Mean-value of the Riemann zeta-function on the critical line

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Abstract. This is an expository article. It is a collection of some important results on the mean-value of $|\zeta(\frac{1}{2} + it)|$.

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1. Introduction

This article is a collection of some important results on the mean value of $|\zeta(\frac{1}{2} + it)|$ and its powers. Section 2 deals with the mean square and §3 with the mean fourth power and the mean twelfth power. Section 4 deals with the general $(2k)$th power where $k > 0$ and §5 gives other important results.

2. Mean square

In 1918 Hardy and Littlewood [12] proved that

$$\int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt \sim T \log T.$$  

In 1926 this was improved by Ingham [18] to

$$\frac{1}{2\pi} \int_0^T |\zeta(\frac{1}{2} + it)|^2 \, dt = \frac{T}{2\pi} \log \frac{T}{2\pi} + (2\gamma - 1) \frac{T}{2\pi} + E(T),$$

where $E(T) = 0 \left(T^{1/2} \log T\right)$. More complicated argument due to Titchmarsh [33] showed that $E(T) = O(T^{5/12} \left(\log T\right)^2)$. The final improvement of this theorem was published in 1978 by Balasubramanian [3] who proved that $E(T) = O(T^{1/3})$, $E(T) = O(T^{27/82})$, and so on. It should be mentioned that subsequent to this discovery Good [10] gave a variant of Balasubramanian's proof. It should also be mentioned that Heath-Brown's powerful method [14] which gives the mean fourth power with error term also gives the theorem of Balasubramanian.

Titchmarsh [34, 35] had introduced a new mean value. He had proved that, as $\delta \to 0$

$$\int_0^\infty |\zeta(\frac{1}{2} + it)|^2 \exp(-\delta t) \, dt \sim \frac{1}{\delta} \log \frac{1}{\delta},$$

and

$$\int_0^\infty |\zeta(\frac{1}{2} + it)|^4 \exp(-\delta t) \, dt \sim \frac{1}{2\pi^2 \delta} \left(\log \frac{1}{\delta}\right)^4.$$
Kober [22] proved that

\[ \int_0^\infty |\zeta(\frac{1}{2} + it)|^2 \exp(-2\delta t) \ dt = \frac{\gamma - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n\delta^n + O(\delta^{N+1}), \]

where the 0-constant depends on \( N \). A simpler proof was given by Atkinson [1]. Regarding \( \Omega \) theorems for \( E(T) \) Good [11] developed a method to prove that \( E(T) = \Omega(T^{1/4}) \). Heath-Brown [15] developed a more powerful method which actually gives

\[ \int_0^T (E(t))^2 \ dt = cT^{3/2} + O(T^{5/4}(\log T)^2), \]

where \( c = \frac{1}{3}(2\pi)^{-1/2}(\zeta(3))^{-4}(\zeta(3))^{-1}. \)

3. Mean fourth power

In 1922 Hardy and Littlewood [13] proved that

\[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 \ dt = o(T(\log T)^4). \]

This was improved in 1926 by Ingham [18] to

\[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 \ dt = \frac{T}{2\pi^2} (\log T)^4 + O(T(\log T)^3) \]

by using the approximate functional equation for \( (\zeta(s))^2 \). A simpler proof of this fact was given recently by Ramachandra [24] using the theorem of Montgomery and Vaughan. Subsequently Heath-Brown [14] showed by deep arguments that

\[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 \ dt = TP(\log T) + O(T^{1+\epsilon}), \]

where \( P(x) \) is a polynomial of degree 4 in \( x \). The coefficient of \( x^4 \) is \( 1/2\pi^2 \) and that of \( x^3 \) is

\[ 2\pi^{-2}[4\gamma - 1 - \log (2\pi) - 12\pi^{-2}\zeta'(-2)]. \]

He also proved another deep result [16] which asserts that

\[ \int_0^T |\zeta(\frac{1}{2} + it)|^{12} \ dt = O(T^2(\log T)^{17}). \]

