Preprints of the 21st IFAC World Congress (Virtual)
Berlin, Germany, July 12-17, 2020

Event-triggered Discrete-time Sliding Mode Control for High-order Systems via Reduced-order Model Approach

Kiran Kumari * **Bijnan Bandyopadhyay** * **Johann Reger** ** **Abhisek K. Behera** ***,****

* *Systems and Control Engineering, Indian Institute of Technology Bombay, India, (kiran@sc.iitb.ac.in, bijnan@sc.iitb.ac.in),*
** *Control Engineering Group, Technische Universität Ilmenau, 98684 Ilmenau, Germany, (johann.reger@tu-ilmenau.de),*
*** *Faculty of Mechanical Engineering, Technion-Israel Institute of Technology, Haifa, 32000, Israel, (a.behera@campus.technion.ac.il),*
**** *Department of Statistics, University of Haifa, Haifa 31905, Israel.*

Abstract: We propose the design of event-triggered (ET) discrete-time sliding mode (DTSM) control for a high-order discrete-time system via a reduced-order model-based approach. This design includes a triggering mechanism using a reduced-order state vector and a controller based on the modified Bartoszewicz' reaching law for a reduced-order model of the system, to stabilize the uncertain high-order system. The main advantages of using a reduced-order vector in the event condition are the low-order synthesis of the controller and the sampling pattern, which may be sparser than the full vector-based design. This motivation arises from the fact that relaxing a few components of the state vector in the triggering mechanism may decrease its rate of violation. An added advantage of the proposal is that the transmission of the reduced-order vector, particularly in a network-based implementation, can outperform the full-order based design due to the severe challenges that exist in the data network. The robust performance for the closed-loop system is achieved using the DTSM control. We show that our proposal guarantees the stability of the full-order plant with the reduced-order triggering mechanism. The control execution is Zeno-free because of the inherent discrete nature of the control. The efficiency of the proposed method is shown using the simulation results of a numerical example.

Keywords: Event-triggered control, Discrete-time sliding mode, Model-order reduction.

1. INTRODUCTION

The analysis of discrete-time systems has become vital for real-time applications where the control is implemented digitally using sophisticated computing platforms. Event-triggering is a prominent technique when controlling sampled-data systems and has grabbed attention of many researchers in recent times (Anta and Tabuada, 2010; Tallapragada and Chopra, 2013). It is an aperiodic technique of control implementation in which states are only transmitted to the controller at certain time instants. These instants are computed with a triggering mechanism that employs some condition on the system states (Cucuzzella and Ferrara, 2016; Lunze and Lehmann, 2010; Tabuada, 2007). An important factor in ET control is the *inter-event time*, which is the time difference between two consecutive control tasks, as it directly affects the sparsity pattern of the triggering sequence. The ET strategy has been extensively studied in the literature (Bandyopadhyay and Behera, 2018; Borgers and Heemels, 2014; Postoyan et al., 2015), but only little emphasis was put on above fact (Girard (2014)). The triggering instants depend on the state trajectory, and all state components contribute towards the occurrence of the event. The violation of the event condition may take longer, when a part of the state vector is used.

In view of this motivation, we propose a triggering mechanism using a lower-dimensional state vector. This idea significantly reduces the number of control tasks. Thus we follow a reduced controller design methodology to realize the reduced-order vector-based triggering mechanism.

Designing control for the high-order systems may be very complicated. Hence it is favorable to design the control via a reduced-order model. Research in the area of model order reduction dominated the 70s, e.g., Routh approximation (Hutton and Friedland, 1975), aggregation method (Aoki, 1968), Padé approximation (Shamash, 1974). Although the main focus of these works is to reduce the computation and complexity of the control design, we may also benefit from it in a network-based implementation. The efficacy of

⋆ This research was partially supported by the German Academic Exchange Service (DAAD) and Department of Science and Technology (DST), Govt. of India through program DAAD-DST PPP (DAAD project-ID 57458791, DST sanction No.-DST/INT/DAAD/P-9/2019). The third author further acknowledges support by the European Union Horizon 2020 research and innovation program under Marie Skłodowska-Curie grant agreement No. 824046.

a reduced-order design is evident in the transmission of the state vector through the feedback network as it reduces the data transmission, and results in low transmission costs. Here the challenge is to ensure that the reduced-order controller also stabilizes the high-order system (Linnemann, 1988). Lamba and Rao (1974) used the aggregation technique in state feedback to stabilize the high-order system. Later Bandyopadhyay et al. (1998) employed the aggregation technique to stabilize the full-order system using the gains of reduced-order state feedback. Contrary to this, we design ET-DTSM control from the reduced-order model to stabilize the high-order uncertain system, exploiting the invariance property of sliding mode control (SMC) to deal with the uncertainty (Utkin, 1977).

In DTSM, the control design is based on the reaching law approach (Gao et al., 1995). Here we use the non-switching reaching law proposed by Bartoszewicz (1998). The design of DTSM for a high-order system using a reduced-order model is given in (Bandyopadhyay et al., 2006, 2007). However, there are only a few papers that investigate event-triggering in SMC (Behera et al., 2018; Cucuzzella et al., 2020) and in the discrete domain (Kumari et al., 2016, 2019b). A continuous SMC approach, which substantially differs from this paper, is presented in (Kumari et al., 2019a).

The question of interest is whether robust performance can be achieved in the discrete-time system using the ET strategy under the same assumptions as in the periodic implementation. We address this problem with ET-DTSM, which renders boundedness of the full-order state trajectory by the use of observed states in (Behera et al., 2016).

