6\alpha \\ & 10\alpha, 10\alpha \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} 6\alpha, 4\alpha, 2\alpha + 1 \\ & 6\alpha + 1, 10\alpha \end{matrix} \middle| 1 \right]; \\ &{}_3F_2 \left[\begin{matrix} 1, 4\alpha, 4\alpha \\ & 8\alpha, 6\alpha + 1 \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} 1, 2\alpha + 1, 2\alpha + 1 \\ & 6\alpha + 1, 6\alpha + 1 \end{matrix} \middle| 1 \right]. \end{aligned}$$

Similarly for the ${}_3F_2$ -series on right-hand side of equation (28), the corresponding parameters are $2a = 2e = 10\alpha + 1$, $2b = 2c = 6\alpha + 1$, $2d = 8\alpha + 2$ and $s = 2\alpha$. Under the action of

the same $\mathfrak{S}_5$, we have also four ${}_3F_2$ -series free from the trivial symmetries:

$${}_3F_2 \left[\begin{matrix} 8\alpha + 1, 6\alpha, 6\alpha \\ 10\alpha + 1, 10\alpha + 1 \end{matrix} \middle| 1 \right], \quad {}_3F_2 \left[\begin{matrix} 6\alpha, 4\alpha + 1, 2\alpha \\ 6\alpha + 1, 10\alpha + 1 \end{matrix} \middle| 1 \right];$$

$${}_3F_2 \left[\begin{matrix} 1, 4\alpha + 1, 4\alpha + 1 \\ 8\alpha + 2, 6\alpha + 1 \end{matrix} \middle| 1 \right]; \quad {}_3F_2 \left[\begin{matrix} 1, 2\alpha, 2\alpha \\ 6\alpha + 1, 6\alpha + 1 \end{matrix} \middle| 1 \right].$$

Observing that the intersection of the last two sets of ${}_3F_2$ -series is empty, we conclude that equation (29) is beyond Thomae's 120 relations for any natural number α .

According to the definition of $I[a, b, c, d, e]$, we can also show that

$$\lim_{\alpha \rightarrow +\infty} I[8\alpha - 2, 0, 6\alpha - 1, 4\alpha - 1, 2\alpha]^{1/\alpha} = \max_{(x,y) \in [0,1]^2} \frac{x^8 y^4 (1-y)^6}{(1-xy)^4} = \left(\frac{4}{27}\right)^2 < 1,$$

which implies the limiting relation:

$$\lim_{\alpha \rightarrow +\infty} I[8\alpha - 2, 0, 6\alpha - 1, 4\alpha - 1, 2\alpha] = 0.$$

Hence, we find from equation (29) infinitely many counter-examples to the Rhin-Viola conjecture.

By employing the same process to Corollary 32, we can further construct another set of infinitely many counter-examples to the Rhin-Viola conjecture:

$$I[4\alpha - 1, 0, 9\alpha - 1, 3\alpha - 1, 6\alpha] = 6I[4\alpha, 0, 9\alpha - 1, 3\alpha + 1, 6\alpha - 1]. \quad (30)$$

The details will not be reproduced due to space limitations.

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Gelfand–Shilov window classes for weighted modulation spaces

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We show that the Gelfand–Shilov algebra $S_{1/2}^{1/2}$ is densely embedded in the weighted modulation space M_m^1 . Here the weight function m is allowed to have a super-exponential growth at infinity. The basic tool is given by an integral transform called *short-time Fourier transform* (STFT). The STFT is used to both define and characterize the previous spaces. Moreover, our result is attained using the properties of the STFT and its adjoint.

Keywords: Ultra-distributions; Gelfand–Shilov type-spaces; Modulation spaces; Short-time Fourier transform

2000 Mathematics Subject Classification: 46F05; 47G30; 35S05; 44A05

1. Introduction

The aim of this paper is to study Gelfand–Shilov spaces as window classes for the weighted modulation spaces $M_m^{p,q}$, $1 \leq p, q \leq \infty$. The definition of modulation spaces is given by means of an integral transform coming from time-frequency analysis – the short-time (STFT). Indeed, we consider the linear operators of translation and modulation (so-called time-frequency shifts) given by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\omega f(t) = e^{2\pi i \omega t} f(t). \quad (1)$$

Next, let g be a non-zero window function in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$, then the STFT of a signal $f \in L^2(\mathbb{R}^d)$ with respect to the window g is given by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i \omega t} dt. \quad (2)$$

We have $V_g f \in L^2(\mathbb{R}^{2d})$. This definition can be extended to pairs of dual topological vector spaces, whose duality, denoted by $\langle \cdot, \cdot \rangle$, extends the inner product on $L^2(\mathbb{R}^d)$.

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The intuitive meaning of the previous ‘time-frequency’, expression can be explained as follows. If $f(t)$ represents a signal varying in time, its Fourier transform $\hat{f}(\omega)$ shows the distribution of its frequency ω , without any additional information about ‘when’ these frequencies appear. To overcome this problem, one may choose a non-negative window function g well localized around the origin. Then, the information of the signal f at the instant x can be obtained by shifting the window g till the instant time x , under consideration, and by computing the Fourier transform of the product $f(x)g(t-x)$, that localizes f around the instant time x .

Once the analysis of the signal f is terminated, we can reconstruct the original signal f by a suitable inversion procedure. Namely, the reproducing formula related to the STFT, for every pair of windows $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ with $\langle \varphi_1, \varphi_2 \rangle \neq 0$, reads as follows

$$\int_{\mathbb{R}^{2d}} V_{\varphi_1} f(x, \omega) M_{\omega} T_x \varphi_2 dx d\omega = \langle \varphi_2, \varphi_1 \rangle f. \quad (3)$$

The function φ_1 is called the *analysis* window, because the STFT $V_{\varphi_1} f$ gives the time-frequency distribution of the signal f , whereas the window φ_2 permits to come back to the original f and, consequently, is called the *synthesis* window. We address in section 2, for a full treatment of the topic.

The modulation space norm $M_m^{p,q}(\mathbb{R}^d)$ of a function f on $\mathbb{R}^d$ is given by the $L_m^{p,q}(\mathbb{R}^{2d})$ norm of its STFT $V_g f$, defined on the time-frequency space $\mathbb{R}^{2d}$, with respect to a *suitable* window function g on $\mathbb{R}^d$. Depending on the growth of the weight function m , different Gelfand–Shilov classes may be chosen as fitting test function spaces for modulation spaces, see [1, 2]. The widest class of weights allowing to define modulation spaces, see [1, 3, 15] is the weight class $\mathcal{N}$. A weight function m on $\mathbb{R}^{2d}$ belongs to $\mathcal{N}$ if it is a continuous, positive function such that

$$m(z) = o(e^{cz^2}), \quad \text{for } |z| \rightarrow \infty, \quad \forall c > 0, \quad (4)$$

with $z \in \mathbb{R}^{2d}$. For instance, every function $m(z) = e^{s|z|^b}$, with $s > 0$ and $0 \leq b < 2$, is in $\mathcal{N}$. Thus, the weight m may grow faster than exponentially at infinity. We notice that there is a limit in enlarging the weight class for modulation spaces, imposed by Hardy’s theorem: if $m(z) \geq Ce^{cz^2}$, for some $c > \pi/2$, then the corresponding modulation spaces are trivial [3].

Our main result (Theorem 3.3) shows that the Gelfand–Shilov class $\mathcal{S}_{1/2}^{1/2}$ is densely embedded in M_m^1 , with $m \in \mathcal{N}$. Some applications of the previous result, restricted to a subclass $\mathcal{M}_v$ of $\mathcal{N}$, are contained in [4, 5]. Namely, let v be a continuous, positive, even function ($v(z) = v(-z)$), with $v(0) = 1$. v is also assumed to be submultiplicative: $v(z_1 + z_2) \leq v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$, and to satisfy: $v(x, \omega) = v(-\omega, x) = v(-x, \omega)$. Then, $m \in \mathcal{M}_v$ (the class of v -moderate weights) if m is a positive, even continuous functions on $\mathbb{R}^{2d}$ that satisfies

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2), \quad \forall z_1, z_2 \in \mathbb{R}^{2d}.$$

Notice that the submultiplicativity and continuity of v imply that the weight v (and, consequently, every v -moderate weight m) is *dominated* by an exponential function, *i.e.*,

$$\exists C, k > 0 \quad \text{such that} \quad v(z) \leq Ce^{k|z|}, \quad z \in \mathbb{R}^{2d}, \quad (5)$$

thereby $\mathcal{M}_v \subset \mathcal{N}$.

The results of references [4, 5] rely on the following property, proved in section 3, by applying the above-mentioned density result.

PROPOSITION 1.1 *For $m \in \mathcal{M}_v$, the space of admissible windows for computing the $M_m^{p,q}$ -norm can be enlarged to M_v^1 .*

This guarantees that every non-zero function $g \in M_v^1$ can be picked up to define the norms of the spaces $M_m^{p,q}$.

Notation The scalar product on $\mathbb{R}^d$ is $xy := x \cdot y$. Hence, for every $c \in \mathbb{R}$, $cx^2 = c(x_1^2 + \dots + x_d^2)$. Given a vector $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, the partial derivative with respect to x_j is denoted by $\partial_j = (\partial/\partial x_j)$. Given a multi-index $p = (p_1, \dots, p_d) \geq 0$, i.e., $p \in \mathbb{N}_0^d$ and $p_j \geq 0$, we write $\partial^p = \partial_1^{p_1} \dots \partial_d^{p_d}$; moreover, we write $x^p = (x_1, \dots, x_d)^{(p_1, \dots, p_d)} = \prod_{i=1}^d x_i^{p_i}$. For $h \in \mathbb{R}$, $\alpha \in \mathbb{R}_+^d$, $h|x|^{1/\alpha} = h \sum_{i=1}^d |x_i|^{1/\alpha_i}$. Moreover, for $p \in \mathbb{N}_0^d$, we set $(p!)^\alpha = (p_1!)^{\alpha_1} \dots (p_d!)^{\alpha_d}$, while as standard $p! = p_1! \dots p_d!$.

The Schwartz class is denoted by $\mathcal{S}$, the space of tempered distributions by $\mathcal{S}'$. We use the brackets $\langle f, g \rangle$ to denote the extension of the inner product on $L^2(\mathbb{R}^d)$ to every pair of dual topological vector spaces. The space of smooth functions is C^∞ while the space of smooth functions with compact support is C_0^∞ . The Fourier transform is normalized to be $\mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t \omega} dt$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2 .

2. Function spaces

2.1 Gelfand–Shilov spaces

The Gelfand–Shilov spaces $\mathcal{S}_\beta^\alpha$ and Σ_β^α were first introduced by Gelfand and Shilov in [6]. They have been applied by many authors, both in time-frequency and in PDE contexts [2, 7–9]. We shall present them in the indices' case: $\alpha = \beta$, required by this study. Besides, instead of their very definition, we shall utilize the following useful characterization.

PROPOSITION 2.1 *The following statements are equivalent [1, 6, 8]:*

- (i) $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $f \in \Sigma_\alpha^\alpha(\mathbb{R}^d)$).
- (ii) $f \in C^\infty(\mathbb{R}^d)$ and there exist real constants $h > 0, k > 0$ such that (resp. for every $h > 0, k > 0$)

$$\|f e^{h|x|^{1/\alpha}}\|_{L^\infty} < \infty \quad \text{and} \quad \|(\mathcal{F}f)e^{k|\omega|^{1/\alpha}}\|_{L^\infty} < \infty. \quad (6)$$

- (iii) $f \in C^\infty(\mathbb{R}^d)$ and there exists $C > 0$ (resp. for every $C > 0$)

$$\|x^p \partial^q f\|_{L^\infty} \leq RC^{|p|+|q|} (p!)^\alpha (q!)^\alpha, \quad \forall p, q \in \mathbb{N}_0^d, \quad (7)$$

for a suitable $R > 0$.

Gelfand–Shilov spaces enjoy the following embeddings

- (i) For $\alpha \geq 0$ [6],

$$\Sigma_\alpha^\alpha \hookrightarrow \mathcal{S}_\alpha^\alpha \hookrightarrow \mathcal{S}. \quad (8)$$

- (ii) For every $0 \leq \alpha_1 < \alpha_2$ [1],

$$\mathcal{S}_{\alpha_1}^{\alpha_1} \hookrightarrow \Sigma_{\alpha_2}^{\alpha_2}. \quad (9)$$

Furthermore, $\mathcal{S}_\alpha^\alpha$ is not trivial if and only if $\alpha \geq 1/2$, whereas $\Sigma_\alpha^\alpha \neq \{0\}$ if and only if $\alpha > 1/2$ (see Pilipović [9]).

The strong duals of Gelfand–Shilov classes $\mathcal{S}_\alpha^\alpha$ and Σ_α^α are spaces of tempered ultra-distributions of Roumieu and Beurling type and will be denoted by $(\mathcal{S}_\alpha^\alpha)'$ and $(\Sigma_\alpha^\alpha)'$, respectively.

The Gelfand–Shilov spaces $\mathcal{S}_\alpha^\alpha$ are invariant under Fourier transform $\mathcal{F}$ and time-frequency shifts

$$\mathcal{F}(\mathcal{S}_\alpha^\alpha) = \mathcal{S}_\alpha^\alpha, \quad T_x(\mathcal{S}_\alpha^\alpha) = \mathcal{S}_\alpha^\alpha \quad \text{and} \quad M_\omega(\mathcal{S}_\alpha^\alpha) = \mathcal{S}_\alpha^\alpha. \quad (10)$$

The same holds true for the spaces Σ_α^α . Therefore, the spaces $\mathcal{S}_\alpha^\alpha$ and Σ_α^α are a family of *Fourier transform and time-frequency shift invariant spaces* contained in the Schwartz class $\mathcal{S}$; the *smallest*, non-trivial one being $\mathcal{S}_{1/2}^{1/2}$. Functions in $\mathcal{S}_{1/2}^{1/2}$ are, e.g., the Gaussian $f(x) = e^{-\pi x^2}$ or the Hermite functions.

Another useful characterization of the spaces $\mathcal{S}_\alpha^\alpha$ and Σ_α^α involves the STFT and is proved in [8, Proposition 4.3]: $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ if and only if $V_g f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d})$, and the same for Σ_α^α . We shall extensively use the case $\alpha = 1/2$: for a non-zero window $g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ we have

$$f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d) \iff V_g f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d}). \quad (11)$$

For a non-zero $g \in L^2(\mathbb{R}^d)$, we write V_g^* for the adjoint of V_g , given by

$$\langle V_g^* F, f \rangle = \langle F, V_g f \rangle, \quad f \in L^2(\mathbb{R}^d), \quad F \in L^2(\mathbb{R}^{2d}).$$

In particular, for $F \in \mathcal{S}(\mathbb{R}^{2d})$, $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(x, \omega) M_\omega T_x g(t) dx d\omega \in \mathcal{S}(\mathbb{R}^d). \quad (12)$$

Take $f \in \mathcal{S}(\mathbb{R}^d)$ and set $F = V_g f$, then

$$f(t) = \frac{1}{\|g\|_{L^2}^2} \int_{\mathbb{R}^{2d}} V_g f(x, \omega) M_\omega T_x g(t) dx d\omega \in \mathcal{S}(\mathbb{R}^d). \quad (13)$$

We refer to [10, Proposition 11.3.2] for a whole treatment of the adjoint operator.

LEMMA 2.2 *Consider a non-zero window $g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. Then, for every $F \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d})$, we have $V_g^* F \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$.*

Proof We set $f = V_g^* F \in \mathcal{S}(\mathbb{R}^d)$ and we shall show that $f \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$ by using the characterization (7). To estimate $\|t^p \partial^q f\|_{L^\infty}$, we first recompute formula [10, Lemma 11.2.1] with the role of the operators ∂^q and t^p interchanged

$$t^p \partial^q (M_\omega T_x g) = \sum_{\delta_1 \leq q} \sum_{\delta_2 \leq p} \binom{q}{\delta_1} \binom{p}{\delta_2} x^{\delta_2} (2\pi i \omega)^{\delta_1} M_\omega T_x (t^{p-\delta_2} \partial^{q-\delta_1} g). \quad (14)$$

Secondly, for $\delta_1 \leq q$ and $\delta_2 \leq p$, we get the following majorization

$$\|M_\omega T_x (t^{p-\delta_2} \partial^{q-\delta_1} g)\|_{L^\infty} = \|t^{p-\delta_2} \partial^{q-\delta_1} g\|_{L^\infty} \leq C^{|p-\delta_2|+|q-\delta_1|+1} ((p-\delta_2)!)^{1/2} ((q-\delta_1)!)^{1/2}$$

The preceding estimates yield the desired result. In fact,

$$\begin{aligned} |t^p \partial^q f| &\leq \int_{\mathbb{R}^{2d}} |F(x, \omega)| |x^p \partial^q (M_\omega T_x g)| dx d\omega \\ &\leq \sum_{\delta_1 \leq q} \sum_{\delta_2 \leq p} \binom{q}{\delta_1} \binom{p}{\delta_2} C^{|p-\delta_2|+|q-\delta_1|+1} ((p-\delta_2)!)^{1/2} ((q-\delta_1)!)^{1/2} \\ &\quad \cdot \int_{\mathbb{R}^{2d}} |F(x, \omega)| |x^{\delta_2} (2\pi i \omega)^{\delta_1}| dx d\omega. \end{aligned}$$

In view of our assumption $F \in S_{1/2}^{1/2}(\mathbb{R}^{2d})$, by using the characterization (7), the last integral can be easily estimated by $\tilde{C}^{|\delta_2|+|\delta_1|+1} (\delta_2!)^{1/2} (\delta_1!)^{1/2}$, hence $\|t^p \partial^q f\|_{L^\infty} \leq K^{|p|+|q|+1} (p!)^{1/2} (q!)^{1/2}$, for every $p, q \in \mathbb{N}_0^d$. ■

2.2 Modulation spaces

Modulation spaces having weights in $\mathcal{M}_v$ (at most sub-exponential growth) were first introduced by Feichtinger [11]. We define them using the Gelfand–Shilov class $S_{1/2}^{1/2}$ as a test function space in this way, hereafter. Let us first argue for weights in the more general class $\mathcal{N}$.

DEFINITION 2.3 *Let $m \in \mathcal{N}$, and g a non-zero window function in $S_{1/2}^{1/2}(\mathbb{R}^d)$. For $1 \leq p, q \leq \infty$, the modulation space $M_m^{p,q}(\mathbb{R}^d)$ consists of all tempered ultra-distributions $f \in (S_{1/2}^{1/2})'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}$ is*

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q},$$

(with obvious changes if either $p = \infty$ or $q = \infty$). If $p = q$, $M_m^p := M_m^{p,p}$, and, if $m \equiv 1$, then $M^{p,q}$ and M^p stand for $M_m^{p,q}$ and M_m^p , respectively.

Notice that:

- (i) If $f, g \in S_{1/2}^{1/2}(\mathbb{R}^d)$, the above integral is convergent as per equations (4), (6) and (11). Namely, if $m \in \mathcal{N}$ we choose $c = h - \epsilon > 0$ in equation (4), for a suitable $\epsilon > 0$, with h being such that $\|V_g f e^{h|\cdot|^2}\|_{L^\infty} < \infty$. Hence

$$\begin{aligned} &\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx \right)^{q/p} d\omega \\ &\leq C \|(V_g f) e^{h|\cdot|^2}\|_{L^\infty} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |m(x, \omega)|^p e^{-hp|(x, \omega)|^2} dx \right)^{q/p} d\omega < \infty. \end{aligned}$$

- (ii) The definition of $M_m^{p,q}$ may depend on the choice of the window function g . However, we shall see in the next section that, if $m \in \mathcal{M}_v$, the definition of $M_m^{p,q}$ does not depend on $g \in S_{1/2}^{1/2}$ – however the class of admissible window can be enlarged to M_v^1 . Since the previous environment is the main concern of this paper, we shall continue to denote $M_m^{p,q}$ for every $m \in \mathcal{N}$, keeping in mind that, if $m \in \mathcal{N} \setminus \mathcal{M}_v$, we have $M_m^{p,q} = M_{m,g}^{p,q}$, where $g \in S_{1/2}^{1/2}(\mathbb{R}^d)$ is the window with respect to which we compute the STFT V_g .

- (iii) For $m \in \mathcal{M}_v$, then $M_m^{p,q}$ is the subspace of ultra-distribution $(\Sigma_1^1)'$ defined in [5, Definition 2.1]
- (iv) If $m \in \mathcal{M}_v$ and fulfills the GRS-condition $\lim_{n \rightarrow \infty} v(nz)^{1/n} = 1$, for all $z \in \mathbb{R}^{2d}$, the definition of modulation spaces is the same as in [12] (because the ‘space of special windows’ $\mathcal{S}_C$ is a subset of $\mathcal{S}_{1/2}^{1/2}$).
- (v) For related constructions of modulation spaces, involving the theory of coorbit spaces, we refer to [13, 14].

3. The density result

First, let us recall the following semigroup property of the Gaussian functions.

LEMMA 3.1 *Let $a, b > 0$, then*

$$e^{-\pi ax^2} * e^{-\pi bx^2} = (a+b)^{-d/2} e^{-\pi abx^2/(a+b)}. \quad (15)$$

Proof It is a straightforward computation by Fourier transform. Namely, recall that, for all $a > 0$, the Fourier transform of a Gaussian is a Gaussian again, given by $\mathcal{F}(e^{-\pi ax^2})(\omega) = a^{-d/2} e^{-\pi \omega^2/a}$. Therefore, we have

$$\begin{aligned} e^{-\pi ax^2} * e^{-\pi bx^2} &= \mathcal{F}^{-1}(\mathcal{F}(e^{-\pi ax^2}) \cdot \mathcal{F}(e^{-\pi bx^2})) \\ &= \mathcal{F}^{-1}(a^{-d/2} e^{-\pi \omega^2/a} \cdot b^{-d/2} e^{-\pi \omega^2/b}) \\ &= (ab)^{-d/2} \mathcal{F}^{-1}(e^{-\pi (a+b)\omega^2/(ab)}) = (a+b)^{-d/2} e^{-\pi abx^2/(a+b)}. \end{aligned} \quad \blacksquare$$

Let $\varphi(z) = e^{-\pi z^2}$, $z \in \mathbb{R}^{2d}$. The family $\{\varphi_n\}_n$, with n positive integer given by

$$\varphi_n(z) = n^{2d} \varphi(nz) \quad (16)$$

is an *approximate identity* and, by the characterization (11), is straightforward showing that $\varphi_n \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d})$, for every $n \in \mathbb{N}$.

LEMMA 3.2 *Let $m \in \mathcal{N}$ and fix a window function $g \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. Consider a function $f \in M_m^1(\mathbb{R}^d)$. Then, for every $l, n \in \mathbb{N}$, $l, n > 0$, the functions defined by*

$$G_{n,l}(z) = [(V_g f)m * \varphi_n](z) e^{-\pi z^2/l}, \quad z \in \mathbb{R}^{2d}, \quad (17)$$

are in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^{2d})$.

Proof We use the characterization in equation (6). We consider $0 < h < \pi/l$. Then, using Young’s inequality for the convolution norm estimate, we have

$$\begin{aligned} \|G_{n,l}(z) e^{hz^2}\|_{L^\infty} &= \|[(V_g f)m * \varphi_n](z) e^{-\pi z^2/l} e^{hz^2}\|_{L^\infty} \\ &= \|(V_g f)m * \varphi_n\|_{L^\infty} \|e^{-\pi z^2/l} e^{hz^2}\|_{L^\infty} \end{aligned}$$

$$\begin{aligned}
&\leq \|(V_g f)m\|_{L^\infty} \|\varphi_n\|_{L^1} \|e^{-(\pi/l-h)z^2}\|_{L^\infty} \\
&= \|(V_g f)m\|_{L^\infty} \|e^{-(\pi/l-h)z^2}\|_{L^\infty} < \infty,
\end{aligned}$$

since $(V_g f)m \in \mathcal{C}_0$. Hence, the left-hand side inequality in equation (6) is fulfilled. Next, we compute the Fourier transform of $G_{n,l}$. Precisely,

$$\begin{aligned}
\mathcal{F}(G_{n,l})(\zeta) &= (\mathcal{F}[(V_g f)m * \varphi_n] * \mathcal{F}(e^{-z^2/l}))(\zeta) \\
&= (\mathcal{F}[(V_g f)m]\mathcal{F}(\varphi_n)) * (l^d e^{-\pi l \zeta^2}) \\
&= l^d (\mathcal{F}[(V_g f)m]e^{-\pi \zeta^2/n^2}) * e^{-\pi l \zeta^2}.
\end{aligned}$$

We look for a positive integer k that satisfies the right-hand side inequality of equation (6). We use equation (15) to compute $e^{-\pi \zeta^2/n^2} * e^{-\pi l \zeta^2} = (1/n^2 + l)^{-d} e^{-\pi l \zeta^2/(1+ln^2)}$. The boundedness of the Fourier transform between the spaces L^1 and L^∞ then gives the desired result

$$\begin{aligned}
\|\mathcal{F}(G_{n,l})(\zeta)e^{k\zeta^2}\|_{L^\infty} &\leq l^d \|\mathcal{F}[(V_g f)m]\|_{L^\infty} \|(e^{-\pi \zeta^2/n^2} * e^{-\pi l \zeta^2})e^{k\zeta^2}\|_{L^\infty} \\
&\leq \left(\frac{ln^2}{1+ln^2}\right)^d \|(V_g f)m\|_{L^1} \|e^{-[\pi l/(1+ln^2)-k]\zeta^2}\|_{L^\infty} < \infty,
\end{aligned}$$

for every k such that $0 < k < \pi l/(1+ln^2)$. ■

Eventually, we have all the pieces in place to prove our main result.

THEOREM 3.3 *Let $m \in \mathcal{N}$ and fix a non-zero window $g \in \mathcal{S}_{1/2}^{1/2}$ to define the modulation space M_m^1 . Then,*

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow M_m^1.$$

Proof For the sake of simplicity, we assume that the window g is normalized, i.e., $\|g\|_{L^2} = 1$. Let $f \in M_m^1(\mathbb{R}^d)$, where its M_m^1 -norm is computed with respect to the fixed window g . Clearly, as pointed out after the definition of the modulation spaces $M_m^{p,q}$, we have $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d) \subset M_m^1(\mathbb{R}^d)$. We shall prove the density of the preceding embedding. Consider the sequence of functions $f_{n,l}$ defined as

$$f_{n,l} := V_g^* G_{n,l},$$

where the functions $G_{n,l}$ are given by equation (17). Using Lemmas 3.2 and 2.2, we obtain that $f_{n,l} \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$.

Since $(V_g f)m \in L^1$, the approximate identity $\{\varphi_n\}_n$ defined in equation (16) gives the L^1 -convergence of the sequence $(V_g f)m * \varphi_n$ to the function $(V_g f)m$.

Next, observe that for every fixed l the functions $G_{n,l}$ converge to $(V_g f)m e^{-\pi z^2/l}$ in L^1 -norm, since

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^{2d}} |(V_g f)m * \varphi_n - (V_g f)m| e^{-\pi z^2/l} dz \leq \|(V_g f)m * \varphi_n - (V_g f)m\|_{L^1}$$

and the right-hand side of the previous inequality tends to 0, when $n \rightarrow \infty$. Now, the Dominated Convergence Theorem gives that the sequence $(V_g f)m e^{-\pi z^2/l}$, both dominated by and pointwise converging to the integrable function $(V_g f)m$, converges in the L^1 -norm to $(V_g f)m$.

Finally, the inversion formula (13) proves that the sequence $f_{n,l}$ converges to f in the M_m^1 -norm. In fact,

$$\begin{aligned}\|f_{n,l} - f\|_{M_m^1} &= \|[V_g(f_{n,l} - f)]m\|_{L^1} = \|G_{n,l} - (V_g f)m\|_{L^1} \\ &\leq \|G_{n,l} - (V_g f)m e^{-\pi z^2 l}\|_{L^1} + \|(V_g f)m e^{-\pi z^2 l} - (V_g f)m\|_{L^1}\end{aligned}$$

and the right-hand side above tends to 0 for $(n, l) \rightarrow +\infty$, independently on the indices' order. ■

Finally, we restrict our attention to v -moderate weights $m \in \mathcal{M}_v$. Then, as shown in [5], modulation spaces can be regarded as subspace of the space of ultra-distributions $(\Sigma_1^1)'$. In this case, in the previous paper it is highlighted that they gain many nice properties. In particular, the definition of modulation spaces is independent of the choice of the non-zero window $g \in \Sigma_1^1$. However, due to our density result, the window class Σ_1^1 can be enlarged to M_v^1 .

Proof of Proposition 1.1. We first notice that $\mathcal{S}_{1/2}^{1/2} \hookrightarrow \Sigma_1^1$, by means of the continuous and dense embedding given in equation (8). Finally, the inclusion $\Sigma_1^1 \subset M_v^1$ [5] yields to the dense embedding $\Sigma_1^1 \hookrightarrow M_v^1$ while the window class extension from Σ_1^1 to M_v^1 follows the same arguments as in [10, Theorem 11.3.7]. ■

For weights $m \in \mathcal{N} \setminus \mathcal{M}_v$, we lose the submultiplicative property and the standard techniques via Young's inequality that cannot be applied henceforth. It seems reasonable to get the independence of the modulation space definition for windows g , at least in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$, but a new approach coming from atomic decomposition theory seems to be more appropriate and will be the object of a forthcoming work.

Since, for $m \in \mathcal{M}_v$, $M_m^1 \hookrightarrow M_m^{p,q}$, $1 \leq p, q < \infty$, see [1, 10], we have the following corollary.

COROLLARY 3.4 *Let $m \in \mathcal{M}_v$, $1 \leq p, q < \infty$, then*

$$\mathcal{S}_{1/2}^{1/2} \hookrightarrow M_m^{p,q}.$$

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Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic

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In this article, necessary and sufficient conditions under which two classes of functions are, respectively, logarithmically completely monotonic and completely monotonic are presented. From these conditions, several two-sided inequalities for the ratio of two gamma functions are deduced.

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1. Introduction

It is well known that Euler's Gamma function $\Gamma(z)$ defined by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \quad (\Re(z) > 0)$$

can be continued *meromorphically* to the whole complex z -plane with *simple* poles at

$$z \in \mathbb{Z}_0^- \quad (\mathbb{Z}_0^- := \{0, -1, -2, \dots\}).$$

In terms of (for example) logarithmic convexity, the function $\Gamma(x)$ ($x \in \mathbb{R}^+$) is characterized by the Bohr–Møllerup (or, more precisely, the Bohr–Møllerup–Artin) Theorem

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(cf., e.g., [1, p. 13, Theorem 1.1]). A closely-related function, the Psi (or Digamma) function $\psi(z)$ is defined by

$$\psi(z) := \frac{d}{dz} \{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log \Gamma(z) = \int_1^z \psi(t) dt.$$

More generally, for the Polygamma function $\psi^{(n)}(z)$, we have

$$\psi^{(n)}(z) := \frac{d^{n+1}}{dz^{n+1}} \{\log \Gamma(z)\} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \mathbb{N} := \{1, 2, 3, \dots\})$$

with, of course,

$$\psi(z) = \psi^{(0)}(z).$$

We recall from [2] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n \in \mathbb{N}_0).$$

Bernstein's Theorem (see [2, p. 160, Theorem 12a]) states that a function f on $[0, \infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ such that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

converges for $x \in [0, \infty)$.

We recall also (see [3–5]) that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \quad (x \in I; n \in \mathbb{N}).$$

It is known [4, 5] that a logarithmically completely monotonic function is also completely monotonic.

For real numbers $\alpha \neq 0$ and β , we define a function $g_{\alpha, \beta}(x)$ by

$$g_{\alpha, \beta}(x) = \left(\frac{e^x \Gamma(x+1)}{(x+\beta)^{x+\beta}} \right)^\alpha \quad (1)$$

in the interval

$$I_\beta \triangleq (\max\{0, -\beta\}, \infty).$$

Guo [6] proved, among other things, that the function $g_{\alpha, \beta}(x)$ is logarithmically completely monotonic if either

$$\alpha > 0 \quad \text{and} \quad \beta \geq 1$$

or

$$\alpha < 0 \quad \text{and} \quad \beta \leq \frac{1}{2}.$$

In this short paper, we shall further prove, by using a method based upon the Hurwitz (or generalized) Zeta function (see, for example, [1, Chapter 2, Section 2.2]), that the above-stated sufficient condition is also necessary. Our main results are given by Theorems 1 and 2 below.

THEOREM 1 Let $\alpha \neq 0$ and β be real parameters. Then the function $g_{\alpha,\beta}(x)$ defined by (1) is logarithmically completely monotonic if and only if either

$$\alpha > 0 \quad \text{and} \quad \beta \geq 1$$

or

$$\alpha < 0 \quad \text{and} \quad \beta \leq \frac{1}{2}.$$

Remark 1 For some functions similar to $g_{\alpha,\beta}(x)$, such as the function $\mathcal{F}_\alpha(x)$ given by

$$\mathcal{F}_\alpha(x) = \frac{e^x \Gamma(x)}{x^{x-\alpha}} \quad (x > 0; \alpha \in \mathbb{R}) \quad (2)$$

and the function $f_{\alpha,\beta}(x)$ given by

$$f_{\alpha,\beta}(x) = \frac{e^x \Gamma(x + \beta)}{x^{x+\beta-\alpha}} \quad (x > 0; \alpha \in \mathbb{R}; \beta \geq 0), \quad (3)$$

the monotonicity, convexity, completely monotonic, and logarithmically completely monotonic properties have been investigated in several earlier works [7–9].

As a direct consequence of Theorem 1, a Kečkić–Vasić–Alzer type two-sided inequality [7, 10] is established as follows.

THEOREM 2 For $y > x > 0$, the inequality:

$$\frac{(x+1)^{x+1}}{(y+1)^{y+1}} e^{y-x} < \frac{\Gamma(x+1)}{\Gamma(y+1)} < \frac{(x+1/2)^{x+1/2}}{(y+1/2)^{y+1/2}} e^{y-x} \quad (4)$$

or, equivalently,

$$I\left(x + \frac{1}{2}, y + \frac{1}{2}\right) < \left(\frac{\Gamma(x+1)}{\Gamma(y+1)}\right)^{1/(x-y)} < I(x+1, y+1) \quad (5)$$

is valid and sharp in the sense that the constants 1 and 1/2 in the very left and the very right sides of the two-sided inequalities (4) and (5) cannot be replaced by smaller and larger ones, respectively, where

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a}\right)^{1/(b-a)} \quad (a > 0; b > 0; a \neq b) \quad (6)$$

is the identric or exponential mean.

Remark 2 Kečkić and Vasić [10] gave the following two-sided inequality:

$$\frac{b^{b-1}}{a^{a-1}} e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b} \quad (b > a > 1), \quad (7)$$

which can be rearranged as follows:

$$\left(\frac{b}{a}\right)^{1/2} \frac{\Gamma(b)}{\Gamma(a)} < [I(a, b)]^{b-a} < \frac{b}{a} \frac{\Gamma(b)}{\Gamma(a)} \quad (b > a \geq 1). \quad (8)$$

Subsequently, Alzer [7, Theorem 4] sharpened the inequality (8) and showed that the following two-sided inequality:

$$\left(\frac{b}{a}\right)^{\alpha} \frac{\Gamma(b)}{\Gamma(a)} < [I(a, b)]^{b-a} < \left(\frac{b}{a}\right)^{\beta} \frac{\Gamma(b)}{\Gamma(a)} \quad (b > a \geq 1) \quad (9)$$

holds true if and only if

$$\alpha \leq \frac{1}{2} \quad \text{and} \quad \beta \geq \gamma,$$

where γ denotes the Euler–Mascheroni constant. Chen and Qi [9], on the other hand, extended the inequalities (7) and (8), and the left-hand side inequality in equation (9), to the case when $b > a > 0$. Therefore, the inequalities (4) and (5) extend and generalize the inequalities (7) to (9).

Finally, a necessary and sufficient condition such that the function

$$\ln g_{-a,b}(x) + c \quad (c \in \mathbb{R})$$

is completely monotonic in $(0, \infty)$ is obtained as follows. Let

THEOREM 3 $a, b, c \in \mathbb{R}$. Then the function $G_{a,b,c}(x)$ given by

$$G_{a,b,c}(x) = a[(x+b) \ln(x+b) - x - \ln \Gamma(x+1)] + c \quad (x \in I_b) \quad (10)$$

is completely monotonic if and only if

$$a > 0, \quad b = \frac{1}{2}, \quad \text{and} \quad c \geq \frac{a}{2} \ln \left(\frac{2\pi}{e} \right).$$

2. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1 Since the sufficiency was established in ref. [6], we only need to prove the necessity.

Suppose that $g_{\alpha,\beta}(x)$ is logarithmically completely monotonic on I_{β} . Then the function $\ln g_{\alpha,\beta}(x)$ is decreasing on I_{β} . Hence, we have

$$\lim_{x \rightarrow \infty} \ln g_{\alpha,\beta}(x) < \infty. \quad (11)$$

It is well known from [11, Chapter 1] that

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right) \quad (x \rightarrow \infty). \quad (12)$$

By equation (12), it is easy to show that

$$\lim_{x \rightarrow \infty} \ln g_{\alpha,\beta}(x) = \infty$$

if either

$$\alpha < 0 \quad \text{and} \quad \beta > \frac{1}{2}$$

or

$$\alpha > 0 \quad \text{and} \quad \beta < \frac{1}{2}.$$

In view of (11), it follows immediately that, if $\alpha < 0$, then $\beta \leq 1/2$ and that, if $\alpha > 0$, then $\beta \geq 1/2$.

Now it remains to show that, if $\alpha > 0$, then $\beta \geq 1$. It is clear that

$$I_\beta = (0, \infty) \quad \text{when} \quad \alpha > 0 \quad \text{and} \quad \beta \geq \frac{1}{2}.$$

We now recall the following familiar relationship (cf. [12, p. 894]; see also [1, p. 22]):

$$\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1, x) \quad (n \in \mathbb{N}; x \notin \mathbb{Z}_0^-), \quad (13)$$

where

$$\zeta(z, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^z} \quad (\Re(z) > 1; x \notin \mathbb{Z}_0^-) \quad (14)$$

is the Hurwitz (or generalized) Zeta function. By equations (13) and (14), direct calculation leads us to

$$\begin{aligned} (-1)^n [\ln g_{\alpha, \beta}(x)]^{(n)} &= (-1)^n \alpha \left(\psi^{(n-1)}(x+1) - \frac{(-1)^{n-2} (n-2)!}{(x+\beta)^{n-1}} \right) \\ &= (-1)^n \alpha \left((-1)^n (n-1)! \zeta(n, x+1) - \frac{(-1)^{n-2} (n-2)!}{(x+\beta)^{n-1}} \right) \\ &= \alpha (n-2)! \left((n-1) \zeta(n, x+1) - \frac{1}{(x+\beta)^{n-1}} \right). \end{aligned}$$

By combining this last result with the hypothesis that

$$(-1)^n [\ln g_{\alpha, \beta}(x)]^{(n)} \geq 0,$$

we conclude that, if $\alpha > 0$ and $n \geq 2$, then the following inequality:

$$(n-1) \zeta(n, x+1) - \frac{1}{(x+\beta)^{n-1}} \geq 0 \quad (15)$$

is valid for $x \in (0, \infty)$. If we proceed to the limit as $x \rightarrow 0+$ in (15), we get

$$(n-1) \zeta(n) \geq \frac{1}{\beta^{n-1}} \geq 0 \quad (n \geq 2),$$

which is equivalent to

$$\beta \geq \left(\frac{1}{n-1} \right)^{1/(n-1)} \left(\frac{1}{\zeta(n)} \right)^{1/(n-1)} \quad (\alpha > 0; n \geq 2). \quad (16)$$

Since $\zeta(n) \geq 1$ is decreasing with $n \geq 2$, we have

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\zeta(n)} \right)^{1/(n-1)} = 1. \quad (17)$$

It is also obvious that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n-1} \right)^{1/(n-1)} = 1, \quad (18)$$

by using l'Hôpital's rule. Taking the limit as $n \rightarrow \infty$ in equation (16) and employing both equations (17) and (18) yields that $\beta \geq 1$ if $\alpha > 0$. The proof of Theorem 1 is thus completed. ■

Proof of Theorem 2. From Theorem 1, it is known that the function $g_{\alpha,\beta}(x)$ is decreasing if $\alpha > 0$ and $\beta \geq 1$. Thus, clearly, the inequality:

$$\frac{e^x \Gamma(x+1)}{(x+\beta)^{x+\beta}} > \frac{e^y \Gamma(y+1)}{(y+\beta)^{y+\beta}}$$

holds true for $y > x > 0$ and $\beta \geq 1$. Rearrangement of the above inequality and the well-known increasingly monotonicity of the identric or exponential mean leads us to the left-hand side inequality in (4) and the right-hand side inequality in (5), as well as to the sharpness of these inequalities.

Similar argument produces the remaining inequalities and their sharpness. This evidently completes our proof of Theorem 2. ■

Proof of Theorem 3 Suppose that

$$a > 0, \quad b = \frac{1}{2}, \quad \text{and} \quad c \geq \frac{a}{2} \ln \left(\frac{2\pi}{e} \right).$$

It is apparent by Theorem 1 that

$$(-1)^n G_{a,b,c}^{(n)}(x) = (-1)^n [\ln g_{-a,b}(x)]^{(n)} \geq 0 \quad (n \in \mathbb{N}; x \in I_b).$$

Therefore, the function $G_{a,b,c}(x)$ is decreasing on I_b . By using equation (12) in (10), we get

$$\lim_{x \rightarrow \infty} G_{a,b,c}(x) = -\frac{a}{2} \ln \left(\frac{2\pi}{e} \right) + c \geq 0, \quad (19)$$

so that

$$G_{a,b,c}(x) \geq 0 \quad (x \in I_b).$$

Consequently, the function $G_{a,b,c}(x)$ is completely monotonic.

Conversely, if $G_{a,b,c}(x)$ is completely monotonic, then $g_{-a,b}(x)$ is logarithmically completely monotonic and $G_{a,b,c}(x)$ is nonnegative and decreasing on I_b . These observations mean that, by Theorem 1, either

$$a < 0 \quad \text{and} \quad b \geq 1$$

or

$$a > 0 \quad \text{and} \quad b \leq \frac{1}{2},$$

and that

$$0 \leq \lim_{x \rightarrow \infty} G_{a,b,c}(x) < \infty. \quad (20)$$

By using equation (12), we can show that

$$\lim_{x \rightarrow \infty} G_{a,b,c}(x) = -\infty$$

if either

$$a < 0 \quad \text{and} \quad b > \frac{1}{2}$$

or

$$a > 0 \quad \text{and} \quad b < \frac{1}{2},$$

which contradicts equation (20). If

$$a > 0 \quad \text{and} \quad b = \frac{1}{2},$$

then, by using equation (12), the inequality (19) is valid, which implies that

$$c \geq \frac{a}{2} \ln \left(\frac{2\pi}{e} \right).$$

The proof of Theorem 3 is thus completed. ■

3. Concluding remarks and observations

In our investigation here, we have found (in Theorems 1 and 3 above) necessary and sufficient conditions under which two given classes of functions are, respectively, logarithmically completely monotonic and completely monotonic. Furthermore, as an interesting consequence of one of our main results (Theorem 1), we have derived some two-sided inequalities which are shown to be sharp (see Theorem 2). Our methodology is based, in part, upon the properties of the Gamma and Psi (or Digamma) functions, the Polygamma function, and the Hurwitz (or generalized) Zeta function $\zeta(z, x)$.

The results presented in this paper have been shown to provide significant improvements over (and extensions of) those considered in several earlier works.

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Integral representations for the Dirichlet L-functions and their expansions in Meixner–Pollaczek polynomials and rising factorials[†]

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In this article we provide integral representations for the Dirichlet beta and Riemann zeta functions, which are obtained by combining Mellin transform with the fractional Fourier transform. As an application of these integral formulas we derive tractable expansions of these L-functions in the series of Meixner–Pollaczek polynomials and rising factorials.

Keywords: Riemann zeta function; Meixner–Pollaczek polynomials; Rising factorials; Confluent hypergeometric function; Mellin transform; Fractional Fourier transform

1. Introduction

In this article, we study the relations between Mellin transform and the fractional powers of Fourier transform and we show how these integral transforms can be used to obtain new results for the Dirichlet L-functions. To illustrate the ideas let us consider the Dirichlet beta function $\beta(s)$, which for $\text{Re}(s) > 0$ is defined as $\beta(s) = \sum_{n \geq 0} ((-1)^n / (2n + 1)^s)$. This function can be obtained as the Mellin transform of $f(x) = \text{sech}(\sqrt{(\pi/2)}x)$, and the functional equation for $\beta(s)$ follows from the fact that $f(x)$ is invariant under cosine transform $\mathcal{F}_c$ and the integral kernel x^{s-1} of the Mellin transform is also ‘invariant’ under $\mathcal{F}_c$, with s replaced by $1 - s$ and some multiplication by functions of s only.

However, more can be said about function $f(x)$: not only it is invariant under $\mathcal{F}_c$, but infinitely many fractional powers $(\mathcal{F}_c^{(m/n)} f)(y)$ have quite simple form. But since the integral kernel of the Mellin transform is not invariant under fractional powers of $\mathcal{F}_c$ (in fact we obtain a transform with the confluent hypergeometric function as an integral kernel), we obtain new integral representations for $\beta(s)$.

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These integral representations (Theorems 4.1 and 5.1) provide a simple way of obtaining tractable expansions of the Dirichlet L-functions in the series of polynomials, such as Meixner–Pollaczek polynomials or rising factorials. The partial sums of these expansions also satisfy functional equation and the coefficients have a very simple form of the exponential generating function. In a companion paper [1] we study in more detail the expansion of the Riemann Ξ -function in Meixner–Pollaczek polynomials and the zeros of the partial sums of this expansion (see also refs. [2–5]).

This article is organized as follows: in section 2, we provide some background material on the fractional Fourier transform and its relation to the Mellin transform. Section 3 is devoted to Mordell integrals, which play the central role in this article since they allow us to compute the fractional Fourier transform of certain hyperbolic functions. In section 4, we study Dirichlet beta function, derive integral representation and expansions in Meixner–Pollaczek polynomials and in rising factorials. Only then in section 5 we study the famous Riemann zeta function $\zeta(s)$: it wasn't given a priority because in this case analysis is somewhat more complicated by the presence of the poles.

At last we would like to mention that everywhere in this article we use the following notation for the rising factorial: $(a)_n = a(a+1) \cdots (a+n-1)$.

2. Fractional Fourier and scaled Mellin transforms

In this section we review some background material on the fractional Fourier sine (cosine) transform and Mellin transform (see refs. [6, 7]) which will be used extensively later. The results are presented in the framework of the fractional Hankel transform $\mathcal{H}_\nu^a$, which includes both fractional cosine and sine transforms as the special cases $\nu = \pm(1/2)$.