Previously Atkinson [2] had proved that

\[ \int_0^\infty |\zeta(\frac{1}{2} + it)|^4 \exp(-\delta t) \ dt = \frac{1}{\delta} Q\left(\log\frac{1}{\delta}\right) + O\left(\left(\frac{1}{\delta}\right)^{4+\epsilon}\right) \]

where \( Q(x) \) is a polynomial of degree 4 in \( x \). The coefficient of \( x^4 \) is \( 1/2\pi^2 \) and that of \( x^3 \) is

\[ -\frac{1}{\pi^2}\left[2 \log(2\pi) - 6\gamma + \frac{24\zeta''(2)}{\pi^2}\right]. \]
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For the fourth power mean Iwaniec [19] proved the following deep result. Let $I_r (r = 1, 2, \ldots, R)$ be $R$ disjoint intervals of length $T_0$ satisfying $T_0^{1/2} \leq T_0 \leq T$. Then

$$\sum_{r=1}^R \int_{I_r} |\zeta(\frac{1}{2} + it)|^4 \, dt \ll \varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$ 

Choosing $R = 1, T_0 = T^{2/3}$ we obtain

$$\int_T^{T^{2/3}} |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T^{2+\varepsilon}.$$ 

The deep result of Iwaniec also implies

$$\int_0^T |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll T^{2+\varepsilon}.$$ 

This can be seen as follows. It suffices to prove that

$$\int_T^{2T} |\zeta(\frac{1}{2} + it)|^{12} \, dt \ll T^{2+\varepsilon}.$$ 

The contributions to the twelfth power moment from those $t$ with $|\zeta(\frac{1}{2} + it)| \leq T^{1+\varepsilon}$ are together $\ll T^{2+9\varepsilon}$ using $\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T (\log T)^4$. So one has to consider the range

$$T^{1+\varepsilon} \leq |\zeta(\frac{1}{2} + it)| \leq T^{1+\varepsilon}$$

since $\mu(\frac{1}{2}) \ll \frac{1}{T}$. We divide the range $[T^{1+\varepsilon}, T^{1+\varepsilon}]$ into $[T^{1+\varepsilon}, 2T^{1+\varepsilon}]$, $[2T^{1+\varepsilon}, 4T^{1+\varepsilon}]$, $[4T^{1+\varepsilon}, 8T^{1+\varepsilon}]$, \ldots the last interval either terminates at $T^{1+\varepsilon}$ or goes just a little beyond. In any typical interval $[V, 2V)$ we have $V \leq |\zeta(\frac{1}{2} + it)| < 2V$. For any given $V$ we consider the set $G(V)$ of all $t$ with $V \leq |\zeta(\frac{1}{2} + it)| < 2V$. We choose a small $\delta > 0$ to be fixed later. We next divide $[T, 2T]$ into disjoint intervals $[T, T + V^{4-\delta}]$, $[T + V^{4-\delta}, T + 2V^{4-\delta}]$, $[T + 2V^{4-\delta}, T + 3V^{4-\delta}]$, \ldots the last interval either terminating at $2T$ or going just a little beyond $2T$. Out of these we retain only those which have at least one point of $G(V)$. With every such interval $[A, B)$ we associate the interval $I : (A - (\log T)^2, B + (\log T)^2)$. The intervals $I$ are not disjoint. But every point of $[A, B)$ is interior to at most two intervals $I$. Hence if we put $T_0 = V^{4-\delta} + 2(\log T)^2$ we have

$$\sum_I \int_I |\zeta(\frac{1}{2} + it)|^4 \, dt \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2})T^\varepsilon.$$ 

By convexity each of the integrals on the left $\geq V^4 (\log T)^{-2}$. Hence

$$RV^4 (\log T)^{-2} \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2})T^\varepsilon.$$ 

We now put $\delta = 16\varepsilon$. It follows that

$$RV^4 (\log T)^{-2} \ll (RV^{4-\delta} + T(R/V^{4-\delta})^{1/2})T^{1+\varepsilon}.$$ 