The contributions of this paper are as follows: Our primary achievement is to use a reduced-order model for designing the triggering mechanism and the control law. Taking the lower-dimensional state vector into the triggering condition, we achieve larger inter-event times. Next, when this triggering mechanism is employed in the feedback loop for the state transmission, the closed-loop high-order system is stabilized. Using the aggregation method, we show that the proposed control for the reduced-order system also stabilizes the equilibrium point of the system in the original coordinates and guarantees the existence of practical quasi-sliding mode (PQSM) in the original system. It is unlike the traditional ET design, in which the full state vector is used to design the control law. The size of the matrices used in the control law is considerably small, which results in moderate computations. The control execution is Zeno-free because of the discrete nature of control law.

The paper is organized as follows: In Section 2 we discuss the problem formulation and give some structural considerations. The main results of the paper are presented in Section 3. It comprises the modified reaching law and the reduced-order model-based design. Simulation results are presented in Section 4, and Section 5 concludes the paper.

Notation: The set of real numbers (n-dim. real vectors) is denoted by $\mathbb{R}$ ($\mathbb{R}^n$). The set of integers (non-negative) is denoted as $\mathbb{Z}$ ($\mathbb{Z}_{\geq 0}$). We denote $S_1 \times S_2$ as the Cartesian product of sets S_1 and S_2. The norm of a vector or induced norm of a matrix is denoted by $\|\cdot\|$, and $|\cdot|$ denotes the absolute value, $\lambda_m(\cdot)$ ($\lambda_M(\cdot)$) is the minimum (maximum) eigenvalue of a matrix.

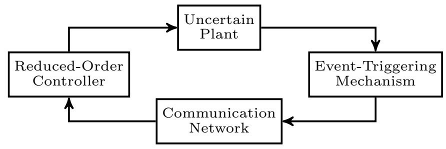

Fig. 1. Plant with event-triggered feedback strategy.

2. PROBLEM FORMULATION

Consider the discrete-time system

$$x_{k+1} = \Phi x_k + \Gamma(u_k + d_k), \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}$ is the control input and $d_k \in \mathbb{R}$ is an unknown disturbance acting on the system at time instant k. The following assumptions are made:

A1. The disturbance is bounded from above, i.e., $|d_k| \leq d_0$ for all $k \in \mathbb{Z}_{\geq 0}$, where d_0 is a known constant.
A2. The pair (Φ, Γ) is controllable.

Assumption A1 may be justified since information on the bound of disturbance is often available from practical knowledge of the engineer. Assumption A2 is very standard for the design of a stabilizing controller. The plant is controlled by a feedback controller implemented over the communication network as shown in Fig. 1, in which the transmission of state to the controller takes place at triggering instant. For the design of the controller in this paper, we consider only dominant modes of the system, as the response of the plant is decided by these modes only. Our goal is to design a reduced-order model-based controller for stabilizing the plant with a reduced communication burden. We accomplish this by adopting an ET transmission policy in the feedback loop for a controller based on the reduced-order system.

2.1 Reduced-order system

Let there exists a transformation matrix P, which transforms system (1) into block diagonal form with dominant and non-dominant modes of the system. Then the transformation $x = Pz$ results in

$$z_{k+1} = \hat{\Phi} z_k + \hat{\Gamma}(u_k + d_k), \tag{2}$$

where $z_k^\top = \begin{bmatrix} z_k^{1\top} & z_k^{2\top} \end{bmatrix} \in \mathbb{R}^n$ with $z_k^1 \in \mathbb{R}^r$ and $z_k^2 \in \mathbb{R}^{n-r}$ being the states with respect to the dominant and non-dominant modes, respectively. The matrices are

$$\hat{\Phi} = P^{-1}\Phi P = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \quad \text{and} \quad \hat{\Gamma} = P^{-1}\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}.$$

The block matrices Φ_1 and Φ_2 contain the dominant and non-dominant modes, respectively. The reduced-order model of (1) is obtained by preserving the dominant modes of the diagonal system (2) as follows

$$z_{k+1}^1 = \Phi_1 z_k^1 + \Gamma_1(u_k + d_k). \tag{3}$$

The dynamics of the non-dominant part, given by

$$z_{k+1}^2 = \Phi_2 z_k^2 + \Gamma_2(u_k + d_k), \tag{4}$$

shall play no role in the controller synthesis. The states of the high-order system (1) and the reduced-order system (3) are related by $z^1 = \overline{C}_a x$, where $\overline{C}_a = [I_r \ 0]P^{-1}$ is known as the *aggregation matrix.* It shall be noted that the control input and external disturbance in the dynamics of the non-dominant part may be interpreted as a perturbation. Moreover, it is also input-to-state stable with respect to these perturbations.

3. MAIN RESULTS

In this section, the design of DTSM control is discussed, which is based on the reduced-order model. This is followed by the stability analysis of the closed-loop system.