Let us define $\mathfrak{A} = L^2([0, \infty), dx)$ to be Hilbert space of the square integrable functions on $[0, \infty)$. We choose a complete orthogonal basis $\phi_n(x) = L_n^\nu(x^2)e^{-(1/2)x^2}x^{\nu+(1/2)}$ ($\nu > -1$), where $L_n^\nu(x)$ are Laguerre polynomials (see [8, 9]). The fractional Hankel transform $\mathcal{H}_\nu^a$ of order a is a unitary operator on $\mathfrak{A}$ defined by $\mathcal{H}_\nu^a \phi_n = e^{-\pi i n a} \phi_n$ (see [7]).

Below we present some of the properties of the fractional Hankel transform which will be used later:

- $\mathcal{H}_\nu^a$ is a unitary operator on $\mathfrak{A}$ such that $\mathcal{H}_\nu^a \mathcal{H}_\nu^b = \mathcal{H}_\nu^{a+b}$ and $(\mathcal{H}_\nu^a)^{-1} = (\mathcal{H}_\nu^a)^* = \mathcal{H}_\nu^{-a}$.
- The integral kernel

$$\begin{aligned} & e^{-(1/2)(x^2+y^2)} (xy)^{\nu+(1/2)} \sum_{n \geq 0} \frac{2n!}{\Gamma(n+\nu+1)} e^{-\pi i n a} L_n^\nu(x^2) L_n^\nu(y^2) \\ &= \frac{e^{(\pi i/2)(\nu+1)(a-\hat{a})}}{|\sin(\pi a/2)|} e^{(i/2) \cot(\pi a/2)(x^2+y^2)} \sqrt{xy} J_\nu \left(\frac{xy}{|\sin(\pi a/2)|} \right), \end{aligned} \quad (1)$$

where $\hat{a} = \text{sign}(\sin(\pi a/2))$ (see ref. [7]).

- When $\nu = -(1/2)$ we obtain the fractional cosine transform $\mathcal{F}_c^a$ with the integral kernel

$$\sqrt{\frac{2}{\pi}} \frac{e^{(\pi i/4)(a-\hat{a})}}{|\sin(\pi a/2)|^{(1/2)}} e^{(i/2) \cot(\pi a/2)(x^2+y^2)} \cos \left(\frac{xy}{|\sin(\pi a/2)|} \right). \quad (2)$$

- When $\nu = (1/2)$ we obtain the fractional sine transform $\mathcal{F}_s^a$ with the integral kernel

$$\sqrt{\frac{2}{\pi}} \frac{e^{(3\pi i/4)(a-\hat{a})}}{|\sin(\pi a/2)|^{(1/2)}} e^{(i/2) \cot(\pi a/2)(x^2+y^2)} \sin \left(\frac{xy}{|\sin(\pi a/2)|} \right). \quad (3)$$

Next we define another Hilbert space $\mathfrak{B} = L^2(\mathbb{R}, |\Gamma(\lambda + (i/2)t)|^2 dt)$. Here and everywhere else in this article $\lambda = ((v+1)/2)$ (note that $\lambda > 0$). The scaled Mellin transform is defined as

$$(\mathcal{M}_\lambda f)(s) = \frac{2^{(1/4)-(s/2)}}{\Gamma(\lambda - (1/4) + (s/2))} (\mathcal{M}f)(s),$$

where $(\mathcal{M}f)(s) = \int_0^\infty f(x)x^{s-1} dx$ is the classical Mellin transform.

The following properties of the scaled Mellin transform will be used later:

- $(\mathcal{M}_\lambda \phi_n)(s) = 2^{\lambda-1}(-i)^n P_n^{(\lambda)}(t/2)$, where $s = (1/2) + it$ and $P_n^{(\lambda)}(t/2)$ are the Meixner–Pollaczek polynomials (see [8])
- Parseval identity: if $g(t) = (\mathcal{M}_\lambda f)(s)$, $s = (1/2) + it$, then $\|f\|_{\mathfrak{A}}^2 = \frac{1}{2\pi} \|g\|_{\mathfrak{B}}^2$.
- The action of $\mathcal{M}_\lambda$ on Fourier cosine $\mathcal{F}_c = \mathcal{H}_{-(1/2)}$ and Fourier sine transforms $\mathcal{F}_s = \mathcal{H}_{(1/2)}$:

$$(\mathcal{M}_{(1/4)} \mathcal{F}_c f)(s) = (\mathcal{M}_{(1/4)} f)(1-s), \quad (\mathcal{M}_{(3/4)} \mathcal{F}_s f)(s) = (\mathcal{M}_{(3/4)} f)(1-s). \quad (4)$$

Next we present a result which will be our main tool in the following sections: it describes the action of the scaled Mellin transform $\mathcal{M}_\lambda$ on the fractional Hankel transform $\mathcal{H}_v^{-(1/2)}$ ($\lambda = ((v+1)/2)$).

PROPOSITION 2.1 *Assume $f(x) \in \mathfrak{A}$ and let $g(t) = (\mathcal{M}_\lambda \mathcal{H}_v^{-(1/2)} f)(t)$. Then $g(t) \in \mathfrak{B}$, $\|f\|_{\mathfrak{A}}^2 = \frac{1}{2\pi} \|g\|_{\mathfrak{B}}^2$ and $g(t)$ can be represented as*

$$g(t) = \frac{e^{(\pi t/4)}}{\Gamma(2\lambda)} \int_0^\infty x^{2\lambda-(1/2)} e^{-(i/2)x^2} {}_1F_1\left(\lambda + \frac{i}{2}t, 2\lambda; ix^2\right) f(x) dx,$$

Furthermore, $f(x) = (\mathcal{H}_v^{(1/2)} \mathcal{M}_\lambda^{-1} g)(x)$ can be expressed as the following integral

$$f(x) = \frac{x^{2\lambda-(1/2)} e^{-(i/2)x^2}}{2\pi \Gamma(2\lambda)} \int_{\mathbb{R}} {}_1F_1\left(\lambda + \frac{i}{2}t, 2\lambda; ix^2\right) e^{(\pi t/4)} g(t) \left| \Gamma\left(\lambda + \frac{i}{2}t\right) \right|^2 dt.$$

Proof The integral kernel of transformation $\mathcal{M}_\lambda \mathcal{H}_v^{-(1/2)}$ is given by

$$\begin{aligned} & 2^{(1/2)} e^{(\pi i/2)\lambda} \frac{2^{(1/4)-(s/2)}}{\Gamma(\lambda - (1/4) + (s/2))} \int_0^\infty y^{s-1} e^{-(i/2)(x^2+y^2)} \sqrt{xy} J_\nu(\sqrt{2xy}) dy \\ &= \frac{1}{\Gamma(2\lambda)} e^{(\pi t/4)} x^{2\lambda-(1/2)} e^{-(i/2)x^2} {}_1F_1\left(\lambda + \frac{i}{2}t, 2\lambda; ix^2\right). \end{aligned}$$

Similarly, the integral kernel of the inverse transformation $\mathcal{H}_v^{(1/2)} \mathcal{M}_\lambda^{-1}$ is given by

$$\begin{aligned} & \frac{1}{2\pi} 2^{(1/2)} e^{-(\pi i/2)\lambda} 2^{-(1/4)+(s/2)} \Gamma\left(\lambda - \frac{1}{4} + \frac{s}{2}\right) \int_0^\infty y^{-s} e^{(i/2)(x^2+y^2)} \sqrt{xy} J_\nu(\sqrt{2xy}) dy \\ &= \frac{1}{2\pi \Gamma(2\lambda)} e^{(\pi t/4)} x^{2\lambda-(1/2)} e^{-(i/2)x^2} {}_1F_1\left(\lambda + \frac{i}{2}t, 2\lambda; ix^2\right) \left| \Gamma\left(\lambda + \frac{i}{2}t\right) \right|^2. \end{aligned}$$

Both integrals were computed with the help of [9]. ■

3. Mordell integrals

In this section, we review several results about function

$$h(y, \tau) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{(i/2)\tau x^2} \cos(xy)}{\cosh(\sqrt{(\pi/2)x})} dx, \quad \text{Im}(\tau) \geq 0.$$

Integrals of this type were used by Riemann to obtain functional equation and asymptotic formula for zeta function (see [10]). Later these integrals were studied by Ramanujan [11] and by Mordell, who analyzed their behavior with respect to modular transformations (see [12, 13]). An extensive collection of facts about function $h(y, \tau)$ can be found in ref. [14].

In the derivation of the integral representations for Dirichlet L-functions we will need explicit formulas for $h(y, \tau)$, which can be obtained using the following functional equations (see [14] for the proof)

$$h(y, \tau) + h(y + i\sqrt{2\pi}, \tau) = \frac{2}{\sqrt{\tau}} e^{(\pi i/4) - (i/2\tau)(y + i\sqrt{\pi/2})^2}, \quad (5)$$

$$h(y, \tau) + e^{-\sqrt{2\pi}y - \pi i\tau} h(y + i\sqrt{2\pi}\tau, \tau) = 2e^{-\sqrt{\pi/2}y - (\pi i\tau/4)}. \quad (6)$$

If $\tau = (m/n)$ is an irreducible fraction, then by iterating equation (5) m times and equation (6) n times we obtain a system of two linear equations in two variables $h_1 = h(y, \tau)$ and $h_2 = h(y + i\sqrt{2\pi}m, \tau)$. After eliminating h_2 from these equations and simplifying the resulting formula for h_1 we obtain

$$h(y, \tau) = \frac{1}{\cosh(n\sqrt{\pi/2}y)} \left[G_{(1/2),-} \left(\sqrt{\frac{\pi}{2}}y, \tau, n \right) + \frac{1}{\sqrt{\tau}} e^{(\pi i/4) - (i/2\tau)y^2} G_{(1/2),-} \left(\sqrt{\frac{\pi}{2}}\frac{y}{\tau}, -\frac{1}{\tau}, m \right) \right], \quad (7)$$

where the quadratic Gauss sum is defined as $G_{a,\pm}(y, \tau, n) = \sum_{k=0}^{n-1} (\pm 1)^k e^{-\pi i\tau(k+a)^2 + (n-2k-2a)y}$. The following two integrals can also be expressed in terms of $h(y, \tau)$ and thus computed explicitly for rational τ :

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{(i/2)\tau x^2} \sin(xy)}{e^{\sqrt{2\pi}x} + 1} dx &= -\frac{1}{2\sqrt{\tau}} e^{-(\pi i/4) - (i/2\tau)y^2} \left(1 + \Phi \left(e^{-(\pi i/4)} \frac{y}{\sqrt{2\tau}} \right) \right) \\ &\quad - \frac{1}{e^{n\sqrt{(\pi/2)y}} + (-1)^n e^{-n\sqrt{(\pi/2)y}}} \left[G_{(1/2),+} \left(\sqrt{\frac{\pi}{2}}y, \tau, n \right) \right. \\ &\quad \left. - \frac{1}{\sqrt{\tau}} e^{-(\pi i/4) - (i/2\tau)y^2} G_{0,-} \left(\sqrt{\frac{\pi}{2}}\frac{y}{\tau}, -\frac{1}{\tau}, m \right) \right] \end{aligned} \quad (8)$$

$$\begin{aligned} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{(i/2)\tau x^2} \sin(xy)}{e^{\sqrt{2\pi}x} - 1} dx &= \frac{1}{2\sqrt{\tau}} e^{-(\pi i/4) - (i/2\tau)y^2} \left(1 + \Phi \left(e^{-(\pi i/4)} \frac{y}{\sqrt{2\tau}} \right) \right) \\ &\quad + \frac{1}{e^{n\sqrt{(\pi/2)y}} + (-1)^{n+m} e^{-n\sqrt{(\pi/2)y}}} \left[G_{0,+} \left(\sqrt{\frac{\pi}{2}}y, \tau, n \right) \right. \\ &\quad \left. - \frac{1}{\sqrt{\tau}} e^{-(\pi i/4) - (i/2\tau)y^2} G_{0,+} \left(\sqrt{\frac{\pi}{2}}\frac{y}{\tau}, -\frac{1}{\tau}, m \right) \right] \end{aligned} \quad (9)$$

4. Dirichlet beta function

In this section we illustrate the interplay between Mellin transform and the fractional cosine transform on the example of the Dirichlet beta function $\beta(s)$, which for $\operatorname{Re}(s) > 0$ can be defined as $\beta(s) = L(s, \chi_4) = \sum_{n \geq 0} ((-1)^n / (2n+1)^s)$. We also define $\xi(s, \chi_4) = 2^s \pi^{-(s/2)} \Gamma((1+s)/2) \beta(s)$ and $\Xi(t, \chi_4) = \xi((1/2) + it, \chi_4)$.

Our main result in this section is the following integral representation for $\Xi(t, \chi_4)$:

THEOREM 4.1 For all $t \in \mathbb{C}$

$$\Xi(t, \chi_4) e^{-(\pi t/4)} = 2 \int_0^\infty e^{-(i/2)y^2} {}_1F_1\left(\frac{1}{4} + \frac{i}{2}t, \frac{1}{2}; iy^2\right) \frac{\sin((y^2/2) + (\pi/8))}{\cosh(\sqrt{\pi}y)} dy. \quad (10)$$

Proof We start with the function $f(x) = \operatorname{sech}(\sqrt{(\pi/2)x})$ and find that

$$(\mathcal{M}_{(1/4)} f)(s) = \frac{2^{(1/4)}}{\sqrt{\pi}} \xi(s, \chi_4), \quad \operatorname{Re}(s) > 0. \quad (11)$$

Note that the functional equation $\xi(s, \chi_4) = \xi(1-s, \chi_4)$ follows at once from equation (4) and the fact that $f(x)$ is invariant under Fourier cosine transform $\mathcal{F}_c$ (see [9]).

Now we use equation (7) and compute $(\mathcal{F}_c^{(1/2)} f)(y)$:

$$\begin{aligned} F(y) &= (\mathcal{F}_c^{(1/2)} f)(y) = \sqrt{\frac{2}{\pi}} C\left(\frac{\pi}{4}, -\frac{1}{2}\right) \int_0^\infty e^{(i/2)(x^2+y^2)} \frac{\cos(\sqrt{2}xy)}{\cosh(\sqrt{(\pi/2)x})} dx \\ &= 2^{(5/4)} \frac{\sin((y^2/2) + (\pi/8))}{\cosh(\sqrt{\pi}y)}, \end{aligned}$$

Next we use the identity $f(x) = (\mathcal{F}_c^{-(1/2)} F)(x)$, equation (11) and Proposition 2.1 to obtain

$$\begin{aligned} \frac{2^{(1/4)}}{\sqrt{\pi}} \xi(s, \chi_4) &= (\mathcal{M}_{(1/4)} \mathcal{F}_c^{-(1/2)} F)(s) \\ &= \frac{e^{(\pi t/4)}}{\Gamma(1/2)} \int_0^\infty e^{-(i/2)y^2} {}_1F_1\left(\frac{1}{4} + \frac{i}{2}t, \frac{1}{2}; iy^2\right) F(y) dy, \end{aligned}$$

which ends the proof. ■

COROLLARY 4.2 For all $t \in \mathbb{C}$

$$\Xi(t, \chi_4) e^{-(\pi t/4)} = \sum_{n \geq 0} a_n P_n^{(1/4)}\left(\frac{t}{2}\right) \quad (12)$$

where the generating function for coefficients $\{a_n\}$ is

$$\sum_{n \geq 0} \frac{a_n}{n!} x^n = \sqrt{2\pi} \frac{\sin(x + (\pi/8))}{\cosh(\sqrt{2\pi}x)}. \quad (13)$$

Proof A rigorous proof (and the integral representation for a_n) can be obtained by expanding the confluent hypergeometric function in equation (10) in Meixner–Pollaczek polynomials

(see [8]):

$$e^{-(iy/2)} {}_1F_1(\lambda + it, 2\lambda; iy) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (2\lambda)_n} P_n^{(\lambda)}(t) y^n. \quad (14)$$

However we decided to present here a more intuitive argument, which shows why the generating function for the coefficients a_n is necessarily the same as the function in the integral representation (10) (up to a simple change of variables).

First we assume that function $\Xi(t, \chi_4)e^{(\pi t/4)}$ lies in the Hilbert space $\mathfrak{B}$ and we expand it in the orthogonal basis given by Meixner–Pollaczek polynomials: $\Xi(t, \chi_4)e^{(\pi t/4)} = \sum_{n \geq 0} (-1)^n a_n P_n^{(1/4)}(t/2)$. Using the orthogonality relation for the Meixner–Pollaczek polynomials (see [8]) we find that the coefficients a_n are given by

$$a_n = \frac{(-1)^n \sqrt{2n!}}{4\pi \Gamma((1/2) + n)} \int_{\mathbb{R}} \Xi(t, \chi_4) e^{(\pi t/4)} P_n^{(1/4)}\left(\frac{t}{2}\right) \left| \Gamma\left(\frac{1}{4} + \frac{i}{2}t\right) \right|^2 dt.$$

The generating function for $\{a_n\}$ is computed using equation (14):

$$\sum_{n \geq 0} \frac{a_n}{n!} x^n = \frac{\sqrt{2} e^{-ix}}{4\pi^{(3/2)}} \int_{\mathbb{R}} {}_1F_1\left(\frac{1}{4} + \frac{i}{2}t, \frac{1}{2}; 2ix\right) \Xi(t, \chi_4) e^{(\pi t/4)} \left| \Gamma\left(\frac{1}{4} + \frac{i}{2}t\right) \right|^2 dt,$$

and using Proposition 2.1 and the integral representation (10) we find that the above integral must be equal to $\sqrt{2\pi}(\sin(x + (\pi/8))/\cosh(\sqrt{2\pi}x))$. ■

COROLLARY 4.3 For all $t \in \mathbb{C}$

$$\Xi(t, \chi_4) e^{-(\pi t/4)} = \sum_{n \geq 0} \left[\frac{b_n (-i)^n}{n!} \left(\frac{1}{4} + \frac{i}{2}t\right)_n + \frac{\bar{b}_n i^n}{n!} \left(\frac{1}{4} - \frac{i}{2}t\right)_n \right], \quad (15)$$

where the generating functions for coefficients $\{b_n\}$ is

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = \sqrt{\pi} e^{(\pi i/4) + i(x/2)} \frac{(\sin(x/2) + (\pi/8))}{\cosh(\sqrt{\pi}x)}.$$

Proof Again we start with the integral representation (10) and rewrite it as $\Xi(t, \chi_4)e^{-(\pi t/4)} = \Psi(t) + \overline{\Psi(t)}$, where

$$\Psi(t) = e^{(3\pi i/8)} \int_0^\infty \frac{e^{-iy^2} {}_1F_1((1/4) + (it/2), (1/2); iy^2)}{\cosh(\sqrt{\pi}y)} dy.$$

Next we use the definition of the confluent hypergeometric function and expand it in the power series in y (see [9]). Integrating term by term we find that the coefficients b_n have the following integral representation:

$$b_n = \frac{e^{(3\pi i/8)} (-1)^n}{(1/2)_n} \int_0^\infty \frac{e^{-iy^2} y^{2n}}{\cosh(\sqrt{\pi}y)} dy. \quad (16)$$

One can find using the above formula that for n large $|b_n| \sim n^\alpha e^{-\sqrt{(\pi/2)n}}$ for some α , thus the series (15) converges for all complex t . The exponential generating function for the coefficients $\{b_n\}$ is computed using equation (7). ■

5. Riemann zeta function

We adopt the following standard definitions (see [15, 10]): $\xi(s) = (1/2)s(s-1)\pi^{-(s/2)}\Gamma(s/2)\zeta(s)$ and $\Xi(t) = \xi((1/2) + it)$. Our main result in this section is the following integral representation for $\Xi(t)$:

THEOREM 5.1 For all $t \in \mathbb{C}$

$$\Xi(t)e^{-(\pi t/4)} = \frac{1}{2} \cos\left(\frac{\pi}{8}\right) - t \sin\left(\frac{\pi}{8}\right) + (1 + 4t^2) \int_0^\infty ye^{-(i/2)y^2} {}_1F_1\left(\frac{3}{4} + \frac{i}{2}t, \frac{3}{2}; iy^2\right) \frac{\sin((y^2/2) + (\pi/8))}{e^{2\sqrt{\pi}y} + 1} dy. \quad (17)$$

Proof Define function $f(x) = (1/(e^{\sqrt{2\pi}x} - 1)) - (1/(\sqrt{2\pi}x))$. The first step is to find that

$$(\mathcal{M}_{(3/4)}f)(s) = \frac{2^{(1/4)}}{\sqrt{\pi}} \frac{\xi(s)}{s(s-1)}, \quad \operatorname{Re}(s) \in (0, 1). \quad (18)$$

Again, the functional equation $\xi(s) = \xi(1-s)$ follows from the fact that f is invariant under Fourier sine transform $\mathcal{F}_s$ (see [9]) and equation (4).

Using equation (9) we find the fractional sine transform $\mathcal{F}_s^{(1/2)}$ of function $f(x)$:

$$F(y) = (\mathcal{F}_s^{(1/2)}f)(y) = 2^{(1/4)} \left[-2 \frac{\sin((y^2/2) + (\pi/8))}{e^{2\sqrt{\pi}y} + 1} + \frac{1}{2i} (\phi(y) - \overline{\phi(y)}) \right], \quad (19)$$

where $\phi(y) = e^{i(y^2/2) + (\pi i/8)}(1 - \Phi(e^{\pi i/4}y))$. Note that function $\phi(y)$ is analytic, and as $y \rightarrow +\infty$ we have (see [9]) $\phi(y) \sim (1/\sqrt{\pi})(e^{-i(y^2/2) - (\pi i/8)}/y) + O(1/y^2)$, thus $\phi(y)$ is in the Hilbert space $\mathfrak{A}$. Next we find that

$$(\mathcal{M}_{3/4}\mathcal{F}_s^{-(1/2)}\phi)(s) = \frac{i}{\sqrt{\pi}} \frac{e^{(\pi i/4)(1-s)}}{s-1}, \quad (\mathcal{M}_{3/4}\mathcal{F}_s^{-(1/2)}\overline{\phi})(s) = \frac{i}{\sqrt{\pi}} \frac{e^{-(\pi i/4)s}}{s}, \quad (20)$$

and to finish the proof we only need to combine equations (18)–(20) and Proposition 2.1. ■

It is interesting to note that equation (17) is essentially equivalent to the integral representation $\Xi(t) = ((1/4) + t^2)[\Upsilon((1/2) + it) + \overline{\Upsilon((1/2) + it)}]$, where $\Upsilon(s)$ is defined by

$$\Upsilon(s) = -\frac{1}{4}e^{(\pi i/2)(s-1)}2^{s-1}\pi^{(s/2)-1}\Gamma\left(\frac{s}{2}\right)\int_L \frac{e^{(ix^2/4\pi)}}{\sinh(x/2)}x^{-s}dx. \quad (21)$$

and the integral is taken along the line $L = e^{(\pi i/4)}\mathbb{R} + \pi i$ (see [10]). One can obtain equation (17) by applying Plancherel theorem for sine transform to the functions inside the integral in equation (21).

It is also of interest to compare integral representation (17) with the well-known Riemann formula (see [10]):

$$\begin{aligned} 2\xi(s) &= 1 + s(s-1) \int_1^\infty (x^{(s/2)} + x^{((1-s)/2)}) \frac{1}{x} \sum_{n \geq 1} e^{-\pi n^2 x} dx \\ &= 1 + s(s-1) \sum_{n \geq 1} \left[\frac{\Gamma((s/2), \pi n^2)}{(\sqrt{\pi}n)^s} + \frac{\Gamma((1-s)/2, \pi n^2)}{(\sqrt{\pi}n)^{1-s}} \right]. \end{aligned} \quad (22)$$

The following proposition shows that equations (22) and (17) are just ‘extreme’ cases ($\alpha = (\pi/2)$ and $\alpha = 0$ correspondingly) of the more general result:

PROPOSITION 5.2 For all $\alpha \in [0, \pi]$

$$2\xi(s) = s e^{(i/2)((\pi/2)-\alpha)(1-s)} + (1-s) e^{-i/2((\pi/2)-\alpha)s} \\ + s(s-1) \sum_{n \geq 1} \left[\frac{\Gamma((s/2), e^{-i((\pi/2)-\alpha)\pi n^2})}{(\sqrt{\pi}n)^s} + \frac{\Gamma((1-s)/2, e^{i((\pi/2)-\alpha)\pi n^2})}{(\sqrt{\pi}n)^{1-s}} \right] \quad (23)$$

and

$$\Xi(t) e^{-((\pi/4)-(\alpha/2))t} = \frac{1}{2} \cos\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) - t \sin\left(\frac{\pi}{8} - \frac{\alpha}{4}\right) \\ + (1+4t^2) \int_0^\infty y e^{-(i/2)y^2} {}_1F_1\left(\frac{3}{4} + \frac{i}{2}t, \frac{3}{2}; iy^2\right) \vartheta(y) dy, \quad (24)$$

where $\vartheta(y) = \operatorname{Re}[e^{(3\pi i/8)+(\alpha i/4)-(i/2)y^2} \sum_{n \geq 1} \exp(\pi i n^2 e^{i\alpha} - 2\sqrt{\pi} n y e^{(i\alpha/2)})]$.

Proof To derive (23) one should start with the function $\psi(y) = \sum_{n \geq 1} e^{\pi i n^2 y} = (1/2)(\theta_3(0, y) - 1)$ and follow the lines of Riemann's proof (see [10]), but take Mellin transform along the line $y \in e^{i\alpha}\mathbb{R}_+$, $\alpha \in [0, \pi]$. Formula (24) is obtained from (23) with the help of expression for the incomplete Gamma function as the Laplace transform of the confluent hypergeometric function (see [9]):

$$\frac{\Gamma((s/2), e^{-i((\pi/2)-\alpha)\pi n^2})}{(\sqrt{\pi}n)^s} = 4i e^{-i((\pi/2)-\alpha)(s/2)} \int_0^\infty y e^{-(i/2)y^2} \\ {}_1F_1\left(\frac{3}{4} + \frac{i}{2}t, \frac{3}{2}; iy^2\right) \exp(\pi i n^2 e^{i\alpha} - 2\sqrt{\pi} n y e^{(i\alpha/2)}) dy. \quad \blacksquare$$

Next we derive an expansion of the Riemann Ξ function in the Meixner–Pollaczek polynomials (see [1] for the detailed analysis of the coefficients of this expansion and zeros of its partial sums).

COROLLARY 5.3 For all $t \in \mathbb{C}$

$$\Xi(t) e^{-(\pi t/4)} = \frac{1}{2} \cos\left(\frac{\pi}{8}\right) - t \sin\left(\frac{\pi}{8}\right) + (1+4t^2) \sum_{n \geq 0} a_n P_n^{(3/4)}\left(\frac{t}{2}\right), \quad (25)$$

where the generating function for coefficients $\{a_n\}$ is

$$\sum_{n \geq 0} \frac{a_n}{n!} x^n = \frac{1}{4} \sqrt{\frac{\pi}{x}} \left[-\sin\left(x + \frac{\pi}{8}\right) \tanh\left(\sqrt{2\pi x}\right) \right. \\ \left. + \frac{1}{2i} \left(\Phi\left(e^{(\pi i/4)} \sqrt{x}\right) e^{ix+(\pi i/8)} - \Phi\left(e^{-(\pi i/4)} \sqrt{x}\right) e^{-ix-(\pi i/8)} \right) \right].$$

Proof Again we start with equation (17), expand the confluent hypergeometric function in the series of Meixner–Pollaczek polynomials (see equation (14)) and integrate term by term to find that the coefficients are given by

$$a_n = \frac{(-1)^n}{(2n+1)!!} \int_0^\infty \frac{\sin((y^2/2) + (\pi/8))}{e^{2\sqrt{\pi}y} + 1} y^{2n+1} dy.$$

The exponential generating function for $\{a_n\}$ is computed using equation (8). ■

Following the lines of the proof of Corollary 4.3 we obtain the following expansion of $\Xi(t)e^{-(\pi t/4)}$ in rising factorials:

COROLLARY 5.4 For all $t \in \mathbb{C}$

$$\begin{aligned} \Xi(t)e^{-(\pi t/4)} &= \frac{1}{2} \cos\left(\frac{\pi}{8}\right) - t \sin\left(\frac{\pi}{8}\right) + (1 + 4t^2) \\ &\quad \times \sum_{n \geq 0} \left[\frac{b_n (-i)^n}{n!} \left(\frac{3}{4} + \frac{i}{2}t\right)_n + \frac{\bar{b}_n i^n}{n!} \left(\frac{3}{4} - \frac{i}{2}t\right)_n \right], \end{aligned}$$

where the generating function for the coefficients $\{b_n\}$ is

$$\sum_{n \geq 0} \frac{b_n}{n!} x^n = \frac{e^{-(3\pi i/8)}}{16} \sqrt{\frac{\pi}{x}} \left[e^{ix} \Phi(e^{(\pi i/4)} \sqrt{x}) + \frac{1 - e^{ix} \cosh(\sqrt{\pi x})}{\sinh(\sqrt{\pi x})} \right].$$

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A function involving gamma function and having logarithmically absolute convexity

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In this paper, the logarithmically complete monotonicity, logarithmically absolute monotonicity and logarithmically absolute convexity of the function $[\Gamma(1+tx)]^s/[\Gamma(1+sx)]^t$ for $x, s, t \in \mathbb{R}$ such that $1+sx > 0$ and $1+tx > 0$ with $s \neq t$ are verified, some known results are generalized.

Keywords: Logarithmically absolutely convex function; Logarithmically completely monotonic function; Logarithmically absolutely monotonic function; Gamma function; Generalization

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1. Introduction

By using a geometrical method in ref. [1], the following double inequality was proved

$$\frac{1}{n!} \leq \frac{[\Gamma(1+x)]^n}{\Gamma(1+nx)} \leq 1, \quad (1)$$

for $x \in [0, 1]$ and $n \in \mathbb{N}$, where $\Gamma(x)$ stands for the classical Euler's gamma function defined for $x > 0$ by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$.

By analytical arguments in ref. [2], it was presented that the function

$$f(x, y) = \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \quad (2)$$

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for all $y \geq 1$ decreases in $x \geq 0$. From this, it is deduced that

$$\frac{1}{\Gamma(1+y)} \leq \frac{[\Gamma(1+x)]^y}{\Gamma(1+xy)} \leq 1, \quad (3)$$

for all $y \geq 1$ and $x \in [0, 1]$, which is a generalization of inequality (1).

In ref. [3], It was shown that if f is a differentiable and logarithmically convex function in $[0, \infty)$, then the function $([f(x)]^a/f(ax))$ for $a \geq 1$ (or $0 < a \leq 1$ respectively) decreases (or increases respectively) in $[0, \infty)$. As one of applications to inequalities involving gamma function, Riemann's zeta function and the complete elliptic integrals of the first kind, inequalities (1) and (3) were deduced.

In ref. [4], an inequality involving a positive linear operator acting on the composition of two continuous functions is presented and, as applications of this inequality, some new inequalities involving the beta, gamma and Riemann's zeta functions and a large family of functions that are Mellin transforms are produced. In particular, for $\beta > \delta > 0$, $\alpha\beta > -1$ and $\alpha\delta > -1$, if either $\alpha < 0$ or $\alpha > 1$, then inequality

$$\frac{[\Gamma(1+\delta)]^\alpha}{\Gamma(1+\alpha\delta)} > \frac{[\Gamma(1+\beta)]^\alpha}{\Gamma(1+\alpha\beta)} \quad (4)$$

holds true; if $0 < \alpha \leq 1$, inequality (4) is reversed. It is not difficult to see that the left hand sides in inequalities (1) and (3) are special cases of inequality (4).

Recall refs. [5–7] that a function f is called completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^k f^{(k)}(x) \geq 0$ for all $k \geq 0$ on I . The set of completely monotonic functions on I is denoted by $\mathcal{C}[I]$. Recall also refs. [8–10] that a positive function f is said to be logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^k [\ln f(x)]^{(k)} \geq 0$ for all $k \in \mathbb{N}$ on I . The set of logarithmically completely monotonic functions on I is denoted by $\mathcal{L}[I]$. Recall ref. [11] that if $f^{(k)}(x)$ for some non-negative integer, k is completely monotonic on an interval I , but $f^{(k-1)}(x)$ is not completely monotonic on I , then $f(x)$ is called a completely monotonic function of k -th order on an interval I . Among other things, it was proved implicitly or explicitly in refs. [6, 8, 10, 12,–14] that a logarithmically completely monotonic function is always completely monotonic, but not conversely. Motivated or stimulated by the papers in refs. [8, 10], among other things, it is further revealed in ref. [12] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L}[(0, \infty)] \subset \mathcal{C}[(0, \infty)]$, where $\mathcal{S}$ denotes the set of Stieltjes transforms. In ref. [12, Theorem 1.1] and refs. [9, 15], it is pointed out that the logarithmically completely monotonic functions on $(0, \infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in ref. [16, Theorem 4.4]. For more information on the logarithmically completely monotonic functions, please refer to refs. [6, 8–12, 14, 15, 17–29] and the references therein.

DEFINITION 1.1 *Let f be a positive function, which has derivatives of all orders on an interval I . If $[\ln f(x)]^{(k)}$ for some non-negative integer k is completely monotonic on I , but $[\ln f(x)]^{(k-1)}$ is not completely monotonic on I , then f is said to be a logarithmically completely monotonic function of k th order on I .*

In refs. [28, 29], the following logarithmically complete monotonicities, as generalizations of the decreasingly monotonic property in [2], are presented

- (i) For given $y > 1$, the function $f(x, y)$ defined by equation (2) decreases and is logarithmically concave with respect to $x \in (0, \infty)$ and $(1/f(x, y))$ is a logarithmically completely monotonic function of second order in $x \in (0, \infty)$.

- (ii) For given $0 < y < 1$, the function $f(x, y)$ defined by equation (2) increases and is logarithmically convex with respect to $x \in (0, \infty)$ and $f(x, y)$ is a logarithmically completely monotonic function of second order in $x \in (0, \infty)$.
- (iii) For given $x \in (0, \infty)$, the function $f(x, y)$ defined by equation (2) is logarithmically concave with respect to $y \in (0, \infty)$, and $(1/f(x, y))$ is a logarithmically completely monotonic function of first order in $y \in (0, \infty)$.
- (iv) For given $x \in (0, \infty)$, let

$$F_x(y) = \frac{\Gamma(1+y)[\Gamma(1+x)]^y}{\Gamma(1+xy)} \quad (5)$$

in $\in (0, \infty)$. If $0 < x < 1$, then $F_x(y)$ is a logarithmically completely monotonic function of second order in $(0, \infty)$; if $x > 1$, then $1/F_x(y)$ is a logarithmically completely monotonic function of second order in $(0, \infty)$.

In ref. [30, Theorem 2.1], it was proved that the function

$$G_{s,t}(x) = \frac{[\Gamma(1+tx)]^s}{[\Gamma(1+sx)]^t} \quad (6)$$

decreases (or increases, respectively) in $x \in [0, \infty)$ if either $s \geq t > 0$ or $0 > s \geq t$ (or both $s > 0$ and $t < 0$ respectively) such that $1+sx > 0$ and $1+tx > 0$. This result generalized and extended the corresponding conclusions in refs. [1, 2, 4].

Recall refs. [5–7, 24, 31, 32] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t) \geq 0$ for $t \in I$ and $k \in \mathbb{N}$. Recall also [13, 33, 34] that a positive function f is said to be logarithmically absolutely monotonic on an interval I if it has derivatives of all orders and $[\ln f(t)]^{(k)} \geq 0$ for $t \in I$ and $k \in \mathbb{N}$. In [13, 33, 34] it was proved that a logarithmically absolutely monotonic function on an interval I is also absolutely monotonic on I , but not conversely.

DEFINITION 1.2 Let f be a positive function which has derivatives of all orders on an interval I . If $[\ln f(x)]^{(k)}$ for some non-negative integer k is absolutely monotonic on I , but $[\ln f(x)]^{(k-1)}$ is not absolutely monotonic on I , then f is said to be a logarithmically absolutely monotonic function of k th order on I .

Recall ref. [5, p. 375, Definition 3] and [6, 7, 31, 35, 36] that a function f which has derivatives of all orders on an interval I is said to be absolutely convex on I if $f^{(2k)}(x) \geq 0$ on I for any nonnegative integer k .

DEFINITION 1.3 A positive function f which has derivatives of all orders on an interval I is said to be logarithmically absolutely convex on I if $[\ln f(x)]^{(2k)} \geq 0$ on I for $k \in \mathbb{N}$.

The main aim of this paper is to generalize and extend some results obtained in refs. [1–4, 28–30], especially those in refs. [28, 29].

Our main result can be stated as the following theorem.

THEOREM 1.4 The function $G_{s,t}(x)$ defined by equation (6) for $x, s, t \in \mathbb{R}$ such that $1+sx > 0$ and $1+tx > 0$ with $s \neq t$ has the following properties:

- (i) $G_{s,t}(x) = (1/G_{t,s}(x))$;
- (ii) For $t > s > 0$ and $x \in (0, \infty)$, $G_{s,t}(x)$ is an increasing function and a logarithmically completely monotonic function of second order in x ;

- (iii) For $t > s > 0$ and $x \in (-(1/t), 0)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x ;
- (iv) For $s < t < 0$ and $x \in (-\infty, 0)$, $G_{s,t}(x)$ is a decreasing function and a logarithmically absolutely monotonic function of second order in x ;
- (v) For $s < t < 0$ and $x \in (0, -(1/s))$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in x ;
- (vi) For $s < 0 < t$ and $x \in (-(1/t), 0)$, $G_{t,s}(x)$ is an increasing function and a logarithmically absolutely convex function in x ;
- (vii) For $s < 0 < t$ and $x \in (0, -(1/s))$, $G_{t,s}(x)$ is a decreasing function and a logarithmically absolutely convex function in x .

2. Proof of Theorem 1.4

It is clear that $G_{s,t}(x) = (1/G_{t,s}(x))$. Therefore, it is sufficient to show Theorem 1.4 only for $s < t$.

Taking logarithm of $G_{s,t}(x)$ yields

$$\ln G_{s,t}(x) = s \ln \Gamma(1 + tx) - t \ln \Gamma(1 + sx)$$

and

$$\begin{aligned} \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} &= st^k \psi^{(k-1)}(1 + tx) - ts^k \psi^{(k-1)}(1 + sx) \\ &= \frac{st}{x^{k-1}} [(tx)^{k-1} \psi^{(k-1)}(1 + tx) - (sx)^{k-1} \psi^{(k-1)}(1 + sx)], \end{aligned} \quad (7)$$

for $k \in \mathbb{N}$.

In ref. [37, Lemma 2.2], it was obtained that the function $x^\alpha |\psi^{(i)}(1 + x)|$ strictly increases in $(0, \infty)$ if and only if $\alpha \geq i$, where $i \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. This was generalized in ref. [38, Theorem 1] as follows: For $\beta \geq (1/2)$, the function $x^\alpha |\psi^{(i)}(x + \beta)|$ strictly increases in $[0, \infty)$ if and only if $\alpha \geq i \in \mathbb{N}$. In particular, the functions $x^{2i} \psi^{(2i)}(1 + x)$ decrease and the functions $x^{2i-1} \psi^{(2i-1)}(1 + x)$ increase in $[0, \infty)$ for $i \in \mathbb{N}$. From this, it is readily obtained for $i \in \mathbb{N}$, $t > s > 0$ and $x > 0$ that

$$\frac{\partial^{2i} [\ln G_{s,t}(x)]}{\partial x^{2i}} > 0 \quad \text{and} \quad \frac{\partial^{2i+1} [\ln G_{s,t}(x)]}{\partial x^{2i+1}} < 0.$$

Since

$$\frac{\partial [\ln G_{s,t}(x)]}{\partial x} = st[\psi(1 + tx) - \psi(1 + sx)] \quad (8)$$

and the psi function ψ increases in $(0, \infty)$, it is easy to see $(\partial [\ln G_{s,t}(x)] / \partial x) > 0$. Consequently, for $t > s > 0$ and $x > 0$, $G_{s,t}(x)$ is an increasing function in $x \in (0, \infty)$ and a logarithmically completely monotonic function of second order in $x \in (0, \infty)$.

If $t > s > 0$ and $0 > x > -(1/t)$, then

$$(-1)^{k-1} \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} = \frac{st}{(-x)^{k-1}} [(tx)^{k-1} \psi^{(k-1)}(1 + tx) - (sx)^{k-1} \psi^{(k-1)}(1 + sx)],$$

for $k \in \mathbb{N}$. Let $h_k(u) = u^k \psi^{(k)}(1 + u)$ for $u \in (-1, 0)$ and $k \in \mathbb{N}$. A straightforward computation gives

$$\begin{aligned} h'_k(u) &= u^{k-1} [k \psi^{(k)}(1 + u) + u \psi^{(k+1)}(1 + u)] \\ &= (-u)^{k-1} \{k [(-1)^{k+1} \psi^{(k)}(1 + u)] + (-u) [(-1)^{k+2} \psi^{(k+1)}(1 + u)]\} > 0. \end{aligned}$$

This means that the function $h_k(u)$ for $u \in (-1, 0)$ and $k \in \mathbb{N}$ increases. Then, it is concluded that $(-1)^{k-1}(\partial^k[\ln G_{s,t}(x)]/\partial x^k) < 0$ for $k \in \mathbb{N}$. As a result, the function $G_{s,t}(x)$ is logarithmically completely monotonic in $x \in (-(1/t), 0)$ for $t > s > 0$.

If $s < t < 0$ and $x < 0$, then

$$(-1)^k \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} = \frac{st}{(-x)^{k-1}} [(sx)^{k-1} \psi^{(k-1)}(1+sx) - (tx)^{k-1} \psi^{(k-1)}(1+tx)],$$

for $k \in \mathbb{N}$, accordingly

$$(-1)^{2i} \frac{\partial^{2i} [\ln G_{s,t}(x)]}{\partial x^{2i}} > 0 \quad \text{and} \quad (-1)^{2i+1} \frac{\partial^{2i+1} [\ln G_{s,t}(x)]}{\partial x^{2i+1}} < 0$$

for $i \in \mathbb{N}$. From equation (8), it follows that $(\partial[\ln G_{s,t}(x)]/\partial x) < 0$. Hence, $G_{s,t}(x)$ is a logarithmically absolutely monotonic function of second order in $x \in (-\infty, 0)$, for $s < t < 0$.

If $s < t < 0$ and $0 < x < -(1/s)$, then formula (7) is valid. In virtue of the increasingly monotonic property of the function $h_k(u)$ for $u \in (-1, 0)$ and $k \in \mathbb{N}$, it is deduced that $(\partial^k[\ln G_{s,t}(x)]/\partial x^k) > 0$ for $k \in \mathbb{N}$. This means that the function $G_{s,t}(x)$ for $s < t < 0$ is logarithmically completely monotonic in $x \in (0, -(1/s))$.

If $s < 0 < t$ and $-1/t < x < 0$, then

$$\begin{aligned} \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} &= \frac{-st}{(-x)^{k-1}} \{(-1)^{k-1}(-tx)^{k-1} [(-1)^k \psi^{(k-1)}(1+tx)] \\ &\quad - (sx)^{k-1} [(-1)^k \psi^{(k-1)}(1+sx)]\}, \end{aligned}$$

for $k \in \mathbb{N}$. Accordingly, it is obtained readily that $(\partial^{2i}[\ln G_{s,t}(x)]/\partial x^{2i}) < 0$ for $i \in \mathbb{N}$. This implies that the function $(1/G_{s,t}(x))$ is logarithmically absolutely convex in $x \in (-(1/t), 0)$ for $s < 0 < t$. Formula (8) implies that the function $G_{s,t}(x)$ increases in $x \in (-(1/t), 0)$ for $s < 0 < t$.

If $s < 0 < t$ and $0 < x < -(1/s)$, then

$$\begin{aligned} \frac{\partial^k [\ln G_{s,t}(x)]}{\partial x^k} &= \frac{-st}{x^{k-1}} (-1)^k \{(sx)^{k-1} [(-1)^k \psi^{(k-1)}(1+sx)] \\ &\quad - (tx)^{k-1} [(-1)^k \psi^{(k-1)}(1+tx)]\}, \end{aligned}$$

for $k \in \mathbb{N}$. Hence, it is deduced apparently that $(\partial^{2i}[\ln G_{s,t}(x)]/\partial x^{2i}) < 0$ for $i \in \mathbb{N}$. This implies that the function $(1/G_{s,t}(x))$ is logarithmically absolutely convex in $x \in (0, -(1/s))$ for $s < 0 < t$. Formula (8) implies that the function $G_{s,t}(x)$ decreases in $x \in (0, -(1/s))$, for $s < 0 < t$. The proof of Theorem 1.4 is complete.

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Stieltjes transform for Boehmians

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To extend, the distributional Stieltjes transform on $s'_\alpha(I)$, to the context of Boehmians, two suitable Boehmian spaces are constructed. The extended Stieltjes transform of a Boehmian is defined as another Boehmian. It is proved that the extended Stieltjes transform is well defined, consistent with the distributional Stieltjes transform and it has all desired properties. We also prove that the extended stieltjes transform and its inverse are continuous with respect to δ -convergence as well as Δ -convergence.

Keywords: Boehmians; Convolution; Distributional Stieltjes transforms

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1. Introduction

The introduction of Boehmian spaces is a remarkable tool in the theory of integral transforms and generalized functions. J. Mikusiński and P. Mikusiński [1] introduced Boehmians as a generalization of distributions by the motivation of regular operators [2]. The construction of an abstract Boehmian space was given in [3] with two notions of convergence. Thereafter, various Boehmian spaces have been defined and also various integral transforms have been extended on them. See [4–11].

To generalize distributional integral transforms, some Boehmian spaces [8, 9, 11] are constructed by using distribution spaces and it is well established that these Boehmian spaces are more general than various distribution spaces and certain Boehmian spaces. Adapting the same technique, in this paper, we construct a Boehmian space $\mathcal{B}_1$, using the distribution space $S'_\alpha(I)$ and extend the Stieltjes transform on it.

On the other hand, many investigations have been going on, regarding the extension of Stieltjes transform and its inversion formulas (See [12–17]). The works in [13–15], [16, §5.4] are using the kernel method and [15], [12, §4.3] are using the adjoint method, to extend the Stieltjes transform to the context of distributions. In this paper, we generalize the Stieltjes transform defined in [13].

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For a fixed $\alpha \in \mathbb{R}$, let $S_\alpha(I)$ be the space of all infinitely differentiable functions on $I = (0, \infty)$ satisfying $\gamma_k(\psi) = \sup_{0 < x < \infty} (1+x)^\alpha x^k |\psi^{(k)}(x)| < \infty$, $\forall k \in \{0, 1, 2, 3, \dots\}$. Each γ_k is a seminorm on $S_\alpha(I)$ and the collection $\{\gamma_k\}$ of seminorms induces a Fréchet space topology on $S_\alpha(I)$. The dual space of $S_\alpha(I)$ is denoted by $S'_\alpha(I)$. We say that (f_n) converges to f in $S'_\alpha(I)$, if for each $\psi \in S_\alpha(I)$, $f_n(\psi) \rightarrow f(\psi)$, as $n \rightarrow \infty$. Let $p \in \Omega = \mathbb{C} \setminus (-\infty, 0]$ with $\alpha < \operatorname{Re} p$. The distributional Stieltjes transform of $f \in S'_\alpha(I)$ is defined by $(\mathfrak{S}_p f)(s) = F(s) = \langle f(x), 1/(s+x)^p \rangle$, $\forall s \in \Omega$.

