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REDUCED-ORDER SLIDING FUNCTION DESIGN FOR A CLASS OF NONLINEAR SYSTEMS

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ABSTRACT

In this paper, the design of first order sliding mode control (SMC) and twisting control based on the reduced order sliding function is proposed for the robust stabilization of an class of uncertain nonlinear single-input system. This method greatly simplifies the control design as the sliding function is linear, which is based on reduced order state vector. The nonlinear system is represented as a cascade interconnection of two subsystems driving and driven subsystems. Sliding surface and SMC are designed for only the driving subsystem that guarantees the asymptotic stability of the entire system. To show the effectiveness of the proposed control schemes, the simulation results of translational oscillator with rotational actuator are illustrated.

Key Words: Reduced order sliding function, sliding mode control, twisting control, uncertain nonlinear system, translational oscillator with rotational actuator.

I. INTRODUCTION

A sliding mode control (SMC) attracts researchers for it's ability to completely annihilate the matched disturbance [25, 9]. SMC design comprises the design of sliding function (surface) and the design of control law that initiates the sliding motion along the surface in finite time [27, 11, 8, 23]. The design of sliding function for nonlinear systems is relatively intricate because of difficulty in proving the stability of the sliding motion, usually for the higher order systems with disturbance.

In this article, we propose a design method for SMC and the twisting control based on reduced order sliding function so that robustness against disturbance can be guaranteed along with the asymptotic stability of a system at equilibrium. As the sliding manifold consists of fewer states, it greatly simplifies the design.

SMC based on reduced order sliding function is rarely found in literature, refer to [29, 20, 1] for the linear systems. An idea of using SMC that is based on reduced order switching function stems from the recent work in [2, 19] for the linear continuous and discrete-time systems in special coordinate basis (SCB) form.

As SMC is discontinuous control, it leads to chattering. This poses implementation issues in certain practical situations [28]. One way to minimize the chattering is to use twisting control by artificially increasing the relative degree of the system. Such control inherits the robustness property of conventional SMC [14, 18, 17].

The proposed method includes the nonlinear coordinate transformation that transforms the system into cascade interconnection of a driving subsystem in phase variable form and a driven subsystem with asymptotically stable dynamics. For the transformed system, a reduced order sliding function is designed that involves only the driving subsystem states. Such control guarantees global asymptotic stability of the

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system. Furthermore, to circumvent the chattering issue, twisting control (second order sliding mode) is designed with the same reduced order switching function.

As an illustrative example, a translational oscillator with rotational actuator (TORA) is considered for the design. TORA was initially studied in [21] to investigate the precession phase lock (PPL) phenomenon encountered in dual-spin spacecraft. Later, in [6, 7], fourth order model of TORA has been considered as a benchmark problem for the nonlinear control system designs. Recently, two-dimensional translational oscillator with rotational actuator model and passivity-based control design for the same has been proposed in [10].

II. PROBLEM FORMULATION

We first illustrate the idea of cascading two subsystems—one is in phase variable form and another is the residual system which is asymptotically stabilizable—which facilitates the design of SMC to stabilize the system dynamics.

2.1. An illustrative example

Consider a nonlinear system

$$\begin{aligned} \dot{x}_1 &= -2x_1^3 + x_2 & \text{(1a)}\\ \dot{x}_2 &= -x_1 + x_2 + x_2^2 + x_3 & \text{(1b)}\\ \dot{x}_3 &= x_4 & \text{(1c)}\\ \dot{x}_4 &= \alpha(x) + \exp(x_1)u + w & \text{(1d)} \end{aligned}$$

where u and w are the control and the disturbance inputs, respectively.

Denote $\mathbf{x}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$. Note that x_3 acts as an input to the $\mathbf{x}_1$-subsystem. Let $\xi(\mathbf{x}_1)$ be a smooth function such that $x_3 = \xi(\mathbf{x}_1)$ makes $\mathbf{x}_1 = 0$ is asymptotically stable for the $\mathbf{x}_1$-subsystem. Though the function $\xi(\mathbf{x}_1)$ can be constructed in many ways, we only discuss the Lyapunov method based design for its relevant to the rest of paper.