3.1 Reduced-order model based design

For designing a SMC, we first transform the reduced-order system (3) into the regular form. Let us write the input matrix in (3) as $\Gamma_1^\top = \begin{bmatrix} \Gamma_{11}^\top & \Gamma_{12} \end{bmatrix}$, where $\Gamma_{12} \in \mathbb{R} \setminus \{0\}$. Now define the transformation matrix

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \quad T_1 = \begin{bmatrix} I_{r-1} & -\Gamma_{11}/\Gamma_{12} \\ 0 & 1 \end{bmatrix}, \quad T_2 = I_{n-r}.$$

Using transformation $[z^{r\top} \ z^{n\top}]^\top = T[z^{1\top} \ z^{2\top}]^\top$, system (2) can be written as

$$z_{k+1}^r = \Phi_r z_k^r + \Gamma_r(u_k + d_k), \tag{5a}$$

$$z_{k+1}^n = \Phi_n z_k^n + \Gamma_n(u_k + d_k), \tag{5b}$$

where $\Phi_r = T_1 \Phi_1 T_1^{-1}$, $\Gamma_r = T_1 \Gamma_1$, $\Phi_n = \Phi_2$ and $\Gamma_n = \Gamma_2$. System (5a) can also be expressed as

$$z_{k+1}^{r_1} = \Phi_{r11} z_k^{r_1} + \Phi_{r12} z_k^{r_2}, \tag{6}$$

$$z_{k+1}^{r_2} = \Phi_{r21} z_k^{r_1} + \Phi_{r22} z_k^{r_2} + \Gamma_{12}(u_k + d_k), \tag{7}$$

where $z_k^r = \begin{bmatrix} z_k^{r_1 \top} & z_k^{r_2} \end{bmatrix}^\top \in \mathbb{R}^r$ is the reduced-order state vector and the matrices are as follows

$$\Phi_r = \begin{bmatrix} \Phi_{r11} & \Phi_{r12} \\ \Phi_{r21} & \Phi_{r22} \end{bmatrix} \text{and } \Gamma_r = \begin{bmatrix} 0 \\ \Gamma_{12} \end{bmatrix}.$$

Since (Φ_{r11}, Φ_{r12}) is a controllable pair, there exists a $c_1 \in \mathbb{R}^{r-1}$ such that the matrix $\Phi_{r11} - \Phi_{r12} c_1^\top$ has all eigenvalues within unit disk. We now define the sliding variable, for $\hat{c}^\top = [c_1^\top \ 1]$, as

$$s_{z_k} = \hat{c}^\top z_k^r. \tag{8}$$

Using transformation $T_r = \begin{bmatrix} 1 & 0 \\ c_1^\top & 1 \end{bmatrix}$, $[z^{r_1 \top} \ s_z]^\top = T_r z^r$, we can rewrite the reduced-order model (5a) as

$$z_{k+1}^{r_1} = \overline{\Phi}_{11} z_k^{r_1} + \overline{\Phi}_{12} s_{z_k}, \tag{9}$$

$$s_{z_k} = \overline{\Phi}_{21} z_k^{r_1} + \overline{\Phi}_{22} s_{z_k} + \Gamma_{12}(u_k + d_k), \tag{10}$$

where $\overline{\Phi}_{11} = \Phi_{r11} - \Phi_{r12} c_1^\top$, $\overline{\Phi}_{12} = \Phi_{r12}$, $\overline{\Phi}_{21} = c_1^\top \Phi_{r11} - c_1^\top \Phi_{r12} c_1^\top + \Phi_{r21} - \Phi_{r22} c_1^\top$ and $\overline{\Phi}_{22} = c_1^\top \Phi_{r12} + \Phi_{r22}$. It is to be noted that $\overline{\Phi}_{11}$ is stable by design.

In DTSM, the sliding variable does not become zero and results in a quasi-sliding mode (QSM) in the system. Thus the state trajectory of the system (5a) is confined in a band around the sliding manifold $\mathcal{S} := \{z^r \in \mathbb{R}^r : \hat{c}^\top z^r = 0\}$. Consequently, the states are bounded in a region around the origin $z^{r_1} = 0$. Indeed, the set $\mathcal{S}$ renders the system (5a) insensitive to the external perturbation. To this end, we design a control law that also exhibits motion similar to QSM.

3.2 Proposed event-triggered control law

Modified Bartoszewicz' reaching law: We write the modified Bartoszewicz' reaching law for ET-DTSM control as

$$s_{z_{k+1}} = \alpha + \tilde{d}_k + s_{d_{k+1}}, \tag{11}$$

where $\tilde{d}_k = \hat{c}^\top \Gamma_r d_k$ and $|\tilde{d}_k| \leq \tilde{d}_o$. The function s_{d_k} is an *a priori* known function and α is a positive constant. According to this law the following holds:

- If $|s_{z_0}| > 2(\alpha + \tilde{d}_o)$, then $s_{d_0} = s_{z_0}$,
$$s_{d_k} s_{z_0} \geq 0, \quad \forall k \geq 0,$$
$$s_{d_k} = 0, \quad \forall k \geq k^\star,$$
$$\text{and} \quad |s_{d_{k+1}}| < |s_{d_k}| - 2(\alpha + \tilde{d}_o), \quad \forall k < k^\star.$$
- Otherwise if $|s_{z_0}| \leq 2(\alpha + \tilde{d}_o)$, then $s_{d_k} = 0$, $\forall k \geq 0$.