It is well known that F is a holomorphic function on Ω . If the image of $S'_\alpha(I)$ is denoted by $\mathcal{H}$, then $\mathfrak{S}_p : S'_\alpha(I) \rightarrow \mathcal{H}$ is linear and one-to-one, and satisfies the property

$$F^{(k)}(s) = \left\langle f(x), \frac{(-1)^k (p)_k}{(s+x)^{p+k}} \right\rangle = (\mathfrak{S}_p(f^{(k)}))(s), \quad (1)$$

where $(p)_k = p(p+1)(p+2) \cdots (p+k-1)$, $\forall k \in \mathbb{N}$.

2. Two Boemian spaces

In this section, we prove some auxiliary results, which are required to construct the two Boehmian spaces $\mathcal{B}_1$ and $\mathcal{B}_2$. Let $\mathcal{B}_1 = \mathcal{B}(S'_\alpha(I), (\mathcal{D}(I), *, \otimes, \Delta_+))$, where $\mathcal{D}(I)$ is the Schwartz testing function space consisting of smooth functions on I with compact supports, $*$ is the convolution defined by

$$(\phi * \varphi)(x) = \int_0^x \phi(x-t)\varphi(t)dt, \quad x > 0; \quad (2)$$

the operation $\otimes$ is defined by

$$(f \otimes \phi)(\psi) = f(\phi \times \psi), \forall \psi \in S_\alpha(I), \quad \text{for } f \in S'_\alpha(I), \quad \phi \in \mathcal{D}(I), \quad (3)$$

where

$$(\phi \times \psi)(x) = \int_0^\infty \phi(t)\psi(x+t)dt, \quad \forall x \in I \quad (4)$$

and Δ_+ is the the collection of all sequences (ϕ_n) from $\mathcal{D}(I)$, satisfying the the following conditions:

$$\Delta_1 \int_0^\infty \phi_n(x)dx = 1, \quad \forall n \in \mathbb{N},$$

$$\Delta_2 \int_0^\infty |\phi_n(x)|dx \leq M, \forall n \in \mathbb{N}, \quad \text{for some } M > 0 \text{ and}$$

$$\Delta_3 \text{ Given } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } \operatorname{supp} \phi_n \subset (0, \epsilon), \quad \forall n \geq N.$$

Each member of Δ_+ is called a delta sequence.

We note that $S'_\alpha(I)$ is a sequential-convergence linear space and $(\mathcal{D}(I), *)$ is a commutative semi-group.

LEMMA 2.1 *If $\phi \in \mathcal{D}(I)$ and $\psi \in S_\alpha(I)$ then $(\phi \times \psi)^{(k)} = \phi \times \psi^{(k)}$ for every $k \in \mathbb{N}$.*

Proof First we claim that $(\phi \times \psi)' = \phi \times \psi'$. For $x \in K = \operatorname{supp} \phi$ and for a small $h \in \mathbb{R}$,

$$\frac{(\phi \times \psi)(x+h) - (\phi \times \psi)(x)}{h} = \int_K \phi(t) \frac{\psi(x+h+t) - \psi(x+t)}{h} dt$$

We note that for each fixed $t \in K$, the integrand tends to $\phi(t)\psi'(x+t)$ as $h \rightarrow 0$. By using mean value theorem, we get that the integrand is dominated by $\phi(t) \sup_{s \in K} |\psi'(s)|$. Now we can

apply the dominated convergence theorem and hence we get $(\phi \times \psi)' = \phi \times \psi'$. By induction on k , we get $(\phi \times \psi)^{(k)} = \phi \times \psi^{(k)}$. ■

LEMMA 2.2 *If $\phi \in \mathcal{D}(I)$ and $\psi \in S_\alpha(I)$ then $\gamma_k(\phi \times \psi) \leq C\gamma_k(\psi)$, for some $C > 0$ and hence $\phi \times \psi \in S_\alpha(I)$.*

Proof Using the previous lemma, for an arbitrary $x \in I$,

$$\begin{aligned} (1+x)^\alpha x^k |(\phi \times \psi)^{(k)}(x)| &= (1+x)^\alpha x^k |(\phi \times \psi^{(k)})(x)| \\ &\leq (1+x)^\alpha x^k \int_0^\infty |\phi(t)\psi^{(k)}(x+t)|dt \\ &\leq \gamma_k(\psi) \int_0^\infty (1+x)^\alpha x^k (x+t)^{-k} (1+(x+t))^{-\alpha} |\phi(t)|dt \\ &\leq \gamma_k(\psi) \int_0^\infty (1+x)^\alpha (1+(x+t))^{-\alpha} |\phi(t)|dt \end{aligned}$$

Let the supp ϕ be contained in (ϵ, M) . If $\alpha \geq 0$, then the last expression is dominated by $\gamma_k(\psi) \int_0^\infty |\phi(t)|dt$. If $\alpha < 0$ then it is dominated by

$$\gamma_k(\psi) \left(\frac{1+x+M}{1+x} \right)^{-\alpha} \int_0^\infty |\phi(t)|dt \leq \gamma_k(\psi)(1+M)^p \int_0^\infty |\phi(t)|dt < \infty.$$

If $C = \max \left\{ \int_0^\infty |\phi(t)|dt, (1+M)^p \int_0^\infty |\phi(t)|dt \right\}$, then it follows that $\gamma_k(\phi \times \psi) \leq C\gamma_k(\psi)$. ■

LEMMA 2.3 *If $f \in S'_\alpha(I)$ and $\phi \in \mathcal{D}(I)$ then $f \otimes \phi \in S'_\alpha(I)$.*

Proof We note that the right hand side of the equation (3) is meaningful, by Lemma 2.2. Obviously, $f \otimes \phi$ is a linear map on $S_\alpha(I)$. Using the fact $f \in S'_\alpha(I)$, and again by Lemma 2.2, we have $|(f \otimes \phi)(\psi)| = |f(\phi \times \psi)| \leq C_1 \gamma_k(\phi \times \psi) \leq C C_1 \gamma_k(\psi)$, for some $C > 0$ and $C_1 > 0$. ■

LEMMA 2.4 *If $\phi, \varphi \in \mathcal{D}(I)$ and $\psi \in S_\alpha(I)$ then $(\phi * \varphi) \times \psi = \phi \times (\varphi \times \psi)$.*

By changing the order of integration and by using a simple change of variable, we can prove the lemma.

LEMMA 2.5 *Let $f, f_1, f_2 \in S'_\alpha(I)$, $\phi, \varphi \in \mathcal{D}(I)$ and $a \in \mathbb{C}$. Then 1. $(f_1 + f_2) \otimes \phi = f_1 \otimes \phi + f_2 \otimes \phi$; 2. $(af) \otimes \phi = a(f \otimes \phi)$; 3. $f \otimes (\phi * \varphi) = (f \otimes \phi) \otimes \varphi$.*

The proof of this lemma is straight forward.

LEMMA 2.6 *If $f_n \rightarrow f$ as $n \rightarrow \infty$ in $S'_\alpha(I)$ and $\phi \in \mathcal{D}(I)$ then $f_n \otimes \phi \rightarrow f \otimes \phi$ in $S'_\alpha(I)$.*

Proof Let $\psi \in S_\alpha(I)$ be arbitrary. By Lemma 2.2, we have $\phi \times \psi \in S_\alpha(I)$. Since $(f_n \otimes \phi - f \otimes \phi)(\psi) = ((f_n - f) \otimes \phi)(\psi) = (f_n - f) \otimes (\phi \times \psi)$, and $f_n \otimes \phi \rightarrow f \otimes \phi$ in $S'_\alpha(I)$, the lemma follows. ■

LEMMA 2.7 *If $f \in S'_\alpha(I)$ and $(\phi_n) \in \Delta_+$ then $f \otimes \phi_n \rightarrow f$ as $n \rightarrow \infty$ in $S'_\alpha(I)$.*

Proof To prove this, first we claim that if $\psi \in S_\alpha(I)$ then $\phi_n \times \psi \rightarrow \psi$, as $n \rightarrow \infty$ in $S_\alpha(I)$. Let $\text{supp } \phi_n \subset (0, \mu_n)$, $\forall n \in \mathbb{N}$ and $\mu = \sup_{n \in \mathbb{N}} \mu_n$. For any $x \in (0, \infty)$, using the first property of (ϕ_n) and mean value theorem, we get

$$\begin{aligned} |(\phi_n \times \psi)^{(k)}(x) - \psi^{(k)}(x)| &= \left| \int_0^\infty \phi_n(t) \psi^{(k)}(x+t) dt - \psi^{(k)}(x) \right| \\ &= \left| \int_0^{\mu_n} \phi_n(t) (\psi^{(k)}(x+t) - \psi^{(k)}(x)) dt \right| \\ &\leq \int_0^{\mu_n} |\phi_n(t) (\psi^{(k)}(x+t) - \psi^{(k)}(x))| dt \\ &\leq \int_0^{\mu_n} t |\phi_n(t)| |\psi^{(k+1)}(x+\zeta)| dt \end{aligned}$$

for some $\zeta \in (0, t)$. Using this estimate, it follows that

$$\begin{aligned} \gamma_k((\phi_n \times \psi) - \psi) &= \sup_{x \in I} (1+x)^\alpha x^k |(\phi_n \times \psi - \psi)^{(k)}(x)| \\ &\leq \sup_{x \in I} (1+x)^\alpha x^k \int_0^{\mu_n} t |\phi_n(t)| |\psi^{(k+1)}(x+\zeta)| dt \\ &\leq T_1 + T_2, \end{aligned}$$

where $T_1 = \mu_n \sup_{0 < x < 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| |\psi^{(k+1)}(x+\zeta)| dt$ and $T_2 = \mu_n \sup_{x \geq 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| |\psi^{(k+1)}(x+\zeta)| dt$. Let $\int_0^\infty |\phi_n(x)| dx \leq M$, $\forall n \in \mathbb{N}$ and $K = \sup_{0 \leq s \leq \mu_n+1} |\psi^{(k)}(s)|$. Then,

$$T_1 \leq \max\{1, 2^\alpha\} \mu_n K \int_0^{\mu_n} |\phi_n(t)| dt \leq \max\{1, 2^\alpha\} \mu_n K M \rightarrow 0 \text{ as } n \rightarrow \infty$$

by the property (Δ_3) of (ϕ_n) .

If $\alpha \geq 0$ then

$$\begin{aligned} T_2 &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| (1+x+\zeta)^{-\alpha} (x+\zeta)^{-(k+1)} dt \\ &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| (1+x)^{-\alpha} x^{-(k+1)} dt \\ &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} x^{-1} M \leq \gamma_{k+1}(\psi) \mu_n \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

If $\alpha < 0$ then

$$\begin{aligned} T_2 &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| (1+x+\zeta)^{-\alpha} (x+\zeta)^{-(k+1)} dt \\ &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} (1+x)^\alpha x^k \int_0^{\mu_n} |\phi_n(t)| (1+x+\mu)^{-\alpha} x^{-(k+1)} dt \\ &\leq \gamma_{k+1}(\psi) \mu_n \sup_{x \geq 1} \left(\frac{1+x+\mu}{1+x} \right)^{-\alpha} x^{-1} \int_0^{\mu_n} |\phi_n(t)| dt \\ &\leq \gamma_{k+1}(\psi) \mu_n (1+\mu)^{-\alpha} M \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence, our claim holds.

Now for each $\psi \in S_\alpha(I)$, $\langle (f \otimes \phi_n)(x), \psi(x) \rangle = \langle f(x), (\phi_n \times \psi)(x) \rangle \rightarrow \langle f(x), \psi(x) \rangle$ as $n \rightarrow \infty$, since $\phi_n \times \psi \rightarrow \psi$ as $n \rightarrow \infty$ in $S_\alpha(I)$. ■

Next, we construct another Boehmian space $\mathcal{B}_2 = \mathcal{B}(\mathcal{H}, (\mathcal{D}(I), *), \otimes, \Delta_+)$, where $\mathcal{H}$ is the space of all analytic functions on $\Omega = \mathbb{C} \setminus (-\infty, 0]$, which are Stieltjes transforms of members of $S'_\alpha(I)$.

In the literature, no convergence is discussed on the range of Stieltjes transform of $S'_\alpha(I)$, and the continuity of the Stieltjes transform is also not discussed. Since we need a notion of convergence to construct a Boehmian space, adapting the technique used in the context of Hilbert transform on $\mathcal{D}(\mathbb{R}^n)$ in [18, p. 159] we introduce a notion of convergence on $\mathcal{H}$ as follows:

$F_n \rightarrow F$ as $n \rightarrow \infty$ in $\mathcal{H}$, if $f_n \rightarrow f$ as $n \rightarrow \infty$ in $S'_\alpha(I)$, where $f_n = \mathfrak{S}_p^{-1} F_n, \forall n \in \mathbb{N}$, $f = \mathfrak{S}_p^{-1} F$, which makes the Stieltjes transform $\mathfrak{S}_p : S'_\alpha(I) \rightarrow \mathcal{H}$ a homeomorphism.

DEFINITION 2.8 For $F \in \mathcal{H}$ and $\phi \in \mathcal{D}(I)$, we define $(F \otimes \phi)(z) = \int_0^\infty F(z+t)\phi(t) dt, \forall z \in \Omega$.

LEMMA 2.9 If $f \in S'_\alpha(I)$ and $\phi \in \mathcal{D}(I)$ then $\mathfrak{S}_p(f \otimes \phi) = (\mathfrak{S}_p f) \otimes \phi$.

Proof Let $z \in \Omega$ be arbitrarily fixed. If $K = \text{supp } \phi$, then $(\mathfrak{S}_p(f \otimes \phi))(z) = \langle (f \otimes \phi)(x), 1/((z+x)^p) \rangle = \langle f(x), \int_K \phi(t) 1/((z+x+t)^p) dt \rangle$.

We note that if $\theta_t(x) = \phi(t) 1/((z+x+t)^p)$, then the map $t \mapsto \theta_t(x)$ is a continuous map from K into $S_\alpha(I)$. Indeed, if $t_n \rightarrow t$ as $n \rightarrow \infty$ in K , then

$$\begin{aligned} \gamma_k(\theta_{t_n} - \theta_t) &= \sup_{x \in I} x^k (1+x)^\alpha \left| \theta_{t_n}^{(k)}(x) - \theta_t^{(k)}(x) \right| \\ &= \sup_{x \in I} x^k (1+x)^\alpha \left| \phi(t_n) \frac{(-1)^k (p)_k}{(z+x+t_n)^{p+k}} - \phi(t) \frac{(-1)^k (p)_k}{(z+x+t)^{p+k}} \right| \\ &= \sup_{x \in I} x^k (1+x)^\alpha \left| (\phi(t_n) - \phi(t)) \frac{(p)_k}{(z+x+t_n)^{p+k}} \right| \\ &\quad + \sup_{x \in I} x^k (1+x)^\alpha \left| \phi(t) \left(\frac{(p)_k}{(z+x+t_n)^{p+k}} - \frac{(p)_k}{(z+x+t)^{p+k}} \right) \right| \\ &= |\phi(t_n) - \phi(t)| \sup_{x \in I} x^k (1+x)^\alpha \left| \frac{(p)_k}{(z+x+t_n)^{p+k}} \right| \\ &\quad + |\phi(t)| |t_n - t| \sup_{x \in I} x^k (1+x)^\alpha \left| \frac{(p)_{k+1}}{(z+x+v)^{p+k+1}} \right| \end{aligned}$$

for some v between t_n and t , using mean value theorem. If

$$w = z + \begin{cases} \min\{t, t_n : n \in \mathbb{N}\} & \text{if } p < 0 \\ \max\{t, t_n : n \in \mathbb{N}\} & \text{if } p \geq 0 \end{cases}$$

and if $\eta_q(x) = 1/((w+x)^q) \forall x \in I$ for any $q \in \mathbb{C}$ with $\alpha \leq \text{Re } q$, then by [16, Lemma 5.3.6], $\eta_q \in S_\alpha(I)$ and, hence, the last expression is dominated by $|\phi(t_n) - \phi(t)| (p)_k \gamma_k(\eta_p) + |\phi(t)| |t_n - t| (p)_{k+1} \gamma_k(\eta_{p+1})$, which tends to zero as $n \rightarrow \infty$.

Therefore, using [19, Theorem 3.27], we get that $\int_K \phi(t) 1/((z+x+t)^p) dt$ exists in $S_\alpha(I)$. Thus, $\langle f(x), \int_K \phi(t) 1/((z+x+t)^p) dt \rangle = \int_K \langle f(x), 1/((z+x+t)^p) \rangle \phi(t) dt = \int_K (\mathfrak{S}_p f)(z+t) \phi(t) dt = \int_0^\infty (\mathfrak{S}_p f)(z+t) \phi(t) dt = ((\mathfrak{S}_p f) \otimes \phi)(z)$. ■

LEMMA 2.10 If $F \in \mathcal{H}$ and $\phi \in \mathcal{D}(I)$ then $F \circledast \phi \in \mathcal{H}$.

Proof For a given $F \in \mathcal{H}$, we can find $f \in S'_\alpha(I)$ such that $F = \mathfrak{S}_p f$. Therefore, $F \circledast \phi = (\mathfrak{S}_p f) \circledast \phi = \mathfrak{S}_p(f \otimes \phi) \in \mathcal{H}$, by using Lemma 2.9 and Lemma 2.3. ■

LEMMA 2.11 If $F \in \mathcal{H}$ and $\phi \in \mathcal{D}(I)$ then $\mathfrak{S}_p^{-1}(F \circledast \phi) = \mathfrak{S}_p^{-1}(F) \otimes \phi$.

Proof Let $\mathfrak{S}_p^{-1}(F) = f \in S'_\alpha(I)$. Then $F = \mathfrak{S}_p f$. Using Lemma 2.9, we have $\mathfrak{S}_p(f \otimes \phi) = F \circledast \phi$. Applying inverse Stieltjes transform on both sides, we get the required identity. ■

LEMMA 2.12 Let $F_1, F_2, F \in \mathcal{H}$, $\phi, \varphi \in \mathcal{D}(I)$ and $a \in \mathbb{C}$. Then 1. $(F_1 + F_2) \circledast \phi = F_1 \circledast \phi + F_2 \circledast \phi$; 2. $(aF) \circledast \phi = a(F \circledast \phi)$ and 3. $F \circledast (\phi * \varphi) = (F \circledast \phi) \circledast \varphi$.

LEMMA 2.13 If $F_n \rightarrow F$, as $n \rightarrow \infty$ in $\mathcal{H}$ and $\phi \in \mathcal{D}(I)$, then $F_n \circledast \phi \rightarrow F \circledast \phi$ in $\mathcal{H}$.

LEMMA 2.14 If $F \in \mathcal{H}$ and $(\phi_n) \in \Delta_+$ then $F \circledast \phi_n \rightarrow F$ as $n \rightarrow \infty$ in $\mathcal{H}$.

We prefer to omit the details of the proofs of the above three lemmas, as they follow immediately, from Lemmas 2.5, 2.6, 2.7 and 2.9. Thus, the Boehmian space $\mathcal{B}_2$ is constructed.

3. Extended Stieltjes transform

DEFINITION 3.1 The extended Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is defined by $\mathfrak{S}_p([f_n/\phi_n]) = [(\mathfrak{S}_p f_n)/\phi_n]$.

LEMMA 3.2 The Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is well defined.

Proof Let $[f_n/\phi_n] \in \mathcal{B}_1$. Then, we have $f_n \otimes \phi_m = f_m \otimes \phi_n$, $\forall m, n \in \mathbb{N}$. Applying Stieltjes transform and using Lemma 2.9 on both sides, we get $(\mathfrak{S}_p f_n) \circledast \phi_m = (\mathfrak{S}_p f_m) \circledast \phi_n$, $\forall m, n \in \mathbb{N}$. This implies that $(\mathfrak{S}_p f_n)/(\phi_n)$ is a quotient in the context of $\mathcal{B}_2$ and hence, $[(\mathfrak{S}_p f_n)/\phi_n] \in \mathcal{B}_2$. Further, if $[f_n/\phi_n] = [g_n/\varphi_n]$, then we have $f_n \otimes \varphi_m = g_m \otimes \phi_n$, $\forall m, n \in \mathbb{N}$. Again, by using the same technique, we get that $(\mathfrak{S}_p f_n) \circledast \varphi_m = (\mathfrak{S}_p g_m) \circledast \phi_n$, $\forall m, n \in \mathbb{N}$. Thus, we get $[(\mathfrak{S}_p f_n)/\phi_n] = [(\mathfrak{S}_p g_n)/\varphi_n]$. This proves that this definition is independent of the representative. Hence the proposition. ■

LEMMA 3.3 (Consistency) The Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is consistent with $\mathfrak{S}_p : S'_\alpha(I) \rightarrow \mathcal{H}$.

Proof Let $f \in S'_\alpha(I)$ and X represent f in $\mathcal{B}_1$. Then, for any $(\phi_n) \in \Delta_+$, $X = [(f \otimes \phi_n)/\phi_n]$. Now using Lemma 2.9, we get that $\mathfrak{S}_p X = \mathfrak{S}_p([(f \otimes \phi_n)/\phi_n]) = [(\mathfrak{S}_p(f \otimes \phi_n))/\phi_n] = [((\mathfrak{S}_p f) \circledast \phi_n)/\phi_n]$, which is the Boehmian representing $\mathfrak{S}_p f$ in $\mathcal{B}_2$. ■

Here, we point out that, $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ extends $\mathfrak{S}_p : S'_\alpha(I) \rightarrow \mathcal{H}$, by the fact $\mathcal{B}_1 \supset S'_\alpha(I)$ and by Lemma 3.3.

PROPOSITION 3.4 The extended Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is linear.

From the linearity of the distributional Stieltjes transform and Lemma 2.9, obviously the lemma follows.

PROPOSITION 3.5 *The extended Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is one-to-one.*

Proof If $[f_n/\phi_n], [g_n/\phi_n] \in \mathcal{B}_1$ with $\mathfrak{S}_p([f_n/\phi_n]) = \mathfrak{S}_p([g_n/\phi_n])$. Then we have $(\mathfrak{S}_p f_n) \otimes \phi_m = (\mathfrak{S}_p g_m) \otimes \phi_n, \forall m, n \in \mathbb{N}$. Using the Lemma 2.9, we get $\mathfrak{S}_p(f_n \otimes \phi_m) = \mathfrak{S}_p(g_m \otimes \phi_n), \forall m, n \in \mathbb{N}$. Since the distributional Stieltjes transform $\mathfrak{S}_p : S'_\alpha(I)$ is one-to-one, we get $f_n \otimes \phi_m = g_m \otimes \phi_n, \forall m, n \in \mathbb{N}$. Hence it follows that $[f_n/\phi_n] = [g_n/\phi_n]$. Therefore, the extended Stieltjes transform is one-to-one. ■

PROPOSITION 3.6 *The extended Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is onto.*

Proof Let $[F_n/\phi_n] \in \mathcal{B}_2$. Then we have $F_n \in \mathcal{H}$ and $F_n \otimes \phi_m = F_m \otimes \phi_n, \forall m, n \in \mathbb{N}$. We choose $f_n \in S'_\alpha(I)$ such that $(\mathfrak{S}_p f_n) = F_n$. Then we have $(\mathfrak{S}_p f_n) \otimes \phi_m = (\mathfrak{S}_p f_m) \otimes \phi_n, \forall m, n \in \mathbb{N}$. Using the Lemma 2.9 and the fact that the distributional Stieltjes transform is one-to-one, we get $f_n \otimes \phi_m = f_m \otimes \phi_n, \forall m, n \in \mathbb{N}$. Thus $[f_n/\phi_n] \in \mathcal{B}_1$. Obviously, we have $\mathfrak{S}_p([f_n/\phi_n]) = [F_n/\phi_n]$. ■

In view of proof of the above proposition, we can define the inverse Stieltjes transform as follows:

DEFINITION 3.7 *The inverse Stieltjes transform $\mathfrak{S}_p^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ by $\mathfrak{S}_p^{-1}([F_n/\phi_n]) = [(\mathfrak{S}_p^{-1} F_n)/\phi_n]$.*

PROPOSITION 3.8 *The Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and the inverse Stieltjes transform $\mathfrak{S}_p^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ are continuous with respect to δ -convergence.*

Proof Let $X_n \xrightarrow{\delta} X$ as $n \rightarrow \infty$ in $\mathcal{B}_1$. Then, by Theorem 2.6 of [3], there exist $f_{n,k}, f_k \in S'_\alpha(I)$ such that $X_n = [f_{n,k}/\phi_k], X = [f_k/\phi_k]$ and for each $k \in \mathbb{N}$, $f_{n,k} \rightarrow f_k$ as $n \rightarrow \infty$ in $S'_\alpha(I)$. Using the continuity of the distributional Stieltjes transform, we get $(\mathfrak{S}_p f_{n,k}) \rightarrow (\mathfrak{S}_p f_k)$, as $n \rightarrow \infty$ in $\mathcal{H}$. We also have $\mathfrak{S}_p X_n = [(\mathfrak{S}_p f_{n,k})/\phi_k]$ and $\mathfrak{S}_p X = [(\mathfrak{S}_p f_k)/\phi_k]$. Again by Theorem 2.6 of [3], we get $\mathfrak{S}_p X_n \xrightarrow{\delta} \mathfrak{S}_p X$, as $n \rightarrow \infty$ in $\mathcal{B}_2$. Hence, $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is continuous.

To prove that $\mathfrak{S}_p^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ is continuous, let $Y_n \xrightarrow{\delta} Y$ as $n \rightarrow \infty$ in $\mathcal{B}_2$. Then, there exist $F_{n,k}, F_k \in \mathcal{H}$ such that $Y_n = [F_{n,k}/\phi_k], Y = [F_k/\phi_k]$ and for each $k \in \mathbb{N}$, $F_{n,k} \rightarrow F_k$ as $n \rightarrow \infty$, in $\mathcal{H}$. As before, using the continuity of the inverse Stieltjes transform, we get $\mathfrak{S}_p^{-1} F_n \rightarrow \mathfrak{S}_p^{-1} F$ as $n \rightarrow \infty$ in $S'_\alpha(I)$ and hence $\mathfrak{S}_p^{-1} Y_n \xrightarrow{\delta} \mathfrak{S}_p^{-1} Y$, as $n \rightarrow \infty$ in $\mathcal{B}_2$. Hence the proposition. ■

LEMMA 3.9 *If $X \in \mathcal{B}_1, Y \in \mathcal{B}_2$ and $\phi \in \mathcal{D}(I)$ then 1. $\mathfrak{S}_p(X \otimes \phi) = (\mathfrak{S}_p X) \otimes \phi$. 2. $\mathfrak{S}_p^{-1}(Y \otimes \phi) = (\mathfrak{S}_p^{-1} Y) \otimes \phi$.*

Proof Let $X = [f_n/\phi_n]$ and $Y = [F_n/\phi_n]$.

1. Using Lemma 2.9, $\mathfrak{S}_p(X \otimes \phi) = \mathfrak{S}_p([f_n/\phi_n] \otimes \phi) = \mathfrak{S}_p[(f_n \otimes \phi)/\phi_n] = [(\mathfrak{S}_p f_n) \otimes \phi]/\phi_n = [(\mathfrak{S}_p f_n)/\phi_n] \otimes \phi$.
2. Similarly using Lemma 2.11, we prove the second part of this lemma. ■

PROPOSITION 3.10 *The Stieltjes transform $\mathfrak{S}_p : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ and the inverse Stieltjes transform $\mathfrak{S}_p^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ are continuous with respect to Δ -convergence.*

Proof Let $X_n \xrightarrow{\Delta} X$ as $n \rightarrow \infty$ in $\mathcal{B}_1$. Then by the definition of Δ -convergence, there exist $f_n \in S'_\alpha(I)$ and $(\phi_n) \in \Delta_+$ such that $(X_n - X) \otimes \phi_n = [(f_n \otimes \phi_k)/\phi_k]$ and $f_n \rightarrow 0$ as $n \rightarrow \infty$ in $S'_\alpha(I)$.

Using Lemma 3.9(1), $((\mathfrak{S}_p X_n) - (\mathfrak{S}_p X)) \otimes \phi_n = (\mathfrak{S}_p(X_n - X)) \otimes \phi_n = \mathfrak{S}_p((X_n - X) \otimes \phi_n) = \mathfrak{S}_p([(f_n \otimes \phi_k)/\phi_k]) = [(\mathfrak{S}_p(f_n \otimes \phi_k))/\phi_k] = [(\mathfrak{S}_p f_n) \otimes \phi_k]/\phi_k = \mathfrak{S}_p f_n \in \mathcal{H}$ and $\mathfrak{S}_p f_n \rightarrow 0$ as $n \rightarrow \infty$ in $\mathcal{H}$. What we have proved is $(\mathfrak{S}_p X_n) \xrightarrow{\Delta} (\mathfrak{S}_p X)$, as $n \rightarrow \infty$ in $\mathcal{B}_2$.

Similarly, by using Lemma 3.9(2), we can prove that $\mathfrak{S}_p^{-1} : \mathcal{B}_2 \rightarrow \mathcal{B}_1$ is continuous with respect to Δ -convergence. ■

LEMMA 3.11 *If $f \in S'_\alpha(I)$ and $\phi \in \mathcal{D}(I)$ then $(f \otimes \phi)^{(k)} = f^{(k)} \otimes \phi$.*

Proof Let $\psi \in S_\alpha(I)$. By using Lemma 2.1, we get $\langle (f \otimes \phi)^{(k)}(x), \psi(x) \rangle = \langle (f \otimes \phi)(x), (-1)^k \psi^{(k)}(x) \rangle = \langle f(x), (-1)^k (\phi \times \psi^{(k)})(x) \rangle = \langle f(x), (-1)^k (\phi \times \psi)^{(k)}(x) \rangle = \langle f^{(k)}(x), (\phi \times \psi)(x) \rangle = \langle (f^{(k)} \otimes \phi)(x), \psi(x) \rangle$. Hence the lemma. ■

DEFINITION 3.12 *For $X = [f_n/\phi_n] \in \mathcal{B}_1$, we define the k th derivative as $X^{(k)} = [f_n^{(k)}/\phi_n]$.*

Using the previous lemma, we can verify that if $[f_n/\phi_n] \in \mathcal{B}_1$ then $[f_n^{(k)}/\phi_n] \in \mathcal{B}_1$ and the definition is independent of the representative f_n/ϕ_n .

LEMMA 3.13 (Consistency) *If $f \in S'_\alpha(I)$ and X represents f in $\mathcal{B}_1$ then $X^{(k)} = f^{(k)}$, $\forall k \in \mathbb{N}_0$.*

Proof Let $X = [(f \otimes \phi_n)/\phi_n] \in \mathcal{B}_1$ for any $(\phi_n) \in \Delta_+$. Then, using the Lemma 3.11, we get that $X^{(k)} = [(f \otimes \phi_n)^{(k)}/\phi_n] = [(f^{(k)} \otimes \phi_n)/\phi_n]$, which represents $f^{(k)}$ in $\mathcal{B}_1$. Hence the lemma. ■

LEMMA 3.14 *If $F \in \mathcal{H}$ and $\phi \in \mathcal{D}(I)$ then $(F \otimes \phi)^{(k)} = F^{(k)} \otimes \phi$.*

The proof of this lemma is similar to that of Lemma 2.1.

DEFINITION 3.15 *For $Y = [F_n/\phi_n] \in \mathcal{B}_1$ we define the k th derivative as $Y^{(k)} = [F_n^{(k)}/\phi_n]$.*

One can verify that this definition is well defined and it is consistent with the classical derivative defined on H .

PROPOSITION 3.16 *If $X \in \mathcal{B}_1$ then $(\mathfrak{S}_p X)^{(k)} = (-1)^k (p)_k \mathfrak{S}_{p+k} X = \mathfrak{S}_p X^{(k)}$.*

Proof Let $X = [f_n/\phi_n] \in \mathcal{B}_1$. Using the property (1) of distributional Stieltjes transform, we get that $(\mathfrak{S}_p X)^{(k)} = (\mathfrak{S}_p[f_n/\phi_n])^{(k)} = ([(\mathfrak{S}_p f_n)/\phi_n])^{(k)} = [(\mathfrak{S}_p f_n)^{(k)}/\phi_n] = [((-1)^k (p)_k \mathfrak{S}_{p+k} f_n)/\phi_n] = (-1)^k (p)_k \mathfrak{S}_{p+k}[f_n/\phi_n] = (-1)^k (p)_k \mathfrak{S}_{p+k} X = (-1)^k (p)_k \mathfrak{S}_{p+k}[f_n/\phi_n] = [((-1)^k (p)_k \mathfrak{S}_{p+k} f_n)/\phi_n] = [(\mathfrak{S}_p f_n^{(k)})/\phi_n] = \mathfrak{S}_p[(f_n^{(k)})/\phi_n] = \mathfrak{S}_p X^{(k)}$. ■

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Properties of multiple Hermite and multiple Laguerre polynomials by the generating function

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A generating function is obtained for multiple Hermite and multiple Laguerre polynomials from the Rodrigues formula, and then interesting recurrence relations are given. We also find a relation between multiple orthogonal polynomials and ordinary orthogonal polynomials such as multiple Hermite polynomials with classical Hermite polynomials and multiple Laguerre polynomials with classical Laguerre polynomials. Lastly, we find a differential equation for the multiple orthogonal polynomials by the generating function, which is a reconstruction of the works by Aptekarev *et al.* [Aptekarev, A.I., Branquinho, A. and Van Assche, W., 2003, Multiple orthogonal polynomials for classical weights. *Transaction of American Mathematical Society*, **355**, 3887–3914.].

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1. Introduction

The topics of multiple orthogonal polynomials are now extensively discussed as a generalization of ordinary orthogonal polynomials [1–4]. The differential equation for multiple orthogonal polynomials is a very important property as it was for ordinary orthogonal polynomials. The differential equation of multiple orthogonal polynomials for classical weights was recently investigated in [5, 6]. In those works, the Rodrigues formula plays a key role in finding the differential equation of multiple orthogonal polynomials. From the Rodrigues formula, we can define [5]

(a) the multiple Hermite polynomials $\{H_n^{(\vec{\alpha})}(x)\}$ by

$$H_n^{(\vec{\alpha})}(x) = w_r^{-1}(w_r w_{r-1}^{-1}(\cdots (w_2 w_1^{-1}(w_1)^{(n_1)}(n_2) \cdots)^{(n_{r-1})}(n_r), \quad (1)$$

where $w_i = e^{(\delta/2)x^2 + \alpha_i x}$, $1 \leq i \leq r$, are the Hermite weights with $\delta < 0$ and $\alpha_i \neq \alpha_j$ for $i \neq j$.

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(b) the multiple Laguerre I polynomials $\{L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\}$ by

$$L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) = w_r^{-1}(w_r w_{r-1}^{-1} x^{n_r} (\cdots (w_2 w_1^{-1} x^{n_2} (w_1 x^{n_1})^{(n_1)})^{(n_2)} \cdots)^{(n_{r-1})})^{(n_r)}, \quad (2)$$

where $w_i = x^{\alpha_i} e^{\beta_i x}$, $\alpha_i > -1$, $1 \leq i \leq r$, are the Laguerre weights with $\beta < 0$ and $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$.

(c) the multiple Laguerre II polynomials $\{L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\}$ by

$$L_{\vec{n}}^{(\alpha;\vec{\beta})}(x) = w_r^{-1}(w_r w_{r-1}^{-1} (\cdots (w_2 w_1^{-1} (w_1 x^{|\vec{n}|})^{(n_1)})^{(n_2)} \cdots)^{(n_{r-1})})^{(n_r)}, \quad (3)$$

where $w_i = x^{\alpha} e^{\beta_i x}$, $\beta_i < 0$, $1 \leq i \leq r$, are the Laguerre weights with $\alpha > -1$ and $\beta_i \neq \beta_j$ for $i \neq j$.

Here r is a positive integer, $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}_0^r$ with $|\vec{n}| = n_1 + n_2 + \cdots + n_r$, $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$, and $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_r) \in \mathbb{R}^r$.

These polynomials have some orthogonality with respect to r weight functions, from which the multiple orthogonal polynomials are named. More precisely, the multiple Hermite polynomials $\{H_{\vec{n}}^{(\vec{\alpha})}(x)\}$ satisfy

$$\int_{-\infty}^{\infty} H_{\vec{n}}^{(\vec{\alpha})}(x) x^k w_i(x) dx = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r,$$

where w_i is the Hermite weight in equation (1). Similarly, the multiple Laguerre I polynomials $\{L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\}$ satisfy

$$\int_0^{\infty} L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) x^k w_i(x) dx = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r,$$

where w_i is the Laguerre I weight in equation (2) and the multiple Laguerre II polynomials $\{L_{\vec{n}}^{(\alpha;\vec{\beta})}(x)\}$ satisfy

$$\int_0^{\infty} L_{\vec{n}}^{(\alpha;\vec{\beta})}(x) x^k w_i(x) dx = 0, \quad k = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, r,$$

where w_i is the Laguerre II weight in (3).

In this paper, a generating function is obtained for multiple Hermite and multiple Laguerre polynomials from the Rodrigues formula, and then interesting recurrence relations are given. We also find a relation between multiple orthogonal polynomials and ordinary orthogonal polynomials such as multiple Hermite polynomials with classical Hermite polynomials and multiple Laguerre polynomials with classical Laguerre polynomials. Lastly, we find a differential equation for the multiple orthogonal polynomials by the generating function, which is a reconstruction of the works by Aptekarev *et al.* [5].

2. Recurrence relations by the generating function

Throughout the paper, r is assumed to be a positive integer. We will use a multi-index $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}_0^r$, $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_r) \in \mathbb{R}^r$, $\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_r) \in \mathbb{R}^r$, and their length $|\vec{n}| = n_1 + n_2 + \cdots + n_r$, $|\vec{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$, $|\vec{\beta}| = \beta_1 + \beta_2 + \cdots + \beta_r$.

The generating function $G(x; t_1, t_2, \dots, t_r)$ for the multiple orthogonal polynomials $\{P_{\vec{n}}(x)\}$ is defined by

$$G(x; t_1, t_2, \dots, t_r) = \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty} \frac{P_{\vec{n}}}{\vec{n}!} t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r}, \quad (4)$$

where $\vec{n}! = n_1! n_2! \cdots n_r!$ is the multi-index factorial function. We also use by $G(x; \mathbf{t}) := G(x; t_1, t_2, \dots, t_r)$, $\sum_{\vec{n}=0}^{\infty} = \sum_{n_r=0}^{\infty} \sum_{n_{r-1}=0}^{\infty} \cdots \sum_{n_1=0}^{\infty}$ and $\mathbf{t}^{\vec{n}} = t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r}$ so that the equation (4) is denoted by

$$G(x; \mathbf{t}) = \sum_{\vec{n}=0}^{\infty} \frac{P_{\vec{n}}(x)}{\vec{n}!} \mathbf{t}^{\vec{n}}.$$

THEOREM 2.1 *Let $\{H_{\vec{n}}^{(\alpha)}(x)\}$ be the multiple Hermite polynomials defined by equation (1). Then the generating function is given by*

$$\begin{aligned} G(x; \mathbf{t}) &= w_r^{-1}(x) \prod_{k=1}^r w_{k-1}^{-1}(x + t_r + \cdots + t_k) w_k(x + t_r + \cdots + t_k) \\ &= \exp \left(\delta x \sum_{i=1}^r t_i + \frac{\delta}{2} \left(\sum_{i=1}^r t_i \right)^2 + \sum_{i=1}^r \alpha_i t_i \right), \end{aligned} \quad (5)$$

where $w_0 = 1$ and w_i , $1 \leq i \leq r$, are the Hermite weights. In particular, if $r = 2$, then

$$G(x; t_1, t_2) = \exp \left(\left(\frac{\delta}{2} \right) (t_1^2 + t_2^2) + \alpha_1 t_1 + \alpha_2 t_2 + \delta x (t_1 + t_2) + \delta t_1 t_2 \right). \quad (6)$$

Proof By the Rodrigues formula and the Cauchy integral formula, we have

$$\begin{aligned} w_r G(x; \mathbf{t}) &= \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} (w_r w_{r-1}^{-1} (\cdots (w_2 w_1^{-1} (w_1)^{(n_1)})^{(n_2)} \cdots)^{(n_{r-1})})^{(n_r)} \mathbf{t}^{\vec{n}} \\ &=: I_r(x; \mathbf{t}), \end{aligned}$$

where

$$I_m(z; t_1, t_2, \dots, t_m) := \sum_{n_m=0}^{\infty} \frac{1}{2\pi i} \oint \frac{t_m^{n_m} w_m(\xi) w_{m-1}^{-1}(\xi)}{(\xi - z)^{n_m+1}} I_{m-1}(\xi; t_1, t_2, \dots, t_{m-1}) d\xi,$$

with $I_0(z) = 1$ and $w_0(z) = 1$. Note first that

$$\begin{aligned} I_1(z; t_1) &= \sum_{n_1=0}^{\infty} \frac{1}{2\pi i} \oint \frac{t_1^{n_1} w_1(\xi)}{(\xi - z)^{n_1+1}} d\xi \\ &= \frac{1}{2\pi i} \oint \frac{w_1(\xi)}{\xi - z - t_1} d\xi, \quad \left| \frac{t_1}{\xi - z} \right| < 1, \\ &= w_1(z + t_1). \end{aligned}$$

Now, assume that

$$\begin{aligned} I_{m-1}(z) &:= I_{m-1}(z; t_1, t_1, \dots, t_{m-1}) \\ &= \prod_{i=1}^{m-1} w_i(z + t_{m-1} + \cdots + t_i) w_{i-1}^{-1}(z + t_{m-1} + \cdots + t_i). \end{aligned}$$

Then, by induction,

$$\begin{aligned} I_m(z; t_1, t_2, \dots, t_m) &= \sum_{n_m=0}^{\infty} \frac{1}{2\pi i} \oint \frac{t_m^{n_m} w_m(\xi) w_{m-1}^{-1}(\xi)}{(\xi - z)^{n_m+1}} I_{m-1}(\xi) d\xi \\ &= \frac{1}{2\pi i} \oint \frac{w_m(\xi) w_{m-1}^{-1}(\xi)}{\xi - z - t_m} I_{m-1}(\xi) d\xi, \quad \left| \frac{t_m}{\xi - z} \right| < 1, \\ &= w_m(z + t_m) w_{m-1}^{-1}(z + t_m) I_{m-1}(z + t_m) \end{aligned}$$

from which the first part of equation (5) is obtained. A simple calculation shows the second part of the equation (5) and the special case of the equation (6). ■

By the same method, we can obtain a generating function for multiple Laguerre polynomials.

THEOREM 2.2 Let $\{L_{\vec{n}}^{(\vec{\alpha}; \vec{\beta})}(x)\}$ be the multiple Laguerre I polynomials defined by equation (2). Then the generating function is given by

$$\begin{aligned} G(x; \mathbf{t}) &= w_r^{-1}(x) \prod_{k=1}^r \frac{1}{1 - t_k} w_{k-1}^{-1}\left(\frac{x}{(1 - t_k) \cdots (1 - t_r)}\right) w_k\left(\frac{x}{(1 - t_k) \cdots (1 - t_r)}\right) \\ &= \left[\prod_{i=1}^r \frac{1}{(1 - t_i)^{\alpha_i+1}} \right] \exp\left(\beta x \left(\frac{1}{(1 - t_1)(1 - t_2) \cdots (1 - t_r)} - 1\right)\right), \end{aligned}$$

where $w_0 = 1$ and w_i , $1 \leq i \leq r$, are the Laguerre I weights. In particular, if $r = 2$, then

$$G(x; t_1, t_2) = \frac{1}{(1 - t_1)^{\alpha_1+1} (1 - t_2)^{\alpha_2+1}} \exp\left(\frac{\beta(t_1 + t_2 - t_1 t_2)x}{(1 - t_1)(1 - t_2)}\right).$$

Proof The proof is just the same as that of Theorem 2.1. ■

THEOREM 2.3 Let $\{L_{\vec{n}}^{(\alpha; \vec{\beta})}(x)\}$ be the multiple Laguerre II polynomials defined by equation (3). Then the generating function is given by

$$\begin{aligned} G(x; \mathbf{t}) &= \frac{1}{1 - |\mathbf{t}|} \prod_{k=1}^r w_k^{-1}\left(\frac{(1 - \sum_{i=1}^k t_i)x}{1 - |\mathbf{t}|}\right) w_k\left(\frac{(1 - \sum_{i=1}^{k-1} t_i)x}{1 - |\mathbf{t}|}\right) \\ &= \frac{1}{(1 - |\mathbf{t}|)^{\alpha+1}} \exp\left(\frac{\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_r t_r}{1 - |\mathbf{t}|} x\right), \end{aligned}$$

where $|\mathbf{t}| = t_1 + t_2 + \cdots + t_r$ and w_i , $1 \leq i \leq r$ are the Laguerre II weights. In particular, if $r = 2$, then

$$G(x; t_1, t_2) = \frac{1}{(1 - t_1 - t_2)^{\alpha+1}} \exp\left(\frac{\beta_1 t_1 + \beta_2 t_2}{1 - t_1 - t_2} x\right).$$

Proof The proof is just the same as that of Theorem 2.1. ■

A simple calculation shows that

$$G_x = \delta(t_1 + t_2 + \cdots + t_r) G \quad (7)$$

and

$$G_{t_i} = (\delta x + \alpha_i) G + G_x, \quad (8)$$

where $G_x = (dG/dx)$, $G_{t_i} = (\partial G/\partial t_i)$, and $G = G(x; \mathbf{t})$ are the generating functions of multiple Hermite polynomials. Hence, we obtain the following.

THEOREM 2.4 For the multiple Hermite polynomials $\{H_n^{(\alpha)}(x)\}$, we have

$$\frac{d}{dx} H_n^{(\vec{\alpha})}(x) = \delta \sum_{i=1}^r H_{\vec{n}-e_i}^{(\vec{\alpha})}(x) = n_i H_{\vec{n}+e_i}^{(\vec{\alpha})}(x) - (\delta x + \alpha_i) H_n^{(\alpha)}(x). \quad (9)$$

Proof From equation (7), we have

$$\begin{aligned} \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \frac{d}{dx} H_n^{(\alpha)}(x) \mathbf{t}^{\vec{n}} &= G_x(x; \mathbf{t}) = \delta(t_1 + t_2 + \cdots + t_r) G(x; \mathbf{t}) \\ &= \delta \sum_{\vec{n}=0}^{\infty} \sum_{i=1}^r t_i \frac{1}{\vec{n}!} H_n^{(\alpha)}(x) \mathbf{t}^{\vec{n}} \\ &= \delta \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \sum_{i=1}^r H_{\vec{n}-e_i}^{(\alpha)}(x) \mathbf{t}^{\vec{n}} \end{aligned}$$

so that the first part of equation (9) is proved. The second part can be proved by the same method with equation (8) and

$$G_{t_i} = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} n_i H_{\vec{n}+e_i}^{(\vec{\alpha})}(x) \mathbf{t}^{\vec{n}},$$

where $G_{t_i} = (\partial G(x; \mathbf{t}) / \partial t_i)$. ■

By Theorem 2.4, we obtain an interesting recurrence relation

$$n_i H_{\vec{n}+e_i}^{(\vec{\alpha})}(x) = (\delta x + \alpha_i) H_{\vec{n}}^{(\vec{\alpha})}(x) + \delta \sum_{j=1}^r H_{\vec{n}-e_j}^{(\vec{\alpha})}(x)$$

which seems to be a two-term recurrence relation. The first part of equation (9) in Theorem 2.4 is very similar to that of classical Hermite polynomials (see [7]). In fact, we can obtain a relation between the multiple Hermite polynomials and the classical Hermite polynomials.