Hence $R \ll \frac{T^{2+\varepsilon}}{V^{12-\delta}} \ll \frac{T^{2+100\varepsilon}}{V^{12}}$. On the other hand by Iwaniec's result we have also

$$\sum_I \int_I |\zeta(\frac{1}{2} + it)|^4 \, dt \ll \varepsilon (RT_0 + T(R/T_0)^{1/2})T^\varepsilon.$$
Here the first term on the right side can be ignored and the left side is

$$\geq \sum I \mu (I) V^4$$

where $\mu (I)$ is the measure of $G (V) \cap I$. Hence

$$\sum I \mu (I) \leq \frac{1}{V^4} T^{1+\varepsilon} (R / T_0)^{1/2}$$

and so

$$\sum I \mu (I) V^{12} \leq V^8 T^{1+\varepsilon} (T^2 + 100 \varepsilon / V^{16-\delta})^{1/2} \leq T^2 + 100 \varepsilon$$

i.e.

$$\int_{G (V)} |\zeta (\frac{1}{2} + it)|^{12} dt \leq T^2 + 100 \varepsilon$$

Summing up over all $V$ the twelfth power moment stated in the beginning is proved.

4. Mean $(2k)$th power

Titchmarsh [36] was the first to prove the following result. For every fixed positive integer $k$,

$$\int_0^\infty |\zeta (\frac{1}{2} + it)|^{2k} \exp (-\delta t) dt \geq \frac{1}{\delta} \left( \log \frac{1}{\delta} \right)^{k^2}$$

This gives

$$\frac{1}{T} \int_T^{2T} |\zeta (\frac{1}{2} + it)|^{2k} dt = \Omega (\log T^{k^2})$$

Put

$$M (k, T, T + H) = \frac{1}{H} \int_T^{T + H} |\zeta (\frac{1}{2} + it)|^{2k} dt$$

where $k$ is a positive real number and

$$T \geq H \geq 100 \log \log T \geq C_0$$

where $C_0$ is a large positive constant. Ramachandra [25] was the first to prove that if $2k$ is a positive integer then $M (k, T, T + H) \gg (\log H)^{1/2}$. This gives

$$\frac{1}{T} \int_T^{2T} |\zeta (\frac{1}{2} + it)|^{2k} dt \gg (\log T)^{k^2}$$

Using Gabriel's convexity theorem this was upheld by Heath-Brown [17] for all rational $k > 0$. Regarding upper bounds Ramachandra [26] was the first to prove that

$$\frac{1}{T} \int_T^{2T} |\zeta (\frac{1}{2} + it)| dt \ll (\log T)^{1/4}.$$
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again by using Gabriel’s convexity theorem that \( M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2} \) where \( m \geq 1 \) is an integer. Jutila [21] proved that

\[
(\log T)^{1/m^2} \ll M\left(\frac{1}{m}, T, 2T\right) \ll (\log T)^{1/m^2}
\]

where the constants implied by the Vinogradov symbol \( \ll \) are independent of \( m \). He applied this to show that the measure of the set \( |\zeta(\frac{1}{2} + it)| \geq V \) is

\[
\leq \exp\left(-\frac{(\log V)^2}{\log \log T}\left(1 + 0\left(\frac{\log V}{\log \log T}\right)\right)\right)
\]

where \( 1 \leq V \leq \log T \).

Regarding lower bounds Ramachandra [27] was the first to prove

\[
M(k, T, T + H) \gg (\log H)^{k^2} (\log \log H)^{-C}
\]

where \( k > 0 \) is any real constant and \( C \) depends only on \( k \). Using Heath-Brown’s idea he [28] proved

\[
M(k, T, T + H) \gg (\log H)^{k^2}
\]

where \( k > 0 \) is any rational constant. Using the same idea he proved [29, 30] that for irrational \( k \)

\[
M(k, T, T + H) \gg g\left(\frac{\log H}{\log \log H}\right)^{k^2},
\]

where \( g \) is a function of \( H \) which tends to infinity with \( H \).