This reaching law ensures that in each time step the sliding variable decreases by a magnitude greater than $2(\alpha + \tilde{d}_o)$. The constant $k^\star$ is a positive integer which confines the rate of convergence of s_{z_k} to the PQSM band (defined later) in $k^\star$ steps. The function s_{d_k} can be defined as

$$s_{d_k} = \frac{k^\star - k}{k^\star} s_{z_0}, \quad \text{for} \ k^\star < \frac{|s_{z_0}|}{2(\alpha + \tilde{d}_o)}.$$

Control law: Let $\{k_i\}_{i \in \mathbb{Z}_{\geq 0}}$ be the sequence of time instants generated by the triggering mechanism (to be designed) at which the control signal is updated. The control which fulfils our design objective is given by

$$u_k = -(\hat{c}^\top \Gamma_r)^{-1} (\hat{c}^\top \Phi_r z_{k_i}^r - s_{d_{k_i+1}}), \tag{12}$$

for all $k \in [k_i, k_{i+1})$ and for any $i \in \mathbb{Z}_{\geq 0}$. Note that $\hat{c}^\top \Gamma_r$ is invertible by design and $k_i + 1 \neq k_{i+1}$. For stability analysis, define the sets: $\Omega_1 = \{z^{r_1} \in \mathbb{R}^{r-1} : z^{r_1 \top} P_1 z^{r_1} \leq \delta_1\}$ and $\Omega_2 = \{s_z \in \mathbb{R} : |s_z| \leq \delta_2\}$ for some positive constants δ_1 and δ_2, and matrix P_1 the solution of

$$\overline{\Phi}_{11}^\top P_1 \overline{\Phi}_{11} - P_1 = -Q_1, \tag{13}$$

where Q_1 is any positive definite matrix. The choice of these constants will be discussed later. For shorthand notation, denote $\Omega = \Omega_1 \times \Omega_2 \times \mathbb{R}^{n-r}$.

3.3 Event-triggering rule

The sampling error can be defined for the reduced-order system as $e_k^z = z_{k_i}^r - z_k^r$, for all $k \in [k_i, k_{i+1})$ and any $i \in \mathbb{Z}_{\geq 0}$. The bound of e_k^z is used to generate the triggering instants as follows:

(i) for $\|\Phi_r\| = 0$,
$$k_0 = 0, \ k_{i+1} = \inf\{k > k_i : \|e_k^z\| \geq \sigma\alpha\}, \tag{14a}$$
(ii) for $\|\Phi_r\| \neq 0$,
$$k_0 = 0, \ k_{i+1} = \inf\{k > k_i : \|\hat{c}\| \|\Phi_r\| \|e_k^z\| \geq \sigma\alpha\}, \tag{14b}$$

where $\alpha > 0$ is the triggering parameter and $\sigma \in (0, 1)$ is a restrictive factor. The constant σ is used to tighten the triggering condition to account for delays and other factors. The case (i) holds for a scalar system with $\Phi_r = 0$, whereas the other one is true for any system in general. In case (i) an event is generated whenever condition $\|e_k^z\| \leq \sigma\alpha$ is violated, whereas in case (ii) an event is generated whenever condition $\|\hat{c}\| \|\Phi_r\| \|e_k^z\| \leq \sigma\alpha$ is violated. The selection of α can decide the frequency of triggering instants which could result from the performance of triggering mechanism. In addition, $k_{i+1} - k_i$ for any i-th control task is always lower bounded by the sampling period. The proposed triggering rule uses the state vector of the reduced-order system instead of the full-order system. The main aim of this triggering mechanism is to achieve larger inter-event time with the help of reduced state vector.

3.4 Analysis of closed-loop system

In the event-triggered implementation of control, we guarantee a PQSM in the system which is different from QSM due to the triggering mechanism and is recalled below.

Definition 1. System (5) is said to be in practical quasi-sliding mode if for some $\varepsilon > \tilde{d}_o$ there exists a $k^\star > 0$ such that $|s_{z_k}| \leq \varepsilon$ for all $k \geq k^\star$.

To analyze the stability of the equilibrium point for the closed-loop system, we first show the existence of the practical quasi sliding motion around the sliding manifold. Thereafter we establish the boundedness of the state trajectories of the full-order system. The following lemma yields a choice of the constants δ_1 and δ_2 under some assumptions.

Lemma 1. Let $\delta_1 > 0$ and $\delta_2 > \tilde{d}_o$ be constants satisfying

$$\delta_1 > \left(1 + \frac{4\lambda_M(P_1)}{\lambda_m(Q_1)}\right) a\delta_2^2, \tag{15}$$

where $a = (2\|\overline{\Phi}_{11}^\top P_1 \overline{\Phi}_{12}\|^2/\lambda_m(Q_1)) + \|\overline{\Phi}_{12}^\top P_1 \overline{\Phi}_{12}\|$. Assume that $s_{z_k} \in \Omega_2$ for all $k \geq 0$. Then $z_k^{r_1} \in \Omega_1$ for all $k \geq 0$.

Proof. Consider $V_1(z_k^{r_1}) = {z_k^{r_1}}^\top P_1 z_k^{r_1}$ as Lyapunov function for the subsystem (9). For simplifying the notation, let us denote $V_{1_k} := V_1(z_k^{r_1})$ and $\Delta V_{1_k} := V_{1_{k+1}} - V_{1_k}$ for any $k \in \mathbb{Z}_{\geq 0}$. Then one can show that

$$\begin{aligned}\Delta V_{1_k} &= {z_k^{r_1}}^\top (\overline{\Phi}_{11}^\top P_1 \overline{\Phi}_{11} - P_1) z_k^{r_1} + \overline{\Phi}_{12}^\top P_1 \overline{\Phi}_{12} s_{z_k}^2 \\ &\quad + 2{z_k^{r_1}}^\top \overline{\Phi}_{11}^\top P_1 \overline{\Phi}_{12} s_{z_k} \\ &= -{z_k^{r_1}}^\top Q_1 z_k^{r_1} + \overline{\Phi}_{12}^\top P_1 \overline{\Phi}_{12} s_{z_k}^2 + 2{z_k^{r_1}}^\top \overline{\Phi}_{11}^\top P_1 \overline{\Phi}_{12} s_{z_k} \\ &\leq -\frac{\lambda_m(Q_1)}{2}\|z_k^{r_1}\|^2 + a s_{z_k}^2. \end{aligned} \tag{16}$$