THEOREM 2.5 For the multiple Hermite polynomials $\{H_n^{(\alpha)}(x)\}$, we have

$$H_m\left(\frac{|\vec{\alpha}|}{2} - \frac{x}{r}\right) = \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{r-1}=0}^{m-|k|} \frac{m!}{(m-|k|-k_{r-1})! k_1! k_2! \cdots k_{r-1}!} \frac{H_{(m-|k|-k_{r-1}, k_1, \dots, k_{r-1})}^{(\vec{\alpha})}(x)}{m!},$$

where $|k| = k_1 + \cdots + k_{r-2}$ and $H_m(x)$ is the classical Hermite polynomials defined by

$$H_m(x) = \frac{1}{m!} e^{x^2} (e^{-x^2})^{(m)}.$$

In particular, if $r = 2$, then

$$H_m\left(\frac{\alpha_1 + \alpha_2 - x}{2}\right) = \sum_{k=0}^m \binom{m}{k} H_{(k, m-k)}^{(\alpha_1, \alpha_2)}(x).$$

Proof If we let $t_1, t_2, \dots, t_r \rightarrow t$ and take $\delta = -(2/r^2)$ in equation (5), then the generating function for multiple Hermite polynomials becomes

$$G(x; t) = e^{-t^2 + 2((|\vec{\alpha}|/2) - (x/r))t}.$$

Since the generating function for the classical Hermite polynomials $\{H_m(x)\}$ is

$$G^H(x; t) = e^{-t^2 + 2xt},$$

we have

$$G(x; t) = G^H\left(\frac{|\vec{\alpha}|}{2} - \frac{x}{r}; t\right).$$

From the definition of generating function, we have

$$\begin{aligned} G(x; t) &= \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} H_{\vec{n}}^{(\alpha)}(x) t^{|\vec{n}|} = \sum_{m=0}^{\infty} \sum_{k_1=0}^m \cdots \sum_{k_{r-1}=0}^{m-|k|} \frac{H_{(m-|k|-k_{r-1}, k_1, \dots, k_{r-1})}^{(\vec{\alpha})}(x)}{(m-|k|-k_{r-1})! k_1! \cdots k_{r-1}!} t^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} H_m\left(\frac{|\vec{\alpha}|}{2} - \frac{x}{r}\right) t^m, \end{aligned}$$

so that the conclusion follows by comparing the coefficients. ■

By the same process, we can obtain a similar result for multiple Laguerre polynomials. Here, we need a lemma.

LEMMA 2.6 $S_k := \sum_{j=[(k-1/r)+1]}^m (-1)^j \binom{m}{j} \binom{rj}{k} = 0$ for $k = 0, 1, 2, \dots, m-1$, where $[x]$ is the greatest integer not greater than x .

Proof First note that $T_i := \sum_{j=0}^m (-1)^j \binom{m}{j} j^i = 0$ for $i = 0, 1, \dots, m-1$. Let $k = rp + q$, where $0 \leq q < r$. Then

$$\begin{aligned} S_k &= \sum_{j=[(k-1/r)+1]}^m (-1)^j \binom{m}{j} \binom{rj}{k} \\ &= \begin{cases} \sum_{j=p+1}^m (-1)^j \binom{m}{j} \binom{rj}{k} & \text{if } q \geq 1 \\ \sum_{j=p}^m (-1)^j \binom{m}{j} \binom{rj}{k} & \text{if } q = 0. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} \binom{rj}{k} &= \frac{1}{k!} rj(rj-1) \cdots (rj-r) \cdots (rj-k+1) \\ &= \frac{1}{k!} \begin{cases} r^p j(j-1)(j-2) \cdots (j-p) \alpha(j) & \text{if } q \geq 1 \\ r^{p-1} j(j-1)(j-2) \cdots (j-p+1) \alpha(j) & \text{if } q = 0, \end{cases} \end{aligned}$$

where $\alpha(j)$ is a polynomial of degree $k - p + \delta_{0,q}$, we have

$$S_k = \sum_{j=0}^m (-1)^j \binom{m}{j} \binom{rj}{k},$$

which can be written by a linear combination of T_i , $i = 0, 1, 2, \dots, k$. Hence, the conclusion follows. ■

THEOREM 2.7 For the multiple Laguerre I polynomials $\{L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)\}$, we have

$$\begin{aligned} \frac{d}{dx} L_{\vec{n}}^{(\vec{\alpha};\beta)}(x) &= \beta [L_{\vec{n}}^{(\vec{\alpha}+1;\beta)}(x) - L_{\vec{n}}^{(\vec{\alpha};\beta)}(x)] \\ n_i L_{\vec{n}+e_i}^{(\vec{\alpha};\beta)}(x) &= (\beta x + \alpha_i + 1) L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x) + x \frac{d}{dx} L_{\vec{n}}^{(\vec{\alpha}+e_i;\beta)}(x) \\ &= (\alpha_i + 1) L_{\vec{n}}^{(\vec{\alpha}+e_i)}(x) + \beta x L_{\vec{n}}^{(\vec{\alpha}+1+e_i;\beta)}(x) \end{aligned} \quad (10)$$

and

$$\begin{aligned} \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{r-1}=0}^{m-|k|} \frac{L_{(m-|k|-k_{r-1}, k_1, \dots, k_{r-1})}^{(\vec{\alpha};-1)}(x)}{(m-|k|-k_{r-1})! k_1! \cdots k_{r-1}!} \\ = \sum_{k=[m-1/r]+1}^m \sum_{j=[m-1/r]+1}^k \frac{(-1)^{j+m}}{k!} \binom{k}{j} \binom{rj}{m} L_k^{(|\vec{\alpha}|/r)}(x), \end{aligned} \quad (11)$$

where $|k| = k_1 + k_2 + \cdots + k_{r-2}$ and $\{L_m^{(\alpha)}(x)\}$ is the classical Laguerre polynomials defined by

$$L_m^{(\alpha)}(x) = \frac{1}{m!} x^{-\alpha} e^x (x^{m+\alpha} e^{-x})^{(m)}. \quad (12)$$

In particular, if $r = 2$, then

$$\frac{1}{m!} \sum_{k=0}^m \binom{m}{k} L_{(k, m-k)}^{(\alpha_1, \alpha_2; -1)}(x) = \sum_{k=[m+1/2]}^m \frac{(-1)^{m+k} 2^{2k-m}}{(m-k)!(2k-m)!} L_k^{(\alpha_1+\alpha_2/2)}(x). \quad (13)$$

Proof Let $G = G^{(\vec{\alpha};\beta)}(x; \mathbf{t})$ be the generating function of the multiple Laguerre I polynomials. Then we have

$$G_x = \beta \left(\frac{1}{(1-t_1)(1-t_2) \cdots (1-t_r)} - 1 \right) G \quad (14)$$

and

$$G_{t_i} = \frac{(\beta x + \alpha_i + 1)}{1-t_i} G + \frac{x}{1-t_i} G_x, \quad 1 \leq i \leq r, \quad (15)$$

from which equation (10) follows. Since the generating function of the classical Laguerre polynomials $\{L_m^{(\alpha)}(x)\}$ (see [7]) is given by

$$G^{(\alpha)}(x; t) = \frac{1}{(1-t)^{\alpha+1}} e^{-(tx/(1-t))}, \quad (16)$$

letting $t_1, t_2, \dots, t_r \rightarrow t$ and $\beta = -1$, we have

$$G^{(|\vec{\alpha}|/r)}(x; 1 - (1 - t)^r) = G.$$

From the definition of generating function,

$$\begin{aligned} G^{(|\vec{\alpha}|/r)}(x; 1 - (1 - t)^r) &= \sum_{m=0}^{\infty} \frac{1}{m!} L_m^{(|\vec{\alpha}|/r)}(x) (1 - (1 - t)^r)^m \\ &= \sum_{m=0}^{\infty} \sum_{j=0}^m \sum_{k=0}^{rj} \frac{(-1)^{j+k}}{m!} \binom{m}{j} \binom{rj}{k} L_m^{(|\vec{\alpha}|/r)}(x) t^k \\ &= \sum_{m=0}^{\infty} \sum_{k=0}^{rm} \sum_{j=\lfloor \frac{k-1}{r} \rfloor + 1}^m \frac{(-1)^{j+k}}{m!} \binom{m}{j} \binom{rj}{k} L_m^{(|\vec{\alpha}|/r)}(x) t^k. \end{aligned}$$

By Lemma 2.6, $S_k := \sum_{j=\lfloor k-1/r \rfloor + 1}^m (-1)^j \binom{m}{j} \binom{rj}{k} = 0$ for $k = 0, 1, \dots, m-1$ so that

$$\begin{aligned} G^{(|\vec{\alpha}|/r)}(x; 1 - (1 - t)^r) &= \sum_{m=0}^{\infty} \sum_{k=m}^{rm} \sum_{j=\lfloor k-1/r \rfloor + 1}^m \frac{(-1)^{j+k}}{m!} \binom{m}{j} \binom{rj}{k} L_m^{(|\vec{\alpha}|/r)}(x) t^k \\ &= \sum_{m=0}^{\infty} \sum_{k=\lfloor m-1/r \rfloor + 1}^m \sum_{j=\lfloor m-1/r \rfloor + 1}^k \frac{(-1)^{j+m}}{k!} \binom{k}{j} \binom{rj}{m} L_k^{(|\vec{\alpha}|/r)}(x) t^m. \end{aligned}$$

Equation (11) follows by comparing the coefficients. If $r = 2$, then (see [8])

$$\sum_{j=\lfloor \frac{m+1}{2} \rfloor}^k (-1)^j \binom{k}{j} \binom{2j}{m} = \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{2j}{m} = (-1)^k \frac{2^{2k-m} k!}{(m-k)!(2k-m)!}$$

so that equation (13) follows. ■

THEOREM 2.8 For the multiple Laguerre II polynomials $\{L_{\vec{n}}^{(\alpha; \vec{\beta})}(x)\}$, we have

$$\begin{aligned} \frac{d}{dx} L_{\vec{n}}^{(\alpha; \vec{\beta})}(x) &= \sum_{i=1}^r \beta_i L_{\vec{n}-e_i}^{(\alpha-1; \vec{\beta})}(x) \\ \frac{d}{dx} L_{\vec{n}}^{(\alpha; \vec{\beta})}(x) - \sum_{i=1}^r \frac{d}{dx} L_{\vec{n}-e_i}^{(\alpha; \vec{\beta})}(x) &= \sum_{i=1}^r \beta_i L_{\vec{n}-e_i}^{(\alpha; \vec{\beta})}(x) \\ n_i L_{\vec{n}+e_i}^{(\alpha; \vec{\beta})}(x) - \sum_{j=1}^r n_j L_{\vec{n}-e_j+e_i}^{(\alpha; \vec{\beta})}(x) &= (\beta_i x + \alpha + 1) L_{\vec{n}}^{(\alpha; \vec{\beta})}(x) + x \frac{d}{dx} L_{\vec{n}}^{(\alpha; \vec{\beta})}(x) \end{aligned} \quad (17)$$

and

$$L_m^{(\alpha)}(-|\vec{\beta}|x) = \frac{1}{r^m} \sum_{k_1=0}^m \sum_{k_2=0}^{m-k_1} \cdots \sum_{k_{r-1}=0}^{m-|k|} \frac{m! L_{(m-|k|-k_{r-1}, k_1, \dots, k_{r-1})}^{(\alpha; \vec{\beta})}(x)}{(m-|k|-k_{r-1})! k_1! \cdots k_{r-1}!}, \quad (18)$$

where $|k| = k_1 + k_2 + \cdots + k_{r-2}$ and $\{L_m^{(\alpha)}(x)\}$ is the classical Laguerre polynomials defined by equation (12). In particular, if $r = 2$, then

$$L_m^{(\alpha)}(-(\beta_1 + \beta_2)x) = 2^{-m} \sum_{k=0}^m \binom{m}{k} L_{(k, m-k)}^{(\alpha; \beta_1, \beta_2)}(x).$$

Proof Let $G = G^{(\alpha; \vec{\beta})}(x; \mathbf{t})$ be the generating function of the multiple Laguerre II polynomials. Then we have

$$G_x = \frac{\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_r t_r}{1 - t_1 - t_2 - \cdots - t_r} G$$

or equivalently

$$(1 - t_1 - t_2 - \cdots - t_r)G_x = (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_r t_r)G \quad (19)$$

and

$$(1 - t_1 - t_2 - \cdots - t_r)G_{t_i} = (\beta_i x + \alpha + 1)G + xG_x, \quad 1 \leq i \leq r, \quad (20)$$

from which equation (17) follows. Since the generating function of the classical Laguerre polynomials $\{L_m^{(\alpha)}(x)\}$ is given by equation (16), letting $t_1, t_2, \dots, t_r \rightarrow t$, we have

$$G(x; t) = G^{(\alpha)}(-(|\vec{\beta}|)x; rt)$$

so that

$$\sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} L_{\vec{n}}^{(\alpha; \vec{\beta})}(x) t^{|\mathbf{n}|} = \sum_{m=0}^{\infty} \frac{1}{m!} L_m^{(\alpha)}(-(|\vec{\beta}|)x) r^m t^m.$$

Equation (18) follows by comparing the coefficients. The case of $r = 2$ can be proved easily. ■

3. Differential equations

THEOREM 3.1 *The multiple Hermite polynomials $\{H_{\vec{n}}^{(\alpha)}(x)\}$ satisfy a differential equation*

$$\begin{aligned} & \delta(|\vec{n}| + r - 1)A_2 A_3 \cdots A_r(H_n) \\ &= (A_2 A_3 \cdots A_r(H_n))_{xx} + (\delta x + \alpha_1)(A_2 A_3 \cdots A_r(H_n))_x \\ &+ \delta \sum_{i=2}^r (n_i + 1)(\alpha_i - \alpha_1)(A_2 A_3 \cdots A_{i-1} A_{i+1} \cdots A_r(H_n)), \end{aligned} \quad (21)$$

where $H_n = H_{\vec{n}}^{(\alpha)}(x)$, and $A_i = (\delta x + \alpha_i) + (d/dx)$ with $A_0 = 1$.

Proof First note that the generating function $G = G(x; \mathbf{t})$ of multiple Hermite polynomials satisfies equations (7) and (8) so that

$$\sum_{i=1}^r t_i G_{t_i} = xG_x + \frac{1}{\delta} G_{xx} + \sum_{i=1}^r \alpha_i t_i G = \frac{1}{\delta} G_{xx} + \left(x + \frac{\alpha_1}{\delta}\right) G_x + \sum_{i=2}^r (\alpha_i - \alpha_1) t_i G. \quad (22)$$

Differentiating both sides by $t_2, t_3, \dots, t_r$, we have

$$\sum_{i=1}^r (t_i G_{t_i})_{t_2 t_3 \cdots t_r} = \left(x + \frac{\alpha_1}{\delta}\right) G_{x t_2 \cdots t_r} + \frac{1}{\delta} G_{x x t_2 \cdots t_r} + \sum_{i=2}^r (\alpha_i - \alpha_1) (t_i G)_{t_2 t_3 \cdots t_r}$$

or equivalently

$$\begin{aligned} \delta t_1(G_{t_2 t_3 \dots t_r})_{t_1} + \delta \sum_{i=2}^r (t_i G_{t_2 \dots t_r})_{t_i} &= (\delta x + \alpha_1)(G_{t_2 \dots t_r})_x + (G_{t_2 \dots t_r})_{xx} \\ &+ \delta \sum_{i=2}^r (\alpha_i - \alpha_1)(t_i G_{t_2 \dots t_{i-1} t_{i+1} \dots t_r})_{t_i}. \end{aligned}$$

Since $G_{t_2 \dots t_r} = A_2 A_3 \dots A_r(G)$, we have

$$t_1(G_{t_2 \dots t_r})_{t_1} = \sum_{\vec{n}=0}^{\infty} \frac{n_1}{\vec{n}!} A_2 A_3 \dots A_r(H_n) \mathbf{t}^{\vec{n}}$$

and

$$(t_i G_{t_2 \dots t_{i-1} t_{i+1} \dots t_r})_{t_i} = \sum_{\vec{n}=0}^{\infty} \frac{n_i + 1}{\vec{n}!} A_2 \dots A_{i-1} A_{i+1} \dots A_r(H_n) \mathbf{t}^{\vec{n}}.$$

Hence, by comparing the coefficients, we obtain

$$\begin{aligned} \delta n_1 A_2 \dots A_r(H_n) + \delta \sum_{i=2}^r (n_i + 1) A_2 \dots A_r(H_n) \\ = (A_2 A_3 \dots A_r(H_n))_{xx} + (\delta x + \alpha_1)(A_2 A_3 \dots A_r(H_n))_x \\ + \delta \sum_{i=2}^r (\alpha_i - \alpha_1)(n_i + 1) A_2 \dots A_{i-1} A_{i+1} \dots A_r(H_n) \end{aligned}$$

from which equation (21) follows. ■

If $r = 2$, then equation (21) becomes

$$\delta(n_1 + n_2 + 1)A_2(H_n) = A_2(H_n)'' + (\delta x + \alpha_1)A_2(H_n)' + (n_2 + 1)(\alpha_2 - \alpha_1)H_n.$$

Since $A_2(H_n) = (\delta x + \alpha_2)H_n + H_n'$, the differential equation becomes

$$\begin{aligned} L[H_n] := H_n''' + (2\delta x + \alpha_1 + \alpha_2)H_n'' + [(\delta x + \alpha_1)(\delta x + \alpha_2) - \delta(n_1 + n_2 - 1)]H_n' \\ - [(n_1 + n_2)\delta x + n_1\alpha_2 + n_2\alpha_1]H_n = 0, \end{aligned} \quad (23)$$

which was already found in [5].

Differentiating equation (22) by $t_1, t_2, \dots, t_r$, we obtain

$$\sum_{i=1}^r (t_i G_{t_i})_{t_1 \dots t_r} = \frac{1}{\delta} G_{xx t_1 \dots t_r} + x G_{x t_1 \dots t_r} + \sum_{i=1}^r \alpha_i (t_i G)_{t_1 \dots t_r}.$$

Since $(t_i G_{t_i})_{t_1 \dots t_r} = (t_i G_{t_1 \dots t_{i-1} t_{i+1} \dots t_r})_{t_i}$, by the same process as the proof of Theorem 3.1, we obtain another differential equation

$$\begin{aligned} \sum_{i=1}^r (n_i + 1) A_1 \dots A_r(H_n) &= \frac{1}{\delta} (A_1 \dots A_r(H_n))_{xx} + x (A_1 \dots A_r(H_n))_x \\ &+ \sum_{i=1}^r \alpha_i (n_i + 1) A_1 \dots A_{i-1} A_{i+1} \dots A_r(H_n). \end{aligned} \quad (24)$$

But, it is not easy to check whether the differential equations (21) and (24) are independent, that is, one equation can be written by a linear combination and iteration of the other equation.

When $r = 2$, then the differential equation (24) becomes

$$\begin{aligned} D[H_n] &:= (A_1 A_2(H_n))_{xx} + \delta x (A_1 A_2(H_n))_x + \alpha_1(n_1 + 1)\delta A_2(H_n) \\ &\quad + \alpha_2(n_2 + 1)\delta A_1(H_n) - (n_1 + n_2 + 2)A_1 A_2(H_n). \end{aligned}$$

Since $A_i(H_n) = (\delta x + \alpha_i)H_n + H'_n$, we obtain

$$\begin{aligned} D[H_n] &= H_n^{(iv)} + (3\delta x + \alpha_1 + \alpha_2)H_n''' \\ &\quad + [\delta x(2\delta x + \alpha_1 + \alpha_2) + (\delta x + \alpha_1) + (\delta x + \alpha_2) - (n_1 + n_2 - 3)\delta]H_n'' \\ &\quad + [\delta x(\delta x + \alpha_1)(\delta x + \alpha_2) + 3\delta^2 x + 2\delta(\delta x + \alpha_1) + 2\delta(\delta x + \alpha_2) \\ &\quad + \alpha_1(n_1 + 1)\delta + \alpha_2(n_2 + 1)\delta - \delta(n_1 + n_2 + 2)(2\delta x + \alpha_1 + \alpha_2)]H_n' \\ &\quad + \delta[\delta x(2\delta x + \alpha_1 + \alpha_2) + 2\delta + \alpha_1(n_1 + 1)(\delta x + \alpha_2) + \alpha_2(n_2 + 1)(\delta x + \alpha_1) \\ &\quad - (n_1 + n_2 + 2)(\delta x + \alpha_1)(\delta x + \alpha_2) - (n_1 + n_2 + 2)\delta]H_n = 0. \end{aligned}$$

By a direct calculation, we can prove that

$$D[H_n] = L[H_n]' + \delta x L[H_n],$$

which is a combination of the operators L and L' in equation (23).

THEOREM 3.2 *The multiple Laguerre I polynomials $\{L_n^{(\vec{\alpha}; \beta)}(x)\}$ satisfy a differential equation*

$$\begin{aligned} \left(\frac{d}{dx} + \beta\right)A_1 A_2 \cdots A_r(L_n) &= \beta F_1 F_2 \cdots F_r(L_n) \\ &\quad + \sum_{k=1}^r \left(\frac{d}{dx} + \beta\right)(A_1 - 1)(A_2 - 1) \cdots (A_{k-1} - 1)A_{k+1} \cdots A_r(L_n), \end{aligned} \quad (25)$$

where $L_n = L_n^{(\vec{\alpha}; \beta)}(x)$, $A_i = (\beta x + \alpha_i + 1) + x(d/dx)$, and $F_i = n_i + A_i = n_i + (\beta x + \alpha_i + 1) + x(d/dx)$.

Proof First note that the generating function $G = G(x; \mathbf{t})$ of multiple Laguerre I polynomials satisfies the equations (14) and (15). Let's define $B_i = (\partial/\partial t_i)$, $\tilde{A}_i = (1 - t_i)B_i = (1 - t_i)(\partial/\partial t_i)$, and $\tilde{F}_i = t_i B_i + (\beta x + \alpha_i + 1) + x(d/dx)$. Then the following can be easily proved.

$$\begin{aligned} \tilde{A}_i(G) &= A_i(G), \quad \tilde{A}_i(G_x) = (A_i(G))_x, \quad \tilde{A}_i(G) = B_i((1 - t_i)G) + G \\ \tilde{A}_i B_j(G) &= B_j \tilde{A}_i(G), \quad \tilde{A}_i B_j((1 - t_j)G) = B_j \tilde{A}_i((1 - t_j)G), \quad i \neq j \\ \tilde{A}_i \tilde{A}_j(G) &= A_i A_j(G), \quad B_i(G) = \tilde{F}_i(G), \quad \tilde{F}_i(G) = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} F(H_n) \mathbf{t}^{\vec{n}}. \end{aligned}$$

Using these results with $(1 - t_1)(1 - t_2) \cdots (1 - t_r)G_{t_1 t_2 \cdots t_r} = \tilde{A}_r \tilde{A}_{r-1} \cdots \tilde{A}_1(G)$, we obtain

$$\begin{aligned} \tilde{A}_r \tilde{A}_{r-1} \cdots \tilde{A}_1(G) &= \tilde{A}_r \tilde{A}_{r-1} \cdots \tilde{A}_2(B_1((1 - t_1)G) + G) \\ &= B_1 \tilde{A}_r \tilde{A}_{r-1} \cdots \tilde{A}_2(G) + \tilde{A}_r \tilde{A}_{r-1} \cdots \tilde{A}_2(G) \\ &= B_1 \cdots B_r((1 - t_1)(1 - t_2) \cdots (1 - t_i)G) + \tilde{A}_2 \tilde{A}_3 \cdots \tilde{A}_r(G) \\ &\quad + \sum_{i=1}^{r-1} B_1 B_2 \cdots B_{i-1} \tilde{A}_{i+1} \tilde{A}_{i+2} \cdots \tilde{A}_r((1 - t_1)(1 - t_2) \cdots (1 - t_i)G), \end{aligned}$$

and so

$$\begin{aligned}
 & B_1 B_2 \cdots B_r ((1-t_1)(1-t_2) \cdots (1-t_r)G_x) \\
 &= B_1 B_2 \cdots B_r (\beta G) - \beta B_1 B_2 \cdots B_r ((1-t_1)(1-t_2) \cdots (1-t_r)G) \\
 &= \beta \tilde{F}_1 \cdots \tilde{F}_r(G) - \beta \left[\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_r(G) \right. \\
 &\quad \left. - \sum_{k=1}^r B_1 \cdots B_{k-1} \tilde{A}_{k+1} \cdots \tilde{A}_r ((1-t_1)(1-t_2) \cdots (1-t_{k-1})G) \right],
 \end{aligned}$$

where A_i is the identity operator if $i \geq r$. By the same method, we have

$$\begin{aligned}
 \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_r(G_x) &= B_1 B_2 \cdots B_r ((1-t_1)(1-t_2) \cdots (1-t_r)G_x) \\
 &\quad + \sum_{k=1}^r B_1 \cdots B_{k-1} \tilde{A}_{k+1} \cdots \tilde{A}_r ((1-t_1)(1-t_2) \cdots (1-t_r)G_x) \\
 &= \beta \tilde{F}_1 \cdots \tilde{F}_r(G) - \beta \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_r(G) + \sum_{k=1}^r B_1 B_2 \cdots B_{k-1} \\
 &\quad \times \tilde{A}_{k+1} \cdots \tilde{A}_r ((1-t_1)(1-t_2) \cdots (1-t_{k-1})(G_x + \beta G))
 \end{aligned}$$

so that

$$\begin{aligned}
 \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_r(G_x + \beta G) &= \beta \tilde{F}_1 \tilde{F}_2 \cdots \tilde{F}_r(G) \\
 &\quad + \sum_{k=1}^r B_1 \cdots B_{k-1} \tilde{A}_{k+1} \cdots \tilde{A}_r ((1-t_1)(1-t_2) \cdots (1-t_{k-1})(G_x + \beta G)).
 \end{aligned} \tag{26}$$

Since $B_i((1-t_i)G) = (1-t_i)G_{t_i} - G = (A_i - 1)(G)$, we have by the definition of generating function,

$$\begin{aligned}
 & B_1 \cdots B_{k-1} \tilde{A}_{k+1} \cdots \tilde{A}_r ((1-t_1)(1-t_2) \cdots (1-t_{k-1})G) \\
 &= \tilde{A}_{k+1} \cdots \tilde{A}_r (A_1 - 1)(A_2 - 1) \cdots (A_{k-1} - 1)(G) \\
 &= \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} (A_1 - 1)(A_2 - 1) \cdots (A_{k-1} - 1) A_{k+1} \cdots A_r(L_n) \mathbf{t}^{\vec{n}}
 \end{aligned} \tag{27}$$

and similarly

$$B_1 \cdots B_{k-1} \tilde{A}_{k+1} \cdots \tilde{A}_r(G_x) = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \frac{d}{dx} [(A_1 - 1) \cdots (A_{k-1} - 1) A_{k+1} \cdots A_r(L_n)] \mathbf{t}^{\vec{n}}. \tag{28}$$

Moreover,

$$\tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_r(G_x + \beta G) = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \left(\frac{d}{dx} + \beta \right) [A_1 A_2 \cdots A_r(L_n)] \mathbf{t}^{\vec{n}} \tag{29}$$

and

$$\tilde{F}_1 \tilde{F}_2 \cdots \tilde{F}_r(G) = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \tilde{F}_1 \tilde{F}_2 \cdots \tilde{F}_r(L_n) \mathbf{t}^{\vec{n}} = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} F_1 F_2 \cdots F_r(L_n) \mathbf{t}^{\vec{n}}. \tag{30}$$

Hence, equation (25) follows by comparing the coefficients of equation (26) using equations (27)–(30). ■

In particular, if $r = 2$, then the differential equation (25) becomes

$$\left(\frac{d}{dx} + \beta\right)A_1A_2(L_n) = \beta F_1F_2(L_n) + \left(\frac{d}{dx} + \beta\right)A_2(L_n) + \left(\frac{d}{dx} + \beta\right)(A_1 - 1)(L_n)$$

so that

$$x^2L_n''' + x(2\beta x + \alpha_1 + \alpha_2 + 3)L_n'' + [(\beta x + \alpha_1 + 1)(\beta x + \alpha_2 + 1) + (1 - n_1 - n_2)\beta x]L_n' - [\beta n_1n_2 + \beta n_1(\beta x + \alpha_2 + 1) + \beta n_2(\beta x + \alpha_1 + 1)]L_n = 0,$$

which was already found in [5].

In order to find a differential equation for multiple Laguerre II polynomials, we need the following lemma.

LEMMA 3.3 *Let $G = G^{(\alpha; \vec{\beta})}(x; \mathbf{t})$ be the generating function of the multiple Laguerre II polynomials. Then for $1 \leq i_1, i_2, \dots, i_m \leq r$, the operator*

$$H_{(i_1, i_2, \dots, i_m)}(G) := \left((1 - t_1 - t_2 - \dots - t_r)^m G \right)_{t_{i_1} t_{i_2} \dots t_{i_m}}$$

can be written by

$$H_{(i_1, i_2, \dots, i_m)}(G) = x^m \frac{d^m G}{dx^m} + a_{m-1}(x) \frac{d^{m-1} G}{dx^{m-1}} + \dots + a_1(x) \frac{dG}{dx} + a_0(x) \quad (31)$$

where a_i 's are polynomials.

Proof From the relation (20), it can be proved by induction. For $m = 1$,

$$H_{i_1}(G) = ((1 - t_1 - t_2 - \dots - t_r)G)_{t_{i_1}} = (\beta_{i_1}x + \alpha + 1)G + xG_{i_1}.$$

Now, assume that the theorem is true for $\leq m - 1$. Then

$$\begin{aligned} H_{(i_1, i_2, \dots, i_m)}(G) &= ((1 - t_1 - t_2 - \dots - t_r)^m G)_{t_{i_1} t_{i_2} \dots t_{i_m}} \\ &= ((1 - t_1 - t_2 - \dots - t_r)^{m-1} H_{i_m}(G))_{t_{i_1} t_{i_2} \dots t_{i_{m-1}}} \\ &\quad - m((1 - t_1 - t_2 - \dots - t_r)^{m-1} G)_{t_{i_1} t_{i_2} \dots t_{i_{m-1}}} \\ &= H_{(i_1, i_2, \dots, i_{m-1})}(H_{i_m}(G)) - mH_{(i_1, i_2, \dots, i_{m-1})}(G) \\ &= H_{i_m}(H_{(i_1, i_2, \dots, i_{m-1})}(G)) - mH_{(i_1, i_2, \dots, i_{m-1})}(G) \end{aligned}$$

from which the conclusion follows. ■

THEOREM 3.4 *The multiple Laguerre II polynomials $\{L_n^{(\alpha; \vec{\beta})}(x)\}$ satisfy a differential equation*

$$\begin{aligned} &\sum_{i=1}^r [(n_i + 1)H_{(1, 2, \dots, r)}(L_n) + (r - \alpha - 1)(n_i + 1)H_{(1, 2, \dots, i-1, i+1, \dots, r)}(L_n)] \\ &= (xH_{(1, 2, \dots, r)}(L_n))_x + \sum_{i=1}^r (n_i + 1)x(H_{(1, 2, \dots, i-1, i+1, \dots, r)}(L_n))_x, \end{aligned} \quad (32)$$

where $L_n = L_n^{(\alpha; \vec{\beta})}(x)$ and $H_{(i_1, \dots, i_m)}$ is the operator defined in equation (31).

Proof Note that the generating function of multiple Laguerre II polynomials satisfies the equations (19) and (20) so that

$$\sum_{i=1}^r (1 - t_1 - \cdots - t_r) t_i G_{t_i} = (1 - t_1 - \cdots - t_r) x G_x + \sum_{i=1}^r t_i (x G_x + (\alpha + 1) G).$$

Multiplying $(1 - t_1 - t_2 - \cdots - t_r)^{r-1}$ and differentiating on both sides, we obtain

$$\begin{aligned} \sum_{i=1}^r \left((1 - |\mathbf{t}|)^r t_i G_{t_i} \right)_{t_1 \cdots t_r} &= [(1 - |\mathbf{t}|)^r x G_x]_{t_1 t_2 \cdots t_r} \\ &+ \sum_{i=1}^r [(1 - |\mathbf{t}|)^{r-1} t_i (x G_x + (\alpha + 1) G)]_{t_1 \cdots t_r}, \end{aligned} \quad (33)$$

where $|\mathbf{t}| = t_1 + t_2 + \cdots + t_r$. By a direct calculation with Lemma 3.3, the left hand side becomes

$$\begin{aligned} &\sum_{i=1}^r [(1 - |\mathbf{t}|)^r t_i G_{t_i}]_{t_1 \cdots t_r} \\ &= \sum_{i=1}^r [t_i ((1 - |\mathbf{t}|)^r G)_{t_1 \cdots t_r} + r t_i ((1 - |\mathbf{t}|)^{r-1} G)_{t_1 \cdots t_{i-1} t_{i+1} \cdots t_r}]_{t_i} \\ &= \sum_{i=1}^r [t_i H_{(1,2,\dots,r)}(G) + t_i r H_{(1,2,\dots,i-1,i+1,\dots,r)}(G)]_{t_i} \\ &= \sum_{\vec{n}=0}^{\infty} \sum_{i=1}^r \frac{1}{\vec{n}!} [(n_i + 1) H_{(1,2,\dots,r)}(L_n) + r(n_i + 1) H_{(1,2,\dots,i-1,i+1,\dots,r)}(L_n)] \mathbf{t}^{\vec{n}}. \end{aligned} \quad (34)$$

Since

$$x[(1 - |\mathbf{t}|)^r G]_{t_1 t_2 \cdots t_r x} = x(H_{(1,2,\dots,r)}(G))_x = \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} x(H_{(1,2,\dots,r)}(L_n))_x \mathbf{t}^{\vec{n}}$$

and

$$\begin{aligned} &\sum_{i=1}^r \left[t_i x \left((1 - |\mathbf{t}|)^{r-1} G_x \right)_{t_1 \cdots t_{i-1} t_{i+1} \cdots t_r} + t_i (\alpha + 1) \left((1 - |\mathbf{t}|)^{r-1} G \right)_{t_1 \cdots t_{i-1} t_{i+1} \cdots t_r} \right]_{t_i} \\ &= \sum_{i=1}^r \left[x t_i \left(H_{(1,\dots,i-1,i+1,\dots,r)}(G) \right)_x + (\alpha + 1) t_i H_{(1,\dots,i-1,i+1,\dots,r)}(G) \right]_{t_i} \\ &= \sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \left[\sum_{i=1}^r (n_i + 1) \left(x \frac{d}{dx} + \alpha + 1 \right) H_{(1,\dots,i-1,i+1,\dots,r)}(L_n) \right] \mathbf{t}^{\vec{n}}, \end{aligned}$$

the right hand side of equation (33) becomes

$$\sum_{\vec{n}=0}^{\infty} \frac{1}{\vec{n}!} \left[x(H_{(1,2,\dots,r)}(L_n))_x + \sum_{i=1}^r (n_i + 1) \left(x \frac{d}{dx} + \alpha + 1 \right) H_{(1,\dots,i-1,i+1,\dots,r)}(L_n) \right] \mathbf{t}^{\vec{n}}. \quad (35)$$

Hence, equation (32) follows by comparing the coefficients of equation (33) with equations (34) and (35). $\blacksquare$

When $r = 2$, the differential equation (32) becomes

$$(n_1 + n_2 + 2)H_{(1,2)}(L_n) + (1 - \alpha)(n_1 + 1)H_2(L_n) + (1 - \alpha)(n_2 + 1)H_1(L_n) \\ = x(H_{(1,2)}(L_n))_x + (n_1 + 1)x(H_2(L_n))_x + (n_2 + 1)(H_1(L_n))_x.$$

Since $H_1 = (\beta_1 x + \alpha) + x(d/dx)$, $H_2 = (\beta_2 x + \alpha) + x(d/dx)$, and $H_{(1,2)}(L_n) = H_1(H_2(L_n)) - H_2(L_n)$, we obtain a differential equation

$$x^2 L_n''' + x[(\beta_1 + \beta_2)x + 2\alpha + 2]L_n'' + [(\beta_1 x + \alpha)(\beta_2 x + \alpha) + (1 - n_2)(\beta_2 x + \alpha) \\ + (1 - n_1)(\beta_1 x + \alpha) + \alpha(n_1 + n_2 - 1)]L_n' - [\beta_1 \beta_2(n_1 + n_2)x \\ - \alpha(\beta_1 n_1 + \beta_2 n_2)]L_n = 0,$$

which was already found in [5].

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A generalization of classical symmetric orthogonal functions using a symmetric generalization of Sturm–Liouville problems

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In this research, usual Sturm–Liouville problems are extended for symmetric functions so that the corresponding solutions preserve the orthogonality property. Two basic examples as special samples of a generalized Sturm–Liouville problem are then introduced. The first example generalizes the associated Legendre functions having extensive applications in physics and engineering and the second example introduces a generic differential equation with various sub-cases having orthogonal solutions. For instance, this generic equation possesses a symmetric differential equation containing a basic solution of symmetric orthogonal polynomials.

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1. Introduction

Let us start our discussion with the boundary value problem

$$(k(x)y'_n(x))' + (\lambda_n \rho(x) - q(x))y_n(x) = 0; \quad k(x) > 0, \quad \rho(x) > 0, \quad (1)$$

defined on an open interval, say (a, b) , which has the boundary conditions

$$\begin{cases} \alpha_1 y_n(a) + \beta_1 y'_n(a) = 0, \\ \alpha_2 y_n(b) + \beta_2 y'_n(b) = 0, \end{cases} \quad (2)$$

where α_1, α_2 and β_1, β_2 are given constants and $k(x), k'(x), q(x)$ and $\rho(x)$ are to be assumed continuous for $x \in [a, b]$.

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In general, this problem is called a regular Sturm–Liouville problem. Also, if one of the boundary points a or b is singular (i.e. $k(a) = 0$ or $k(b) = 0$), the problem is called a singular Sturm–Liouville problem.

Suppose $y_n(x)$ and $y_m(x)$ are two sequences of the solutions (eigenfunctions) of equation (1). According to Sturm–Liouville theory [1, 2], they are orthogonal with respect to the weight function $\rho(x)$ on (a, b) under the given conditions (2) and we have

$$\int_a^b \rho(x) y_n(x) y_m(x) dx = \left(\int_a^b \rho(x) y_n^2(x) dx \right) \delta_{n,m} \quad \text{if} \quad \delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \quad (3)$$

Many important special functions in theoretical and mathematical physics are the solutions of a regular (or singular) Sturm–Liouville problem that satisfy the orthogonality condition (3). For instance, one can point to the associated Legendre functions [1], Bessel functions [1, 2], Fourier trigonometric sequences [3], ultraspherical functions [4, 5], Hermite functions [4, 5] and so on. Fortunately, most of these functions have the symmetry property, namely $y_n(-x) = (-1)^n y_n(x)$, and have found important applications in physics and engineering, see e.g. [1, 2] for more details. Therefore, if we can here generalize the mentioned examples symmetrically and preserve their orthogonality property, it seems that we will be able to find some new applications in physics and engineering that logically extend the previous applications. In this article, by doing this task, we introduce some generalized classical symmetric orthogonal functions and obtain their orthogonality properties.

2. A symmetric generalization of Sturm–Liouville problems

Without loss of generality, let $y = \Phi_n(x)$ be a sequence of symmetric functions that satisfies the following differential equation

$$A(x) \Phi_n''(x) + B(x) \Phi_n'(x) + \left(\lambda_n C(x) + D(x) + \frac{1 - (-1)^n}{2} E(x) \right) \Phi_n(x) = 0, \quad (4)$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$ are arbitrary functions and $\{\lambda_n\}$ is a sequence of constants. Clearly choosing $E(x) = 0$ in equation (4) is equivalent to the usual Sturm–Liouville equation (1). Since $\Phi_n(x)$ is a symmetric sequence, by substituting the property $\Phi_n(-x) = (-1)^n \Phi_n(x)$ in equation (4), one can conclude that the functions $A(x)$, $C(x)$, $D(x)$ and $E(x)$ are *even* while $B(x)$ must be an *odd* function. We will frequently use this note in the paper.

To prove the orthogonality property of $\Phi_n(x)$, first both sides of equation (4) should be multiplied by the positive function

$$R(x) = \exp \left(\int \frac{B(x) - A'(x)}{A(x)} dx \right) = \frac{1}{A(x)} \exp \left(\int \frac{B(x)}{A(x)} dx \right), \quad (5)$$

in order to convert it in the form of a self-adjoint equation. Note that $R(x)$ in relation (5) is an even function, because $B(x)$ is odd and $A(x)$ is even. Therefore, the self-adjoint form of equation (4) becomes

$$\begin{aligned} \frac{d}{dx} \left(A(x) R(x) \frac{d\Phi_n(x)}{dx} \right) &= -(\lambda_n C(x) + D(x)) R(x) \Phi_n(x) \\ &\quad - \frac{1 - (-1)^n}{2} E(x) R(x) \Phi_n(x). \end{aligned} \quad (6)$$

Since $A(x) R(x)$ is an even function, the orthogonality interval corresponding to equation (6) should be considered symmetrically, say $[-\theta, \theta]$. Hence, by assuming that $A(x) R(x)$ vanishes at $x = \theta$ and applying the Sturm–Liouville theorem on equation (6) we have

$$\begin{aligned} & \left[A(x) R(x) (\Phi'_n(x) \Phi_m(x) - \Phi'_m(x) \Phi_n(x)) \right]_{-\theta}^{\theta} \\ &= (\lambda_m - \lambda_n) \int_{-\theta}^{\theta} C(x) R(x) \Phi_n(x) \Phi_m(x) dx \\ & \quad - \left(\frac{(-1)^m - (-1)^n}{2} \right) \int_{-\theta}^{\theta} E(x) R(x) \Phi_n(x) \Phi_m(x) dx. \end{aligned} \quad (7)$$

Obviously the left-hand side of equality (7) is zero. So, to prove the orthogonality property, it is enough to show that the expression

$$F(n, m) = \frac{(-1)^m - (-1)^n}{2} \int_{-\theta}^{\theta} E(x) R(x) \Phi_n(x) \Phi_m(x) dx, \quad (8)$$

is always equal to zero for every $m, n \in \mathbb{Z}^+$. For this purpose, in general four cases should be considered for m and n :

- a) If both m and n are even (or odd), then $F(n, m) = 0$, because we have $F(2i, 2j) = F(2i + 1, 2j + 1) = 0$.
- b) If one of the two mentioned values is odd and the other one is even (or conversely) then

$$F(2i, 2j + 1) = - \int_{-\theta}^{\theta} E(x) R(x) \Phi_{2i}(x) \Phi_{2j+1}(x) dx. \quad (9)$$

But in relation (9), $E(x)$, $R(x)$ and $\Phi_{2i}(x)$ are even functions and $\Phi_{2j+1}(x)$ is odd. So, the integrand of (9) is an odd function and consequently $F(2i, 2j + 1) = 0$. This result similarly holds for the case $n = 2i + 1$ and $m = 2j$, i.e. $F(2i + 1, 2j) = 0$.

By noting these themes, the main theorem of the paper can be expressed as follows:

THEOREM 1 *The symmetric sequence $\Phi_n(x) = (-1)^n \Phi_n(-x)$, as a specific solution of differential equation (4), satisfies the orthogonality relation*

$$\int_{-\theta}^{\theta} W^*(x) \Phi_n(x) \Phi_m(x) dx = \left(\int_{-\theta}^{\theta} W^*(x) \Phi_n^2(x) dx \right) \delta_{n,m}, \quad (10)$$

where

$$W^*(x) = C(x) R(x) = C(x) \exp \left(\int \frac{B(x) - A'(x)}{A(x)} dx \right), \quad (11)$$

denotes the corresponding weight function and is a positive (for $C(x) > 0$) and even function on $[-\theta, \theta]$.

Here is a good position to present some practical examples that generalize the well-known symmetric orthogonal functions and preserve their orthogonality property. Sections 3 and 4 introduce two basic examples in this regard.