Let $V = 0.5\mathbf{x}_1^\top \mathbf{x}_1$ be the Lyapunov function for the $\mathbf{x}_1$-subsystem. Then, time derivative of V along the solution of $\mathbf{x}_1$-subsystem can be given by

$$\begin{aligned} \dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= x_1(-2x_1^3 + x_2) + x_2(-x_1 + x_2 + x_2^2 + x_3) \quad (2)\\ &= -2x_1^4 + x_2^2 + x_2^3 + x_2\,\xi(\mathbf{x}_1). \end{aligned}$$

Clearly, $\xi(\mathbf{x}_1) = -(k_1 x_2 + x_2^2)$ for any $k_1 > 1$ makes $\dot{V} < 0$. Thus, the $\mathbf{x}_1$-subsystem is asymptotically stabilizable to the origin.

Define $T(x)$ be a diffeomorphic map given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 - \xi \\ x_4 - \dot{\xi} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 + k_1 x_2 + x_2^2 \\ x_4 + (k_1 + 2x_2)(-x_1 + x_2 + x_2^2 + x_3) \end{bmatrix}.$$

Then, the system (1) can be transformed into under the transformation $z = T(x)$,

$$\begin{aligned} \dot{z}_1 &= -2z_1^3 + z_2 & \text{(3a)}\\ \dot{z}_2 &= -z_1 - (k_1 - 1)z_2 + z_3 & \text{(3b)}\\ \dot{z}_3 &= z_4 & \text{(3c)}\\ \dot{z}_4 &= a(z) + b(z)u + w & \text{(3d)} \end{aligned}$$

where $b(z) = \exp(z_1)$ and

$$\begin{aligned} a(z) = (k_1 + 2z_2)((k_1 - 1)z_1 - 2k_1 z_2 - (k_1 - 1)z_3 \\ + z_4 + k_1^2 z_2 + 2z_1^3) + \alpha(T^{-1}(z)) \\ + 2(z_1 - z_2 - z_3 + k_1 z_2)^2. \end{aligned}$$

It can be easily verified that $V(T^{-1}(z))$ is an input-to-state stable (ISS) Lyapunov function ([24, 12]) for the $\mathbf{z}_1$-subsystem if z_3 is imagined to be the input. So, the trajectories of $\mathbf{z}_1$-subsystem are bounded when z_3 is bounded and converge to $\mathbf{z}_1 = 0$ only when $z_3 \equiv 0$. So, the goal of the designer is to design a stabilizing the feedback law $u(z)$ to stabilize the chained form subsystem. The synthesis technique in the motivating example is very standard in the literature particularly in the case of nonlinear systems.

2.2. Problem statement

Consider a class of single input nonlinear system

$$\begin{aligned} \dot{\mathbf{x}}_1 &= f_{11}(\mathbf{x}_1) + f_{12}(x_{21}) & \text{(4a)}\\ \dot{x}_2 &= A_{21}x_2 + A_{22}x_3 & \text{(4b)}\\ \dot{x}_3 &= f_{31}(x) + g_{32}(x)u + \mathbf{w} & \text{(4c)} \end{aligned}$$

where $\mathbf{x}_1 \in \mathcal{X}_1 \subset \mathbb{R}^p$, $x_2 = \begin{bmatrix} x_{21} & x_{22} & \cdots & x_{2q} \end{bmatrix}^\top \in \mathcal{X}_2 \subset \mathbb{R}^q$ and $x_3 \in \mathcal{X}_3 \subset \mathbb{R}$ are the states. Here, $x = (\mathbf{x}_1, x_2, x_3) \in \mathcal{X} \subset \mathbb{R}^n$ where $n = p + q + 1$ and $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. u and $\mathbf{w}$ are the control and unknown disturbance inputs respectively. The vector fields $f_{11}(\cdot)$, $f_{12}(\cdot)$, and $f_{31}(\cdot)$ are smooth on the set $\mathcal{X}$ which contains $x = 0$ and $g_{32}(x) \neq 0$ for all $x \in \mathcal{X}$. The matrices A_{21} and A_{22} are given by

$$A_{21} = \begin{bmatrix} 0 & I_{q-1} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We observe that the matrix pair (A_{21}, A_{22}) is controllable. In this paper, we study the stabilization of the system (4) by designing a control law such that the system trajectories converge to the origin in spite of the disturbance. As it can be seen that the subsystem (4a) is driven by x_{21}, we make the following assumptions to achieve the robust stabilization of system (4).