The last inequality is obtained by applying Young's inequality to the term $2{z_k^{r_1}}^\top \overline{\Phi}_{12}^\top P_1 \overline{\Phi}_{12} s_{z_k}$. We now use $V_{1_k} \leq \lambda_M(P_1)\|z_k^{r_1}\|^2$ and the fact that $|s_{z_k}| \leq \delta_2$ for all $k \in \mathbb{Z}_{\geq 0}$, to deduce

$$\Delta V_{1_k} \leq -\frac{\lambda_m(Q_1)}{2\lambda_M(P_1)} V_{1_k} + a\delta_2^2. \tag{17}$$

Note that $\delta_1' := \delta_1 - a\delta_2^2 > 0$, which follows from (15). Then it is easy to verify that

$$\Delta V_{1_k} < -\frac{\lambda_m(Q_1)}{4\lambda_M(P_1)} V_{1_k},$$

for all $V_{1_k} > \delta_1'$ because of (15), and also for $V_{1_k} \leq \delta_1'$, we have

$$V_{1_{k+1}} \leq V_{1_k} - \frac{\lambda_m(Q_1)}{2\lambda_M(P_1)} V_{1_k} + a\delta_2^2 < \delta_1' + a\delta_2^2 = \delta_1.$$

Thus the trajectory $z_k^{r_1}$ starting within Ω_1 enters the set $\{z^{r_1} : V_1(z^{r_1}) \leq \delta_1'\}$ in finite number of steps, and eventually can never leave the set Ω_1. Therefore Ω_1 is a positively invariant set. This proves our claim. □

We now discuss the convergence of the state trajectories to some bounded region starting within the set $\Omega_1 \times \Omega_2$. Before proceeding further, we fix δ_1 and δ_2 as constants according to (15). Note that we choose $\delta_2 > \tilde{d}_o$, because the upper bound on disturbance contributes to the ultimate band of the sliding trajectory. Hence the set of interest, to which the initial condition $s_z(0)$ belongs, should be bigger than $\tilde{d}_o$. In our analysis, we show the convergence of s_z to the vicinity of $\mathcal{S}$ in finite time. The existence of PQSM in the system for some appropriate design parameters is presented below.

Proposition 1. Consider system (5a), control law (12), and triggering mechanism (14). Let $(z^{r_1}(0), s_z(0)) \in \Omega_1 \times \Omega_2$. Then for any given $\tilde{d}_o < \varepsilon < \delta_2$, there exists a triggering parameter $\alpha > 0$ such that $s_{z_k} \in \Omega_2$, for all $k \in \mathbb{Z}_{\geq 0}$. Also PQSM is enforced in the system.

Proof. We define the set $\mathcal{S}_\varepsilon = \{s_z \in \mathbb{R} : |s_z| \leq \varepsilon\}$ for the given $\varepsilon > 0$. Now we write the closed-loop dynamics of the sliding variable (8) in the interval $k \in [k_i, k_{i+1})$, for any $i \in \mathbb{Z}_{\geq 0}$ as

$$\begin{aligned} s_{z_{k+1}} &= \hat{c}^\top \Phi_r z_k^r - \hat{c}^\top \Phi_r z_{k_i}^r + \tilde{d}_k + s_{d_{k_i+1}} \\ &= -\hat{c}^\top \Phi_r e_k^z + \tilde{d}_k + s_{d_{k_i+1}}. \end{aligned} \tag{18}$$

The triggering law ensures either $\|\hat{c}\|\|\Phi_r\|\|e_k^z\| < \alpha$ or $\|e_k^z\| < \alpha$, for all $k \in [k_i, k_{i+1})$. As a result in case (i), upper bound on the first term is zero and in case (ii),

$$|\hat{c}^\top \Phi_r e_k^z| \leq \|\hat{c}\|\|\Phi_r\|\|e_k^z\| < \alpha.$$

Also the term $s_{d_{k_i+1}}$ goes to zero in $k^\star$ steps by design and the remaining term is bounded by $\tilde{d}_o$. So from (18), $|s_{z_{k+1}}| \leq \alpha + \tilde{d}_o$ for all $k \geq k^\star$. Using $\alpha < \varepsilon - \tilde{d}_o$, we get $|s_{z_{k+1}}| < \varepsilon$. From this it can be seen that s_{z_k} starting within $\Omega_2 \setminus \mathcal{S}_\varepsilon$ enters the set $\mathcal{S}_\varepsilon$ in the steps no longer than $k^\star$. Thus the trajectory of (5a) enters the set $\left\{z^r \in \mathbb{R}^r : |s_z| = |\hat{c}^\top z^r| \leq \varepsilon\right\}$ in finite time $k^\star$, and subsequently does not leave thereafter, which finally proves our claim. □

Remark 1. Proposition 1 shows the existence of PQSM in the coordinate of the reduced-order system. However, a similar result also holds for the full-order system with the same sliding variable. That is to say, the state trajectories of the full-order plant are bounded around the sliding manifold $\{x \in \mathbb{R}^n : \hat{c}^\top T_1 \overline{C}_a x = 0\}$ in some finite time.