3. A symmetric generalization of the associated Legendre functions

The associated Legendre functions satisfy the differential equation

$$(1-x^2) \frac{d^2 \Psi}{dx^2} - 2x \frac{d\Psi}{dx} + \left(\mu_n - \frac{\nu}{1-x^2} \right) \Psi = 0, \quad (12)$$

where μ_n are eigenvalues and ν is a constant parameter [3,1]. These functions play a key role in the potential theory. In general, there are three types of solution for equation (12) depending on different values of μ_n and ν :

i) If $\mu_n = (n + \alpha)(n + \alpha + 1)$ and $\nu = \alpha^2$; $n \in Z^+$, $\alpha > -1$ are considered in equation (12) then the corresponding solution is indicated by

$$\Psi(x) = U_n^{(\alpha)}(x) = (1-x^2)^{\frac{\alpha}{2}} P_n^{(\alpha, \alpha)}(x), \quad (13)$$

where $P_n^{(\alpha, \alpha)}(x)$, known as ultraspherical polynomials, is a special case of Jacobi polynomials [4,5]

$$P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^n \binom{n + \alpha + \beta + k}{k} \binom{n + \alpha}{n - k} \left(\frac{x - 1}{2} \right)^k, \quad (14)$$

for $\alpha = \beta$. In this way, solution (13) satisfies the orthogonality relation

$$\begin{aligned} \int_{-1}^1 U_n^{(\alpha)}(x) U_m^{(\alpha)}(x) dx &= \left(\int_{-1}^1 (U_n^{(\alpha)}(x))^2 dx \right) \delta_{n,m} \\ &= \frac{2^{2\alpha+1} \Gamma^2(n + \alpha + 1)}{n! (2n + 2\alpha + 1) \Gamma(n + 2\alpha + 1)} \delta_{n,m}. \end{aligned} \quad (15)$$

ii) If $\mu_n = n(n + 1)$ and $\nu = m^2$; $m, n \in Z^+$ then equation (12) has a solution as

$$\Psi(x) = P_n^m(x) = (1-x^2)^{m/2} \frac{d^m(P_n(x))}{dx^m}, \quad (16)$$

in which $P_n(x) = P_n^{(0,0)}(x)$ denotes the Legendre polynomials [2]. Moreover, according to [1], solution (16) satisfies the orthogonality relation

$$\int_{-1}^1 P_i^m(x) P_j^m(x) dx = \left(\int_{-1}^1 (P_i^m(x))^2 dx \right) \delta_{i,j} = \frac{2(i+m)!}{(2i+1)(i-m)!} \delta_{i,j}. \quad (17)$$

iii) Finally if $\mu_n = n(n + 1)$ and $\nu = \alpha^2$; $-1 < \alpha < 1$ are selected in equation (12), the related solution takes the form

$$\Psi(x) = V_n^{(\alpha)}(x) = \left(\frac{1-x}{1+x} \right)^{\frac{\alpha}{2}} P_n^{(\alpha, -\alpha)}(x), \quad (18)$$

and satisfies the relation

$$\begin{aligned} \int_{-1}^1 V_n^{(\alpha)}(x) V_m^{(\alpha)}(x) dx &= \left(\int_{-1}^1 (V_n^{(\alpha)}(x))^2 dx \right) \delta_{n,m} \\ &= \frac{2 \Gamma(n + 1 + \alpha) \Gamma(n + 1 - \alpha)}{(n!)^2 (2n + 1)} \delta_{n,m}. \end{aligned} \quad (19)$$

Now, to extend the associated Legendre functions, it is sufficient in the main equation (4) to choose

$$\begin{aligned} A(x) &= 1 - x^2 = A(-x); & B(x) &= -2x = -B(-x), \\ C(x) &= 1 = C(-x); & D(x) &= -\frac{\nu}{1-x^2} = D(-x), \\ \lambda_n &= \mu_n; & E(x) &= E(-x) \text{ Arbitrary,} \end{aligned} \quad (20)$$

and then apply the theorem 1 to establish the orthogonality property on the same interval $[-1, 1]$. To do this, let $\Phi_n(x) = Q_n(x; \nu, E(x))$ be a known solution of differential equation

$$(1-x^2) \Phi_n''(x) - 2x \Phi_n'(x) + \left(\mu_n - \frac{\nu}{1-x^2} + \frac{1-(-1)^n}{2} E(x) \right) \Phi_n(x) = 0. \quad (21)$$

According to theorem 1, we have

$$\int_{-1}^1 Q_n(x; \nu, E(x)) Q_m(x; \nu, E(x)) dx = \left(\int_{-1}^1 Q_n^2(x; \nu, E(x)) dx \right) \delta_{n,m}, \quad (22)$$

provided that the arbitrary function $E(x)$ is even. Evidently various options can be selected for $E(x)$, which are directly related to the orthogonal sequence $Q_n(x; \nu, E(x))$. For example, choosing the even function $E(x) = 0$ gives one of the three cases of usual associated Legendre functions, stated previously. A further special example is when $E(x) = E(-x) = -2/x^2$. This case is introduced and studied in section 4, part (a) in more detail.

The sequence $Q_n(x; \nu, E(x))$ (as a known solution of equation (21)) is now orthogonal under the predetermined condition $E(-x) = E(x)$. Hence, it can be applied in the theory of the expansion of functions, as many boundary value problems of mathematical physics are solved by using the expansion of functions in terms of the eigenfunctions of a Sturm–Liouville problem. So, if we assume

$$f(x) = \sum_{n=0}^{\infty} q_n Q_n(x; \nu, E(x)), \quad (23)$$

then according to property (22), the unknown coefficients q_n are found as

$$q_n = \frac{\int_{-1}^1 f(x) Q_n(x; \nu, E(x)) dx}{\int_{-1}^1 (Q_n(x; \nu, E(x)))^2 dx}. \quad (24)$$

4. A generic differential equation and its orthogonal solutions

In this section, by using the main differential equation (4) and theorem 1, we are going to introduce a fundamental class of symmetric orthogonal functions that generates almost all known symmetric sequences of orthogonal polynomials. Thus, let us start with a second-order

differential equation, which is introduced in ref. [6], as

$$x^2(px^2 + q) \Phi_n''(x) + x(rx^2 + s) \Phi_n'(x) - (n(r + (n-1)p)x^2 + (1 - (-1)^n)s/2) \Phi_n(x) = 0. \quad (25)$$

It is clear that equation (25) is a special sub-case of equation (4) for

$$\begin{aligned} A(x) &= x^2(px^2 + q); & \text{an even function,} \\ B(x) &= x(rx^2 + s); & \text{an odd function,} \\ C(x) &= x^2; & \text{an even function,} \\ D(x) &= 0; & \text{an even function,} \\ E(x) &= -s; & \text{an even function,} \end{aligned} \quad (26)$$

and $\lambda_n = -n(r + (n-1)p)$. According to [6], the polynomial solution of equation (25) is as

$$\begin{aligned} \Phi_n(x) &= S_n \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) \\ &= \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{(2i + (-1)^{n+1} + 2[n/2])p + r}{(2i + (-1)^{n+1} + 2)q + s} \right) x^{n-2k}, \end{aligned} \quad (27)$$

where neither both values q and s nor both values p and r can vanish together. From the solution (27) one can conclude that the leading coefficient of polynomials is

$$K_n = \prod_{i=0}^{[n/2]-1} \frac{(2i + (-1)^{n+1} + 2[n/2])p + r}{(2i + (-1)^{n+1} + 2)q + s}. \quad (28)$$

Therefore, for instance, the monic polynomials of (27) for $n = 0, 1, \dots, 5$ are respectively

$$\begin{aligned} \bar{S}_0 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) &= 1, \quad \bar{S}_1 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) = x, \\ \bar{S}_2 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) &= x^2 + \frac{q+s}{p+r}, \quad \bar{S}_3 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) = x^3 + \frac{3q+s}{3p+r}x, \\ \bar{S}_4 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) &= x^4 + 2\frac{3q+s}{5p+r}x^2 + \frac{(3q+s)(q+s)}{(5p+r)(3p+r)}, \\ \bar{S}_5 \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) &= x^5 + 2\frac{5q+s}{7p+r}x^3 + \frac{(5q+s)(3q+s)}{(7p+r)(5p+r)}x. \end{aligned} \quad (29)$$

Moreover, in [6] we showed that the monic polynomials

$$\bar{S}_n \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right) = \frac{1}{K_n} S_n \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| x \right), \quad (30)$$

satisfy the recurrence relation

$$\bar{S}_{n+1}(x) = x \bar{S}_n(x) + C_n \left(\begin{matrix} r & s \\ p & q \end{matrix} \right) \bar{S}_{n-1}(x); \quad \bar{S}_0(x) = 1, \bar{S}_1(x) = x, \quad n = 1, 2, \dots, \quad (31)$$

in which

$$C_n \begin{pmatrix} r & s \\ p & q \end{pmatrix} = \frac{pq n^2 + ((r-2p)q - (-1)^n ps) n + (r-2p)s(1 - (-1)^n)/2}{(2pn + r - p)(2pn + r - 3p)}. \quad (32)$$

On the other hand, since the form of recurrence relation (31) is explicitly known, the Favard theorem [4] can here be employed to design a generic orthogonality relation for the monic polynomials (30) as follows

$$\begin{aligned} & \int_{-\theta}^{\theta} W \begin{pmatrix} r & s \\ p & q \end{pmatrix} \bigg| x \bar{S}_n \begin{pmatrix} r & s \\ p & q \end{pmatrix} \bigg| x \bar{S}_m \begin{pmatrix} r & s \\ p & q \end{pmatrix} \bigg| x \, dx \\ &= \left((-1)^n \prod_{i=1}^n C_i \begin{pmatrix} r & s \\ p & q \end{pmatrix} \int_{-\theta}^{\theta} W \begin{pmatrix} r & s \\ p & q \end{pmatrix} \bigg| x \, dx \right) \delta_{n,m}, \end{aligned} \quad (33)$$

where

$$W \begin{pmatrix} r & s \\ p & q \end{pmatrix} \bigg| x = x^2 \exp \left(\int \frac{(r-4p)x^2 + (s-2q)}{x(px^2 + q)} \, dx \right) = \exp \left(\int \frac{(r-2p)x^2 + s}{x(px^2 + q)} \, dx \right), \quad (34)$$

is the related weight function. Note that according to relations (5) and (26), the function $A(x)R(x) = (px^2 + q) \exp \left(\int ((r-2p)x^2 + s)/x(px^2 + q) \, dx \right)$ must vanish at $x = \theta$ (and subsequently at $x = -\theta$). The standard values of this point can be either $\theta = 1$ or $\theta = \infty$. Here we should add that the weight function (34), as an analogue of Pearson distributions family [2, 5], can be investigated from a statistical point of view. This function satisfies a first-order differential equation of the form

$$x \frac{d}{dx} ((px^2 + q) W(x)) = (rx^2 + s) W(x). \quad (35)$$

After the introduction of this section, we can now introduce a generic differential equation and investigate its orthogonal solutions. Without loss of generality, let us consider the following differential equation

$$x^2(a x^\lambda + b) y'' + x(c x^\lambda + d) y' + (\alpha_n x^\lambda + \beta(1 - (-1)^n)/2) y = 0, \quad (36)$$

in which a, b, c, d, λ and β are real numbers and $\{\alpha_n\}$ is a sequence of constants. First it is clear that equation (36) is a special case of the main equation (4) for

$$\begin{aligned} A(x) &= x^2(a x^\lambda + b); & B(x) &= x(c x^\lambda + d); \\ C(x) &= x^\lambda; & D(x) &= 0; & E(x) &= \beta. \end{aligned} \quad (37)$$

But, our main aim of defining equation (36) is to find some conditions under which the general solution of the foresaid equation satisfies the conditions of theorem 1. For this purpose, we should reconsider equation (25) and suppose that $x = t^{\lambda/2}$. By substituting

$$\begin{aligned} y' &= \frac{dy}{dx} = \frac{2}{\lambda} t^{-(\lambda/2)+1} \frac{dy}{dt}, \\ y'' &= \frac{d^2y}{dx^2} = \frac{4}{\lambda^2} t^{-\lambda+2} \frac{d^2y}{dt^2} + \left(\frac{4}{\lambda^2} - \frac{2}{\lambda} \right) t^{-\lambda+1} \frac{dy}{dt}, \end{aligned} \quad (38)$$

into equation (25) and considering its general solution in relation to (27), we get

$$t^2(pt^\lambda + q) \frac{d^2y}{dt^2} + t \left(\left(\frac{\lambda}{2}r + \left(1 - \frac{\lambda}{2}\right)p \right) t^\lambda + \frac{\lambda}{2}s + \left(1 - \frac{\lambda}{2}\right)q \right) \frac{dy}{dt} - \left(\frac{\lambda^2}{4}n(r + (n-1)p)t^\lambda + \frac{\lambda^2}{4}s \left(\frac{1 - (-1)^n}{2} \right) \right) y = 0 \iff y = S_n \left(\begin{matrix} r & s \\ p & q \end{matrix} \middle| t^{\frac{\lambda}{2}} \right). \quad (39)$$

Here, for convenience, let us assume in equation (39) that

$$p = a, \quad q = b, \quad \frac{\lambda}{2}r + \left(1 - \frac{\lambda}{2}\right)p = c, \quad \frac{\lambda}{2}s + \left(1 - \frac{\lambda}{2}\right)q = d. \quad (40)$$

Hence, the equation is transformed to the same equation as (36) with parameters

$$\alpha_n = -\frac{\lambda}{4}n(2c + (\lambda n - 2)a) \quad \text{and} \quad \beta = -\frac{\lambda}{4}(2d + (\lambda - 2)b), \quad (41)$$

and has the general solution

$$y = S_n \left(\begin{matrix} \frac{2}{\lambda}c + \left(1 - \frac{2}{\lambda}\right)a, & \frac{2}{\lambda}d + \left(1 - \frac{2}{\lambda}\right)b \\ a, & b \end{matrix} \middle| x^{\lambda/2} \right) \\ = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{(2i + (-1)^{n+1} + 2[n/2] + 1 - 2/\lambda)a + 2c/\lambda}{(2i + (-1)^{n+1} + 3 - 2/\lambda)b + 2d/\lambda} \right) x^{(n-2k)(\lambda/2)}. \quad (42)$$

Now, the necessary conditions for the solution of the equation corresponding to (41) in order to satisfy the conditions of theorem 1 are respectively that:

- i) The general solution (42) must be symmetric.
- ii) The coefficients of equation (36) must alternatively be even and odd.

Consequently we should have

$$\begin{cases} (-x)^{\lambda/2} = -x^{\lambda/2} \\ (-x)^\lambda = x^\lambda \end{cases} \implies \begin{cases} (-1)^{\lambda/2} = -1 \\ (-1)^\lambda = +1 \end{cases} ; \quad \lambda \in \mathbb{R}. \quad (43)$$

Under these conditions, many options can be selected for λ . For instance, if $\lambda = 2/3$ in relation to (42), then the differential equation

$$x^2(ax^{(2/3)} + b)y'' + x(cx^{(2/3)} + d)y' - \left(\frac{n}{3} \left(c + \left(\frac{n}{3} - 1 \right) a \right) x^{(2/3)} + \frac{1}{6} \left(d - \frac{2}{3}b \right) (1 - (-1)^n) \right) y = 0, \quad (44)$$

has a general solution as

$$y = S_n \left(\begin{matrix} 3c - 2a, & 3d - 2b \\ a, & b \end{matrix} \middle| \sqrt[3]{x} \right), \quad (45)$$

which is symmetric and satisfies the conditions of theorem 1. Therefore, it has a corresponding weight function as

$$W_1(x) = x^{(2/3)} \exp \left(\int \frac{(c - 8a/3)x^{(2/3)} + (d - 2b)}{ax^{5/3} + bx} dx \right). \quad (46)$$

Moreover, substituting $x = t^{1/3}$ into relation (29) gives the orthogonality property of this sequence of symmetric orthogonal functions as

$$\begin{aligned} & \int_{-\theta^3}^{\theta^3} W_1(t) \bar{S}_n \left(\begin{matrix} 3c - 2a, & 3d - 2b \\ a, & b \end{matrix} \middle| \sqrt[3]{t} \right) \bar{S}_m \left(\begin{matrix} 3c - 2a, & 3d - 2b \\ a, & b \end{matrix} \middle| \sqrt[3]{t} \right) dt \\ &= \left((-1)^n \prod_{i=1}^n C_i \left(\begin{matrix} 3c - 2a, & 3d - 2b \\ a, & b \end{matrix} \right) \int_{-\theta^3}^{\theta^3} W_1(t) dt \right) \delta_{n,m}. \end{aligned} \quad (47)$$

Note that the right-hand side of (47) can be computed by referring to (32).

But there are generally four special sub-classes of polynomials (27) (and consequently the main sequence (42) for $\lambda = 2$) that are orthogonal with respect to four special sub-solutions of equation (35). The main properties of these sub-classes are respectively summarized in the following sections (a), (b), (c) and (d).

a) First subclass: generalized Ultraspherical polynomials (GUP)

These polynomials were first studied by Chihara [4]. He obtained their main properties by applying a direct relation between them and Jacobi orthogonal polynomials. To study GUP individually we refer the readers to [7–9].

By replacing the characteristic vector

$$(a, b, c, d, \lambda) = (-1, 1, -2u - 2v - 2, 2u, 2), \quad (48)$$

in the main sequence (42), the explicit form of the monic GUP is revealed as

$$\begin{aligned} \bar{S}_n \left(\begin{matrix} -2u - 2v - 2, & 2u \\ -1, & 1 \end{matrix} \middle| x \right) &= \prod_{i=0}^{[n/2]-1} \frac{2i + 2u + 2 - (-1)^n}{-2i - (2v + 2u + 2 - (-1)^n + 2[n/2])} \\ &\times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{-2i - (2v + 2u + 2 - (-1)^n + 2[n/2])}{2i + 2u + 2 - (-1)^n} \right) x^{n-2k}. \end{aligned} \quad (49)$$

These polynomials correspond to the weight function

$$W \left(\begin{matrix} -2u - 2v - 2 & 2u \\ -1 & 1 \end{matrix} \middle| x \right) = x^{2u} (1 - x^2)^v, \quad (50)$$

and satisfy the orthogonality relation

$$\begin{aligned} & \int_{-1}^1 x^{2u} (1 - x^2)^v \bar{S}_n \left(\begin{matrix} -2u - 2v - 2, & 2u \\ -1, & 1 \end{matrix} \middle| x \right) \bar{S}_m \left(\begin{matrix} -2u - 2v - 2, & 2u \\ -1, & 1 \end{matrix} \middle| x \right) dx \\ &= \left((-1)^n \prod_{i=1}^n C_i \left(\begin{matrix} -2u - 2v - 2, & 2u \\ -1, & 1 \end{matrix} \right) \int_{-1}^1 x^{2u} (1 - x^2)^v dx \right) \delta_{n,m}, \end{aligned} \quad (51)$$

where

$$\int_{-1}^1 x^{2u} (1 - x^2)^v dx = B(u + \frac{1}{2}, v + 1) = \frac{\Gamma(u + 1/2)\Gamma(v + 1)}{\Gamma(u + v + 3/2)}. \quad (52)$$

From relation (52), one can deduce that the constraints on the parameters u and v are respectively $u + 1/2 > 0$, $(-1)^{2u} = 1$ and $v + 1 > 0$. Note that $B(u, v)$ in this relation denotes the Beta integral [2] and $\Gamma(u)$ denotes the gamma function [1].

Finally the polynomials GUP satisfies the differential equation

$$x^2(-x^2 + 1) \Phi_n''(x) - 2x((u + v + 1)x^2 - u) \Phi_n'(x) + (n(2u + 2v + n + 1)x^2 + ((-1)^n - 1)u) \Phi_n(x) = 0 \quad (53)$$

which is evidently a special case of the generic differential equation (36) for the initial vector (48) and parameters (41).

But, as we reminded in section 3, there is a special case for $E(x)$ in equation (21) that can be derived from equation (53). To generate this special case, first we should define the following sequence

$$G_n(x; a, b) = x^a(1 - x^2)^{b/2} \bar{S}_n \left(\begin{matrix} -2a - 2b - 2, & 2a \\ -1, & 1 \end{matrix} \middle| x \right); \quad a > -\frac{1}{2}, \quad b > -1. \quad (54)$$

If this sequence is replaced into equation (53), after some calculations we get

$$(1 - x^2) G_n''(x) - 2x G_n'(x) + \left((n + a + b)(n + a + b + 1) - \frac{b^2}{1 - x^2} + \frac{a((-1)^n - a)}{x^2} \right) G_n(x) = 0. \quad (55)$$

By comparing equation (55) with (21) one can conclude that the sequence

$$Q_n \left(x; b^2, -\frac{2}{x^2} \right) = G_n(x; 1, b); \quad b > -1, \quad (56)$$

is a general solution of equation (24) for $\mu_n = (n + b + 1)(n + b + 2)$, $\nu = b^2$ and $E(x) = -2/x^2$. Consequently, substituting these values into orthogonality (22) and noting the relation (51) yields

$$\begin{aligned} & \int_{-1}^1 Q_n \left(x; b^2, -\frac{2}{x^2} \right) Q_m \left(x; b^2, -\frac{2}{x^2} \right) dx \\ &= \left(\prod_{i=1}^n \frac{(i + 1 - (-1)^i)(i + 1 - (-1)^i + 2b)}{(2i + 2b + 1)(2i + 2b + 3)} \right) \frac{\sqrt{\pi} \Gamma(b + 1)}{2\Gamma(b + 5/2)} \delta_{n,m} \end{aligned} \quad (57)$$

b) Second subclass: generalized Hermite polynomials

GHP differential equation is a special case of the generic equation (36) for $\lambda = 2$, $p = 0$, $q = 1$, $r = -2$ and $s = 2u$ so that we have

$$x^2 \Phi_n''(x) - 2x(x^2 - u) \Phi_n'(x) + (2n x^2 + ((-1)^n - 1)u) \Phi_n(x) = 0. \quad (58)$$

According to relations (27) and (28), the explicit form of the monic GHP is as

$$\begin{aligned} \bar{S}_n \left(\begin{matrix} -2 & 2u \\ 0 & 1 \end{matrix} \middle| x \right) &= \prod_{i=0}^{[n/2]-1} \frac{2i + (-1)^{n+1} + 2 + 2u}{-2} \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \\ &\quad \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{-2}{2i + (-1)^{n+1} + 2 + 2u} \right) x^{n-2k}, \end{aligned} \quad (59)$$

that corresponds to the weight function

$$W \left(\begin{matrix} -2, & 2u \\ 0, & 1 \end{matrix} \middle| x \right) = x^{2u} e^{-x^2}. \quad (60)$$

Thus, substituting the relations (59) and (60) in (33) when $\theta = \infty$ yields

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{2u} e^{-x^2} \bar{S}_n \left(\begin{matrix} -2 & 2u \\ 0 & 1 \end{matrix} \middle| x \right) \bar{S}_m \left(\begin{matrix} -2 & 2u \\ 0 & 1 \end{matrix} \middle| x \right) dx \\ &= \left((-1)^n \prod_{i=1}^n C_i \left(\begin{matrix} -2 & 2u \\ 0 & 1 \end{matrix} \right) \right) \Gamma \left(u + \frac{1}{2} \right) \delta_{n,m}. \end{aligned} \quad (61)$$

It is clear that the orthogonality (61) is valid when $u + 1/2 > 0$ and $(-1)^{2u} = 1$. To study GHP individually see [4, 9, 5].

c) Third subclass: a finite class of symmetric orthogonal polynomials on $(-\infty, \infty)$

There are two finite classes of symmetric orthogonal polynomials that are particular sub-cases of functions (42) and can be considered as two independent generalized Sturm–Liouville problems. Usually finite orthogonal polynomials are less known in the theory of special functions. To study this kind of polynomials see e.g. [10]. The first sequence of finite symmetric orthogonal polynomials can be defined by the initial vector

$$(a, b, c, d, \lambda) = (1, 1, -2u - 2v + 2, -2u, 2), \quad (62)$$

which results the differential equation

$$\begin{aligned} & x^2(x^2 + 1) \Phi_n''(x) - 2x((u + v - 1)x^2 + a) \Phi_n'(x) \\ & + (n(2u + 2v - (n + 1))x^2 + (1 - (-1)^n)u) \Phi_n(x) = 0 \end{aligned} \quad (63)$$

This equation is evidently a special case of the generic equation (36) and therefore has an explicit polynomial solution as

$$\begin{aligned} \bar{S}_n \left(\begin{matrix} -2u - 2v + 2, & -2u \\ 1, & 1 \end{matrix} \middle| x \right) &= \prod_{i=0}^{[n/2]-1} \frac{2i + (-1)^{n+1} + 2 - 2u}{2i + 2[n/2] + (-1)^{n+1} + 2 - 2u - 2v} \\ &\times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{2i + 2[n/2] + (-1)^{n+1} + 2 - 2u - 2v}{2i + (-1)^{n+1} + 2 - 2u} \right) x^{n-2k}. \end{aligned} \quad (64)$$

According to theorem 1, if the equation (63) is written in self-adjoint form, the following term must vanish on $(-\infty, \infty)$, i.e. we should have

$$[x^{-2u}(1 + x^2)^{-v+1}(\Phi_n'(x)\Phi_m(x) - \Phi_m'(x)\Phi_n(x))]_{-\infty}^{\infty} = 0, \quad (65)$$

in which

$$x^{-2u}(1 + x^2)^{-v} = W \left(\begin{matrix} -2u - 2v + 2, & -2u \\ 1 & 1 \end{matrix} \middle| x \right), \quad (66)$$

denotes the corresponding weight function. On the other hand, since $\Phi_n(x)$ is a polynomial of degree n , we have

$$\max \deg(\Phi_n'(x)\Phi_m(x) - \Phi_m'(x)\Phi_n(x)) = n + m - 1. \quad (67)$$

Therefore, from relation (65) it is concluded that the condition

$$-2u + 2(-v + 1) + n + m - 1 \leq 0, \quad (68)$$

must always be satisfied. This leads to the following corollary.

COROLLARY *The finite polynomial set $\{S_n(p, q, r, s; x)\}_{n=0}^{n=N}$ for $p = 1$, $q = 1$, $r = -2u - 2v + 2$ and $s = -2u$ is orthogonal with respect to the weight function $x^{-2u}(1+x^2)^{-v}$ on $(-\infty, \infty)$ if and only if $N \leq u + v - 1/2$, $u < 1/2$ and $v > 0$.*

The above corollary and using Favard theorem yield the orthogonality relation

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{x^{-2u}}{(1+x^2)^v} \bar{S}_n \left(\begin{matrix} -2u-2v+2, & -2u \\ 1, & 1 \end{matrix} \middle| x \right) \bar{S}_m \left(\begin{matrix} -2u-2v+2, & -2u \\ 1, & 1 \end{matrix} \middle| x \right) dx \\ &= \left((-1)^n \prod_{i=1}^n C_i \left(\begin{matrix} -2u-2v+2, & -2u \\ 1, & 1 \end{matrix} \right) \right) \frac{\Gamma(u+v-1/2)\Gamma(-u+1/2)}{\Gamma(u+v)} \delta_{n,m} \end{aligned} \quad (69)$$

if and only if $N = \max\{m, n\} \leq u + v - 1/2$, $u < 1/2$ and $v > 0$.

d) Fourth subclass: another finite class of symmetric orthogonal polynomials on $(-\infty, \infty)$.

Similar to the previous case, we can define another finite sequence of symmetric orthogonal polynomials on the real line having the following main properties:

Characteristic vector

$$(a, b, c, d, \lambda) = (1, 0, -2u + 2, 2, 2), \quad (70)$$

Differential equation:

$$x^4 \Phi_n''(x) + 2x((1-u)x^2 + 1) \Phi_n'(x) - (n(n+1-2u)x^2 + 1 - (-1)^n) \Phi_n(x) = 0. \quad (71)$$

Explicit form of monic polynomials:

$$\begin{aligned} \bar{S}_n \left(\begin{matrix} -2u+2 & 2 \\ 1 & 0 \end{matrix} \middle| x \right) &= \prod_{i=0}^{[n/2]-1} \frac{2}{2i + 2[n/2] + (-1)^{n+1} + 2 - 2u} \\ &\times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{2i + 2[n/2] + (-1)^{n+1} + 2 - 2u}{2} \right) x^{n-2k}. \end{aligned} \quad (72)$$

Weight function:

$$W \left(\begin{matrix} -2u+2 & 2 \\ 1 & 0 \end{matrix} \middle| x \right) = x^{-2u} \exp \left(-\frac{1}{x^2} \right). \quad (73)$$

Orthogonality relation:

$$\begin{aligned} & \int_{-\infty}^{\infty} x^{-2u} \exp \left(-\frac{1}{x^2} \right) \bar{S}_n \left(\begin{matrix} -2u+2 & 2 \\ 1 & 0 \end{matrix} \middle| x \right) \bar{S}_m \left(\begin{matrix} -2u+2 & 2 \\ 1 & 0 \end{matrix} \middle| x \right) dx \\ &= \left((-1)^n \prod_{i=1}^n C_i \left(\begin{matrix} -2u+2 & 2 \\ 1 & 0 \end{matrix} \right) \right) \Gamma \left(u - \frac{1}{2} \right) \delta_{n,m} \end{aligned} \quad (74)$$

if and only if $N = \max\{m, n\} \leq u - 1/2$.

As observed, each of polynomial classes of sections (a), (b), (c) and (d) were particular sub-cases of a symmetric generalized Sturm–Liouville problem. However, since all weight functions of presented polynomials were even functions, the condition $(-1)^{2u} = 1$ is always necessary. Hence, we can also consider the mentioned weights respectively as $|x|^{2u}(1-x^2)^v$, $|x|^{2u}e^{-x^2}$, $|x|^{-2u}(1+x^2)^{-v}$ and $|x|^{-2u}e^{-1/x^2}$.

5. Conclusion

In this paper, we proved that the usual Sturm–Liouville problem

$$\frac{d}{dx} \left(k(x) \frac{dy}{dx} \right) + (\lambda_n \rho(x) - q(x))y = 0; \quad k(x) > 0, \quad \rho(x) > 0, \quad a < x < b,$$

under the boundary conditions

$$\begin{cases} \alpha_1 y(a) + \beta_1 y'(a) = 0, \\ \alpha_2 y(b) + \beta_2 y'(b) = 0, \end{cases}$$

can be extended to the problem

$$\frac{d}{dx} \left(k(x) \frac{dy}{dx} \right) + (\lambda_n \rho(x) - q(x) + \frac{1 - (-1)^n}{2} E(x))y = 0,$$

with the boundary condition $\alpha_3 y(\theta) + \beta_3 y'(\theta) = 0$; $-\theta < x < \theta$, provided that the solution of latter differential equation has the symmetry property $y_n(-x) = (-1)^n y_n(x)$. Then, we presented two basic examples to show the advantage of extending Sturm–Liouville problems for symmetric functions. Finally note that the extension of Sturm–Liouville problems for discrete variables is also possible.

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On boundedness of the generalized B -potential integral operators in the Lorentz spaces

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In this paper, we study the convolution operator (B -convolution), and the generalized B -potential integral and fractional integral (B -fractional integral) with rough kernel, associated with the Laplace–Bessel differential operator $\Delta_B = \sum_{i=1}^k B_i + \sum_{j=k+1}^n (\partial^2/\partial x_j^2)$, $B_i = (\partial^2/\partial x_i^2) + (\gamma_i/x_i)(\partial/\partial x_i)$. We get O’Neil type inequality for the B -convolution. By using O’Neil type inequality we obtain a pointwise rearrangement estimate of the generalized B -potential integral. We prove the boundedness of the generalized B -potential integral operator in the Lorentz spaces, and the proof is based on the pointwise estimate of the rearrangement of this operator.

Keywords: Laplace–Bessel differential operator; O’Neil type inequality; Generalized B -potential integrals; B -fractional integral; Lorentz spaces

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1. Main results

The fractional integral operators play an important role in the theory of harmonic analysis, differentiation theory and PDE’s. The fractional integrals and related topics associated with the Laplace–Bessel differential operator have been research areas for many mathematicians such as Gadzhiev and Aliev [1], Kipriyanov [2], Lyakhov [3], Guliyev [4, 5], Serbetci and Ekinçioğlu [6] and others.

In this paper, we realize some estimations of the generalized B -fractional integrals with rough kernels generated by the generalized shift operator of the form

$$T^\gamma f(x) = C_{k,\gamma} \int_0^\pi \cdots \int_0^\pi f((x', y')_\alpha, x'' - y'') \, d\nu(\alpha),$$

where $C_{k,\gamma} = \pi^{-(k/2)} \prod_{i=1}^k (\Gamma((\gamma_i + 1)/2)/(\Gamma(\gamma_i/2)))$, $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $(x', y')_\alpha = ((x_1, y_1)_{\alpha_1}, \dots, (x_k, y_k)_{\alpha_k})$, $(x_i, y_i)_{\alpha_i} = \sqrt{x_i^2 - 2x_i y_i \cos \alpha_i + y_i^2}$, $1 \leq i \leq k$, $d\nu(\alpha) =$

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$\prod_{i=1}^k \sin^{\gamma_i-1} \alpha_i d\alpha_i$, $1 \leq k \leq n$. Note that the generalized shift operator T^γ is closely related to the Δ_B Laplace–Bessel differential operator ([2, 3, 7, 8])

$$\Delta_B = \sum_{i=1}^k B_i + \sum_{i=k+1}^n \frac{\partial^2}{\partial x_i^2}, \quad k = 1, \dots, n, \quad \text{where } B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_i > 0.$$

Furthermore, T^γ generates the corresponding B -convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}_{k,+}^n} f(y) T^\gamma g(x) (y')^\gamma dy,$$

where $\mathbb{R}_{k,+}^n$ denotes the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, \dots, x_n)$ defined by the inequalities $x_1 > 0, \dots, x_k > 0$, $1 \leq k \leq n$, $(x')^\gamma = x_1^{\gamma_1} \dots x_k^{\gamma_k}$, and $\gamma = (\gamma_1, \dots, \gamma_k)$ is a multi-index consisting of fixed positive numbers.

Let $K_\alpha \in WL_{Q/(Q-\alpha), \gamma}(\mathbb{R}_{k,+}^n)$, $\Omega \in L_{Q/(Q-\alpha), \gamma}(S_{k,+}^{n-1})$, and let Ω be homogeneous of degree zero on $\mathbb{R}_{k,+}^n$, i.e., $\Omega(tx) = \Omega(x)$ for all $t > 0$, $x \in \mathbb{R}_{k,+}^n$, where $S_{k,+}^{n-1} = \{x \in \mathbb{R}_{k,+}^n : |x| = 1\}$, and $0 < \alpha < Q$, $Q = n + |\gamma|$, $|\gamma| = \gamma_1 + \dots + \gamma_k$.

We define the generalized B -potential integral by

$$(K_\alpha \otimes f)(x) = \int_{\mathbb{R}_{k,+}^n} K_\alpha(y) T^\gamma f(x) (y')^\gamma dy, \quad (1)$$

and the B -fractional integral by

$$I_{\Omega, \alpha, \gamma} f(x) = \int_{\mathbb{R}_{k,+}^n} \frac{\Omega(y)}{|y|^{Q-\alpha}} T^\gamma f(x) (y')^\gamma dy. \quad (2)$$

It is clear that when $\Omega \equiv 1$, $I_{\Omega, \alpha, \gamma}$ is the usual B -Riesz potential $I_{\alpha, \gamma}$ ([1, 3, 4]).

In the following theorems we give our main results. In Theorem 1.1 we get the O'Neil type inequality ([9]) for the B -convolutions. In Theorem 1.2 we obtain a pointwise rearrangement estimate of the generalized B -potential integral $K_\alpha \otimes f$ by using O'Neil type inequality for the B -convolutions. Finally, in Theorem 1.3 we show the validity of the Hardy–Littlewood–Sobolev inequality for $K_\alpha \otimes f$ in the Lorentz spaces. The proof of Theorem 1.3 is based on the pointwise rearrangement estimate of $K_\alpha \otimes f$ obtained in Theorem 1.2

THEOREM 1.1 *Let f, g be two positive measurable functions on $\mathbb{R}_{k,+}^n$, then for all $t > 0$ the following inequality holds*

$$(f \otimes g)_\gamma^{**}(t) \leq C_{k, \gamma} \left(f_\gamma^{**}(t) \int_0^t g_\gamma^{**}(u) du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u) du \right). \quad (3)$$

THEOREM 1.2 *Suppose that $K_\alpha \in WL_{Q/(Q-\alpha), \gamma}(\mathbb{R}_{k,+}^n)$, $0 < \alpha < Q$. Then for the generalized B -potential $K_\alpha \otimes f$ the following inequalities hold*

$$(K_\alpha \otimes f)_\gamma^*(t) \leq (K_\alpha \otimes f)_\gamma^{**}(t) \leq A_1 \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right), \quad (4)$$

where $A_1 = C_{k, \gamma} (Q/\alpha)^2 \|K_\alpha\|_{WL_{Q/(Q-\alpha), \gamma}}$.

THEOREM 1.3 *(Hardy–Littlewood–Sobolev theorem for the generalized B -potential integral in the Lorentz spaces) Let $0 < \alpha < Q$, $1 \leq p < q < \infty$, and $K_\alpha \in WL_{Q/(Q-\alpha), \gamma}(\mathbb{R}_{k,+}^n)$. Then*

- (i) If $1 < p < Q/\alpha$, $1 \leq r \leq s \leq \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1/p - 1/q = \alpha/Q$, then $K_\alpha \otimes f \in L_{q,s,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|K_\alpha \otimes f\|_{L_{q,s,\gamma}} \leq A_1 K(p, q, r, s) \|f\|_{L_{p,r,\gamma}},$$

where $K(p, q, r, s) = ((p')^{1/s} (p's'/r')^{1/r'} + (qr/s)^{1/s} q^{1/r'})$, $p' = p/(p-1)$.

- (ii) If $p = Q/\alpha$, $r = 1$, and $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}_{k,+}^n)$, then $K_\alpha \otimes f \in L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|K_\alpha \otimes f\|_{L_{\infty,\gamma}} \leq 2A_1 \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

- (iii) If $p = 1$, $1 \leq r \leq \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1 - 1/q = \alpha/Q$, then $K_\alpha \otimes f \in WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|K_\alpha \otimes f\|_{WL_{q,\gamma}} \leq 2A_1 \|f\|_{L_{1,r,\gamma}}.$$

As a consequence of Theorem 1.3 we have the following corollaries.

COROLLARY 1.4 (Hardy–Littlewood–Sobolev theorem of rough B -fractional integrals in the Lorentz spaces) Let $0 < \alpha < Q$, $1 \leq p < q < \infty$, and let Ω be homogeneous of degree zero on $\mathbb{R}_{k,+}^n$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S_{k,+}^{n-1})$. Then

- (i) If $1 < p < Q/\alpha$, $1 \leq r \leq s \leq \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1/p - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in L_{q,s,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,s,\gamma}} \leq A_2 K(p, q, r, s) \|f\|_{L_{p,r,\gamma}}.$$

where

$$A_2 = C_{k,\gamma} (Q/\alpha)^2 (A/Q)^{(Q-\alpha)/Q}, A = \|\Omega\|_{L_{Q/(Q-\alpha),\gamma}(S_{k,+}^{n-1})}^{Q/(Q-\alpha)}.$$

- (ii) If $p = Q/\alpha$, $r = 1$, $q = s = \infty$, and $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}_{k,+}^n)$, then $I_{\Omega,\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{\infty,\gamma}} \leq 2A_2 \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

- (iii) If $p = 1$, $1 \leq r \leq \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1 - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \leq 2A_2 \|f\|_{L_{1,r,\gamma}}.$$

COROLLARY 1.5 (Hardy–Littlewood–Sobolev theorem of B -Riesz potential in the Lorentz spaces) Let $0 < \alpha < Q$ and $1 \leq p < q < \infty$. Then

- (i) If $1 < p < Q/\alpha$, $1 \leq r \leq s \leq \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1/p - 1/q = \alpha/Q$, then $I_{\alpha,\gamma} f \in L_{q,s,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\alpha,\gamma} f\|_{L_{q,s,\gamma}} \leq A_3 K(p, q, r, s) \|f\|_{L_{p,r,\gamma}},$$

where $A_3 = C_{k,\gamma} (Q/\alpha)^2 \omega(n, k, \gamma)^{(Q-\alpha)/Q}$.

- (ii) If $p = Q/\alpha$, $r = 1$, and $f \in L_{Q/\alpha,1,\gamma}(\mathbb{R}_{k,+}^n)$, then $I_{\alpha,\gamma} f \in L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\alpha,\gamma} f\|_{L_{\infty,\gamma}} \leq 2A_3 \|f\|_{L_{Q/\alpha,1,\gamma}}.$$

- (iii) If $p = 1$, $1 \leq r \leq \infty$, $f \in L_{1,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1 - 1/q = \alpha/Q$, then $I_{\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\alpha,\gamma} f\|_{WL_{q,\gamma}} \leq 2A_3 \|f\|_{L_{1,r,\gamma}}.$$

COROLLARY 1.6 Let $0 < \alpha < Q$, Ω be homogeneous of degree zero on $\mathbb{R}_{k,+}^n$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S_{k,+}^{n-1})$. Then

- (i) If $1 < p < Q/\alpha$, $1 \leq r \leq s \leq \infty$, $f \in L_{p,\gamma}(\mathbb{R}_{k,+}^n)$ and $1/p - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in L_{q,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\Omega,\alpha,\gamma} f\|_{L_{q,\gamma}} \leq A_4 K(p, q) \|f\|_{L_{p,\gamma}},$$

where $A_4 = C_{k,\gamma}(Q/\alpha)^2(A/Q)^{(Q-\alpha)/Q}$, $K(p, q) \equiv K(p, q, p, q) = (p^{1/q} q^{1/p'} + (p')^{1/q} (q')^{1/p'})$.

- (ii) If $p = 1$, $f \in L_{1,\gamma}(\mathbb{R}_{k,+}^n)$ and $1 - 1/q = \alpha/Q$, then $I_{\Omega,\alpha,\gamma} f \in WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$ and

$$\|I_{\Omega,\alpha,\gamma} f\|_{WL_{q,\gamma}} \leq 2A_4 \|f\|_{L_{1,\gamma}}.$$

Note that, in [10] Corollary 1.6 has been proved. Furthermore, in the case $\Omega \equiv 1$ Corollary 1.6 was proved in [1, 3, 5], but in these works the constants were not determined.

2. Preliminaries

Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}_{k,+}^n)$ the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}_{k,+}^n} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \leq p < \infty.$$

If $p = \infty$, we assume

$$L_{\infty,\gamma}(\mathbb{R}_{k,+}^n) = L_\infty(\mathbb{R}_{k,+}^n) = \{f : \|f\|_{L_{\infty,\gamma}} = \operatorname{ess\,sup}_{x \in \mathbb{R}_{k,+}^n} |f(x)| < \infty\}.$$

For any measurable set $E \subset \mathbb{R}_{k,+}^n$, let $|E|_\gamma = \int_E (x')^\gamma dx$. Suppose $f : \mathbb{R}_{k,+}^n \rightarrow \mathbb{R}$ is a measurable function, then the decreasing γ -rearrangement of f defined on $[0, \infty)$ by

$$f_\gamma^*(t) = \inf \{s > 0 : f_{*,\gamma}(s) \leq t\}, \quad (t \geq 0)$$

where $f_{*,\gamma}$ is the γ -distribution function of f defined by

$$f_{*,\gamma}(s) = |\{x \in \mathbb{R}_{k,+}^n : |f(x)| > s\}|_\gamma \quad (s \geq 0).$$

Some properties of γ -rearrangements of functions are given as follows ([11–13]):

- (i) if $0 < p < \infty$, then

$$\int_{\mathbb{R}_{k,+}^n} |f(x)|^p (x')^\gamma dx = \int_0^\infty (f_\gamma^*(t))^p dt; \quad (5)$$

- (ii) for any $t > 0$,

$$\sup_{|E|_\gamma=t} \int_E |f(x)| (x')^\gamma dx = \int_0^t f_\gamma^*(s) ds; \quad (6)$$

- (iii)

$$\int_{\mathbb{R}_{k,+}^n} |f(x)g(x)| (x')^\gamma dx \leq \int_0^\infty f_\gamma^*(t) g_\gamma^*(t) dt. \quad (7)$$

We denote by $WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$ the weak $L_{p,\gamma}$ space of all measurable functions f with finite norm

$$\|f\|_{WL_{p,\gamma}} = \sup_{t>0} t^{1/p} f_{\gamma}^*(t) < \infty, \quad 1 \leq p < \infty.$$

We define a function f_{γ}^{**} on $(0, \infty)$ by

$$f_{\gamma}^{**}(t) = \frac{1}{t} \int_0^t f_{\gamma}^*(s) ds, \quad t > 0.$$

DEFINITION 2.1 If $0 < p, q < \infty$, then the Lorentz space $L_{p,q,\gamma}(\mathbb{R}_{k,+}^n) = L_{p,q}(\mathbb{R}_{k,+}^n, (x')^{\gamma} dx)$ is the set of all classes of measurable functions f with the finite quasi-norm

$$\|f\|_{p,q,\gamma} \equiv \|f\|_{L_{p,q,\gamma}} = \left(\int_0^{\infty} (t^{1/p} f_{\gamma}^*(t))^q \frac{dt}{t} \right)^{1/q}.$$

If $0 < p \leq \infty, q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}_{k,+}^n) = WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$. If $1 \leq q \leq p$ or $p = q = \infty$, then the functional $\|f\|_{p,q,\gamma}$ is a norm ([11, 13]). If $p = q = \infty$, then the space $L_{\infty,\infty,\gamma}(\mathbb{R}_{k,+}^n)$ is denoted by $L_{\infty,\gamma}(\mathbb{R}_{k,+}^n)$.

In the case $1 < p, q < \infty$ we define

$$\|f\|_{(p,q),\gamma} = \left(\int_0^{\infty} (t^{1/p} f_{\gamma}^{**}(t))^q \frac{dt}{t} \right)^{1/q},$$

(with the usual modification if $0 < p \leq \infty, q = \infty$) which is a norm on $L_{p,q,\gamma}(\mathbb{R}_{k,+}^n)$ for $1 < p < \infty, 1 \leq q \leq \infty$ or $p = q = \infty$. If $1 < p \leq \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q,\gamma} \leq \|f\|_{(p,q),\gamma} \leq p' \|f\|_{p,q,\gamma}$$

that is, the quasi-norms $\|f\|_{p,q,\gamma}$ and $\|f\|_{(p,q),\gamma}$ are equivalent.

For the B -shift operator T^{γ} the following lemma is valid.

LEMMA 2.2 For any measurable set $A = (A', A'') \subset \mathbb{R}_{k,+}^n$, $A' = A_1 \times \cdots \times A_k \subset (0, \infty)^k$, $A'' \subset \mathbb{R}^{n-k}$ and for any $y \in \mathbb{R}_{k,+}^n$ the following equality holds

$$\int_A T^{\gamma} g(x) (x')^{\gamma} dx = C_{k,\gamma} \int_{(y,0)+\bar{A}} g \left(\sqrt{z_1^2 + \bar{z}_1^2}, \dots, \sqrt{z_k^2 + \bar{z}_k^2}, z'' \right) d\mu(z, \bar{z}'), \quad (8)$$

where

$$(x, 0) = (x, \underbrace{0, \dots, 0}_{k\text{-times}}), \bar{z}' = (\bar{z}_1, \dots, \bar{z}_k), d\mu(z, \bar{z}') = \bar{z}'^{\gamma-1} dz d\bar{z}', d\bar{z}' = d\bar{z}_1 \cdots d\bar{z}_k,$$

$$\bar{z}'^{\gamma-1} = \bar{z}_1^{\gamma_1-1} \cdots \bar{z}_k^{\gamma_k-1}, (z, \bar{z}') \in \mathbb{R}_{k,+}^n \times (0, \infty)^k, m_i = \sup A_i, i = 1, \dots, k,$$

$$\bar{A} = ((-m_1, m_1) \times [0, m_1) \times \cdots \times (-m_k, m_k) \times [0, m_k)) \times A''.$$

Proof of the Lemma 2.2 is straightforward after applying the following substitutions

$$\begin{aligned} z'' &= x'', z_i = x_i \cos \alpha_i, \bar{z}_i = x_i \sin \alpha_i, \quad 0 \leq \alpha_i < \pi, \\ i &= 1, \dots, k, \bar{z}' = (\bar{z}_1, \dots, \bar{z}_k), (z, \bar{z}') \in \mathbb{R}_{k,+}^n \times (0, \infty)^k. \end{aligned} \quad (9)$$

We need the following two generalized Hardy inequalities ([14]) which are to be used in the proof of the Theorem 1.3.