Assumption 1 *The disturbance $\mathbf{w}(x,t)$ is a continuously differentiable function. Moreover, there exist some positive constants μ_1 and μ_2 such that $|\mathbf{w}(x,t)| \leq \mu_1$ and $|\dot{\mathbf{w}}(x,t)| \leq \mu_2$ for all $x \in \mathcal{X}$ and all $t \geq 0$.*

Assumption 2 *The subsystem* (4a) *is* locally *ISS.*

Remark 1 *The above assumption, that* (4a) *is* locally *ISS, ensures the system trajectory remains bounded for every admissible x_{21}. However, the conservatism can be overcome if the bounded input is affine in* (4a) *([24]), i.e., $f_{21}(x_{21}) = f_0 x_{21}$ for some nonzero scalar f_0. Although this result is stated for an ISS system, the similar argument can be made under some assumption for the* locally *ISS system.*

Our goal of the paper is to design the sliding mode based control laws for a subsystem of (4) such that the stability of closed loop system is guaranteed. Two sliding based design of control laws—one is the first order or the classical SMC law and other one is a continuous control law via higher order sliding mode—are used in our work. We present the stability and design approaches in the following section of the paper.

III. DESIGN OF SLIDING MODE CONTROL

In this section, the design of SMC for a subsystem is presented such that the whole system is stabilizable by the reduced order control law. At first, we introduce a coordinate transformation that transforms the original system into a cascaded form in which the control input drives only a subsystem (driving system) while another (driven) subsystem is driven by the state of driving subsystem. This is followed by the design of SMC with stability analysis of the system.

3.1. Coordinate transformation

Let $\xi_0 : \mathcal{X}_1 \to \mathcal{X}_3$ be a smooth function such that $\mathbf{x}_1 = 0$ of the driven subsystem

$$\dot{\mathbf{x}}_1 = f_{11}(\mathbf{x}_1) + f_{12}(\xi_0(\mathbf{x}_1)) \tag{5}$$

is asymptotically stable. Then, from Assumption 2 there exists a smooth function (generally a different $\xi_0(\mathbf{x}_1)$) $\xi(\mathbf{x}_1)$ such that (4a) with $x_{21} = \xi(\mathbf{x}_1) + \nu$ is *locally* ISS for any ν in some compact domain. Now, we define a map $T : \mathcal{X} \to \mathbb{R}^n$,

$$T(x) = \begin{bmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 - D\xi(\mathbf{x}_1) \end{bmatrix} \tag{6}$$

where $\mathbf{x}_2 := \begin{bmatrix} x_2^\top & x_3 \end{bmatrix}^\top$ and $D = \begin{bmatrix} 1 & d/dt & \cdots & d^q/dt^q \end{bmatrix}^\top$ is a vector differential operator. We assume that the map $x \mapsto T(x)$ is diffeomorphic for all $x \in \mathcal{X}$.

Then, the system (4) under the transformation $\tilde{x} = T(x)$ in the new coordinate can be represented as

$$\dot{\tilde{\mathbf{x}}}_1 = \tilde{f}_{11}(\tilde{\mathbf{x}}_1) + \tilde{f}_{12}(\tilde{x}_{21} + \xi(\tilde{\mathbf{x}}_1)) \tag{7a}$$
$$\dot{\tilde{x}}_2 = A_{21}\tilde{x}_2 + A_{22}\tilde{x}_3 \tag{7b}$$
$$\dot{\tilde{x}}_3 = \tilde{f}_{31}(\tilde{x}) + \tilde{g}_{32}(\tilde{x})u + \mathbf{w} \tag{7c}$$