Now from Lemma 1, it is also evident that z^{r_1} remains bounded within Ω_1. The main result of the paper is:

Theorem 1. Consider system (5), control law (12), and event-triggering mechanism (14). Suppose $(z^{r_1}(0), s_z(0), z^2(0)) \in \Omega$, then it follows that

- there exists $\alpha > 0$, such that $z_k^r \in \Omega_1$, for all $k \geq 0$, and it becomes ultimately bounded.
- the non-dominant state trajectory z_k^n is ultimately bounded.

Proof. We start the proof by choosing any $\varepsilon > 0$ satisfying $\tilde{d}_0 < \varepsilon < \delta_2$. Then Proposition 1 implies the existence of $\alpha > 0$, such that $s_{z_k} \in \Omega_2$ for all $k \in \mathbb{Z}_{\geq 0}$. As a consequence of this, Lemma 1 guarantees that $z_k^{r_1} \in \Omega_1$ for all $k \geq 0$. Also there exist an integer $k^\star > 0$ and $\alpha > 0$, such that $|s_{z_k}| \leq \varepsilon$ for all $k \geq k^\star$ by Proposition 1.

We now consider the evolution of $z_k^{r_1}$ for all $k \geq k^\star$ in the subsequent analysis. One can show the inequality

$$\Delta V_{1_k} \leq -\frac{\lambda_m(Q_1)}{4}\|z^{r_1}\|^2, \quad \forall \|z^{r_1}\| \geq \frac{2\sqrt{a}}{\sqrt{\lambda_m(Q_1)}}\varepsilon,$$

which guarantees the convergence of trajectories to the set $\{z^{r_1} : \|z^{r_1}\| < 2\varepsilon\sqrt{a}/\sqrt{\lambda_m(Q_1)}\}$ in finite number of steps. By recalling the fact that

$$\|z_k^{r_1}\| < \frac{2\sqrt{a}}{\sqrt{\lambda_m(Q_1)}}\varepsilon \implies V_{1_k} < \frac{4\lambda_M(P_1)a}{\lambda_m(Q_1)}\varepsilon^2,$$

it can be shown that for $V_{1_k} \le 4\lambda_M(P_1)a\varepsilon^2/\lambda_m(Q_1)$,

$$V_{1_{k+1}} \le V_{1_k} - \frac{\lambda_m(Q_1)}{2}\|z_k^{r_1}\|^2 + a\varepsilon^2 < ba\varepsilon^2,$$

where $b = 1 + (4\lambda_M(P_1)/\lambda_m(Q_1))$. Thus there exists a $\hat{k}^\star > 0$, such that $z_k^{r_1} \in \{z^{r_1} : \|z^{r_1}\|^2 \le ab\varepsilon^2/\lambda_m(P_1)\}$ for all $k \ge \hat{k}^\star$. Using these bounds in the inequality

$$\|z^r\| \le \|T_r^{-1}\|(\|z^{r_1}\| + |s_z|),$$

one achieves that $\|z_k^r\| \le \left\|T_r^{-1}\right\|(1 + \sqrt{ab/\lambda_m(P_1)})\varepsilon =: \varepsilon_1$ for all $k \ge \hat{k}^\star$.

The boundedness of the non-dominant state trajectories can be shown using the Lyapunov function $V_2(z_k^n) = z_k^{n\top} P_2 z_k^n$, where the positive definite matrix P_2 is a solution to the equation $\Phi_n^\top P_2 \Phi_n - P_2 = -Q_2$ for any $Q_2 > 0$. We denote $\Delta V_{2_k} := V_{2_{k+1}} - V_{2_k}$ for $V_2(z_{k+1}^n) = V_{2_{k+1}}$ and $V_2(z_k^n) = V_{2_k}$. It can be shown that

$$\begin{aligned}\Delta V_{2_k} &= -z_k^{n\top} Q_2 z_k^n + 2z_k^{n\top}\Phi_n^\top P_2 \Gamma_n(u_k + d_k)\\ &\quad + (u_k + d_k)^\top \Gamma_n^\top P_2 \Gamma_n(u_k + d_k)\\ &\le -\lambda_m(Q_2)\|z_k^n\|^2 + 2\|\Phi_n^\top P_2 \Gamma_n\|\|z_k^n\|(|u_k| + d_0)\\ &\quad + \|\Gamma_n^\top P_2 \Gamma_n\|(|u_k| + d_0)^2.\end{aligned}$$

Let us analyze each term separately. The second term can be written as

$$\begin{aligned}2\|\Phi_n^\top P_2\Gamma_n\|\|z_k^n\|(|u_k| + d_0) &\le \frac{\lambda_m(Q_2)}{2}\|z_k^n\|^2\\ &\quad + \frac{2\|\Phi_n^\top P_2\Gamma_n\|^2}{\lambda_m(Q_2)}(|u_k| + d_0)^2.\end{aligned}$$

Since $z_k^{r_1}$ is bounded for all $k \in \mathbb{Z}_{\ge 0}$, there exists a constant $U_M > 0$, such that $|u_k| \le U_M$ for all $k \in \mathbb{Z}_{\ge 0}$. Therefore one can show the boundedness of z_k^n from

$$\begin{aligned}\Delta V_{2_k} &\le -\frac{\lambda_m(Q_2)}{2}\|z_k^n\|^2\\ &\quad + \left(\frac{2\|\Phi_n^\top P_2\Gamma_n\|^2}{\lambda_m(Q_2)} + \|\Gamma_n^\top P_2\Gamma_n\|\right)(U_M + d_0)^2\\ &\le -\frac{\lambda_m(Q_2)}{4}\|z_k^n\|^2, \qquad \forall \|z_k^n\| \ge \mu, \end{aligned} \tag{19}$$

where

$$\mu = \sqrt{\frac{4}{\lambda_m(Q_2)}\left(\frac{2\|\Phi_n^\top P_2\Gamma_n\|^2}{\lambda_m(Q_2)} + \|\Gamma_n^\top P_2\Gamma_n\|\right)}(U_M + d_0).$$