LEMMA 2.3 Let $1 \leq r \leq s \leq \infty$ and let v and w be two functions such that measurable and positive a.e. on $(0, \infty)$. Then there exists a constant C independent of the function φ such that

$$\left(\int_0^\infty \left(\int_0^t \varphi(\tau) d\tau \right)^s w(t) dt \right)^{1/s} \leq C \left(\int_0^\infty \varphi(t)^r v(t) dt \right)^{1/r}, \quad (10)$$

if and only if

$$K = \sup_{r>0} \left(\int_r^\infty w(t) dt \right)^{1/s} \left(\int_0^r v(t)^{1-r'} dt \right)^{1/r'} < \infty, \quad (11)$$

where $r + r' = rr'$. Moreover, if C is the best constant in (10) and K is defined by (11), then

$$K \leq C \leq k(r, s)K. \quad (12)$$

Here the constant $k(r, s)$ in equation (12) can be written in various forms. For example ([15])

$$k(r, s) = r^{1/s} (r')^{1/r'} \text{ or } k(r, s) = s^{1/s} (s')^{1/r'} \text{ or } k(r, s) = \left(1 + \frac{s}{r'}\right)^{1/s} \left(1 + \frac{r'}{s}\right)^{1/r'}.$$

LEMMA 2.4 Let $1 \leq r \leq s \leq \infty$ and let v and w be two functions such that measurable and positive a.e. on $(0, \infty)$. Then there exists a constant C independent of the function φ such that

$$\left(\int_0^\infty \left(\int_t^\infty \varphi(\tau) d\tau \right)^s w(t) dt \right)^{1/s} \leq C \left(\int_0^\infty \varphi(t)^r v(t) dt \right)^{1/r} \quad (13)$$

if and only if

$$K_1 = \sup_{r>0} \left(\int_0^r w(t) dt \right)^{1/s} \left(\int_r^\infty v(t)^{1-r'} dt \right)^{1/r'} < \infty.$$

Moreover, the best constant C in equation (13) satisfies the inequalities $K_1 \leq C \leq k(r, s)K_1$.

In the following lemma, we give a relation between the generalized shift operator $T^\gamma f$ and γ -rearrangement of the function f .

LEMMA 2.5 For any measurable set $A \subset \mathbb{R}_{k,+}^n$ and $y \in \mathbb{R}_{k,+}^n$ the following equality holds

$$\sup_{|A|_\gamma=t} \int_A T^\gamma |f(x)| (x')^\gamma dx = C_{k,\gamma} \int_0^t f_\gamma^*(s) ds. \quad (14)$$

Proof By Lemma 2.2 we have

$$\int_A T^\gamma |f(x)| (x')^\gamma dx = C_{k,\gamma} \int_{(y,0)+\bar{A}} |\bar{f}(z, \bar{z}')| d\mu(z, \bar{z}'), \quad (15)$$

where $\bar{f}(z, \bar{z}') = f(\sqrt{z_1^2 + \bar{z}_1'^2}, \dots, \sqrt{z_k^2 + \bar{z}_k'^2}, z'')$. For the function $\bar{f}(z, \bar{z}')$ the analogue of the equality (6) is valid ([12]):

$$\sup_{\mu(\bar{A})=t} \int_{\bar{A}} |\bar{f}(z, \bar{z}')| d\mu(z, \bar{z}') = \int_0^t (\bar{f})_\mu^*(s) ds, \quad (16)$$

where $(\bar{f})_\mu^*(s) = \inf \{t > 0 : \mu(\{(z, \bar{z}') : |\bar{f}(z, \bar{z}')| > t\}) \leq s\}$.

Note that $\mu((y, 0) + \bar{A})) = |A|_\gamma$ and $(\bar{f})_\mu^*(s) = f_\gamma^*(s)$. From the equalities (15) and (16) we have

$$\begin{aligned} \sup_{|A|_\gamma=t} \int_A T^\gamma |f(x)|(x')^\gamma dx &= C_{k,\gamma} \sup_{\mu(\bar{A})=t} \int_{(y,0)+\bar{A}} |\bar{f}(z, \bar{z}')| d\mu(z, \bar{z}') \\ &= C_{k,\gamma} \int_0^t (\bar{f})_\mu^*(s) ds = \int_0^t f_\gamma^*(s) ds. \end{aligned}$$

Thus Lemma 2.5 is proved. ■

3. Proof of the theorems

Proof of Theorem 1.1 Note that the methods of proof used here are closer to that in ref. [16]. We choose a measurable set E_t , $t > 0$, such that

$$\{x \in \mathbb{R}_{k,+}^n : |f(x)| > f_\gamma^*(t)\} \subset E_t \subset \{x \in \mathbb{R}_{k,+}^n : |f(x)| \geq f_\gamma^*(t)\}.$$

Set

$$f_1(x) = (f(x) - f_\gamma^*(t))\chi_{E_t}(x), \quad f_2(x) = f(x) - f_1(x).$$

For any measurable set A in $\mathbb{R}_{k,+}^n$ with measure $|A|_\gamma = t$, we have

$$\int_A (g \otimes f_1)(x)(x')^\gamma dx = \int_{\mathbb{R}_{k,+}^n} f_1(y)(y')^\gamma dy \int_A T^\gamma g(x)(x')^\gamma dx.$$

Hence, from Lemma 2.2 we obtain

$$\begin{aligned} \int_A (g \otimes f_1)(x)(x')^\gamma dx &\leq C_{k,\gamma} \int_0^t g_\gamma^*(u) du \int_{\mathbb{R}_{k,+}^n} f_1(y)(y')^\gamma dy \\ &\leq C_{k,\gamma} \int_0^t g_\gamma^{**}(u) du \int_{\mathbb{R}_{k,+}^n} f_1(y)(y')^\gamma dy \\ &= C_{k,\gamma} \left(\int_{E_t} f(y)(y')^\gamma dy - t f_\gamma^*(t) \right) \int_0^t g_\gamma^{**}(u) du. \end{aligned}$$

Thus from equation (6) we have

$$\begin{aligned} (g \otimes f_1)_\gamma^{**}(t) &= \frac{1}{t} \sup_{|A|_\gamma=t} \int_A (g \otimes f_1)_\gamma(x)(x')^\gamma dx \\ &\leq C_{k,\gamma} (f_\gamma^{**}(t) - f_\gamma^*(t)) \int_0^t g_\gamma^{**}(u) du. \end{aligned}$$

Next, estimate $(g \otimes f_2)_\gamma^{**}(t)$. By using Lemma 2.5 and equality (6), we get

$$(T^\gamma g(x))_\gamma^*(s) \leq (T^\gamma g(x))_\gamma^{**}(s) = \frac{1}{s} \sup_{|A|_\gamma=s} \int_A T^\gamma g(x)(y')^\gamma dy = C_{k,\gamma} g_\gamma^{**}(s), \quad (17)$$

hence from equations (7) and (17) we obtain

$$\begin{aligned}(g \otimes f_2)(x) &\leq \int_0^\infty (f_2)_\gamma^*(u) (T^u g(x))_\gamma^*(u) du \\ &\leq C_{k,\gamma} \int_0^\infty (f_2)_\gamma^*(u) g_\gamma^{**}(u) du \\ &= C_{k,\gamma} \left(f_\gamma^*(t) \int_0^t g_\gamma^{**}(u) du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u) du \right).\end{aligned}$$

From equation (6) we have

$$(g \otimes f_2)_\gamma^{**}(t) \leq C_{k,\gamma} \left(f_\gamma^*(t) \int_0^t g_\gamma^{**}(u) du + \int_t^\infty f_\gamma^*(u) g_\gamma^{**}(u) du \right).$$

Then we get equation (3) and the proof of the theorem is completed. ■

Proof of Theorem 1.2 Since $K_\alpha \in WL_{Q/Q-\alpha,\gamma}(\mathbb{R}_{k,+}^n)$, we have

$$(K_\alpha)_\gamma^*(t) \leq \|K_\alpha\|_{WL_{Q/Q-\alpha,\gamma}} t^{\alpha/Q-1}, \quad (K_\alpha)_\gamma^{**}(t) \leq \frac{Q}{\alpha} \|K_\alpha\|_{WL_{Q/Q-\alpha,\gamma}} t^{\alpha/Q-1}.$$

By using inequality (3) we get the inequality (4). Hence the proof of the theorem is completed. ■

COROLLARY 3.1 Suppose that $s\Omega$ is homogeneous of degree zero on $\mathbb{R}_{k,+}^n$ and $\Omega \in L_{Q/(Q-\alpha),\gamma}(S_{k,+}^{n-1})$, $0 < \alpha < Q$. Then for the B -fractional integral $I_{\Omega,\alpha,\gamma}$ the following inequalities hold

$$(I_{\Omega,\alpha,\gamma} f)_\gamma^*(t) \leq (I_{\Omega,\alpha,\gamma} f)_\gamma^{**}(t) \leq A_2 \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right).$$

COROLLARY 3.2 For the B -Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}_{k,+}^n} T^\gamma |x|^{\alpha-Q} f(y) (y')^\gamma dy, \quad 0 < \alpha < Q$$

the following inequalities hold

$$(I_{\alpha,\gamma} f)_\gamma^*(t) \leq (I_{\alpha,\gamma} f)_\gamma^{**}(t) \leq A_3 \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right).$$

Proof of Theorem 1.3

(i) Let $1 < p < Q/\alpha$, $1 \leq r \leq s \leq \infty$, $f \in L_{p,r,\gamma}(\mathbb{R}_{k,+}^n)$ and $1/p - 1/q = \alpha/Q$. By using inequality (4) we have

$$\begin{aligned} \|K_\alpha \otimes f\|_{L_{q,s,\gamma}} &= \|(K_\alpha \otimes f)_\gamma^*(t)t^{1/q-1/s}\|_{L_s(0,\infty)} \\ &\leq A_1 \left(\int_0^\infty t^{s(\alpha/Q-1)+s/q-1} \left(\int_0^t f_\gamma^*(s)ds \right)^s dt \right)^{1/s} \\ &\quad + A_1 \left(\int_0^\infty \left(\int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s)ds \right)^s t^{s/q-1} dt \right)^{1/s}. \end{aligned}$$

From Lemma 2.3, for the validity of the inequality

$$\left(\int_0^\infty t^{s(\alpha/Q-1)+s/q-1} \left(\int_0^t f_\gamma^*(\tau)d\tau \right)^s dt \right)^{1/s} \leq C_1 \left(\int_0^\infty (t^{1/p} f_\gamma^*(t))^r \frac{dt}{t} \right)^{1/r}$$

necessary and sufficient condition is

$$\begin{aligned} &\sup_{t>0} \left(\int_t^\infty \tau^{s(\alpha/Q-1)+s/q-1} d\tau \right)^{1/s} \left(\int_0^t \tau^{(r/p-1)(1-r')} d\tau \right)^{1/r'} \\ &= s^{-1/s} \left(1 - \frac{\alpha}{Q} - \frac{1}{q} \right)^{-1/s} \left(\frac{p'}{r'} \right)^{1/r'} \sup_{t>0} t^{\alpha/Q-1+1/q+1-1/p} < \infty \\ &\iff \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}, \end{aligned}$$

where $C_1 \leq s^{-1/s} (1 - \alpha/Q - 1/q)^{-1/s} (p'/r')^{1/r'}$ $s^{1/s} (s')^{1/r'} = (p')^{1/s}$
 $(p's'/r')^{1/r'}$.

Furthermore, from Lemma 2.4, for the validity of the inequality

$$\left(\int_0^\infty \left(\int_t^\infty \tau^{\alpha/Q-1} f_\gamma^*(\tau)d\tau \right)^s t^{s/q-1} dt \right)^{1/s} \leq C_2 \left(\int_0^\infty (t^{1/p} f_\gamma^*(t))^r \frac{dt}{t} \right)^{1/r}$$

necessary and sufficient condition is

$$\begin{aligned} &\sup_{t>0} \left(\int_0^t \tau^{s/q-1} d\tau \right)^{1/s} \left(\int_t^\infty \tau^{(\alpha/Q-1)r'-r'/p+r'/r} d\tau \right)^{1/r'} \\ &= \left(\frac{q}{s} \right)^{1/s} (r')^{-1/r'} \left(\frac{1}{p} - \frac{\alpha}{Q} \right)^{-1/r'} \sup_{t>0} t^{\alpha/Q-(1/p-1/q)} < \infty \\ &\iff \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}, \end{aligned}$$

where $C_2 \leq (q/s)^{1/s} (r')^{-1/r'} (1/p - \alpha/Q)^{-1/r'} r^{1/s} (r')^{1/r'} = (qr/s)^{1/s} q^{1/r'}$.

By using these inequalities and applying equality (5) we obtain

$$\|K_\alpha \otimes f\|_{L_{q,s,\gamma}} \leq A_1 (C_1 + C_2) \|f\|_{L_{p,r,\gamma}}.$$

(ii) Let $p = Q/\alpha$, $r = 1$, and $f \in L_{Q/\alpha, 1, \gamma}(\mathbb{R}_{k, +}^n)$.

By using inequality (4) we have

$$\begin{aligned}\|K_\alpha \otimes f\|_{L_{\infty, \gamma}} &= \sup_{t>0} (K_\alpha \otimes f)_\gamma^*(t) \\ &\leq A_1 \sup_{t>0} \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right) \\ &\leq 2A_1 \int_0^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds = 2A_1 \|f\|_{L_{Q/\alpha, 1, \gamma}}.\end{aligned}$$

(iii) Let $p = 1$, $1 - 1/q = \alpha/Q$, $1 \leq r \leq \infty$ and $f \in L_{1, r, \gamma}(\mathbb{R}_{k, +}^n)$.

By using inequality (4) we have

$$\begin{aligned}\|K_\alpha \otimes f\|_{WL_{q, \gamma}} &= \sup_{t>0} t^{1/q} (K_\alpha \otimes f)_\gamma^*(t) \\ &\leq A_1 \sup_{t>0} t^{1/q} \left(t^{\alpha/Q-1} \int_0^t f_\gamma^*(s) ds + \int_t^\infty s^{\alpha/Q-1} f_\gamma^*(s) ds \right) \\ &= A_1 \sup_{t>0} \int_0^t f_\gamma^*(s) ds + A_1 \sup_{t>0} t^{1/q} \int_t^\infty s^{-1/q} f_\gamma^*(s) ds \\ &\leq 2A_1 \|f_\gamma^*\|_{L_1(0, \infty)} = 2A_1 \|f\|_{L_{1, r, \gamma}}.\end{aligned}$$

Thus the proof of Theorem 1.3 is completed. ■

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n -Dimensional Hankel transform and complex powers of Bessel operator

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In this paper, we study a version of the n -dimensional Hankel transform on certain spaces $\mathcal{H}_\mu$ which were studied in [Molina, S., 2003, A generalization of the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$ and the space of multipliers. *Actas del VII Congreso Dr. Antonio Monteiro*, pp. 49–56.]. Moreover, we introduce an n -dimensional generalization of Bessel operator and we have studied its properties in relation to the Hankel transform. Moreover, we study some application to the study of the fractional powers of Bessel operator on $L^2(\mathbb{R}_+^n)$.

Keywords: Hankel transform; Fractional powers; Bessel operator

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1. Introduction

The Hankel transformation of μ th order for functions of one dimension is defined by

$$\hbar_\mu f(y) = \int_0^\infty f(x) \sqrt{xy} J_\mu(xy) dx,$$

where $0 < y < \infty$, $\mu \in \mathbb{R}$, $\mu \geq -1/2$ and J_μ designates the well-known Bessel function of first kind and order μ . This transformation has been studied in the Zemanian space $\mathcal{H}_\mu$ and was extended to $\mathcal{H}'_\mu$ in ref. [1].

In this work, we extend the properties of n -dimensional $\hbar_\mu$ given by

$$h_\mu \phi(y) = \int_{\mathbb{R}_+^n} \phi(x_1, \dots, x_n) \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\mu_i}(x_i y_i)\} dx_1, \dots, dx_n, \quad (1)$$

where $\mu = (\mu_1, \dots, \mu_n)$, with $\mu_i \geq -1/2$ ($\mu_i \in \mathbb{R}$) for $i = 1, \dots, n$, and we obtain that is an automorphism on the n -dimensional spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$. In ref. [2] a generalization to

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n -dimensional of $\hbar_\mu$ on spaces $\mathcal{H}_\mu$ has been studied, but considering $\mu \in \mathbb{R}$. The importance of using n -tuples μ lies in the fact that we can study the operators $N_{\mu,i}$ and $M_{\mu,i}$ introduced in section 3. Moreover, we can define a generalization n -dimensional of the Bessel operators S_μ as

$$S_\mu f(x) = \Delta f(x) - \sum \frac{4\mu_i^2 - 1}{4x_i^2} f(x),$$

where $x = (x_1, \dots, x_n)$ and Δ is the Laplacian operator.

In section 5, we obtain a property of interaction between S_μ and $\hbar_\mu$ similar to the interaction between Δ and Fourier transform as

$$\hbar_\mu(S_\mu f)(y) = -|y|^2 \hbar_\mu(f)(y).$$

In section 7, we obtain some applications of those results to find solutions to some partial differential equations involving the Bessel operator.

In section 8, we consider the operator $-S_{\mu,2}$, the negative of Bessel operator in $L^2(\mathbb{R}_+^n)$ and we establish the non-negativity. Next, we study the fractional complex powers and we deduce some interesting properties of the operator $(-S_{\mu,2})^\alpha$, $\alpha \in \mathbb{C}$.

2. Preliminaries

In this section, we repeat the relevant material from ref. [3] without proofs. By $\mathbb{R}_+^n$, we shall denote the n -tuples of real numbers in $(0, \infty)$. By $\mathbb{N}_0$, we shall denote $\mathbb{N} \cup \{0\}$. If $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we shall put $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$ and the notations $x < y$ and $x \leq y$ will mean, respectively, $x_i < y_i$ and $x_i \leq y_i$ for $i = 1, \dots, n$. By $x = a$ ($x \in \mathbb{R}^n$, $a \in \mathbb{R}$), we mean $x_1 = x_2 = \dots = x_n = a$. We shall use the usual notation e_j for $j = 1, \dots, n$ for the members of the canonical basis of $\mathbb{R}^n$. For $k, m \in \mathbb{N}_0^n$, we set $|k| = k_1 + \dots + k_n$, $k! = k_1! \dots k_n!$,

$$\binom{k}{m} = \binom{k_1}{m_1} \dots \binom{k_n}{m_n},$$

$$\sum_{j=m}^k f(j) = \sum_{j_1=m_1}^{k_1} \sum_{j_2=m_2}^{k_2} \dots \sum_{j_n=m_n}^{k_n} f(j_1, \dots, j_n).$$

For $m \in \mathbb{R}^n$ and $x \in \mathbb{R}_+^n$, we set $x^m = x_1^{m_1} \dots x_n^{m_n}$. If $D_j = \partial/(\partial x_j)$, $j = 1, \dots, n$, we shall write

$$D^k = D_1^{k_1} \dots D_n^{k_n} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}.$$

By T_i , we denote the linear operators given by $T_i = x_i^{-1} \partial/\partial x_i$. Thus, $T_i T_j = T_j T_i$ and $T_i^n T_j^m = T_j^m T_i^n$ for any $n, m \in \mathbb{N}$. For $k \in \mathbb{N}_0^n$, we set $T^k = T_n^{k_n} T_{n-1}^{k_{n-1}} \dots T_1^{k_1}$. Moreover, if $k \in \mathbb{N}_0^n$, the following equality is valid

$$T^k \{\theta \cdot \varphi\} = \sum_{j=0}^k \binom{k}{j} T^{k-j} \theta \cdot T^j \varphi, \quad (2)$$

where ‘ $\cdot$ ’ denote the usual product.

For $\mu \in \mathbb{R}^n$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, the space $\mathcal{H}_\mu$ is the space of functions $\phi \in C^\infty(\mathbb{R}_+^n)$ such that for all $m, k \in \mathbb{N}_0^n$ verifies

$$\gamma_{m,k}^\mu(\phi) = \sup_{x \in \mathbb{R}_+^n} |x^m T^k \{x^{-\mu-1/2} \phi(x)\}| < \infty, \quad (3)$$

where $-\mu - 1/2 = (-\mu_1 - 1/2, -\mu_2 - 1/2, \dots, -\mu_n - 1/2)$, $\mathcal{H}_\mu$ is endowed with the topology generated by the family (3) of seminorms. Thus, $\mathcal{H}_\mu$ is a Fréchet space. The space $\mathcal{H}_\mu$ verifies that $\mathcal{H}_{\mu+qe_i} \subset \mathcal{H}_\mu$ for $q \in \mathbb{N}$ even and $\{e_i\}_{i=1,\dots,n}$ the canonical basis of $\mathbb{R}^n$. Moreover, the functions $\phi \in \mathcal{H}_\mu$ verify that $D^k \phi(x)$ is a rapid descent as $|x| \rightarrow \infty$ for every $k \in \mathbb{N}_0^n$. So, $\mathcal{H}_\mu \subset L^1(\mathbb{R}_+^n) \cap L^2(\mathbb{R}_+^n)$ for $\mu \geq -1/2$.

$\mathcal{H}'_\mu$ denotes the dual space of $\mathcal{H}_\mu$ and is also complete. If $f : \mathbb{R}_+^n \rightarrow \mathbb{C}$ be a locally integrable function on $\mathbb{R}_+^n$, such that f is of slow growth at infinite ($\exists r \in \mathbb{N}$ such that $|f(x)| = O(|x|^{-r})$ as $|x| \rightarrow \infty$) and $x^{\mu+1/2} f(x)$ is integrable on $Q = (0, 1)^n \subset \mathbb{R}_+^n$, then, f define a regular generalized function in $\mathcal{H}'_\mu$ given by

$$(f, \phi) = \int_{\mathbb{R}_+^n} f(x) \phi(x) dx$$

for all $\phi \in \mathcal{H}_\mu$. $\mathcal{H}_\mu$ can be identified with a subspace of $\mathcal{H}'_\mu$ if $\mu \geq -1/2$.

For all $r \in \mathbb{Z}^n$ and $\mu \in \mathbb{R}^n$, the mapping given by $h(\phi)(x) = x^r \phi(x)$ is an isomorphism from $\mathcal{H}_\mu$ onto $\mathcal{H}_{\mu+r}$. Moreover, the adjoint operator h' given by

$$(h'(f), \phi) = (f(x), x^r \phi(x)), \quad (f \in \mathcal{H}'_{\mu+r}, \phi \in \mathcal{H}_\mu)$$

is an isomorphism from $\mathcal{H}'_{\mu+r}$ onto $\mathcal{H}'_\mu$.

Let $\mathcal{O}$ denote the space of functions $\theta \in C^\infty(\mathbb{R}_+^n)$ with the property that for every $k \in \mathbb{N}_0^n$ there exists an integer n_k and $C \in \mathbb{R}_+$ such that

$$|(1 + |x|^2)^{n_k} T^k \theta(x)| < C \quad \forall x \in \mathbb{R}_+^n.$$

$\mathcal{O}$ is a space of multipliers of $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$.

If P and Q be polynomials of one variable such that Q has no zeros on $[0, \infty)$, then $P(|x|^2)/Q(|x|^2) \in \mathcal{O}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

3. The operators $N_{\mu,i}$, $M_{\mu,i}$ and S_μ

Let $\mu = (\mu_1, \dots, \mu_n)$, where $\mu_i \in \mathbb{R}, i = 1, \dots, n, x = (x_1, \dots, x_n)$. We define the operators $N_{\mu,i} = x_i^{\mu_i+1/2} \partial / \partial x_i x_i^{-\mu_i-1/2}$ and $M_{\mu,i} = x_i^{-\mu_i-1/2} \partial / \partial x_i x_i^{\mu_i+1/2}$.

LEMMA 3.1 *Let $\mu \in \mathbb{R}^n$, then*

(a) $N_{\mu,i} : \mathcal{H}_\mu \rightarrow \mathcal{H}_{\mu+e_i}$ is continuous linear mappings for $i = 1, \dots, n$. Also, $N_{\mu,i}$ is an isomorphism with $N_{\mu,i}^{-1}$ given by

$$N_{\mu,i}^{-1} \phi(x) = x_i^{\mu_i+1/2} \int_{\infty}^{x_i} t^{-\mu_i-1/2} \phi(x_1, \dots, t, \dots, x_n) dt.$$

(b) $M_{\mu,i} : \mathcal{H}_{\mu+e_i} \rightarrow \mathcal{H}_\mu$ is continuous linear mappings.

Proof (a) Let $\phi \in \mathcal{H}_\mu$ and $m, k \in \mathbb{N}_0^n$, it is easy to check that

$$\gamma_{m,k}^{\mu+e_i}(N_{\mu,i}\phi) = \gamma_{m,k+e_i}^\mu(\phi),$$

hence $N_{\mu,i}\phi \in \mathcal{H}_{\mu+e_i}$ and $N_{\mu,i}$ is continuous.

To prove that $N_{\mu,i}^{-1} : \mathcal{H}_{\mu+e_i} \rightarrow \mathcal{H}_\mu$ is continuous, we first consider $k \geq e_i$. Let $\phi \in \mathcal{H}_{\mu+e_i}$ then

$$T^k \{x^{-\mu-1/2} N_{\mu,i}^{-1} \phi(x)\} = T^{k-e_i} \{x^{-(\mu+e_i)-1/2} \phi(x)\},$$

hence that,

$$\gamma_{m,k}^\mu(N_{\mu,i}^{-1}\phi) = \gamma_{m,k-e_i}^{\mu+e_i}(\phi),$$

for all $m \in \mathbb{N}_0^n$. Now, we consider $k \in \mathbb{N}_0^n, k = (k_1, \dots, k_n)$ such that $k_i = 0$ and we denote $x_t = (x_1, \dots, t, \dots, x_n)$. Then,

$$\begin{aligned} \gamma_{m,k}^\mu(N_{\mu,i}^{-1}\phi) &= \sup_{x \in \mathbb{R}_+^n} |x^m T^k \{x^{-\mu-1/2} N_{\mu,i}^{-1} \phi\}| = \sup_{x \in \mathbb{R}_+^n} \left| x^m T^k \left\{ \int_\infty^{x_i} x_t^{-\mu-1/2} \phi(x_t) dt \right\} \right| \\ &\leq \sup_{x \in \mathbb{R}_+^n} \int_{x_i}^\infty |x^m T^k \{x_t^{-\mu-1/2} \phi(x_t)\}| dt \\ &\leq \sup_{x \in \mathbb{R}_+^n} \int_{x_i}^\infty |x_t^m t T^k \{x_t^{-\mu-1/2} t^{-1} \phi(x_t)\}| dt. \end{aligned} \quad (4)$$

Taking into account that

$$x_t^m t = \frac{1}{1+t^2} (x_t^{m+e_i} + x_t^{m+3e_i}). \quad (5)$$

From (4) and (5) we obtain

$$\begin{aligned} \gamma_{m,k}^\mu(N_{\mu,i}^{-1}\phi) &\leq \sup_{x \in \mathbb{R}_+^n} \int_{x_i}^\infty \left| \frac{1}{1+t^2} (x_t^{m+e_i} + x_t^{m+3e_i}) T^k \{x_t^{-(\mu+e_i)-1/2} \phi(x_t)\} \right| dt \\ &\leq C [\gamma_{m+e_i,k}^{\mu+e_i}(\phi) + \gamma_{m+3e_i,k}^{\mu+e_i}(\phi)]. \end{aligned}$$

We conclude that $N_{\mu,i}^{-1}\phi \in \mathcal{H}_\mu$ and $N_{\mu,i}^{-1}$ are continuous. Moreover $N_{\mu,i}^{-1} N_{\mu,i}\phi = \phi$, for all $\phi \in \mathcal{H}_\mu$ and $N_{\mu,i} N_{\mu,i}^{-1}\Psi = \Psi$, for all $\Psi \in \mathcal{H}_{\mu+e_i}$.

(b) Let $\phi \in \mathcal{H}_{\mu+e_i}$ and $m, k \in \mathbb{N}_0^n$. Then,

$$\gamma_{m,k}^\mu(M_{\mu,i}\phi) = \sup_{x \in \mathbb{R}_+^n} \left| x^m T^k \left\{ x^{-2\mu-1} \frac{\partial}{\partial x_i} \{x^{\mu+1/2} \phi(x)\} \right\} \right|. \quad (6)$$

We observe that

$$\begin{aligned} \frac{\partial}{\partial x_i} \{x^{\mu+1/2} \phi(x)\} &= \frac{\partial}{\partial x_i} \{x^{2\mu+e_i+1} x^{-(\mu+e_i)-1/2} \phi(x)\} = \\ &= (2\mu_i + 2) x^{2\mu+1} x^{-(\mu+e_i)-1/2} \phi(x) + x_i x^{2\mu+1} \frac{\partial}{\partial x_i} \{x^{-(\mu+e_i)-1/2} \phi(x)\}. \end{aligned} \quad (7)$$

From (6) and (7) we obtain

$$\gamma_{m,k}^\mu(M_{\mu,i}\phi) \leq (2\mu_i + 2) \gamma_{m,k}^{\mu+e_i}(\phi) + \sup_{x \in \mathbb{R}_+^n} \left| x^m T^k \left\{ x_i \frac{\partial}{\partial x_i} \{x^{-(\mu+e_i)-1/2} \phi(x)\} \right\} \right|. \quad (8)$$

Taking into account the equality (2), we obtain

$$\begin{aligned} T^k \left\{ x_i \frac{\partial}{\partial x_i} \{ x^{-(\mu+e_i)-1/2} \phi(x) \} \right\} &= x_i^2 T^{k+e_i} \{ x^{-(\mu+e_i)-1/2} \phi(x) \} \\ &\quad + 2k_i T^k \{ x^{-(\mu+e_i)-1/2} \phi(x) \}. \end{aligned} \quad (9)$$

By (8) and (9) we conclude

$$\begin{aligned} \gamma_{m,k}^\mu(M_{\mu,i}\phi) &\leq (2\mu_i + 2)\gamma_{m,k}^{\mu+e_i}(\phi) + \sup_{x \in \mathbb{R}_+^n} |x^{m+2e_i} T^{k+e_i} \{ x^{-(\mu+e_i)-1/2} \phi(x) \}| \\ &\quad + \sup_{x \in \mathbb{R}_+^n} |2k_i x^m T^k \{ x^{-(\mu+e_i)-1/2} \phi(x) \}| \\ &= (2\mu_i + 2k_i + 2)\gamma_{m,k}^{\mu+e_i}(\phi) + \gamma_{m+2e_i, k+e_i}^{\mu+e_i}(\phi); \end{aligned}$$

and this proves the lemma. ■

Now, we introduce the operator $S_\mu = \sum_{i=1}^n M_{\mu,i} N_{\mu,i}$. This operator is continuous in $\mathcal{H}_\mu$ into itself and verifies

$$S_\mu = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n \left(\frac{4\mu_i^2 - 1}{4x_i^2} \right).$$

Now, we consider the adjoint operators $N'_{\mu,i} : \mathcal{H}'_{\mu+e_i} \rightarrow \mathcal{H}'_\mu$ and $M'_{\mu,i} : \mathcal{H}'_\mu \rightarrow \mathcal{H}'_{\mu+e_i}$ defined by

$$(N'_{\mu,i} f, \phi) = (f, N_{\mu,i} \phi), \quad \text{for } \phi \in \mathcal{H}_\mu, f \in \mathcal{H}'_{\mu+e_i}$$

and

$$(M'_{\mu,i} g, \varphi) = (g, M_{\mu,i} \varphi), \quad \text{for } \varphi \in \mathcal{H}_{\mu+e_i}, g \in \mathcal{H}'_\mu.$$

By Lemma 3.1, $N'_{\mu,i}$ is an isomorphism. It is not hard to see that when $-M'_{\mu,i}$ is restricted to $\mathcal{H}_\mu$ for $\mu \geq -1/2$ coincide with $N_{\mu,i}$. A similar conclusion is obtained for $M_{\mu,i}$ and $-N'_{\mu,i}$. Then, we can define $N_{\mu,i}$ and $M_{\mu,i}$ as generalized differential operators on $\mathcal{H}'_\mu$ and $\mathcal{H}'_{\mu+e_i}$, respectively, as follows:

$$N_{\mu,i} = -M'_{\mu,i} \quad \text{and} \quad M_{\mu,i} = -N'_{\mu,i}. \quad (10)$$

4. Hankel transform in $\mathcal{H}_\mu$

LEMMA 4.1 *Let $\mu \geq -1/2$, $\phi \in \mathcal{H}_\mu$ and $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$, then*

- (a) $h_{\mu+e_i}(-x_i \phi) = N_{\mu,i} h_\mu(\phi)$,
- (b) $h_{\mu+e_i}(N_{\mu,i} \phi) = -y_i h_\mu(\phi)$,
- (c) $h_\mu(-x_i^2 \phi) = M_{\mu,i} N_{\mu,i} h_\mu(\phi)$ and $h_\mu(-|x|^2 \phi) = S_\mu h_\mu(\phi)$,
- (d) $h_\mu(M_{\mu,i} N_{\mu,i} \phi) = -y_i^2 h_\mu(\phi)$ and $h_\mu(S_\mu \phi) = -|y|^2 h_\mu(\phi)$. If $\phi \in \mathcal{H}_{\mu+e_i}$, then
- (e) $h_\mu(x_i \phi) = M_{\mu,i} h_{\mu+e_i}(\phi)$,
- (f) $h_\mu(M_{\mu,i} \phi) = y_i h_{\mu+e_i}(\phi)$.

Proof The proof is similar to one dimension (see ref. [1]). ■

In order to prove that the Hankel transformation is an automorphism on $\mathcal{H}_\mu$, we shall give some definitions and observations.

Let $\mu = (\mu_1, \dots, \mu_n)$, $r \in \mathbb{N}$ a positive integer, $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$, then, for each i , $1 \leq i \leq n$, we define the operators

$$N_{\mu,i}^r = N_{\mu+(r-1)e_i,i} N_{\mu+(r-2)e_i,i} \cdots N_{\mu+e_i,i} N_{\mu,i}. \quad (11)$$

If $r = 0$, we define,

$$N_{\mu,i}^0 \phi = \phi, \quad \phi \in \mathcal{H}_\mu. \quad (12)$$

Thus, by Lemma 3.1, $N_{\mu,i}^r \phi \in \mathcal{H}_{\mu+me_i}$ for $\phi \in \mathcal{H}_\mu$.

Taking into account the above definition, we define for $m \in \mathbb{N}_0^n$, $m = (m_1, \dots, m_n)$ the following operator

$$N_\mu^m = N_{\mu+m_1e_1+\dots+m_{n-1}e_{n-1},n}^{m_n} \cdots N_{\mu+m_1e_1,2}^{m_2} N_{\mu,1}^{m_1}, \quad (13)$$

It is clear that $N_\mu^m \phi \in \mathcal{H}_{\mu+m}$, for $\phi \in \mathcal{H}_\mu$.

LEMMA 4.2 *Let $\phi \in \mathcal{H}_\mu$ and $m \in \mathbb{N}_0^n$, then*

$$N_\mu^m \phi(x) = x^{\mu+m+1/2} T^m \{x^{-\mu-(1/2)} \phi(x)\}. \quad (14)$$

Proof First observe that

$$N_{\mu',i}^{m_i} \phi(x) = x^{\mu'+m_ie_i+1/2} T_i^{m_i} \{x^{-\mu'-1/2} \phi(x)\} \quad (15)$$

which is obtained by induction on m_i . Next, we obtain (14) by applying (15) n -times. ■

LEMMA 4.3 *Let $\phi \in \mathcal{H}_\mu$ and $k \in \mathbb{N}_0^n$, then*

- (a) $N_{\mu+k,i} \{x^k \phi\} = x^k N_{\mu,i} \phi$ for $1 \leq i \leq n$.
- (b) $N_{\mu+k,i}^m \{x^k \phi\} = x^k N_{\mu,i}^m \phi$ for $1 \leq i \leq n$ and $m \in \mathbb{N}_0$.
- (c) $N_{\mu+k}^m \{x^k \phi\} = x^k N_\mu^m \phi$ for all $m \in \mathbb{N}_0^n$.

Proof (a) First observe that $x^k \phi \in \mathcal{H}_{\mu+k}$, for $\phi \in \mathcal{H}_\mu$ and $k \in \mathbb{N}_0^n$. So,

$$N_{\mu+k,i} \{x^k \phi(x)\} = x^{\mu+k+1/2} \frac{\partial}{\partial x_i} \{x^{-(\mu+k)-1/2} x^k \phi(x)\} = x^k N_{\mu,i} \phi(x).$$

By applying (11) and (a) we obtain (b) by applying (13) and (b) we obtain (c). ■

THEOREM 4.1 *Let $\mu \in \mathbb{R}^n$, $\mu = (\mu_1, \dots, \mu_n)$, $\mu \geq -1/2$. Then, the Hankel transformation h_μ given by (1), is an automorphism on $\mathcal{H}_\mu$.*

Proof Let $\phi \in \mathcal{H}_\mu$, $h_\mu(\phi)(y) = \Phi(y)$, we shall prove that $\Phi \in \mathcal{H}_\mu$. Let $m, k \in \mathbb{N}_0$ then by application of (a) and (b) of Lemma 4.1 we obtain that

$$h_{\mu+m+k}(N_{\mu+k}^m\{x^k\phi\})(y) = (-y_1)^{m_1}(-y_2)^{m_2} \cdots (-y_n)^{m_n} h_{\mu+k}(x^k\phi)(y). \quad (16)$$

and

$$h_{\mu+k}(x^k\phi)(y) = (-1)^k N_\mu^k\{h_\mu\phi\}(y) = (-1)^k N_\mu^k\Phi(y). \quad (17)$$

By applying $h_{\mu+m+k}$ in both members of (c) in Lemma 4.3 we obtain that

$$(-1)^k(-y_1)^{m_1}(-y_2)^{m_2} \cdots (-y_n)^{m_n} N_\mu^k\Phi(y) = h_{\mu+m+k}(x^k N_\mu^m\phi)(y).$$

Taking into account (14), we obtain from the last equation

$$\begin{aligned} & (-1)^{k+m} y^{m+\mu+k+(1/2)} T^k\{y^{-\mu-(1/2)}\Phi(y)\} \\ &= \int_{\mathbb{R}_+^n} x^k N_\mu^m\phi(x) \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\mu_i+m_i+k_i}(x_i y_i)\} dx_1 \cdots dx_n \\ &= \int_{\mathbb{R}_+^n} x^{\mu+m+k+1/2} T^m\{x^{-\mu-1/2}\phi(x)\} \prod_{i=1}^n \{\sqrt{x_i y_i} J_{\mu_i+m_i+k_i}(x_i y_i)\} dx_1 \cdots dx_n, \end{aligned}$$

then,

$$\begin{aligned} & (-1)^{m+k} y^m T^k\{y^{-\mu-(1/2)}\Phi(y)\} \\ &= \int_{\mathbb{R}_+^n} x^{2\mu+2k+m+1} T^m\{x^{-\mu-(1/2)}\phi(x)\} \prod_{i=1}^n \left\{ \frac{J_{\mu_i+m_i+k_i}(x_i y_i)}{(x_i y_i)^{\mu_i+k_i}} \right\} dx_1 \cdots dx_n. \end{aligned} \quad (18)$$

Since $z^{-\nu} J_{\nu+n}$ is bounded in $(0, \infty)$ for $\nu > -1/2$ and $n \in \mathbb{N}$ [4, p. 50, Eq. (1)] then

$$\begin{aligned} & \left| x^{2\mu+2k+m+1} T^m\{x^{-\mu-(1/2)}\phi(x)\} \prod_{i=1}^n \left\{ \frac{J_{\mu_i+m_i+k_i}(x_i y_i)}{(x_i y_i)^{\mu_i+k_i}} \right\} \right| \\ & \leq C |x^{2\mu+2k+m+1} T^m\{x^{-\mu-1/2}\phi(x)\}|. \end{aligned}$$

From (2) we have that

$$T^m\{x^{-\mu-(1/2)}\phi(x)\} = x^{-\mu-(1/2)} \sum_{j=0}^m b_{m,j} \frac{D^j\phi(x)}{x^{2m-j}}, \quad \text{where } b_{m,j} \text{ are constants.} \quad (19)$$

Since $\phi \in \mathcal{H}_\mu$ and the fact that $D^j\phi$ is a rapid descent as $|x| \rightarrow \infty$ for all $j \in \mathbb{N}_0$, we conclude that the integral (18) is absolutely convergent. Then, $y^m T^k\{y^{-\mu-1/2}\Phi(y)\}$ is continuous for all $m, k \in \mathbb{N}_0$ and from (19), we deduce the continuity of $D^k\Phi$ for all $k \in \mathbb{N}_0$. Then, we conclude

that $\Phi \in C^\infty(\mathbb{R}_+^n)$. For each $1 \leq i \leq n$, choose $v_i \in \mathbb{N}$ such that $v_i > \mu_i + k_i + (m_i + 1)/2$, then

$$x^{2\mu+2k+m+1} < \prod_{i=1}^n (1 + x_i^2)^{v_i}.$$

Then,

$$\begin{aligned} \gamma_{m,k}^\mu(\Phi) &= \sup_{\mathbb{R}_+^n} |y^m T^k \{y^{-\mu-(1/2)} \Phi(y)\}| \leq C \int_{\mathbb{R}_+^n} |x^{2\mu+2k+m+1} T^m \{x^{-\mu-(1/2)} \phi(x)\}| dx \leq \\ &\leq C \int_{\mathbb{R}_+^n} \prod_{i=1}^n (1 + x_i^2)^{v_i} |T^m \{x^{-\mu-(1/2)} \phi(x)\}| dx \\ &\leq C \sup_{x \in \mathbb{R}_+^n} \left| \prod_{i=1}^n (1 + x_i^2)^{v_i+1} T^m \{x^{-\mu-(1/2)} \phi(x)\} \right| \int_{\mathbb{R}_+^n} \frac{1}{\prod_{i=1}^n (1 + x_i^2)} dx \\ &= C' \sum_{j \in \Gamma} c_j \gamma_{j,m}^\mu(\phi), \end{aligned}$$

for some finite subset Γ of $\mathbb{N}_0$. Thus, we conclude that h_μ is continuous from $\mathcal{H}_\mu$ into itself. Since $h_\mu = h_\mu^{-1}$ the proof is complete. ■

5. Hankel transform in $L^2(\mathbb{R}_+^n)$

LEMMA 5.1 Let $\mu > -1/2$ and $\phi, \varphi \in \mathcal{H}_\mu$, then

$$\int_{\mathbb{R}_+^n} \phi(y) \overline{h_\mu \varphi(y)} dy = \int_{\mathbb{R}_+^n} h_\mu \phi(y) \overline{\varphi(y)} dy.$$

Proof For $\mu > -1/2$ the kernel in the Hankel transform (1) are bounded in $\mathbb{R}_+^n$ and $\mathcal{H}_\mu \subset L^2(\mathbb{R}_+^n)$. From this fact and Fubini's theorem the lemma follows. ■

COROLLARY 5.1 Let $\mu > -1/2$ and $\phi \in \mathcal{H}_\mu$, then

$$\|h_\mu \phi\|_2 = \|\phi\|_2.$$

For $\alpha > -1$ the Laguerre polynomials L_k^α are defined by

$$L_k^\alpha(x) = x^{-\alpha} e^x \frac{1}{k!} \frac{d^k}{dx^k} \{e^{-x} x^{k+\alpha}\},$$

for $x > 0$ and $k \in \mathbb{N}_0$. The Laguerre polynomials satisfy the following orthogonality properties ([5], p. 7):

$$\int_0^\infty L_k^\alpha(x) L_j^\alpha(x) e^{-x} x^\alpha dx = \frac{\Gamma(k + \alpha + 1)}{\Gamma(k + 1)} \delta_{jk}.$$

The sequence of polynomials $\{\tilde{L}_k^\alpha\}$ given by

$$\tilde{L}_k^\alpha = \left(\frac{\Gamma(k + 1)}{\Gamma(k + \alpha + 1)} \right)^{1/2} L_k^\alpha$$

is a complete orthonormal system of $L^2(\mathbb{R}_+, e^{-x} x^\alpha dx)$ ([6], p. 61).

The functions given by

$$e^{-x^2/2} x^{\mu+(1/2)} \tilde{L}_k^\mu(x^2) \quad (k \in \mathbb{R}_0), \quad (20)$$

for $x > 0$, $\mu > -1$, form a complete orthogonal system in $L^2(\mathbb{R}_+)$ and verifies that

$$\int_0^\infty e^{-x^2} x^{2\mu+1} \tilde{L}_k^\mu(x^2) \tilde{L}_j^\mu(x^2) dx = \frac{1}{2} \delta_{kj}.$$

We consider for $x \in \mathbb{R}_+^n$, $x = (x_1, \dots, x_n)$, $\mu \in \mathbb{R}^n$, $\mu = (\mu_1, \dots, \mu_n)$, $\mu \geq -1/2$ and $k \in \mathbb{R}_0$, $k = (k_1, \dots, k_n)$ the following functions:

$$\Psi_k^\mu(x) = \prod_{i=1}^n e^{-x_i^2/2} x_i^{\mu_i+(1/2)} \tilde{L}_{k_i}^{\mu_i}(x_i^2) = e^{-|x|^2/2} x^{\mu+(1/2)} \prod_{i=1}^n \tilde{L}_{k_i}^{\mu_i}(x_i^2). \quad (21)$$

The sequence $\{\Psi_k^\mu\}$ is a complete orthogonal system in $L^2(\mathbb{R}_+^n)$ ([5], pp. 153–154). Note that the functions (21) are in $\mathcal{H}_\mu$ so, this is a dense subspace of $L^2(\mathbb{R}_+^n)$.

Thus, by Corollary 5.1 and the density of $\mathcal{H}_\mu$ in $L^2(\mathbb{R}_+^n)$, we consider the extension of Hankel transform to $L^2(\mathbb{R}_+^n)$ as the only continuous extension to $L^2(\mathbb{R}_+^n)$ of the operator $h_\mu : \mathcal{H}_\mu \rightarrow L^2(\mathbb{R}_+^n)$.

LEMMA 5.2 *Let $\mu > -1/2$ and $f, g \in L^2(\mathbb{R}_+^n)$, then*

$$\int_{\mathbb{R}_+^n} f(y) \overline{h_\mu g(y)} dy = \int_{\mathbb{R}_+^n} h_\mu f(y) \overline{g(y)} dy$$

Proof It follows directly by density of $\mathcal{H}_\mu$ in $L^2(\mathbb{R}_+^n)$ and by jointly continuity of interior product in $L^2(\mathbb{R}_+^n)$. ■

COROLLARY 5.2 *Let $\mu > -1/2$ and $f \in L^2(\mathbb{R}_+^n)$, then*

$$\|h_\mu f\|_2 = \|f\|_2.$$

6. Hankel transform in $\mathcal{H}'_\mu$

Throughout this section, we consider $\mu \in \mathbb{R}^n$, $\mu \geq -1/2$. We denote by h'_μ , the generalized Hankel transform defined by transposition as

$$(h'_\mu f, \phi) = (f, h_\mu \phi), \quad (f \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu). \quad (22)$$

We have that the generalized Hankel transform h'_μ is an automorphism on $\mathcal{H}'_\mu$ and

$$(h'_\mu)^{-1} = (h_\mu^{-1})' = h'_\mu.$$

LEMMA 6.1 *Let $\mu = (\mu_1, \dots, \mu_n)$, $\mu \geq -1/2$, $e_1, \dots, e_n$ the canonical basis of $\mathbb{R}^n$ and $f \in \mathcal{H}'_\mu$, then*

- (a) $h'_{\mu+e_i}(-x_i f) = N_{\mu,i} h'_\mu f$,
- (b) $h'_{\mu+e_i}(N_{\mu,i} f) = -y_i h'_\mu f$,
- (c) $h'_\mu(-x_i^2 f) = M_{\mu,i} N_{\mu,i} h'_\mu f$,

$$(d) \ h'_\mu(M_{\mu,i}N_{\mu,i}f) = -y_i^2 h'_\mu f.$$

If $f \in \mathcal{H}'_{\mu+e_i}$, then

$$(e) \ h'_\mu(x_i f) = M_{\mu,i} h'_{\mu+e_i} f,$$

$$(f) \ h'_\mu(M_{\mu,i}f) = y_i h'_{\mu+e_i} f.$$

Proof The proof is a simple application of definition of generalized Hankel transform given by (22) and the definitions of generalized operators $N_{\mu,i}$ and $M_{\mu,i}$ given by (10). ■

Remark 6.1 $L^2(\mathbb{R}_+^n) \subset \mathcal{H}'_\mu$. Indeed, the functions in $L^2(\mathbb{R}_+^n)$ verifies the conditions given in section 2 for a function $f : \mathbb{R}_+^n \rightarrow \mathbb{C}$ defines a regular generalized function in $\mathcal{H}'_\mu$. Thus, for $f \in L^2(\mathbb{R}_+^n)$ have sense two definitions of $h_\mu f$. In the following proposition we shall see the coincidence of both definitions. We denote with T_g to the regular generalized distribution defined by the function g .