where $\tilde{f}_{11} \equiv f_{11}$, $\tilde{f}_{12} \equiv f_{12}$ and $\tilde{f}_{31}(\tilde{x}) = f_{31}(T^{-1}(\tilde{x})) - d^{q+1}\xi(\tilde{\mathbf{x}}_1)/dt^{q+1}$, and $\tilde{g}_{32}(\tilde{x}) = g_{32}(T^{-1}(\tilde{x}))$. It may be noted that for the transformed system (7), we can have $\tilde{\mathbf{x}}_1 \in \widetilde{\mathcal{X}}_1$, $\tilde{x}_2 \in \widetilde{\mathcal{X}}_2$ and $\tilde{x}_3 \in \widetilde{\mathcal{X}}_3$ with $\widetilde{\mathcal{X}} = \widetilde{\mathcal{X}}_1 \times \widetilde{\mathcal{X}}_2 \times \widetilde{\mathcal{X}}_3$. Also, $\tilde{g}_{32}(\tilde{x}) \neq 0$ for all $\tilde{x} \in \widetilde{\mathcal{X}}$. Here, $\tilde{x}_2 = \begin{bmatrix} \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2q} \end{bmatrix}^\top$. Note that in the transformed system (7), the subsystem (7a) still satisfies Assumption 2.

3.2. Sliding mode control design

As the $\tilde{\mathbf{x}}_2$-subsystem in (7b) is in regular form [16], we can directly begin to design the SMC for this subsystem. Consider the sliding function given by

$$\sigma(\tilde{\mathbf{x}}_2) = c^\top \tilde{\mathbf{x}}_2 = \begin{bmatrix} c_1^\top & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \tag{8}$$

and the manifold $\mathsf{S}_1 = \left\{\tilde{\mathbf{x}}_2 \in \mathbb{R}^{q+1} : \sigma(\tilde{\mathbf{x}}_2) = 0\right\}$. Here, c_1 is designed such that all the eigenvalues of the matrix $A_{21} - A_{22}c_1^\top$ have negative real parts. This is always possible because the pair (A_{21}, A_{22}) is controllable. The task is now to design the control law

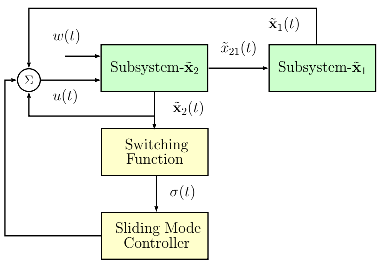

Fig. 1. Control scheme with reduced order sliding function.

which brings the sliding mode in the $\tilde{\mathbf{x}}_2$-subsystem. The SMC for this subsystem is now given as

$$\begin{aligned} u &= -\tilde{g}_{32}^{-1}(\tilde{x})(c_1^\top A_{21}\tilde{x}_2 + c_1^\top A_{22}\tilde{x}_3 + \tilde{f}_{31}(\tilde{x}) \\ &\quad + k\sigma + Q\mathrm{sgn}(\sigma(\tilde{\mathbf{x}}_2))) \end{aligned} \tag{9}$$

where k and Q are some positive constants. Fig.1 shows the idea of using the reduced order based SMC.

Remark 2 *The control law* (9) *becomes discontinuous for* $\tilde{\mathbf{x}}_2 \in \mathsf{S}_1$ *which may cause some adverse effects in the practical systems due to chattering. To avoid this issue, some approximations to the discontinuous component are used in the real time implementation. However, in such attempts the system response may be compromised compared to that of with discontinuous control law.*

It may be noted that the closed loop system stability can be shown by first establishing the sliding motion for $\tilde{\mathbf{x}}_2$-subsystem. So, we show that the control law (9) achieves the sliding mode in some finite-time.