The non-dominant state trajectories will enter the set $\{z^n : \|z^n\| \le \mu\}$ in finite number of steps (say $k_2^\star$). Then one can again show that for $V_{2_k} \le \lambda_M(P_2)\mu^2$,

$$V_{2_{k+1}} \le V_{2_k} - \frac{\lambda_m(Q_2)}{2\lambda_M(P_2)}V_{2_k} + \frac{\lambda_m(Q_2)}{4}\mu^2 < \kappa\mu^2,$$

where $\kappa = \lambda_M(P_2) + (\lambda_m(Q_2)/4)$. This further implies $\|z_{k+1}^n\| \le \sqrt{\kappa/\lambda_m(P_2)}\mu$. It may be noted that $\sqrt{\kappa/\lambda_m(P_2)} > 1$. Hence the trajectories again converge to this set due to (19) and thus we conclude that $\{z^n : \|z^n\| \le \sqrt{\kappa/\lambda_m(P_2)}\mu\}$ is a positively invariant set. □

4. SIMULATION RESULTS

Consider the system in (Bandyopadhyay et al., 2006)

$$\Phi = \begin{bmatrix}1 & 0.198 & -0.908 & -0.059\\ 0 & 0.997 & -9.698 & -0.908\\ 0 & -0.012 & 1.488 & 0.231\\ 0 & -0.125 & 5.394 & 1.488\end{bmatrix}, \ \Gamma = \begin{bmatrix}0.0228\\ 0.0922\\ 0.0964\\ 1.0292\end{bmatrix},$$

and $d_k = 0.05\sin(2k)$. This system has two dominant modes. Using a proper transformation, the reduced-order model is obtained as

$$z_{k+1}^1 = \begin{bmatrix}1 & 0\\ 0 & 2.7547\end{bmatrix}z_k^1 + \begin{bmatrix}1.7055\\ 0.6945\end{bmatrix}(u_k + d_k).$$

We choose $\hat{c} = [-0.1625 \ \ 1]^\top$, such that $\Phi_{r11} - \Phi_{r12}c_1^\top$ has eigenvalues inside the unit disk, where matrices are mentioned in (6), and the aggregation matrix is

$$\overline{C}_a = \begin{bmatrix}1 & 0.7770 & 0.0034 & 1.5650\\ 0 & -0.0487 & 2.3341 & 0.4606\end{bmatrix}.$$

In order to illustrate our proposed strategy, we choose $Q_1 = 2$ and satisfying relation (13) gives $P_1 = 2.1978$. Then for defining set Ω_2, we fix $\delta_2 = 8$ and using relation (15) we get $\delta_1 = 1.2003 \times 10^4$. Further, by taking proper transformation and choosing $z^2(0) = [2 \ \ 1]^\top$ we get the initial condition as $x(0) = [59.1815 \ -44.9639 \ 0.2269 \ 10.6018]^\top$. The practical sliding mode band is chosen as $\varepsilon = 0.6$, which satisfies relation $\tilde{d}_o < \varepsilon < \delta_2$. This results in the triggering parameter $\alpha = 0.5087$, and we choose $\sigma = 0.96$.

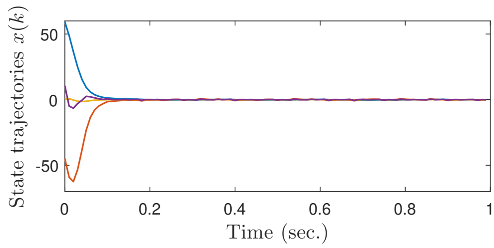

Fig. 2. State trajectories.

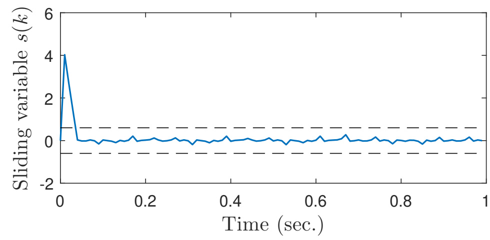

Fig. 3. Sliding variable trajectory.

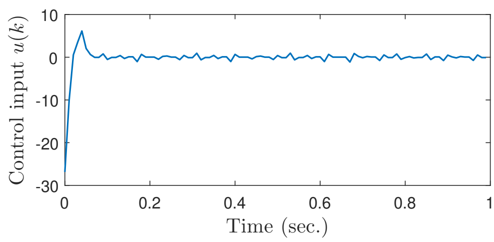

Fig. 4. Control input.

The evolution of states of the high-order system is given in Fig. 2, and it is observed that the states are bounded. The sliding variable is plotted in Fig. 3. This figure shows that applying ET-DTSM control, when designed using reduced-order model, results in PQSM in the high-order system.

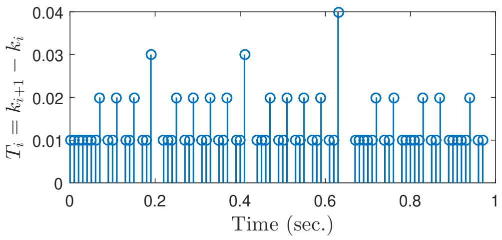

Fig. 5. Evolution of inter-event time.