PROPOSITION 6.1 *Let $f \in L^2(\mathbb{R}_+^n)$ then $h'_\mu T_f = T_{h_\mu f}$*

Proof Let $\varphi \in \mathcal{H}_\mu$, then

$$(T_{h_\mu f}, \varphi) = \int_{\mathbb{R}_+^n} h_\mu f(x) \varphi(x) dx = \int_{\mathbb{R}_+^n} f(x) h_\mu \varphi(x) = (T_f, h_\mu \varphi) = (h'_\mu T_f, \varphi). \quad \blacksquare$$

COROLLARY 6.1 *Let $f \in \mathcal{H}'_\mu$. Then $f \in L^2(\mathbb{R}_+^n)$ if and only if $h_\mu f \in L^2(\mathbb{R}_+^n)$.*

7. Application

Let $\mu \geq -1/2$ and S_μ^k be the operator S_μ iterated k -times. Then, for $f \in \mathcal{H}'_\mu$ we obtain

$$h_\mu(S_\mu f) = h_\mu \left(\left[\sum_{i=1}^n M_{\mu,i} N_{\mu,i} \right] f \right) = \sum_{i=1}^n -y_i^2 h_\mu f = -|y|^2 h_\mu f$$

and

$$h_\mu(S_\mu^k f) = (-|y|^2)^k h_\mu f. \quad (23)$$

Suppose that we want to compute a function $f \in \mathcal{H}'_\mu$ which satisfies the following differential equation defined on $\mathbb{R}_+^n$

$$P(S_\mu) f = g, \quad (24)$$

where $g \in \mathcal{H}'_\mu$, the polynomial P has no zeros on $(-\infty, 0]$. By applying the Hankel transform in both members of (24) and taking into account (23), we obtain

$$P(-|y|^2) h_\mu f = h_\mu g.$$

Since P has no roots on $(-\infty, 0]$ then the function $1/(P(-|y|^2))$ is a member of $\mathcal{O}$ (see section 2). We conclude that

$$f = h_\mu \left(\frac{1}{P(-|y|^2)} h_\mu g \right)$$

is a solution of equation (24). Moreover, this solution is unique in $\mathcal{H}'_\mu$.

8. Fractional powers of the Bessel operator in $L^2(\mathbb{R}_+^n)$

Throughout this section we consider $\mu \in \mathbb{R}^n$, $\mu \geq -1/2$. We consider the part of the Bessel operator in $L^2(\mathbb{R}_+^n)$, i.e. the operator $S_{\mu,2}$ with domain

$$D(S_{\mu,2}) = \{f \in L^2(\mathbb{R}_+^n) : S_{\mu}f \in L^2(\mathbb{R}_+^n)\}.$$

Taking into account Corollary 6.1 we have that

$$D(S_{\mu,2}) = \{f \in L^2(\mathbb{R}_+^n) : -|y|^2 h_{\mu} f \in L^2(\mathbb{R}_+^n)\}$$

and

$$S_{\mu,2}f = -h_{\mu}|y|^2 h_{\mu}f, \quad (f \in D(S_{\mu,2})).$$

PROPOSITION 8.1 *For $S_{\mu,2}$ given as above, then*

- (a) $S_{\mu,2}$ is injective.
- (b) $S_{\mu,2}$ is closed.
- (c) $-S_{\mu,2}$ is *m*-accretive.

Proof

- (a) Let $f \in D(S_{\mu,2})$ such that $S_{\mu,2}f = 0$. Taking Hankel transform we obtain that $-|y|^2 h_{\mu} f(y) = 0$, so $f = 0$.
- (b) Let $\{f_n\}_{n \in \mathbb{N}} \subset D(S_{\mu,2})$ such that

$$\| \cdot \|_2 - \lim_{n \rightarrow \infty} f_n = f \quad \text{and} \quad \| \cdot \|_2 - \lim_{n \rightarrow \infty} S_{\mu,2}f_n = g.$$

Since the convergence in $L^2(\mathbb{R}_+^n)$ implies the weak convergence in $\mathcal{D}'(\mathbb{R}_+^n)$, we have

$$(S_{\mu,2}f, \varphi) = \lim_{n \rightarrow \infty} (f_n, S_{\mu,2}\varphi) = \lim_{n \rightarrow \infty} (S_{\mu,2}f_n, \varphi) = (g, \varphi), \quad (\varphi \in \mathcal{D}(\mathbb{R}_+^n)).$$

Hence $S_{\mu,2}f = g$ and $S_{\mu,2}$ is closed.

- (c) Let $\lambda > 0$, we will prove that $\lambda - S_{\mu,2}$ is bijective. In fact, given $f \in L^2(\mathbb{R}_+^n)$, taking Hankel transforms we obtain that the equation

$$(\lambda - S_{\mu,2})u = f$$

has a unique solution u given by

$$u(x) = h_{\mu}((\lambda + |y|^2)^{-1} h_{\mu} f(y))(x).$$

As h_{μ} is an isomorphism in $L^2(\mathbb{R}_+^n)$ and $(\lambda + |y|^2)^{-1} \in L^{\infty}(\mathbb{R}_+^n)$ then $u \in L^2(\mathbb{R}_+^n)$. Moreover, taking into account that h_{μ} is an isometry of $L^2(\mathbb{R}_+^n)$ into itself, then

$$\begin{aligned} \|(\lambda - S_{\mu,2})^{-1}f\|_2 &= \|u\|_2 = \|h_{\mu}((\lambda + |y|^2)^{-1} h_{\mu} f)\|_2 \\ &= \|(\lambda + |y|^2)^{-1} h_{\mu} f\|_2 \leq \lambda^{-1} \|h_{\mu} f\|_2 = \lambda^{-1} \|f\|_2 \end{aligned}$$

Hence $S_{\mu,2}$ is *m*-accretive. ■

PROPOSITION 8.2 *The spectral angle of $-S_{\mu,2}$ is $\omega(-S_{\mu,2}) = 0$. Moreover, (the spectrum) $\sigma(-S_{\mu,2}) = [0, \infty)$.*

Proof As the operator $-S_{\mu,2}$ is m-acretive we know that $-S_{\mu,2}$ is sectorial (see ref. [7]) and $\omega(-S_{\mu,2}) \leq \pi/2$.

Fix $z \in \mathbb{C} \setminus [0, \infty)$ with $\operatorname{Re} z \geq 0$, then

$$z + S_{\mu,2} : D(S_{\mu,2}) \subset L^2(\mathbb{R}_+^n) \longrightarrow L^2(\mathbb{R}_+^n)$$

is bijective. We first observe that if $z = |z| e^{i\theta}$, then

$$|z - |y|^2|^{-1} \leq |z|^{-1} |\sin \theta|^{-1}. \quad (25)$$

Given $g \in L^2(\mathbb{R}_+^n)$, as in the proof of Proposition 8.1 we obtain that

$$f = h_\mu((z - |y|^2)^{-1} h_\mu g) \quad (26)$$

is the unique solution of equation

$$(z + S_{\mu,2})f = g,$$

and from (25) we obtain that $(z - |y|^2)^{-1} \in L^\infty(\mathbb{R}_+^n)$ and $f \in L^2(\mathbb{R}_+^n)$. Therefore $\sigma(-S_{\mu,2}) \subset [0, \infty)$. Let us prove that $-S_{\mu,2}$ is sectorial of type ε for all $\varepsilon > 0$. Let θ be such that $\varepsilon < |\theta| < \pi/2$, $z = \lambda e^{i\theta}$ and $g \in L^2(\mathbb{R}_+^n)$. From (26) we obtain that

$$\|(z + S_{\mu,2})^{-1} g\|_2 = \|f\|_2 = \|h_\mu((z - |y|^2)^{-1} h_\mu g)\|_2 \leq |z|^{-1} |\sin \theta|^{-1} \|g\|_2$$

hence

$$\|\lambda(\lambda e^{i\theta} - S_{\mu,2})^{-1}\|_2 \leq |\sin \theta|^{-1}$$

We conclude that $\omega(-S_{\mu,2}) = 0$.

Finally, we must prove that $\sigma(-S_{\mu,2}) = [0, \infty)$. Let $M_{|y|^2}$ be the multiplication operator

$$\begin{aligned} M_{|y|^2} : D(M_{|y|^2}) &\subset L^2(\mathbb{R}_+^n) \longrightarrow L^2(\mathbb{R}_+^n) \\ D(M_{|y|^2}) &= \{h \in L^2(\mathbb{R}_+^n) : |y|^2 h \in L^2(\mathbb{R}_+^n)\} \end{aligned}$$

We know that

$$-S_{\mu,2} = h_\mu |y|^2 h_\mu \quad (27)$$

then

$$\begin{aligned} z \notin \sigma(M_{|y|^2}) &\iff \text{the function } y \longmapsto (z - |y|^2)^{-1} \in L^\infty(\mathbb{R}_+^n) \\ &\iff h_\mu(z - |y|^2)^{-1} h_\mu \text{ is bijection on } L^2(\mathbb{R}_+^n) \\ &\iff z + S_{\mu,2} \text{ is bijection on } L^2(\mathbb{R}_+^n) \iff z \notin \sigma(-S_{\mu,2}), \end{aligned}$$

hence $\sigma(M_{|y|^2}) = \sigma(-S_{\mu,2})$. Since $\sigma(M_{|y|^2}) = [0, \infty)$ and the assertion follows. $\blacksquare$

We can now study the fractional powers $(-S_{\mu,2})^\alpha$, in view of the non-negativity of $-S_{\mu,2}$. From (27), it is clear that

$$D(-S_{\mu,2}) = \{f \in L^2(\mathbb{R}_+^n) : |y|^2 h_\mu f \in L^2(\mathbb{R}_+^n)\}.$$

It is to be expected that

$$(-S_{\mu,2})^\alpha f = h_\mu |y|^{2\alpha} h_\mu f \quad (28)$$

and

$$D((-S_{\mu,2})^\alpha) = \{f \in L^2(\mathbb{R}_+^n) : |y|^{2\alpha} h_\mu f \in L^2(\mathbb{R}_+^n)\}. \quad (29)$$

We will see that (28) and (29) are valid for all $\alpha \in \mathbb{C}$.

For this purpose, we first consider the case of complex fractional powers with positive real part.

In the following α will be a complex number with $\operatorname{Re} \alpha > 0$ and m will be an integer such that $m > \operatorname{Re} \alpha$. Consider the Balakrishnan operator J^α defined on $D((-S_{\mu,2})^m)$ given by

$$J^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1} (-S_{\mu,2}(\lambda - S_{\mu,2})^{-1})^m \phi \, d\lambda \quad (\phi \in D((-S_{\mu,2})^m)). \quad (30)$$

Under the conditions stated above, we have the following proposition:

PROPOSITION 8.3 *Given $\phi \in D((-S_{\mu,2})^m)$, then*

$$J^\alpha \phi = h_\mu |y|^{2\alpha} h_\mu \phi. \quad (31)$$

Proof Applying Hankel transform in (30) and taking into account the following equality

$$h_\mu (-S_{\mu,2}(\lambda - S_{\mu,2})^{-1})^m = (|y|^2)^m (\lambda + |y|^2)^{-m} h_\mu$$

we obtain that

$$h_\mu J^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1} (|y|^2)^m (\lambda + |y|^2)^{-m} h_\mu \phi \, d\lambda.$$

From the representation of fractional powers of multiplication operator by z with $z \in \mathbb{C} \setminus \mathbb{R}_-$ given by

$$z^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1} (z(\lambda + z))^{-m} \phi \, d\lambda.$$

(see ref. [7], Remark 3.1.1), the assertion follows. ■

By the theorem of representation of the fractional powers (see ref. [7], Theorem 5.2.1) the operator $(-S_{\mu,2})^\alpha$ can be described as the operator

$$(-S_{\mu,2} + 1)^n J^\alpha (-S_{\mu,2} + 1)^{-n} \quad (32)$$

with $\operatorname{Re} \alpha > 0$ and $n > \operatorname{Re} \alpha$, defined on its natural domain

$$\{f \in L^2(\mathbb{R}_+^n) : J^\alpha [(-S_{\mu,2} + 1)^{-n}] f \in D((-S_{\mu,2})^n)\}.$$

PROPOSITION 8.4

$$D((-S_{\mu,2})^\alpha f \in L^2(\mathbb{R}_+^n) : |y|^{2\alpha} h_\mu f \in L^2(\mathbb{R}_+^n))\}$$

and

$$(-S_{\mu,2})^\alpha = h_\mu |y|^{2\alpha} h_\mu$$

Proof By induction on n , it is easy to check that

$$h_\mu (-S_\mu + 1)^n = (|y|^2 + 1)^n h_\mu \quad (33)$$

and

$$h_\mu (-S_\mu + 1)^{-n} = (|y|^2 + 1)^{-n} h_\mu \quad (34)$$

are valid in $\mathcal{H}'_\mu$.

By (31), (33) and (34) we obtain that

$$\begin{aligned}
 D((-S_{\mu,2})^\alpha) &= \{f \in L^2(\mathbb{R}_+^n) : J^\alpha[(-S_{\mu,2} + 1)^{-n}]f \in D((-S_{\mu,2})^n)\} \\
 &= \{f \in L^2(\mathbb{R}_+^n) : h_\mu[(-S_{\mu,2} + 1)^n J^\alpha(-S_{\mu,2} + 1)^{-n} f] \in L^2(\mathbb{R}_+^n)\} \\
 &= \{f \in L^2(\mathbb{R}_+^n) : (|y|^2 + 1)^n |y|^{2\alpha} (|y|^2 + 1)^{-n} h_\mu f \in L^2(\mathbb{R}_+^n)\} \\
 &= \{f \in L^2(\mathbb{R}_+^n) : |y|^{2\alpha} h_\mu f \in L^2(\mathbb{R}_+^n)\}
 \end{aligned}$$

and

$$(-S_{\mu,2})^\alpha = h_\mu |y|^{2\alpha} h_\mu. \quad \blacksquare$$

Let us now consider the case of complex powers with negative real part. Let $\alpha \in \mathbb{C}$, $\operatorname{Re} \alpha > 0$. By definition 7.1.1 of ref. [7], the power $(-S_{\mu,2})^{-\alpha}$ is defined as the inverse of the injective operator $(-S_{\mu,2})^\alpha$.

PROPOSITION 8.5

$$D(-S_{\mu,2})^{-\alpha} = \{f \in L^2(\mathbb{R}_+^n) : |y|^{-2\alpha} h_\mu f \in L^2(\mathbb{R}_+^n)\}$$

and

$$(-S_{\mu,2})^{-\alpha} = h_\mu |y|^{-2\alpha} h_\mu.$$

Proof By the definition of the power $(-S_{\mu,2})^{-\alpha}$ and Proposition 8.4, we have that

$$\begin{aligned}
 f \in D((-S_{\mu,2})^{-\alpha}) &\iff \exists g \in D((-S_{\mu,2})^\alpha) \text{ and } f = (-S_{\mu,2})^\alpha g \\
 &\iff g \in L^2(\mathbb{R}_+^n), |y|^{2\alpha} h_\mu g \in L^2(\mathbb{R}_+^n) \text{ and } f = h_\mu [|y|^{2\alpha} h_\mu g] \\
 &\iff f \in L^2(\mathbb{R}_+^n), \text{ and } g = h_\mu |y|^{-2\alpha} h_\mu f
 \end{aligned}$$

and

$$(-S_{\mu,2})^{-\alpha} f = g = h_\mu (|y|^{-2\alpha} h_\mu f)$$

which proves the proposition. \blacksquare

Finally, we consider imaginary powers of Bessel operator. According to the definition 7.1.2 of ref. [7], the power $(-S_{\mu,2})^{i\tau}$ is defined as

$$(-S_{\mu,2})^{i\tau} = (-S_{\mu,2} + 1)^2 (-S_{\mu,2})^{-1} (-S_{\mu,2})^{1+i\tau} (-S_{\mu,2} + 1)^{-2}.$$

Thus we have the following characterization for the power $(-S_{\mu,2})^{i\tau}$.

PROPOSITION 8.6

$$D(-S_{\mu,2})^{i\tau} = \{f \in L^2(\mathbb{R}_+^n) : |y|^{2i\tau} h_\mu f \in L^2(\mathbb{R}_+^n)\}$$

and

$$(-S_{\mu,2})^{i\tau} = h_\mu |y|^{2i\tau} h_\mu.$$

Proof The proof is a direct consequence of Proposition 8.4 by using (33) and (34). \blacksquare

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A large family of semi-classical polynomials of class one

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The aim of this work is to study the family of semi-classical sequences of polynomials fulfilling the three-term recurrence relation with $\beta_n = (-1)^n \beta_0$, $n \geq 0$, where $\beta_0 \in \mathbb{C}$. Some information about the shape of polynomials intervening in the distributional equation in which the corresponding linear functional (or form) satisfies are given, and two identities satisfied by the appropriate structure relation coefficients are established. These results are determinative to work out in detail the particular case when $\beta_0 \neq 0$ and the class is equal to one. In this case, we give explicitly the recurrence coefficients, the structure relation coefficients and an integral representation.

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1. Introduction

It is a matter of fact that a semi-classical sequence of polynomials of class $s \in \mathbb{N}$ satisfies two statements: a three-term recurrence relation and the corresponding normalized linear functional satisfies a distributional equation [1–7].

A crucial problem aims to deduce the recurrence coefficients, denoted generally by β_n and γ_{n+1} , $n \geq 0$, as a function of the coefficients contained in polynomials intervening in the corresponding distributional equation. This leads to a coupled algebraic system of equations giving β_n and γ_{n+1} that are called Laguerre–Freud equations (see [2, 8] for $s \leq 2$). Because of nonlinearity of the above-mentioned system, recurrence coefficients are explicitly known only in two cases: classical case ($s = 0$) [9] and symmetric (i.e. $\beta_n = 0$, $n \geq 0$) semi-classical case of class one [1]. Thus, many authors have used some elementary transformations under classical polynomials or under their corresponding linear functionals in order to give some solutions when $s \geq 1$. For instance, we can mention the adjunction of a finite number of Dirac’s masses to classical linear functionals [10–12], the product and the division of a linear functional by a polynomial [8, 12, 13], evaluation of Hankel determinants [14], and so on [15, 16, 18].

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Semi-classical sequences given in these cited references are just isolated examples either in a symmetrical case or in a nonsymmetrical case.

In this paper, we are interested in a whole family of semi-classical sequences $\{B_n\}_{n \geq 0}$ of class $s \geq 0$, verifying the following three-term recurrence relation:

$$\begin{cases} B_{n+2}(x) = (x - (-1)^{n+1}\beta_0)B_{n+1}(x) - \gamma_{n+1}B_n(x), & n \geq 0, \\ B_0(x) = 1, \quad B_1(x) = x - \beta_0, \end{cases} \quad (1)$$

with $\gamma_{n+1} \in \mathcal{C} - \{0\}$, $n \geq 0$, and $\beta_0 \in \mathcal{C}$.

This family is wide enough to accommodate all symmetric semi-classical sequences of polynomials as well as some particular non-symmetric semi-classical sequences. Indeed, if we take beside $\beta_0 = 0$, we meet the symmetric semi-classical sequences.

Sequences of polynomials fulfilling equation (1) have been the subject of some works: for instance, Maroni [16] characterized such sequences by a particular quadratic decomposition. But in his paper, Maroni did not treat the case where those sequences are semi-classical, so no semi-classical example of these sequences was given. Later, Alaya and Maroni [1] gave all symmetric semi-classical sequences of polynomial of class one. They obtained them as a particular case of the so-called Laguerre–Hahn sequences.

The layout of this paper will be as follows: in section 2, we find preliminaries and notations used in the sequel. Some characterizations of sequences fulfilling equation (1) are also proved. Section 3 is devoted to formalize a shape of polynomial, noted by Φ , intervening in the distributional equation, when the sequence is semi-classical. Two relations between structure relation coefficients are also established. In section 4, as an exploitation of the results in section 3, we treat in detail all semi-classical sequences of class one satisfying equation (1) with $\beta_0 \neq 0$ (since the symmetric case is treated in ref. [1]). We show that only one canonical case appear: $\Phi(x) = x^3 - x$. In such a case, the recurrence coefficients, the structure relation coefficients and an integral representation are explicitly given. The method used here, unlike in ref. [3], consists of directly solving a nonlinear system.

2. Preliminaries and notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathcal{C}$, and $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f \rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$; in particular, we denote by $(u)_n = \langle u, x^n \rangle$, $n \geq 0$, the moments of u . We have the following operations on $\mathcal{P}'$.

The left multiplication of a linear functional by a polynomial is given by

$$\langle gu, f \rangle := \langle u, gf \rangle, \quad f, g \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (2)$$

The derivative of a linear functional is

$$\langle u', f \rangle := -\langle u, f' \rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (3)$$

The homothetic and translation of a linear functional are given by

$$\langle h_a u, f \rangle := \langle u, h_a f \rangle, \quad \langle \tau_{-b} u, f \rangle := \langle u, \tau_b f \rangle, \quad a \in \mathcal{C} - \{0\}, \quad b \in \mathcal{C}, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \quad (4)$$

where

$$(h_a f)(x) := f(ax) \quad \text{and} \quad (\tau_b f)(x) = f(x - b). \quad (5)$$

The division of a linear functional by a polynomial of first degree is

$$\langle (x - c)^{-1}u, f \rangle := \langle u, \theta_c f \rangle, \quad (6)$$

where

$$\theta_c f(x) := \frac{f(x) - f(c)}{x - c}, \quad c \in \mathcal{C}, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}'. \quad (7)$$

The even part of a linear functional is

$$\langle \sigma u, f \rangle := \langle u, \sigma f \rangle, \quad f \in \mathcal{P}, \quad u \in \mathcal{P}', \quad (8)$$

where

$$(\sigma f)(x) = f(x^2). \quad (9)$$

DEFINITION 2.1 A sequence of polynomials $\{B_n\}_{n \geq 0}$ is said to be a monic orthogonal polynomial sequence (MOPS) with respect to a linear functional u if:

- (i) $\deg B_n = n$ and the leading coefficient of $B_n(x)$ is equal to 1.
- (ii) $\langle u, B_n B_m \rangle = r_n \delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$.

In such conditions, we say that u is regular. In the following, we take all regular linear functionals u normalized, that is, $(u)_0 = 1$.

According to Favard's theorem [17], a sequence of monic orthogonal polynomials satisfies a three-term recurrence relation

$$\begin{cases} B_{n+2}(x) = (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), & n \geq 0, \\ B_0(x) = 1, \quad B_1(x) = x - \beta_0, \end{cases} \quad (10)$$

with $(\beta_n, \gamma_{n+1}) \in \mathcal{C} \times \mathcal{C} - \{0\}$, $n \geq 0$.

A shifting leaves the property of orthogonality invariant; the sequence $\{\tilde{B}_n\}_{n \geq 0}$, where $\tilde{B}_n(x) = a^{-n}B_n(ax + b)$, $n \geq 0$, $a \in \mathcal{C} - \{0\}$, $b \in \mathcal{C}$, satisfies the recurrence relation [5]

$$\begin{cases} \tilde{B}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{B}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{B}_n(x), & n \geq 0, \\ \tilde{B}_0(x) = 1, \quad \tilde{B}_1(x) = x - \tilde{\beta}_0, \end{cases}$$

with $\tilde{\beta}_n = (\beta_n - b)/a$, $\tilde{\gamma}_{n+1} = \gamma_{n+1}/a^2$, and $n \geq 0$.

We recall that a regular linear functional u is said to be symmetric if $(u)_{2n+1} = 0$, $n \geq 0$, or equivalently in equation (10) $\beta_n = 0$, $n \geq 0$ [17].

The following lemma gives some characterizations of a MOPS satisfying equation (1).

LEMMA 2.2 Let $\{B_n\}_{n \geq 0}$ be a MOPS relative to a linear functional u verifying equation (10). The following statements are equivalent:

- (a) $\{B_n\}_{n \geq 0}$ satisfies equation (1).
- (b) $(x - \beta_0)B_n(-x) = (-1)^n \left(x - (-1)^n \beta_0 \right) B_n(x)$, $n \geq 0$.
- (c) $(u)_{2n+1} = \beta_0(u)_{2n}$, $n \geq 0$.

Proof (a) $\Rightarrow$ (b). The proof will be done by induction over n . Since $B_0(x) = 1$ and $B_1(x) = x - \beta_0$, the cases $n = 0, 1$ are true. Assume that the result is true until $n + 1$ and let us prove it for $n + 2$.

Multiplying both sides of equation (1), where $x \leftarrow (-x)$, by $x - \beta_0$, and using recurrence hypothesis, we have

$$(x - \beta_0)B_{n+2}(-x) = (-1)^n \left(x - (-1)^n \beta_0 \right) \left\{ \left(x - (-1)^{n+1} \beta_0 \right) B_{n+1}(x) - \gamma_{n+1} B_n(x) \right\}.$$

The result ensues by using equation (1) again.

(b) $\Rightarrow$ (c). We have $\langle u, B_1(x) \rangle = 0$, then $(u)_1 = \beta_0$. Thus (c) is true for $n = 0$.

On the other hand, we can write

$$x^{2n} = \sum_{k=0}^{2n} \theta_{2n,k} B_k(x), \quad \theta_{2n,k} \in \mathcal{C}, \quad n \geq 0. \quad (11)$$

Taking into account equation (11) and the orthogonality, one has

$$\begin{aligned} (u)_{2n+1} - \beta_0(u)_{2n} &= \langle u, (x - \beta_0)x^{2n} \rangle \\ &= \sum_{k=0}^{2n} \theta_{2n,k} \langle u, B_1(x) B_k(x) \rangle \\ &= \theta_{2n,1} \langle u, B_1^2(x) \rangle. \end{aligned}$$

On the other hand, using equation (11), where $x \leftarrow (-x)$, the hypothesis, and the orthogonality, we have

$$\begin{aligned} (u)_{2n+1} - \beta_0(u)_{2n} &= \sum_{k=0}^{2n} \theta_{2n,k} \langle u, (x - \beta_0) B_k(-x) \rangle \\ &= \sum_{k=0}^n \theta_{2n,2k} \langle u, B_1(x) B_{2k}(x) \rangle - \sum_{k=0}^{n-1} \theta_{2n,2k+1} \langle u, (x + \beta_0) B_{2k+1}(x) \rangle, \quad n \geq 1 \\ &= -\theta_{2n,1} \langle u, x^2 - \beta_0^2 \rangle. \end{aligned}$$

Hence, $\theta_{2n,1} \langle u, B_1^2(x) \rangle = -\theta_{2n,1} \langle u, x^2 - \beta_0^2 \rangle$,

or equivalently,

$$\theta_{2n,1} [(u)_2 - \beta_0(u)_1] = 0. \quad (12)$$

But we have $(u)_1 = \beta_0$, it follows that

$$\langle u, B_1^2 \rangle = \langle u, x^2 - 2\beta_0 x + \beta_0^2 \rangle = (u)_2 - \beta_0(u)_1.$$

Since $\langle u, B_1^2 \rangle \neq 0$, the relation (12) implies that $\theta_{2n,1} = 0$. The assertion (c) holds.

(c) $\Rightarrow$ (a). First, using equation (8), it is easy to show that statement (c) is equivalent to

$$\sigma(xu) = \beta_0 \sigma u. \quad (13)$$

Second, in order to prove (a), we shall prove first by induction over n in which we have

$$\begin{cases} B_{2n}(x) = P_n(x^2), & n \geq 0, \\ B_{2n+1}(x) = (x - \beta_0) R_n(x^2), & n \geq 0, \end{cases} \quad (14)$$

where $P_n(x)$ and $R_n(x)$ are monic polynomials of degree n .

Indeed, since $B_0(x) = 1$ and $B_1(x) = x - \beta_0$, the relations (14) are true for $n = 0$.

We suppose that they are true until n and prove them for $n + 1$.

On the one hand, from equation (10), one has

$$B_{2n+2}(x) = (x - \beta_{2n+1})B_{2n+1}(x) - \gamma_{2n+1}B_{2n}(x). \quad (15)$$

Multiplying the two members of equation (15) by $B_{2n+1}(x)$ and applying the form u , we obtain

$$\langle u, xB_{2n+1}^2(x) \rangle = \beta_{2n+1} \langle u, B_{2n+1}^2(x) \rangle,$$

or again, taking into account the recurrence hypothesis, we have

$$\langle (x^3 - 2\beta_0x^2 + \beta_0^2x)u, R_n^2(x^2) \rangle = \beta_{2n+1} \langle u, B_{2n+1}^2(x) \rangle.$$

Using equation (13), this relation becomes

$$-\beta_0 \langle (x^2 - \beta_0^2)u, R_n^2(x^2) \rangle = \beta_{2n+1} \langle u, B_{2n+1}^2(x) \rangle,$$

or again,

$$-\beta_0 \langle u, (x + \beta_0)R_n(x^2)B_{2n+1}(x) \rangle = \beta_{2n+1} \langle u, B_{2n+1}^2(x) \rangle.$$

Taking into account $(x + \beta_0)R_n(x^2) = B_{2n+1}(x) + \text{lower-degree terms}$ and the orthogonality of $\{B_n\}_{n \geq 0}$, we have

$$-\beta_0 \langle u, B_{2n+1}^2(x) \rangle = \beta_{2n+1} \langle u, B_{2n+1}^2(x) \rangle,$$

hence,

$$\beta_{2n+1} = -\beta_0. \quad (16)$$

Taking into account the recurrence hypothesis and equation (16), the relation (15) becomes

$$B_{2n+2}(x) = (x^2 - \beta_0^2)R_n(x^2) - \gamma_{2n+1}P_n(x^2) = P_{n+1}(x^2). \quad (17)$$

On the other hand, from equation (10), one has

$$B_{2n+3}(x) = (x - \beta_{2n+2})B_{2n+2}(x) - \gamma_{2n+2}B_{2n+1}(x). \quad (18)$$

Multiplying the two members of equation (18) by $B_{2n+2}(x)$ and applying the form u , we obtain

$$\langle u, xB_{2n+2}^2(x) \rangle = \beta_{2n+2} \langle u, B_{2n+2}^2(x) \rangle,$$

or again, thanks to equation (17)

$$\langle \sigma(xu), P_{n+1}^2(x) \rangle = \beta_{2n+2} \langle u, B_{2n+2}^2(x) \rangle.$$

Thanks to equation (13), the last relation can be written as

$$\beta_0 \langle u, B_{2n+2}^2(x) \rangle = \beta_{2n+2} \langle u, B_{2n+2}^2(x) \rangle,$$

hence,

$$\beta_{2n+2} = \beta_0. \quad (19)$$

Taking into account the recurrence hypothesis and equation (19), the relation (18) becomes

$$B_{2n+3}(x) = (x - \beta_0)[P_{n+1}(x^2) - \gamma_{2n+2}R_n(x^2)] = (x - \beta_0)R_{n+1}(x^2).$$

Finally, equation (14) hold and (c) holds as a consequence. ■

Remark 2.3 When $\beta_0 = 0$, we again recover characterizations of symmetric MOPS [13].

DEFINITION 2.4 [17, 5] *A linear functional u is semi-classical of class s , if and only if u is regular and there exist two polynomials Ψ of degree $p \geq 1$ and Φ monic of degree $t \geq 0$, such that*

$$(\Phi u)' + \Psi u = 0 \quad (20)$$

and

$$\prod_{c \in Z_\Phi} (|\Psi(c) + \Phi'(c)| + |\langle u, \theta_c \Psi + \theta_c^2(\Phi) \rangle|) > 0, \quad (21)$$

where Z_Φ is the set of zeros of Φ .

The class of u is then given by $s = \max(t - 2, p - 1)$ and the corresponding orthogonal sequence $\{B_n\}_{n \geq 0}$ is called semi-classical of class s .

A semi-classical linear functional of class zero is called classical linear functional [5].

If u is a semi-classical of class s fulfilling equations (20) and (21), then the form $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ is also semi-classical with the same class and it verifies the equation $(\tilde{\Phi}\tilde{u})' + \tilde{\Psi}\tilde{u} = 0$, where $\tilde{\Phi}(x) = a^{-t}\Phi(ax + b)$ and $\tilde{\Psi}(x) = a^{1-t}\Psi(ax + b)$, $a \in \mathcal{C} - \{0\}$, $b \in \mathcal{C}$.

Let us recall the following results.

LEMMA 2.5 [4, 5] *Let u be a semi-classical linear functional satisfying $(\Phi_1 u)' + \Psi_1 u = 0$, and $(\Phi_2 u)' + \Psi_2 u = 0$, where Φ_1 , Ψ_1 , Φ_2 , and Ψ_2 are polynomials, Φ_1 and Φ_2 monic, $\deg \Psi_1 \geq 1$, $\deg \Psi_2 \geq 1$.*

Denoting $s_1 = \max(\deg \Phi_1 - 2, \deg \Psi_1 - 1)$ and $s_2 = \max(\deg \Phi_2 - 2, \deg \Psi_2 - 1)$. Let $\Phi = \gcd(\Phi_1, \Phi_2)$. Then, there exists a polynomial Ψ , $\deg \Psi \geq 1$ such that

$$(\Phi u)' + \Psi u = 0,$$

with

$$s = \max(\deg \Phi - 2, \deg \Psi - 1) = s_1 - \deg \Phi_1 + \deg \Phi = s_2 - \deg \Phi_2 + \deg \Phi.$$

PROPOSITION 2.6 [4, 5] *For any semi-classical linear functional u , the pair (Φ, Ψ) that realizes the class (i.e. equations (20) and (21)) is unique.*

We prove the following proposition.

PROPOSITION 2.7 *Let u be a semi-classical linear functional of class s , verifying equations (20) and (21). If besides, there exist $\hat{\Phi}$ and $\hat{\Psi}$ such that $(\hat{\Phi}u)' + \hat{\Psi}u = 0$, then Φ divides $\hat{\Phi}$.*

Proof The form u verifies

$$(\Phi u)' + \Psi u = 0, \quad s = \max(\deg \Phi - 2, \deg \Psi - 1). \quad (22)$$

and

$$(\hat{\Phi}u)' + \hat{\Psi}u = 0.$$

We denote by $\varphi = \gcd(\Phi, \hat{\Phi})$. According to Lemma 2.5, there exists ψ , $\deg \psi \geq 1$, such that

$$(\varphi u)' + \psi u = 0. \quad (23)$$

We can write $\Phi = \varphi\tilde{\Phi}$, $\tilde{\Phi} \in \mathcal{P}$. Multiplying both sides of equation (23) by $\tilde{\Phi}$, we get

$$(\Phi u)' + (\psi\tilde{\Phi} - \varphi\tilde{\Phi}')u = 0. \quad (24)$$

Identifying equations (22) and (24) and taking into account the regularity of u , one has

$$\psi\tilde{\Phi} = \Psi + \varphi\tilde{\Phi}'.$$

Then,

$$\deg(\psi\tilde{\Phi}) \leq \max(\deg \Psi, \deg(\varphi\tilde{\Phi}')) \leq \max(s+1, s+1) = s+1,$$

in particular, $\deg \psi \leq s+1$. This implies that $\max(\deg \varphi - 2, \deg \psi - 1) \leq s$.

But u is of class s , then $\max(\deg \varphi - 2, \deg \psi - 1) = s$.

Hence, pairs (Φ, Ψ) and (φ, ψ) verify the class s . Thanks to Proposition 2.6, necessarily $\varphi = \Phi$, which ends the proof. ■

PROPOSITION 2.8 [1] *A linear functional u is semi-classical of class s , if and only if it is regular and there exist polynomials A_0 (monic), C_0 , and D_0 , with no non-trivial common factor such that:*

$$A_0(z)S'(u)(z) = C_0(z)S(u)(z) + D_0(z), \quad (25)$$

where $S(u)(z)$ is the formal Stieltjes function of the linear functional u defined by $S(u)(z) := -\sum_{n \geq 0} (u)_n / z^{n+1}$.

The class s is given by

$$s = \max(\deg C_0 - 1, \deg D_0). \quad (26)$$

If u satisfies equation (20), then we have

$$\begin{cases} A_0(z) = \Phi(z), \\ C_0(z) = -\Psi(z) - \Phi'(z), \\ D_0(z) = -(u\theta_0\Phi)'(z) - (u\theta_0\Psi)(z). \end{cases} \quad (27)$$

REMARK 2.9 [4, 5] *Polynomials A_0 , C_0 , and D_0 can not have a common root, otherwise equation (25) will be simplified and then the class is less than s .*

PROPOSITION 2.10 [4, 5] *Let $\{B_n\}_{n \geq 0}$ be a MOPS relative to a linear functional u . Then u is semi-classical of class s and verifies equation (20), if and only if there exist two polynomial sequences $\{C_n\}_{n \geq 0}$ and $\{D_n\}_{n \geq 0}$ with coefficients dependent on n and $\deg C_n \leq s+1$ and $\deg D_n \leq s$, $n \geq 0$, for which we have the structure relation:*

$$\Phi(x)B'_{n+1}(x) = \frac{1}{2}(C_{n+1}(x) - C_0(x))B_{n+1}(x) - \gamma_{n+1}D_{n+1}(x)B_n(x), \quad n \geq 0, \quad (28)$$

where

$$C_{n+1}(x) = -C_n(x) + 2(x - \beta_n)D_n(x), \quad n \geq 0, \quad (29)$$

$$\gamma_{n+1}D_{n+1}(x) = -\Phi(x) + \gamma_n D_{n-1}(x) + (x - \beta_n)^2 D_n(x) - (x - \beta_n)C_n(x), \quad n \geq 0, \quad (30)$$

with $D_{-1}(x) = 0$ and C_0 and D_0 are given by equation (27).

Remark 2.11

- (i) The structure relation gives information about the multiplicity of the zeros of orthogonal polynomials.
- (i) Each D_{n+1} , $n \geq 0$, is a non-zero polynomial [12, p. 236].

3. Semi-classical polynomials verifying equation (1)

In the sequel, $\{B_n\}_{n \geq 0}$ is supposed to be a semi-classical sequence of class s fulfilling equations (28)–(29) and that its corresponding linear functional u satisfies equation (20).

PROPOSITION 3.1 *The following identities hold:*

$$\Phi(x) = x^{r_1}(x + \beta_0)^{r_2}(x - \beta_0)^{r_3}\widehat{\Phi}(x^2), \quad (31)$$

and for $n \geq 0$,

$$(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}D_n(x) = (x - \beta_0)\left(x + (-1)^n\beta_0\right)^2 D_n(-x) \quad (32)$$

and

$$\begin{aligned} & (-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}C_n(x) \\ &= -(x - \beta_0)^2(x + \beta_0)C_n(-x) - 2(-1)^n\beta_0(x - \beta_0)\Phi(-x), \end{aligned} \quad (33)$$

where $r_1, r_2 \in \{0, 1\}$, $r_3 \in \{0, 1, 2\}$ with $r_3 = 0$ when $r_2 = 1$, and $\widehat{\Phi}(x) \in \mathcal{P}$.

Proof According to Lemma 2.2, we have

$$(x - \beta_0)B_{n+1}(-x) = (-1)^{n+1}\left(x - (-1)^{n+1}\beta_0\right)B_{n+1}(x), \quad n \geq 0. \quad (34)$$

Taking the derivatives in equation (34), then multiplying both sides by $(x - \beta_0)$, we get, taking into account equation (34),

$$\begin{aligned} (x - \beta_0)^2 B'_{n+1}(-x) &= \beta_0 \left[(-1)^{n+1} - 1 \right] B_{n+1}(x) \\ &+ (-1)^n (x - \beta_0) \left(x - (-1)^{n+1}\beta_0 \right) B'_{n+1}(x), \quad n \geq 0. \end{aligned} \quad (35)$$

Multiplying both sides of equation (28), where $x \leftarrow (-x)$, by $(x - \beta_0)^2$, we get

$$\begin{aligned} \Phi(-x)(x - \beta_0)^2 B'_{n+1}(-x) &= \frac{1}{2} \left(C_{n+1}(-x) - C_0(-x) \right) (x - \beta_0)^2 B_{n+1}(-x) \\ &- \gamma_{n+1} D_{n+1}(-x) (x - \beta_0)^2 B_n(-x), \quad n \geq 0. \end{aligned}$$

Using Lemma 2.2 (b) and equation (35) in the last identity, we get

$$\begin{aligned} & (x - \beta_0) \left(x - (-1)^{n+1}\beta_0 \right) \Phi(-x) B'_{n+1}(x) \\ &= - \left\{ \frac{1}{2} (x - \beta_0) \left(x - (-1)^{n+1}\beta_0 \right) \left(C_{n+1}(-x) - C_0(-x) \right) \right. \\ &\quad \left. - \beta_0 \left[1 + (-1)^n \right] \Phi(-x) \right\} B_{n+1}(x) \\ &\quad - \gamma_{n+1} (x - \beta_0) \left(x - (-1)^n\beta_0 \right) D_{n+1}(-x) B_n(x), \quad n \geq 0. \end{aligned} \quad (36)$$

Multiplying the two members of equation (36) by $(x - (-1)^n \beta_0)$, we obtain

$$\begin{aligned}
 & (x - \beta_0)^2 (x + \beta_0) \Phi(-x) B'_{n+1}(x) \\
 &= \left\{ -\frac{1}{2} (x - \beta_0)^2 (x + \beta_0) (C_{n+1}(-x) - C_0(-x)) \right. \\
 &\quad \left. + \beta_0 [1 + (-1)^n] (x - (-1)^n \beta_0) \Phi(-x) \right\} B_{n+1}(x) \\
 &\quad - \gamma_{n+1} (x - \beta_0) (x - (-1)^n \beta_0)^2 D_{n+1}(-x) B_n(x), \quad n \geq 0.
 \end{aligned} \tag{37}$$

According to the Proposition 2.10, equation (37) implies that there exists a polynomial $\widehat{\Psi}$ such that

$$\left((x - \beta_0)^2 (x + \beta_0) \Phi(-x) u \right)' + \widehat{\Psi} u = 0,$$

Therefore, following Proposition 2.7, $\Phi(x)$ divides $(x - \beta_0)^2 (x + \beta_0) \Phi(-x)$. Then, there exists $A \in \mathcal{P}$, such that

$$A(x) \Phi(x) = (x - \beta_0)^2 (x + \beta_0) \Phi(-x). \tag{38}$$

If x_0 is an eventual root of $\Phi(x)$, $x_0 \notin \{-\beta_0, \beta_0\}$, then $-x_0$ also. Thus, if besides $x_0 \neq 0$, it follows that $(x^2 - x_0^2)$ divides $\Phi(x)$. Hence, we can write

$$\Phi(x) = x^{r_1} (x + \beta_0)^{r_2} (x - \beta_0)^{r_3} \widehat{\Phi}(x^2), \quad r_1, r_2, r_3 \in \mathbb{N}, \quad \widehat{\Phi}(x) \in \mathcal{P}. \tag{39}$$

By considering all the eventual power of x^2 or $x^2 - \beta_0^2$ in the factorized expression of $\Phi(x)$, as a factor of $\widehat{\Phi}(x^2)$, we can choose integers r_1, r_2 , and r_3 such that $r_1, r_2 \in \{0, 1\}, r_3 \in \{0, 1, 2\}$, and $r_3 = 0$ when $r_2 = 1$. Indeed, if β_0 is a root of $\Phi(x)$ of order $n \geq 3$, then according to equation (38), β_0 is a root of $(x - \beta_0)^2 (x + \beta_0) \Phi(-x)$ of order $\geq n$. Then β_0 is a root of $\Phi(-x)$ of order $\geq n - 2$, or equivalently $-\beta_0$ is a root of $\Phi(x)$ of order $\geq n - 2$, therefore $(x^2 - \beta_0^2)^{n-2}$ is a divisor of $\Phi(x)$. Hence $r_3 \leq 2$. The same thing for r_2 . Thus equation (31) holds.