Lemma 1 *Consider the subsystem* (7b) *and* (7c)*, and the control law* (9)*. Then, the sliding mode in the system is enforced by the control law* (9) *in a finite-time if* $k > 0$ *and* $Q > \mu_1$*.*

Proof. Omitted due to page limitation. ■

Proposition 1 *Consider the system* (7) *and the control law* (9)*. Let the conditions in Lemma 1 hold for the controller gain. Then, there exists a subset of* $\widetilde{\mathcal{X}}$ *such that the trajectories of* (7) *starting within this region remain within* $\widetilde{\mathcal{X}}$ *for all time and converge to zero asymptotically.*

Proof. Since $\tilde{\mathbf{x}}_1$-subsystem is locally ISS, this is equivalent to say that there exists a Lyapunov function $V_{\tilde{\mathbf{x}}_1}$ such that

$$\dot{V}_{\tilde{\mathbf{x}}_1} \leq -\alpha_1(\|\tilde{\mathbf{x}}_1\|) + \gamma_1(|\tilde{x}_{21}|) \tag{10}$$

for some class-$\mathcal{K}_\infty$ functions α_1 and γ_1. For any $a_1 > 0$ define the set $\widetilde{\mathcal{X}}_{10} = \{\tilde{\mathbf{x}}_1 \in \widetilde{\mathcal{X}}_1 : V_{\tilde{\mathbf{x}}_1}(\tilde{\mathbf{x}}_1) \leq a_1\}$ such that $\widetilde{\mathcal{X}}_{10} \subset \widetilde{\mathcal{X}}_1$. We now rewrite the dynamics (7b) and (7c) with the control (9) as

$$\begin{aligned} \dot{\tilde{x}}_2 &= (A_{21} - A_{22}c_1^\top)\tilde{x}_2 + A_{22}\sigma \\ \dot{\sigma} &= -k\sigma - Q\,\mathrm{sgn}(\sigma) + \mathbf{w}. \end{aligned}$$

Let P_2 and Q_2 be the symmetric and positive definite matrices for which $(A_{21} - A_{22}c_1^\top)^\top P_2 + P_2(A_{21} - A_{22}c_1^\top) = -Q_2$ holds. Then, using the Lyapunov function $V_{\tilde{x}_2} = \tilde{x}_2^\top P_2 \tilde{x}_2$, one can easily arrive at

$$\dot{V}_{\tilde{x}_2} \leq -\frac{\lambda_{\min}\{Q_2\}}{2}\|\tilde{x}_2\|^2 + \frac{2\|P_2A_{22}\|^2}{\lambda_{\min}\{Q_2\}}\sigma^2. \tag{11}$$

Construct the set $\widetilde{\mathcal{X}}_{20} = \Pi_1 \cap \Pi_2$ where $\Pi_1 = \{\tilde{x}_2 \in \widetilde{\mathcal{X}}_2 : V_{\tilde{x}_2}(\tilde{x}_2) \leq a_2\}$ for any $a_2 > 0$ such that $\Pi_1 \subset \widetilde{\mathcal{X}}_2$ and

$$\Pi_2 = \left\{\tilde{x}_2 \in \widetilde{\mathcal{X}}_2 : \gamma_1(\|\tilde{x}_2\|) \leq \min_{\tilde{\mathbf{x}}_1 \in \partial\widetilde{\mathcal{X}}_{10}} \alpha_1(\|\tilde{\mathbf{x}}_1\|)\right\}.$$

Finally, we let $\widetilde{\mathcal{X}}_{30} = \{\sigma \in \mathbb{R} : V_\sigma(\sigma) \leq a_3\}$ where $V_\sigma(\sigma) = \sigma^2/2$ and $a_3 \leq \lambda_{\min}^2\{Q_2\}/(8\|P_2A_{22}\|^2)\min_{\tilde{x}_2 \in \partial\widetilde{\mathcal{X}}_{20}}\|\tilde{x}_2\|^2$. With this, let us denote $\widetilde{\mathcal{X}}_0 = \widetilde{\mathcal{X}}_{10} \times \widetilde{\mathcal{X}}_{20} \times \widetilde{\mathcal{X}}_{30}$ and assume that $(\tilde{\mathbf{x}}_1(0), \tilde{x}_2(0), \sigma(0)) \in \widetilde{\mathcal{X}}_0$.