The size of this band is 0.6. The control input is shown in Fig. 4, and the evolution of inter-event time with time is given in Fig. 5. The total number of triggering instants for a run time of 1 sec is 76.

5. CONCLUSION

We present an ET-DTSM control for stabilizing the high-order system using a reduced-order model of the system. The triggering mechanism is designed using the reduced-order state, which results in a reduction of control tasks as compared to the full-order design. We obtain the desired steady-state performance in terms of the PQSM band and also found regions in which states are bounded. The control law and triggering rule use the system matrix and states of the reduced-order system, so it reduces computations. The ET strategy also decreases the computational burden. Hence the proposed strategy is useful for the systems of higher dimensions.

REFERENCES

Anta, A. and Tabuada, P. (2010). Preliminary results on state-triggered stabilizing control tasks. In *Proc. 45th IEEE Proc. Conf. Decision and Control, San Diego, USA*, 892–897.

Aoki, M. (1968). Control of large-scale dynamic systems by aggregation. *IEEE Transactions on Automatic Control*, 13(3), 246–253.

Bandyopadhyay, B., Abera, A., and Janardhanan, S. (2006). Sliding mode control design via reduced order model approach. In *Proc. IEEE International Conference Ind. Technology, Mumbai, India*, 1538–1541.

Bandyopadhyay, B. and Behera, A.K. (2018). *Event-Triggered Sliding Mode Control: A New Approach to Control System Design*, volume 139. Studies in Systems, Decision and Control, Cham, Switzlerland, Springer International Publishing.

Bandyopadhyay, B., Janardhanan, S., Sreeram, V., et al. (2007). Sliding mode control design via reduced order model approach. *International Journal of Automation and Computing*, 4(4), 329–334.

Bandyopadhyay, B., Unbehauen, H., and Patre, B. (1998). A new algorithm for compensator design for higher-order system via reduced model. *Automatica*, 34(7), 917–920.

Bartoszewicz, A. (1998). Discrete-time quasi-sliding-mode control strategies. *IEEE Transactions on Industrial Electronics*, 45(4), 633–637.

Behera, A.K., Bandyopadhyay, B., and Yu, X. (2018). Periodic event-triggered sliding mode control. *Automatica*, 96, 61–72.

Behera, A.K., Bandyopadhyay, B., and Reger, J. (2016). Discrete event-triggered sliding mode control with fast output sampling feedback. In *Proc. 14th International Workshop on Variable Structure Systems, Nanjing, China*, 148–153.

Borgers, D.N. and Heemels, W.M. (2014). Event-separation properties of event-triggered control systems. *IEEE Transactions on Automatic Control*, 59(10), 2644–2656.

Cucuzzella, M. and Ferrara, A. (2016). Event-triggered second order sliding mode control of nonlinear uncertain systems. In *IEEE 2016 European Control Conference*, 295–300.

Cucuzzella, M., Incremona, G.P., and Ferrara, A. (2020). Event-triggered variable structure control. *International Journal of Control*, 93(2), 252–260.

Gao, W., Wang, Y., and Homaifa, A. (1995). Discrete-time variable structure control systems. *IEEE Transactions on Industrial Electronics*, 42(2), 117–122.

Girard, A. (2014). Dynamic triggering mechanisms for event-triggered control. *IEEE Transactions on Automatic Control*, 60(7), 1992–1997.

Hutton, M. and Friedland, B. (1975). Routh approximations for reducing order of linear, time-invariant systems. *IEEE Transactions on Automatic Control*, 20(3), 329–337.

Kumari, K., Bandyopadhyay, B., Reger, J., and Behera, A.K. (2019a). Event-triggered sliding mode control for a higher-order system via reduced-order model based design. *revised in Automatica*.

Kumari, K., Bandyopadhyay, B., Behera, A.K., and Reger, J. (2016). Event-triggered sliding mode control for delta operator systems. In *Proc. 42nd Annual Conference of IEEE Ind. Electronics Society, Florence, Italy*, 148–153.

Kumari, K., Bandyopadhyay, B., Kim, K.S., and Shim, H. (2019b). Output feedback based event-triggered sliding mode control for delta operator systems. *Automatica*, 103, 1–10.

Lamba, S. and Rao, S. (1974). On suboptimal control via the simplified model of davison. *IEEE Transactions on Automatic Control*, 19(4), 448–450.

Linnemann, A. (1988). Existence of controllers stabilizing the reduced-order model and not the plant. *Automatica*, 24(5), 719.

Lunze, J. and Lehmann, D. (2010). A state-feedback approach to event-based control. *Automatica*, 46(1), 211–215.

Postoyan, R., Tabuada, P., Nešić, D., and Anta, A. (2015). A framework for the event-triggered stabilization of nonlinear systems. *IEEE Transactions on Automatic Control*, 60(4), 982–996.

Shamash, Y. (1974). Stable reduced-order models using padé-type approximations. *IEEE Transactions on Automatic Control*, 19(5), 615–616.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *IEEE Transactions on Automatic Control*, 52(9), 1680–1685.

Tallapragada, P. and Chopra, N. (2013). On event triggered tracking for nonlinear systems. *IEEE Transactions on Automatic Control*, 58(9), 2343–2348.

Utkin, V. (1977). Variable structure systems with sliding modes. *IEEE Transactions on Automatic control*, 22(2), 212–222.