Ramachandra [31] has proved (see also Balasubramanian and Ramachandra [4]) that if \( k \) is an integer subject to \( 1 \leq k \leq \log H \), we have uniformly in \( k, T, H \)

\[
M(k, T, T + H) \gg \sum_{n \leq H/100} \frac{(d_k(n))^2}{n} \left(1 - \frac{\log n}{\log H} + \frac{1}{\log \log H}\right).
\]

Put \( \Lambda = (\text{RHS})^{1/2k} \). The quantity \( \Lambda \) has been studied as a function of \( k \) by Balasubramanian. It was shown by Balasubramanian and Ramachandra [5] that the maximum of \( \Lambda \) as \( k \) varies is attained in \( 1 \leq k \leq \log H \) and that

\[
\log \max \Lambda \gg \left(\frac{\log H}{\log \log H}\right)^{1/2}.
\]

By an ingeneous argument Balasubramanian [6] showed that

\[
\log \max \Lambda \sim \alpha\left(\frac{\log H}{\log \log H}\right)^{1/2},
\]

where \( \alpha = \frac{1}{4} \max_{l > 0, \text{\( l \) is real}} \{ (\exp(-2l) + \int_{2l}^\infty e^{-\theta} d\theta)(2l \exp(2l^{1/2})) \} \).

Richert showed by his pocket calculator that \( \alpha = 0.75 \ldots \). Thus

\[
\max_{T < t < T + H} |\zeta(\frac{1}{2} + it)| > \exp\left(\frac{3}{4}\left(\frac{\log H}{\log \log H}\right)^{1/2}\right)
\]
where \( T \geq H \geq 100 \log \log T \) and \( T > C_0 \). For the earlier history of the problem see \([32]\) and \([5]\). This is an improvement on the result

\[
\max_{0 < t \leq T} |\zeta(\frac{1}{2} + it)| > \exp \left( \frac{1}{20} \left( \frac{\log T}{\log \log T} \right)^{1/2} \right)
\]

for \( T \geq T_0 \) due to Montgomery \([23]\) who proved this on the assumption of the Riemann hypothesis.

5. Other important results

Let

\[
M(s) = \sum_{m \leq M} a_m m^{-s}
\]

and

\[
S(T, M) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right)M(it)\right|^2 \, dt.
\]

Iwaniec \([21]\) was the first to prove that

\[
S(T, M) \ll T^{1+\varepsilon} \sum_{m \leq M} |a_m|^2
\]

for \( M \leq T^{1/2} \) unconditionally and for \( M \leq T^{4/7} \) conditionally. In the case \( a_m = \mu(m) \) Heath-Brown has proved this result unconditionally for \( M \leq T^{8/15} \) (see p. 306 of \([8]\)).

In a series of papers Deshouillers and Iwaniec \([8, 9]\) have proved this unconditionally when \( a_m = \Lambda(m) \) or \( a_m = \mu(m) \) for \( M \leq T^{4/7} \).

Iwaniec \([20]\) was the first to prove that

\[
\int_0^T |\zeta(\frac{1}{2} + it)|^4 |M(it)|^2 \, dt \ll T^{1+\varepsilon} \sum_{m \leq M} |a_m|^2
\]

for \( M \leq T^{1/10} \). Deshouillers and Iwaniec prove this for \( M \leq T^{1/5} \). Asymptotic formulae are also important (e.g. for the discussion of the zeros on the critical line). In this connection Balasubramanian and Conrey \([7]\) have proved that

\[
\int_0^T \left| \zeta\left(\frac{1}{2} + it\right)M\left(\frac{1}{2} + it\right)\right|^2 \, dt \sim T \sum_{m_1, m_2 \leq M} a_{m_1} a_{m_2} \left( \log \frac{T(m_1, m_2)}{2\pi m_1 m_2} + 2\gamma - 1 \right)
\]

Provided \( a_m = \mu(m)f(m) \) where \( f \) is a smooth function \( \ll 1 \) and \( M \leq T^{8/5} \). Using this and refining the method of Levinson they proved that 38% of the zeros lie on the critical line.

In the theory of the Riemann zeta-function an indispensable book is \([37]\). This has been very useful in the preparation of this article.

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