We now prove the Lyapunov stability first. It follows immediately from Lemma 1 that $\widetilde{\mathcal{X}}_{30}$ is a positively invariant set, i.e., $\sigma(t)$ belongs to this set for all $t \geq 0$. Similarly, when $\sigma \in \widetilde{\mathcal{X}}_{30}$, (11) implies that $\widetilde{\mathcal{X}}_{20}$ is also a positively invariant set. And, finally from (10) we achieve that the trajectories of (7a) remain within $\widetilde{\mathcal{X}}_{10}$ whenever $\tilde{x}_2 \in \widetilde{\mathcal{X}}_{20}$. This all together shows that (7) is Lyapunov stable.

Asymptotic convergence can be argued easily by observing the sliding mode $\sigma(\tilde{\mathbf{x}}_2) = 0$ in the $\tilde{\mathbf{x}}_2$-subsystem. From the fact that $A_{21} - A_{22}c_1^\top$ is Hurwitz, the trajectories, $\tilde{x}_2(t)$, of the system

$$\dot{\tilde{x}}_2(t) = (A_{21} - A_{22}c_1^\top)\tilde{x}_2(t)$$

converge to zero as $t \to \infty$. This also implies that $\tilde{x}_3(t)$ goes to zero asymptotically. Recall that the subsystem (7a) is locally ISS, so the trajectories of this system remain bounded $\widetilde{\mathcal{X}}_{10}$ for all time and goes to zero when $\tilde{x}_{21} \equiv 0$. Since by design $\tilde{x}_{21}(t)$ goes to zero as $t \to \infty$, $\tilde{\mathbf{x}}_1(t)$ also converges to zero as $t \to \infty$. So, we achieve that the trajectories of the closed loop system (7) approach to $\tilde{x} = 0$ asymptotically. ■

IV. DESIGN VIA TWISTING CONTROL

The conventional SMC (9), that ensures the sliding mode in the $\tilde{\mathbf{x}}_2$-subsystem, suffers from the *chattering* effects which may not be desirable for practical systems. One way to address this issue is by designing the continuous SMC via higher order sliding mode algorithm. In this case, we use twisting control algorithm ([14]) to make the control law continuous and thereby minimizing the chattering effects ([3]). It may be noted that in this synthesis process the derivative of the sliding function can be obtained by employing a robust exact differentiator [15, 4, 13].

4.1. Design of sliding mode control via twisting algorithm

First, we see that the system (7b) can be written as

$$\dot{\tilde{x}}_2 = (A_{21} - A_{22}c_1^\top)\tilde{x}_2 + A_{22}\sigma. \tag{12}$$

Then, by denoting $\sigma_1 \equiv \sigma_1(\tilde{\mathbf{x}}_2) = \sigma(\tilde{\mathbf{x}}_2)$ and $\sigma_2 \equiv \sigma_2(\tilde{\mathbf{x}}) = \dot{\sigma}(\tilde{\mathbf{x}}_2)$, we obtain

$$\begin{aligned}\dot{\sigma}_1 &= \sigma_2\\ \dot{\sigma}_2 &= M(\tilde{x}) + \tilde{g}_{32}(\tilde{x})v + \dot{\mathbf{w}}\end{aligned}$$

where $v = \dot{u}$ is the new virtual control input and

$$\begin{aligned}M(\tilde{x}) = c_1^\top(A_{21} - A_{22}c_1^\top)^2\tilde{x}_2 + c_1^\top A_{22}\sigma_2 + \dot{\tilde{f}}_{31}(\tilde{x})\\ + c_1^\top(A_{21} - A_{22}c_1^\top)A_{22}\sigma_1 + \dot{\tilde{g}}_{32}(\tilde{x})u.\end{aligned}$$