Now, on the one hand, using equation (31) in equation (37), we get

$$\begin{aligned}
 & (-1)^{r_1+r_2+r_3} x^{r_1} (x + \beta_0)^{r_3+1} (x - \beta_0)^{r_2+2} \widehat{\Phi}(x^2) B'_{n+1}(x) \\
 &= \left\{ -\frac{1}{2} (x - \beta_0)^2 (x + \beta_0) (C_{n+1}(-x) - C_0(-x)) \right. \\
 &\quad \left. + \beta_0 [1 + (-1)^n] (x - (-1)^n \beta_0) \Phi(-x) \right\} B_{n+1}(x) \\
 &\quad - \gamma_{n+1} (x - \beta_0) (x - (-1)^n \beta_0)^2 D_{n+1}(-x) B_n(x), \quad n \geq 0.
 \end{aligned} \tag{40}$$

On the other hand, multiplying both sides of equation (28) by $(-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}$ and taking into account equation (31), we get

$$\begin{aligned} & (-1)^{r_1+r_2+r_3}x^{r_1}(x+\beta_0)^{r_3+1}(x-\beta_0)^{r_2+2}\widehat{\Phi}(x^2)B'_{n+1}(x) \\ &= (-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}\left(\frac{C_{n+1}(x)-C_0(x)}{2}\right)B_{n+1}(x) \\ &\quad - (-1)^{r_1+r_2+r_3}\gamma_{n+1}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}D_{n+1}(x)B_n(x), \quad n \geq 0. \end{aligned} \quad (41)$$

By the identification of equations (40) and (41), we obtain

$$M(n, x)B_{n+1}(x) = N(n, x)B_n(x), \quad n \geq 0,$$

where

$$\begin{aligned} M(n, x) &= -(-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}\left(\frac{C_{n+1}(x)-C_0(x)}{2}\right) \\ &\quad - (x-\beta_0)^2(x+\beta_0)\left(\frac{C_{n+1}(-x)-C_0(-x)}{2}\right) \\ &\quad + \beta_0[1+(-1)^n]\left(x-(-1)^n\beta_0\right)\Phi(-x), \quad n \geq 0 \end{aligned}$$

and

$$\begin{aligned} N(n, x) &= \gamma_{n+1}\left\{-(-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}D_{n+1}(x) \right. \\ &\quad \left. + (x-\beta_0)\left(x-(-1)^n\beta_0\right)^2D_{n+1}(-x)\right\}, \quad n \geq 0. \end{aligned}$$

But the recurrence equation (10) implies that polynomials $B_n(x)$ and $B_{n+1}(x)$ do not have a common root. So, we must have for $n \geq \deg(N(n, x))$, $M(n, x) = 0$ and $N(n, x) = 0$. Or $\deg(N(n, x)) \leq s+3$, then we have

$$\begin{aligned} & (-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}\left(\frac{C_{n+1}(x)-C_0(x)}{2}\right) \\ &= -(x-\beta_0)^2(x+\beta_0)\left(\frac{C_{n+1}(-x)-C_0(-x)}{2}\right) \\ &\quad + \beta_0[1+(-1)^n]\left(x-(-1)^n\beta_0\right)\Phi(-x), \quad n \geq s+3 \end{aligned} \quad (42)$$

and

$$\begin{aligned} & (-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}D_{n+1}(x) \\ &= (x-\beta_0)\left(x-(-1)^n\beta_0\right)^2D_{n+1}(-x), \quad n \geq s+3. \end{aligned} \quad (43)$$

Shifting $(n+1) \leftarrow n$ in equations (42) and (43) gives, respectively,

$$\begin{aligned} & (-1)^{r_1+r_2+r_3}(x+\beta_0)^{r_3-r_2+1}(x-\beta_0)^{r_2-r_3+2}\left(\frac{C_n(x)-C_0(x)}{2}\right) \\ &= -(x-\beta_0)^2(x+\beta_0)\left(\frac{C_n(-x)-C_0(-x)}{2}\right) \\ &\quad + \beta_0[1-(-1)^n]\left(x+(-1)^n\beta_0\right)\Phi(-x), \quad n \geq s+4 \end{aligned} \quad (44)$$

and

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} D_n(x) \\ &= (x - \beta_0)(x + (-1)^n \beta_0)^2 D_n(-x), \quad n \geq s + 4. \end{aligned} \quad (45)$$

On the other hand, since $\beta_n = (-1)^n \beta_0$, $n \geq 0$, gives

$$\begin{cases} C_{n+1}(x) + C_n(x) = 2(x - (-1)^n \beta_0) D_n(x), & n \geq 0, \\ C_{n+1}(-x) + C_n(-x) = -2(x + (-1)^n \beta_0) D_n(-x), & n \geq 0, \end{cases} \quad (46)$$

Adding equations (42) and (44), we get thanks to equation (46):

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} \left((x - (-1)^n \beta_0) D_n(x) - C_0(x) \right) \\ &= -(x - \beta_0)^2 (x + \beta_0) \left(-(x + (-1)^n \beta_0) D_n(-x) - C_0(-x) \right) \\ &+ 2\beta_0 (x - \beta_0) \Phi(-x), \quad n \geq s + 4. \end{aligned} \quad (47)$$

By multiplying both sides of equation (45) by $(x - (-1)^n \beta_0)$, we obtain

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} (x - (-1)^n \beta_0) D_n(x) \\ &= (x - \beta_0)^2 (x + \beta_0) (x + (-1)^n \beta_0) D_n(-x), \quad n \geq s + 4, \end{aligned} \quad (48)$$

thus, using equation (48) in equation (47), we get

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} C_0(x) \\ &= -(x - \beta_0)^2 (x + \beta_0) C_0(-x) - 2\beta_0 (x - \beta_0) \Phi(-x). \end{aligned} \quad (49)$$

Taking account of equation (49), equation (42) gives

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} C_{n+1}(x) \\ &= -(x - \beta_0)^2 (x + \beta_0) C_{n+1}(-x) + 2(-1)^n \beta_0 (x - \beta_0) \Phi(-x), \quad n \geq s + 3. \end{aligned} \quad (50)$$

In the following, we shall prove, by decreasing recurrence, that equations (43) and (50) remain true for all integer n , $0 \leq n \leq s + 2$. Indeed, these relations are true for $n = s + 3$ and $n = s + 4$. We suppose that they are true for $n + 2$ and $n + 1$, and we will prove them for n .

First, from equation (29), where $n \leftarrow (n + 2)$, we have

$$\begin{aligned} D_{n+1}(x) &= \frac{1}{\gamma_{n+2}} \left\{ \gamma_{n+3} D_{n+3}(x) + \Phi(x) - \left(x - (-1)^n \beta_0 \right)^2 D_{n+2}(x) \right. \\ &\quad \left. + \left(x - (-1)^n \beta_0 \right) C_{n+2}(x) \right\}, \quad n \geq 0. \end{aligned} \quad (51)$$

Multiplying both sides of the last equation by $(-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2}$ and using recurrence hypothesis, one has

$$\begin{aligned} & (-1)^{r_1+r_2+r_3} (x + \beta_0)^{r_3-r_2+1} (x - \beta_0)^{r_2-r_3+2} D_{n+1}(x) \\ &= \frac{1}{\gamma_{n+2}} \left\{ \gamma_{n+3} (x - \beta_0) \left(x + (-1)^{n+1} \beta_0 \right)^2 D_{n+3}(-x) \right. \\ &\quad \left. - (x - \beta_0)^3 (x + \beta_0)^2 D_{n+2}(-x) + E(x) + F(x) \right\}, \end{aligned} \quad (52)$$

where

$$E(x) = -(x + (-1)^{n+1}\beta_0)(x - \beta_0)^2(x + \beta_0)C_{n+2}(-x)$$

and

$$\begin{aligned} F(x) &= (-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}\Phi(x) \\ &\quad + 2(-1)^{n+1}\beta_0(x - (-1)^n\beta_0)(x - \beta_0)\Phi(-x). \end{aligned}$$

Thanks to equation (31) and to the obvious equality $(x - \beta_0)(x + \beta_0) = (x - (-1)^n\beta_0)(x + (-1)^n\beta_0)$, we have

$$F(x) = (x - \beta_0)(x + (-1)^{n+1}\beta_0)^2\Phi(-x) \quad (53)$$

and

$$E(x) = -(x + (-1)^{n+1}\beta_0)^2(x + (-1)^n\beta_0)(x - \beta_0)C_{n+2}(-x). \quad (54)$$

Using equations (53) and (54) in (52), we get

$$\begin{aligned} &(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}D_{n+1}(x) \\ &= \frac{1}{\gamma_{n+2}}(x - \beta_0)\left(x + (-1)^{n+1}\beta_0\right)^2\left\{\gamma_{n+3}D_{n+3}(-x) - \left(x + (-1)^n\beta_0\right)^2D_{n+2}(-x)\right. \\ &\quad \left.- \left(x + (-1)^n\beta_0\right)C_{n+2}(-x) + \Phi(-x)\right\}. \end{aligned} \quad (55)$$

Thanks to equation (51), where $x \leftarrow (-x)$, the last equality becomes

$$\begin{aligned} &(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}D_{n+1}(x) \\ &= (x - \beta_0)\left(x + (-1)^{n+1}\beta_0\right)^2D_{n+1}(-x), \end{aligned}$$

hence equation (43) holds for the order n .

On the other hand, from equation (28) and the fact that $\beta_n = (-1)^n\beta_0$, $n \geq 0$, we have

$$\begin{aligned} &(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}C_{n+1}(x) \\ &= 2(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}\left(x - (-1)^{n+1}\beta_0\right)D_{n+1}(x) \\ &\quad - (-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}C_{n+2}(x). \end{aligned}$$

Since equation (43) is true for n , recurrence hypothesis allow us to rewrite the preceding relation as

$$\begin{aligned} &(-1)^{r_1+r_2+r_3}(x + \beta_0)^{r_3-r_2+1}(x - \beta_0)^{r_2-r_3+2}C_{n+1}(x) \\ &= (x - \beta_0)^2(x + \beta_0)\left\{2\left(x + (-1)^{n+1}\beta_0\right)D_{n+1}(-x) + C_{n+2}(-x)\right\} \\ &\quad + 2\beta_0(-1)^n(x - \beta_0)\Phi(-x). \end{aligned}$$

The proof ends thanks to (28), where $x \leftarrow (-x)$. ■

Remark 3.2 If we take $\beta_0 = 0$ in equations (31) and (33) where $n = 0$, we meet, thanks to equation (27), the following result established in ref. [1].

COROLLARY 3.3 *Let u be a symmetric semi-classical linear functional of class s , satisfying equations (20) and (21). Then if s is even, Φ is even and Ψ is odd; if s is odd, Φ is odd and Ψ is even.*

4. Semi-classical polynomials of class one verifying equation (1)

In this paragraph, u denotes a semi-classical linear functional of class $s = 1$ satisfying

$$(\Phi u)' + \Psi u = 0, \quad \Phi \text{ monic},$$

with

$$\Phi(x) = \sum_{k=0}^3 a_k x^k, \quad \sum_{k=0}^3 |a_k| \neq 0, \quad (56)$$

$$\Psi(x) = \sum_{k=0}^2 b_k x^k, \quad |b_1| + |b_2| \neq 0. \quad (57)$$

We suppose, besides, that the corresponding MOPS $\{B_n\}_{n \geq 0}$ satisfies the three-term recurrence relation (1). Since the case $\beta_0 = 0$ is well known [1], we suppose that $\beta_0 \neq 0$. Thus, thanks to a suitable homothetic, we can take $\beta_0 = 1$.

Before seeking for all possible expressions of Φ , let us give a technical Lemma,

LEMMA 4.1 *Putting $C_0(x) = e_0 x^2 + f_0 x + g_0$, we have*

$$\Psi(x) = -(3a_3 + e_0)x^2 - (2a_2 + f_0)x - (a_1 + g_0),$$

and

$$D_0(x) = (a_3 + e_0)x + 2a_2 + a_1 + f_0 + g_0$$

(C_0 and D_0 are the polynomials in Proposition 2.8).

Proof Immediate application of equation (26). Note that $(u)_1 = \beta_0 = 1$. ■

4.1 Expression of the polynomial Φ

Since $s = 1$ and $\beta_0 = 1$, relations (31)–(33) will be written successively as

$$\Phi(x) = x^{r_1}(x+1)^{r_2}(x-1)^{r_3}\widehat{\Phi}(x^2), \quad (58)$$

where $r_1, r_2 \in \{0, 1\}$, $r_3 \in \{0, 1, 2\}$, with $r_3 = 0$ when $r_2 = 1$, $\deg \widehat{\Phi}(x) \leq 1$, and for $n \geq 0$:

$$(-1)^{r_1+r_2+r_3}(x+1)^{r_3-r_2+1}(x-1)^{r_2-r_3+2}D_n(x) = (x-1)\left(x+(-1)^n\right)^2 D_n(-x), \quad (59)$$

$$\begin{aligned} (-1)^{r_1+r_2+r_3}(x+1)^{r_3-r_2+1}(x-1)^{r_2-r_3+2}C_n(x) &= -(x-1)^2(x+1)C_n(-x) \\ &\quad + 2(-1)^{n+1}(x-1)\Phi(-x). \end{aligned} \quad (60)$$

From Proposition 2.10, we have $\deg D_0 \leq 1$. Then, either D_0 is a nontrivial constant or D_0 is written as $D_0(x) = \lambda(x - \alpha)$, $\lambda, \alpha \in \mathcal{L}$.

First case. D_0 is a nontrivial constant.

Taking $n = 0$ in equation (59), we obtain $r_1 = 1$ and $r_3 = r_2 + 1$.

We have to distinguish two subcases:

- $r_2 = 0$. In this case $r_3 = 1$. Then, from equation (59), one has

$$(x + 1)^2 D_{2n+1}(x) = (x - 1)^2 D_{2n+1}(-x), \quad n \geq 0.$$

Hence, 1 is a double root of $D_{2n+1}(x)$. This is impossible since $\deg(D_{2n+1}) \leq 1$.

- $r_2 = 1$. Then $r_3 = 0$. Then, from equation (59), one has

$$(x - 1)^2 D_{2n}(x) = (x + 1)^2 D_{2n}(-x), \quad n \geq 0.$$

Hence, -1 is a double root of $D_{2n}(x)$. This is impossible since $\deg(D_{2n}) \leq 1$.

Second case. $D_0(x) = \lambda(x - \alpha)$, $\lambda, \alpha \in \mathbb{C}$.

We distinguish two cases:

Case A: $\lambda \neq 0$.

We have to consider two subcases:

(A1) $\alpha = 0$.

From equation (59), where $n = 0$, one has $r_3 = r_2 + 1$ and $r_1 + r_2 + r_3$ is odd. Then $r_1 = 0$.

Therefore, equation (59) gives

$$-(x + 1)^2 D_{2n+1}(x) = (x - 1)^2 D_{2n+1}(-x), \quad n \geq 0.$$

Hence, 1 is a double root of $D_{2n+1}(x)$. This is impossible since $\deg(D_{2n+1}) \leq 1$.

(A2) $\alpha \neq 0$.

For $n = 0$, equation (59) drags that

$$(-1)^{r_1+r_2+r_3+1}(x + 1)^{r_3-r_2+1}(x - 1)^{r_2-r_3+2}(x - \alpha) = (x - 1)(x + 1)^2(x + \alpha).$$

Then, necessarily $\alpha \in \{-1, 1\}$.

Precisely $\alpha = -1$. Indeed if $\alpha = 1$, then $\begin{cases} r_3 = r_2 + 2, \\ r_1 + r_2 + r_3 \text{ is odd.} \end{cases}$

Thus $r_1 = 1$, $r_2 = 0$, $r_3 = 2$, and therefore $\Phi(x) = x(x - 1)^2$.

For $n = 0$, equation (60) becomes $-(x + 1)^2 C_0(x) = -(x - 1)^2 C_0(-x) + 2x(x + 1)(x - 1)$, in particular, 1 is a root of C_0 .

Then, 1 is a common root of D_0 , C_0 , and Φ . Thanks to Remark 2.9, this is impossible.

Therefore $\alpha = -1$, then $\begin{cases} r_2 = r_3 \\ r_1 + r_2 + r_3 \text{ is odd,} \end{cases}$

hence,

$$r_1 = 1, r_3 = r_2 = 0. \quad (61)$$

For $n = 0$, equation (60) becomes

$$(x + 1)(x - 1)C_0(x) = (x - 1)(x + 1)C_0(-x) + 2\Phi(-x), \quad (62)$$

then 1 and -1 are roots of $\Phi(x)$.

Taking into account equation (61), it follows that $\Phi(x) = x^3 - x$. Hence equation (62) becomes $C_0(x) = C_0(-x) - 2x$. Consequently, $C_0(x) = e_0 x^2 - x + g_0$.

Then, from Lemma 4.1, we get

$$\begin{cases} \Psi(x) = -(e_0 + 3)x^2 + x + 1 - g_0, \\ D_0(x) = (e_0 + 1)(x + 1), \\ e_0 \neq -1. \end{cases}$$

Case B: $\lambda = 0$.

Thanks to Lemma 4.1, we have $a_3 = -e_0$ and $a_2 - e_0 + f_0 = 0$.

On the other hand, $s = 1 = \max(\deg C_0 - 1, \deg D_0)$, then $\deg C_0 = 2$; thus $e_0 \neq 0$ implies that $\deg \Phi = 3$ and, in particular, $a_3 = 1$.

Hence,

$$\begin{cases} \Phi(x) = x^3 - (f_0 + 1)x^2 + a_1x + a_0, \\ C_0(x) = -x^2 + f_0x + g_0, \\ \Psi(x) = -2x^2 + (f_0 + 2)x - (a_1 + g_0). \end{cases}$$

By comparison of monomials of the highest degree in equation (60), where $n = 0$, we remark that $r_1 + r_2 + r_3$ is odd.

We distinguish two subcases:

- $r_2 = 0$

We have to investigate three situations:

- $r_3 = 0$.

Then $r_1 = 1$. Therefore equation (58) becomes $\Phi(x) = x\widehat{\Phi}(x^2)$.

For $n = 0$, equation (60) can be written as

$$(x + 1)(x - 1)C_0(x) = (x - 1)(x + 1)C_0(-x) + 2\Phi(-x).$$

So, 1 and -1 are the roots of Φ .

Hence, $\Phi(x) = x(x^2 - 1)$, $C_0(x) = -x^2 - x + g_0$, and $\Psi(x) = -2x^2 + x + 1 - g_0$.

- $r_3 = 1$.

In this case, $r_1 = 0$ and equation (58) becomes $\Phi(x) = (x - 1)\widehat{\Phi}(x^2)$, or equivalently, since $\deg(\Phi) = 3$

$$\Phi(x) = (x - 1)(x^2 - \omega^2), \quad \omega \in \mathcal{C}. \quad (63)$$

For $n = 0$, the relation (60) gives

$$(x + 1)C_0(x) = (x - 1)C_0(-x) - 2(x^2 - \omega^2), \quad (64)$$

Since, 1 is a root of Φ , then $-f_0 + a_1 + a_0 = 0$ implies that

$$\Phi(x) = x^3 - (f_0 + 1)x^2 + (f_0 - a_0)x + a_0. \quad (65)$$

By comparison of equations (63)–(65), we get $f_0 = 0$ and $a_0 = \omega^2$.

Then $\Phi(x) = x^3 - x^2 - \omega^2x + \omega^2$ and $C_0(x) = -x^2 + g_0$.

The relation (64) drags that $g_0 = \omega^2$.

Finally, ω is a common root of C_0 , D_0 , and Φ , which is impossible.

- $r_3 = 2$.

In this case $r_1 = 1$ and we can write $\Phi(x) = x(x - 1)^2$.

For $n = 0$, equation (60) becomes

$$(x + 1)^2C_0(x) = (x - 1)^2C_0(-x) - 2x(x + 1)(x - 1).$$

Hence, 1 is a common root of Φ , C_0 , and D_0 .

- $r_2 = 1$

Necessarily $r_3 = 0$. Then $r_1 = 0$.

Thus, the relation equation (59) can be written as

$$-(x-1)^2 D_{2n}(x) = (x+1)^2 D_{2n}(-x), \quad n \geq 0.$$

Hence, -1 is a double root of $D_{2n}(x)$. This is impossible since $\deg(D_{2n}) \leq 1$.

In the following table, we summarize all possible situations.

$\Phi(x) = x(x^2 - 1), \quad (r_1 = 1, r_2 = r_3 = 0)$	
$\Psi(x) = -(e_0 + 3)x^2 + x + (1 - g_0),$	
$C_0(x) = e_0 x^2 - x + g_0,$	
$D_0(x) = (e_0 + 1)(x + 1), e_0, g_0 \in \mathbb{C}.$	(66)

4.2 Expressions of the recurrence coefficients and the structure relation coefficients

By Proposition 2.10, we can write

$$C_n(x) = e_n x^2 + f_n x + g_n, \quad n \geq 0 \quad \text{and} \quad D_n(x) = k_n x + l_n, \quad n \geq 0,$$

with $e_n, f_n, g_n, k_n, l_n \in \mathbb{C}$.

Taking account of equation (66), the relation (59) gives

$$l_n = (-1)^n k_n, \quad n \geq 0. \quad (67)$$

Hence

$$D_n(x) = k_n(x + (-1)^n), \quad n \geq 0. \quad (68)$$

On the other hand, taking account of equation (66), the relation (60) gives

$$f_n = (-1)^{n+1}, \quad n \geq 0. \quad (69)$$

Thus

$$C_n(x) = e_n x^2 + (-1)^{n+1} x + g_n, \quad n \geq 0. \quad (70)$$

Substituting equations (68) and (70) in equation (28), we obtain

$$e_{n+1} = -e_n + 2k_n, \quad n \geq 0, \quad (71)$$

and

$$g_{n+1} = -g_n - 2k_n, \quad n \geq 0. \quad (72)$$

In the same way, substituting equations (68) and (70) in equation (29), we get, taking into account equation (66),

$$k_n = e_n + 1, \quad n \geq 0, \quad (73)$$

and

$$\gamma_{n+1} k_{n+1} = \gamma_n k_{n-1} - g_n - k_n, \quad n \geq 0 \quad (k_{-1} = 0). \quad (74)$$

From equations (71) and (73), we have $e_{n+1} = e_n + 2$, $n \geq 0$, which implies that

$$e_n = 2n + e_0, \quad n \geq 0. \quad (75)$$

Then, from equation (73)

$$k_n = 2n + 1 + e_0, \quad n \geq 0. \quad (76)$$

As, from equations (71) and (72), we get

$$e_{n+1} + g_{n+1} = -(e_n + g_n), \quad n \geq 0.$$

Then, by iteration

$$e_n + g_n = (-1)^n(e_0 + g_0), \quad n \geq 0.$$

Thanks to equation (75), the last relation gives

$$g_n = -2n + (-1)^n(e_0 + g_0) - e_0, \quad n \geq 0. \quad (77)$$

Then equation (74) becomes

$$(2n + 3 + e_0)\gamma_{n+1} = (2n - 1 + e_0)\gamma_n - (-1)^n(e_0 + g_0) - 1, \quad n \geq 0 \quad (\gamma_0 = 0). \quad (78)$$

Or, equivalently,

$$(4n + 3 + e_0)\gamma_{2n+1} = (4n - 1 + e_0)\gamma_{2n} - e_0 - g_0 - 1, \quad n \geq 0 \quad (79)$$

and

$$(4n + 5 + e_0)\gamma_{2n+2} = (4n + 1 + e_0)\gamma_{2n+1} + e_0 + g_0 - 1, \quad n \geq 0. \quad (80)$$

By eliminating γ_{2n+1} , we obtain

$$\begin{aligned} & (4n + 3 + e_0)(4n + 5 + e_0)\gamma_{2n+2} - (4n - 1 + e_0)(4n + 1 + e_0)\gamma_{2n} \\ & = -8n - 4 + 2g_0, \quad n \geq 0. \end{aligned}$$

Let

$$\Gamma_n := (4n - 1 + e_0)(4n + 1 + e_0)\gamma_{2n}, \quad n \geq 0 \quad (\Gamma_0 = 0).$$

Then, we have $\Gamma_{n+1} - \Gamma_n = -8n - 4 + 2g_0, n \geq 0$.

Thus, by iteration

$$\Gamma_n = -2n(2n - g_0), \quad n \geq 0.$$

Hence

$$\gamma_{2n} = \frac{-2n(2n - g_0)}{(4n - 1 + e_0)(4n + 1 + e_0)}, \quad n \geq 0,$$

and from equation (79),

$$\gamma_{2n+1} = -\frac{(2n + e_0 + 1)(2n + e_0 + g_0 + 1)}{(4n + 1 + e_0)(4n + 3 + e_0)}, \quad n \geq 0.$$

In summary, we have

Semi-classical MOPS $\{B_n\}_{n \geq 0}$ of class 1 with $\beta_n = (-1)^n, n \geq 0$.

$$\begin{aligned}\Phi(x) &= x^3 - x, \\ \Psi(x) &= -(e_0 + 3)x^2 + x + (1 - g_0). \\ \gamma_n &= -\frac{(2n + e_0 + (-1)^{n+1}e_0)(2n + e_0 + (-1)^{n+1}(e_0 + 2g_0))}{4(2n - 1 + e_0)(2n + 1 + e_0)}, n \geq 1. \\ e_0 &\neq -(2n + 1), n \geq 0, g_0 \neq 2n, n \geq 1, e_0 + g_0 \neq -(2n + 1), n \geq 0. \\ C_n(x) &= (2n + e_0)x^2 + (-1)^{n+1}x + [-2n + (-1)^n(e_0 + g_0) - e_0], n \geq 0. \\ D_n(x) &= (2n + 1 + e_0)(x + (-1)^n), e_0 \neq -(2n + 1), n \geq 0, n \geq 0.\end{aligned}$$

4.3 An integral representation

An integral representation of u is given by

$$\begin{aligned}\langle u, f \rangle &= k \int_{-1}^1 \frac{1}{|x|^{g_0}} |x - 1|^{(e_0 + g_0 - 1)/2} |x + 1|^{(e_0 + g_0 + 1)/2} f(x) dx, \\ &\quad - (1 + \Re(e_0)) < \Re(g_0) < 1,\end{aligned}\tag{81}$$

where k is a normalization factor; it will be chosen such that $(u)_0 = 1$. $\Re$ is the real part of a complex number.

Proof A solution of equation (8) has the integral representation

$$\int_C U(x) f(x) dx, \quad f \in \mathcal{P}$$

if the following conditions hold

$$\begin{aligned}&((x^3 - x)U)' + (- (e_0 + 3)x^2 + x + (1 - g_0))U = 0 \\ &[(x^3 - x)U(x)f(x)]_C = 0, \quad f \in \mathcal{P}\end{aligned}$$

where C is an acceptable path. We solve the first condition as a differential equation

$$((x^3 - x)U)' + (- (e_0 + 3)x^2 + x + (1 - g_0))U = 0$$

equivalent to

$$\frac{U'}{U} = -\frac{g_0}{x} + \frac{e_0 + g_0 - 1}{2(x - 1)} + \frac{e_0 + g_0 + 1}{2(x + 1)},$$

thus

$$U(x) = k \frac{1}{|x|^{g_0}} |x - 1|^{(e_0 + g_0 - 1)/2} |x + 1|^{(e_0 + g_0 + 1)/2}, \quad |x| < 1, \quad U(x) = 0, \quad |x| > 1.$$

Assuming $-(1 + \Re(e_0)) < \Re(g_0) < 1$, condition $[(x^3 - x)U(x)f(x)]_{-1}^1 = 0, f \in \mathcal{P}$ holds. ■

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A new class of analytic functions defined by means of a convolution operator involving the Hurwitz–Lerch Zeta function

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In this paper, we investigate a new class of analytic functions defined by the convolution operator $J_{s,b}(f)$ involving the Hurwitz–Lerch Zeta function, which was introduced recently by H. M. Srivastava and A. A. Attiya [*Integral Transforms and Special Functions*, **18**, 2007, 207–216]. Coefficient inequalities, distortion theorems and other properties of functions belonging to this class are derived. We also indicate briefly the relevant connections of some of the results presented here with those that were proven in earlier investigations.

Keywords: Analytic functions; Starlike functions; Hurwitz–Lerch Zeta function; Linear operators; Hadamard product (or convolution); Srivastava–Attiya operator

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

A function f in the class $\mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of *starlike functions of order α* in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; \quad 0 \leq \alpha < 1). \quad (2)$$

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With a view to introducing the Srivastava–Attiya convolution operator $J_{s,b}$, we begin by recalling a general Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [1, p. 121 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s \in \mathbb{C} \quad \text{when} \quad |z| < 1; \quad \Re(s) > 1 \quad \text{when} \quad |z| = 1), \quad (3)$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}; \quad \mathbb{N} := \{1, 2, 3, \dots\}).$$

Several interesting properties and characteristics of the Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [2], Ferreira and López [3], Garg *et al.* [4], Lin *et al.* [5] and [6], Luo and Srivastava [7], and others.

Srivastava and Attiya [8] introduced the linear operator

$$J_{s,b}(f) : \mathcal{A} \longrightarrow \mathcal{A}$$

defined, in terms of the Hadamard product (or convolution), by

$$J_{s,b}(f)(z) := G_{s,b}(z) * f(z)$$

$$(z \in \mathbb{U}; \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s \in \mathbb{C}), \quad (4)$$

where, for convenience,

$$G_{s,b}(z) := (1+b)^s [\Phi(z, s, b) - b^{-s}] \quad (z \in \mathbb{U}). \quad (5)$$

It is easy to observe from equation (1) and the definitions (3) and (4) that

$$J_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^s a_k z^k. \quad (6)$$

The Srivastava–Attiya operator $J_{s,b}(f)$ contains, among its special cases, the integral operators introduced and investigated earlier by (for example) Alexander [9], Libera [10], Bernardi [11], and Jung *et al.* [12].

We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{s,b}^*(\alpha)$, if $J_{s,b}(f)$ is in the class $\mathcal{S}^*(\alpha)$, i.e., if

$$\Re \left(\frac{z J'_{s,b}(f)(z)}{J_{s,b}(f)(z)} \right) > \alpha$$

$$(z \in \mathbb{U}; \quad 0 \leq \alpha < 1; \quad b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \quad s \in \mathbb{C}). \quad (7)$$

The main object of this paper is to present a systematic investigation of the analytic function class $\mathcal{S}_{s,b}^*(\alpha)$ defined above by means of the Srivastava–Attiya convolution operator $J_{s,b}(f)$. In particular, for this (presumably new) function class, we derive the coefficient inequalities, distortion theorems and other interesting properties of functions of this class. We also indicate briefly the relevant connections of some of the results presented here with those that were proven in earlier investigations.

2. Coefficient inequalities for the function class $\mathcal{S}_{s,b}^*(\alpha)$

THEOREM 1 Let $\alpha \in [0, 1)$. If $f(z) \in \mathcal{A}$ satisfies the following inequality:

$$\sum_{k=2}^{\infty} (k - \alpha) \left| \left(\frac{1+b}{k+b} \right)^s \right| |a_k| \leq 1 - \alpha, \quad (8)$$

then

$$f(z) \in \mathcal{S}_{s,b}^*(\alpha).$$

Proof Suppose that the condition (8) is satisfied for $\alpha \in [0, 1)$. For $f(z) \in \mathcal{A}$, we define the function $F(z)$ by

$$F(z) := \frac{z J'_{s,b}(f)(z)}{J_{s,b}(f)(z)} - \alpha. \quad (9)$$

To prove that $f(z) \in \mathcal{S}_{s,b}^*$, it suffices to show that

$$\left| \frac{F(z) - 1}{F(z) + 1} \right| < 1 \quad (z \in \mathbb{U}).$$

Indeed, we have

$$\begin{aligned} |\mathcal{F}(z)| &:= \left| \frac{F(z) - 1}{F(z) + 1} \right| = \left| \frac{[z J'_{s,b}(f)(z)]/[J_{s,b}(f)(z)] - \alpha - 1}{[z J'_{s,b}(f)(z)]/[J_{s,b}(f)(z)] - \alpha + 1} \right| \\ &= \left| \frac{z J'_{s,b}(f)(z) - \alpha J_{s,b}(f)(z) - J_{s,b}(f)(z)}{z J'_{s,b}(f)(z) - \alpha J_{s,b}(f)(z) + J_{s,b}(f)(z)} \right| \\ &= \left| \frac{z + \sum_{k=2}^{\infty} k[(1+b)/(k+b)]^s a_k z^k - \alpha z - \alpha}{\sum_{k=2}^{\infty} [(1+b)/(k+b)]^s a_k z^k - z - \sum_{k=2}^{\infty} [(1+b)/(k+b)]^s a_k z^k} \right| \\ &= \left| \frac{z + \sum_{k=2}^{\infty} k[(1+b)/(k+b)]^s a_k z^k - \alpha z - \alpha}{\sum_{k=2}^{\infty} [(1+b)/(k+b)]^s a_k z^k + z + \sum_{k=2}^{\infty} [(1+b)/(k+b)]^s a_k z^k} \right| \\ &= \left| \frac{\alpha z + \sum_{k=2}^{\infty} [(1+b)/(k+b)]^s (\alpha + 1 - k) a_k z^k}{(2 - \alpha) z - \sum_{k=2}^{\infty} [(1+b)/(k+b)]^s (\alpha - 1 - k) a_k z^k} \right|, \end{aligned}$$

so that

$$\begin{aligned} |\mathcal{F}(z)| &\leq \frac{\alpha |z| + \sum_{k=2}^{\infty} |[(1+b)/(k+b)]^s| |\alpha + 1 - k| \cdot |a_k| \cdot |z|^k}{(2 - \alpha) |z| - \sum_{k=2}^{\infty} |[(1+b)/(k+b)]^s| |\alpha - 1 - k| \cdot |a_k| \cdot |z|^k} \\ &< \frac{\alpha + \sum_{k=2}^{\infty} |[(1+b)/(k+b)]^s| (k - 1 - \alpha) \cdot |a_k|}{(2 - \alpha) - \sum_{k=2}^{\infty} |[(1+b)/(k+b)]^s| (k + 1 - \alpha) \cdot |a_k|} \\ &\leq 1, \end{aligned}$$

provided that

$$\begin{aligned} \alpha + \sum_{k=2}^{\infty} \left| \left(\frac{1+b}{k+b} \right)^s \right| (k-1-\alpha) \cdot |a_k| \\ \leq (2-\alpha) - \sum_{k=2}^{\infty} \left| \left(\frac{1+b}{k+b} \right)^s \right| (k+1-\alpha) \cdot |a_k|, \end{aligned}$$

which is equivalent to the hypothesis (8) of Theorem 1. We thus have completed the proof of Theorem 1. $\blacksquare$

Next, in Theorem 2 below, we present the coefficient inequalities for functions $f(z)$ which are in the class $\mathcal{S}_{s,b}^*(\alpha)$.

THEOREM 2 *Let $\alpha \in [0, 1)$. If $f(z) \in \mathcal{S}_{s,b}^*(\alpha)$, then*

$$|a_n| \leq \frac{2(1-\alpha)}{n-1} \left| \left(\frac{n+b}{1+b} \right)^s \right| \prod_{j=2}^{n-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (n \in \mathbb{N} \setminus \{1\}). \quad (10)$$

The result is sharp.

Proof We set

$$p(z) := \frac{(zJ'_{s,b}(f)(z)/J_{s,b}(f)(z)) - \alpha}{1-\alpha} = 1 + c_1z + c_2z^2 + \cdots.$$

Then $p(z)$ is analytic with

$$p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

Since

$$zJ'_{s,b}(f)(z) = [(1-\alpha)p(z) + \alpha]J_{s,b}(f)(z),$$

by virtue of equation (6) we have

$$(n-1) \left(\frac{1+b}{n+b} \right)^s a_n = (1-\alpha) \left[c_{n-1} + \sum_{m=2}^{n-1} \left(\frac{1+b}{m+b} \right)^s a_m c_{n-m} \right] \quad (n \in \mathbb{N} \setminus \{1\}). \quad (11)$$

By applying Carathéodory's Lemma (see ref. [13]), we obtain

$$(n-1) \left| \left(\frac{1+b}{n+b} \right)^s \right| \cdot |a_n| \leq 2(1-\alpha) \left(1 + \sum_{m=2}^{n-1} \left| \left(\frac{1+b}{m+b} \right)^s \right| \cdot |a_m| \right). \quad (12)$$

We shall prove, by using the principle of mathematical induction, that the inequality (10) is satisfied for $n \in \mathbb{N} \setminus \{1\}$. If $n = 2$, then we have

$$|a_2| \leq 2(1 - \alpha) \left| \left(\frac{2+b}{1+b} \right)^s \right|.$$

Now suppose that the inequality (10) is satisfied for $n \leq k$. Then, from equations (10) and (12), we find that

$$\begin{aligned} k \left| \left(\frac{1+b}{k+1+b} \right)^s \right| \cdot |a_{k+1}| &\leq 2(1 - \alpha) \left[1 + \sum_{m=2}^k \left| \left(\frac{1+b}{m+b} \right)^s \right| \cdot |a_m| \right] \\ &\leq 2(1 - \alpha) \left[1 + \sum_{m=2}^k \frac{2(1 - \alpha)}{m-1} \prod_{j=2}^{m-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) \right] \\ &= 2(1 - \alpha) \prod_{j=2}^k \left(1 + \frac{2(1 - \alpha)}{j-1} \right). \end{aligned}$$

Hence

$$|a_n| \leq \frac{2(1 - \alpha)}{n-1} \left| \left(\frac{n+b}{1+b} \right)^s \right| \prod_{j=2}^{n-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) \quad (n \in \mathbb{N} \setminus \{1\}).$$

Finally, the result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{2(1 - \alpha)}{n-1} \left(\frac{n+b}{1+b} \right)^s \prod_{j=2}^{n-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) z^n \quad (n \in \mathbb{N} \setminus \{1\}). \quad \blacksquare$$

3. Distortion inequalities for the function class $\mathcal{S}_{s,b}^*(\alpha)$

We first state the distortion inequalities for functions in the class $\mathcal{S}_{s,b}^*(\alpha)$ given by Theorem 3 below.

THEOREM 3 *Let $f(z) \in \mathcal{S}_{s,b}^*(\alpha)$ and $0 \leq \alpha < 1$. Then*

$$\begin{aligned} r - 2(1 - \alpha)r^2 \sum_{k=2}^{\infty} \frac{1}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) &\leq |f(z)| \\ &\leq r + 2(1 - \alpha)r^2 \sum_{k=2}^{\infty} \frac{1}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) \quad (|z| = r < 1) \end{aligned}$$

and

$$\begin{aligned} 1 - 2(1 - \alpha)r \sum_{k=2}^{\infty} \frac{k}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) &\leq |f'(z)| \\ &\leq 1 + 2(1 - \alpha)r \sum_{k=2}^{\infty} \frac{k}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1 - \alpha)}{j-1} \right) \quad (|z| = r < 1). \end{aligned}$$

Proof Let $f(z) \in \mathcal{A}$ be given by equation (1). Then, by using Theorem 2, we have

$$\begin{aligned} |f(z)| &\leq |z| + \sum_{k=2}^{\infty} |a_k| \cdot |z|^k \\ &\leq r + r^2 \sum_{k=2}^{\infty} \frac{2(1-\alpha)}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \\ &= r + 2(1-\alpha)r^2 \sum_{k=2}^{\infty} \frac{1}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1) \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq |z| - \sum_{k=2}^{\infty} |a_k| \cdot |z|^k \\ &\geq r - 2(1-\alpha)r^2 \sum_{k=2}^{\infty} \frac{1}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1). \end{aligned}$$

We also find from equation (1) that

$$\begin{aligned} |f'(z)| &\leq 1 + \sum_{k=2}^{\infty} k|a_k| \cdot |z|^{k-1} \\ &\leq 1 + 2(1-\alpha)r \sum_{k=2}^{\infty} \frac{k}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1) \end{aligned}$$

and

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k| \cdot |z|^{k-1} \\ &\geq 1 - 2(1-\alpha)r \sum_{k=2}^{\infty} \frac{k}{k-1} \left| \left(\frac{k+b}{1+b} \right)^s \right| \prod_{j=2}^{k-1} \left(1 + \frac{2(1-\alpha)}{j-1} \right) \quad (|z| = r < 1). \end{aligned}$$

We thus have completed our proof of Theorem 3. ■

4. Extreme points of the function class $\mathcal{S}_{s,b}^*(\alpha)$

Let $\tilde{\mathcal{S}}_{s,b}^*(\alpha)$ be the subclass of $\mathcal{S}_{s,b}^*(\alpha)$ consisting of all functions $f(z) \in \mathcal{A}$, which satisfy the condition (8). We shall determine the extreme points of the function class $\tilde{\mathcal{S}}_{s,b}^*(\alpha)$.

THEOREM 4 *Let*

$$f_1(z) := z$$

and

$$f_k(z) := z + \frac{(1-\alpha) \cdot |(k+b)^s|}{(k-\alpha) \cdot |(1+b)^s|} z^k \quad (k \in \mathbb{N} \setminus \{1\}).$$

Then

$$f \in \tilde{\mathcal{S}}_{s,b}^*(\alpha) \quad (0 \leq \alpha < 1),$$

if and only if it can be expressed in the following form:

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z) \quad \left(\lambda_k > 0; \quad \sum_{k=2}^{\infty} \lambda_k = 1 \right).$$

Proof Suppose that

$$\begin{aligned} f(z) &= \sum_{k=2}^{\infty} \lambda_k f_k(z) \\ &= z + \sum_{k=2}^{\infty} \lambda_k \frac{(1-\alpha) \cdot |(k+b)^s|}{(k-\alpha) \cdot |(1+b)^s|} z^k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} (k-\alpha) \frac{|(1+b)^s|}{|(k+b)^s|} \lambda_k \frac{(1-\alpha) \cdot |(k+b)^s|}{(k-\alpha) \cdot |(1+b)^s|} \\ &= (1-\alpha) \sum_{k=2}^{\infty} \lambda_k \\ &= (1-\alpha)(1-\lambda_1) \leq 1-\alpha. \end{aligned}$$

Thus, by the definition of the function class $\tilde{\mathcal{S}}_{s,b}^*(\alpha)$, we have

$$f \in \tilde{\mathcal{S}}_{s,b}^*(\alpha) \quad (0 \leq \alpha < 1).$$

Conversely, we suppose that

$$f \in \tilde{\mathcal{S}}_{s,b}^*(\alpha) \quad (0 \leq \alpha < 1).$$

Then, by using equation (8), we may set

$$\lambda_k = \frac{(k-\alpha) \cdot |(1+b)^s|}{(1-\alpha) \cdot |(k+b)^s|} |a_k| \quad (k \in \mathbb{N} \setminus \{1\}) \quad \text{and} \quad \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k,$$

which would readily imply that

$$f(z) = \sum_{k=2}^{\infty} \lambda_k f_k(z),$$

and the proof of Theorem 4 is thus completed. ■

5. The Fekete–Szegő problem for the function class $\mathcal{S}_{s,b}^*(\alpha)$

In this section, we shall obtain the Fekete–Szegő inequality for functions in the class $\mathcal{S}_{s,b}^*(\alpha)$ when

$$s > 0, \quad b > 0 \quad \text{and} \quad 0 \leq \alpha < 1.$$

In order to prove our main result involving the Fekete–Szegő inequality, we need the following lemma.

LEMMA (see ref. [14]). *If*

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

is an analytic function in $\mathbb{U}$ such that

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}),$$

then

$$|c_2 - v c_1^2| \leq \begin{cases} -4v + 2 & (v \leq 0) \\ 2 & (0 \leq v \leq 1) \\ 4v - 2 & (v \geq 1). \end{cases}$$

When $v < 0$ or $v > 1$, the equality holds true if and only if

$$p(z) = \frac{1+z}{1-z}$$

or one of its rotations. If $0 < v < 1$, then the equality holds true if and only if

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations. If $v = 0$, then the equality holds true if and only if

$$p(z) = \left(\frac{1+\lambda}{2} \right) \left(\frac{1+z}{1-z} \right) + \left(\frac{1-\lambda}{2} \right) \left(\frac{1-z}{1+z} \right) \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, then the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds true in the case $v = 0$.

THEOREM 5 *Let*

$$s > 0, \quad b > 0 \quad \text{and} \quad 0 \leq \alpha < 1.$$

If $f(z) \in \mathcal{S}_{s,b}^*(\alpha)$ is given by equation (1), then

$$|a_3 - \mu a_2^2| \leq \begin{cases} (1-\alpha)^2 \left[(1-\alpha)^{-1} \left(\frac{3+b}{1+b} \right)^s - 4\mu \left(\frac{2+b}{1+b} \right)^{2s} + 2 \left(\frac{3+b}{1+b} \right)^s \right] & (\mu \leq \sigma_1) \\ (1-\alpha) \left(\frac{3+b}{1+b} \right)^s & (\sigma_1 \leq \mu \leq \sigma_2) \\ -(1-\alpha)^2 \left[(1-\alpha)^{-1} \left(\frac{3+b}{1+b} \right)^s - 4\mu \left(\frac{2+b}{1+b} \right)^{2s} + 2 \left(\frac{3+b}{1+b} \right)^s \right] & (\mu \geq \sigma_2), \end{cases}$$

where

$$\sigma_1 := \frac{1}{2} \left(\frac{1+b}{2+b} \right)^s \left(\frac{3+b}{2+b} \right)^s \quad \text{and} \quad \sigma_2 := \frac{2-\alpha}{2(1-\alpha)} \left(\frac{1+b}{2+b} \right)^s \left(\frac{3+b}{2+b} \right)^s.$$

The result is sharp.

Remark For solutions of the Fekete-Szegő problem associated with other analytic function classes, see the recent work by (for example) Srivastava *et al.* [15].

Proof For $f(z) \in \mathcal{S}_{s,b}^*(\alpha)$, let

$$p(z) = \frac{(zJ'_{s,b}(f)(z)/J_{s,b}(f)(z)) - \alpha}{1-\alpha} = 1 + c_1 z + c_2 z^2 + \dots$$

Then, by virtue of equation (6), we have

$$a_2 = (1-\alpha)c_1 \left(\frac{2+b}{1+b} \right)^s \quad \text{and} \quad a_3 = \frac{1-\alpha}{2} \left(\frac{3+b}{1+b} \right)^s [c_2 + (1-\alpha)c_1^2].$$

Therefore, we obtain

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1-\alpha}{2} \left(\frac{3+b}{1+b} \right)^s [c_2 + (1-\alpha)c_1^2] - \mu(1-\alpha)^2 c_1^2 \left(\frac{2+b}{1+b} \right)^{2s} \\ &= \frac{1-\alpha}{2} \left(\frac{3+b}{1+b} \right)^s \left(c_2 - c_1^2(1-\alpha) \left[2\mu \left(\frac{2+b}{1+b} \right)^s \left(\frac{2+b}{3+b} \right)^s - 1 \right] \right). \end{aligned}$$

We write

$$a_3 - \mu a_2^2 = \frac{1-\alpha}{2} \left(\frac{3+b}{1+b} \right)^s (c_2 - v c_1^2),$$

where

$$v := (1-\alpha) \left[2\mu \left(\frac{2+b}{1+b} \right)^s \left(\frac{2+b}{3+b} \right)^s - 1 \right].$$

Our result (asserted by Theorem 5) now follows by the application of the above Lemma. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if

$$J_{s,b}(f)(z) = \frac{z}{(1 - e^{i\theta}z)^{2(1-\alpha)}} \quad (\theta \in \mathbb{R}).$$

When $\sigma_1 < \mu < \sigma_2$, the equality holds true if and only if

$$J_{s,b}(f)(z) = \frac{z}{(1 - e^{i\theta}z^2)^{1-\alpha}} \quad (\theta \in \mathbb{R}).$$

If $\mu = \sigma_1$, then the equality holds true if and only if

$$\begin{aligned} J_{s,b}(f)(z) &= \left(\frac{z}{(1 - e^{i\theta}z)^{2(1-\alpha)}} \right)^{[(1+\lambda)/2]} \left(\frac{z}{(1 + e^{i\theta}z)^{2(1-\alpha)}} \right)^{[(1-\lambda)/2]} \\ &= \frac{z}{[(1 - e^{i\theta}z)^{1+\lambda}(1 + e^{i\theta}z)^{1-\lambda}]^{1-\alpha}} \quad (0 \leq \lambda \leq 1). \end{aligned}$$

Finally, when $\mu = \sigma_2$, the equality holds true if and only if $J_{s,b}(f)(z)$ satisfies the following condition:

$$\frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} = (1 - \alpha)p(z) + \alpha,$$

where

$$\frac{1}{p(z)} = \left(\frac{1 + \lambda}{2} \right) \left(\frac{1 + z}{1 - z} \right) + \left(\frac{1 - \lambda}{2} \right) \left(\frac{1 - z}{1 + z} \right) \quad (0 < \lambda < 1).$$

We conclude our present investigation by mentioning that, by suitably specializing the parameters involved, some of our main results (Theorems 1 to 5 above) would yield a number of (known or new) results for relatively much simpler function classes, which were studied in several earlier works by employing many of the aforementioned special cases of the Srivastava–Attiya convolution operator $J_{s,b}(f)(z)$.

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