Note that $\sigma_2 = c_1^\top(A_{21} - A_{22}c_1^\top)\tilde{x}_2 + c_1^\top A_{22}\sigma_1 + \tilde{f}_{31}(\tilde{x}) + \tilde{g}_{32}(\tilde{x})u + \mathbf{w}$. The twisting control law which enforces $\sigma_1 = \sigma_2 = 0$ in a finite-time, is given by

$$v = -\tilde{g}_{32}^{-1}(\tilde{x})(M(\tilde{x}) + \epsilon_1\text{sgn}(\sigma_1) + \epsilon_2\text{sgn}(\sigma_2))$$

where ϵ_1 and ϵ_2 are some positive constants. We see that with the virtual control the actual control signal now becomes continuous. It is worthy to note that the control law (13) requires the information of σ_2 which depends on the disturbance, $\mathbf{w}$. So, this control law may not be possible to implement unless the information of uncertainty is known (i.e., σ_2 is known). However, this difficulty can be avoided by a robust exact differentiator as given below for some $\kappa_1 > 0$ and $\kappa_2 > 0$,

$$\begin{aligned}\dot{s}_1 &= -\kappa_1|s_1 - \sigma_1|^{\frac{1}{2}}\text{sgn}(s_1 - \sigma_1) + s_2\\ \dot{s}_2 &= -\kappa_2\text{sgn}(s_1 - \sigma_1)\end{aligned}$$

which provides σ_2 exactly in the presence of uncertainty in a finite-time. Then, the control law can be given by

$$u(t) = \int_0^t \hat{v}(\tau)d\tau \tag{13}$$

where $\hat{v} = -\tilde{g}_{32}^{-1}(\tilde{x})(M(\tilde{x}) + \epsilon_1\text{sgn}(\sigma_1) + \epsilon_2\text{sgn}(s_2))$. Now, using the control law (13) one can write the closed loop system (7) as

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_1 &= \tilde{f}_{11}(\tilde{\mathbf{x}}_1) + \tilde{f}_{12}(\tilde{x}_{21} + \xi(\tilde{\mathbf{x}}_1)) && \text{(14a)}\\ \dot{\tilde{x}}_2 &= (A_{21} - A_{22}c_1^\top)\tilde{x}_2 + A_{22}\sigma_1 && \text{(14b)}\\ \dot{\sigma}_1 &= \sigma_2 && \text{(14c)}\\ \dot{\sigma}_2 &= -\epsilon_1\text{sgn}(\sigma_1) - \epsilon_2\text{sgn}(s_2) + \dot{\mathbf{w}} && \text{(14d)}\\ \dot{\tilde{\sigma}}_1 &= -\kappa_1|\tilde{\sigma}_1|^{\frac{1}{2}}\text{sgn}(\tilde{\sigma}_1) + \tilde{\sigma}_2 && \text{(14e)}\\ \dot{\tilde{\sigma}}_2 &= -\kappa_2\text{sgn}(\tilde{\sigma}_1) - \dot{\sigma}_2 && \text{(14f)}\end{aligned}$$

where $\tilde{\sigma}_1 = s_1 - \sigma_1$ and $\tilde{\sigma}_2 = s_2 - \sigma_2$. The stability of the closed loop system is presented in the next subsection.

4.2. Stability analysis

Let us denote $\sigma_{\mathbf{v}} = \begin{bmatrix}\sigma_1 & \sigma_2\end{bmatrix}^\top$, $s_{\mathbf{v}} = \begin{bmatrix}s_1 & s_2\end{bmatrix}^\top$ and $\tilde{\sigma}_{\mathbf{v}} = \begin{bmatrix}\tilde{\sigma}_1 & \tilde{\sigma}_2\end{bmatrix}^\top$.

Lemma 2 *Consider the subsystems* (14e) *and* (14f). *Assume that σ_1 is available and there exists $\mu > 0$ such that $|\dot{\sigma}_2(t)| \leq \mu$ for all $t \geq 0$. Then, $\tilde{\sigma}_{\mathbf{v}} = 0$ is finite-time stable if $\kappa_1 > 1.5\sqrt{\mu}$ and $\kappa_2 > \mu$.*

Proof. Refer to [15]. ■

Remark 3 *The assumption in the above lemma, that $|\dot{\sigma}_2|$ is bounded, is not an additional constraint imposed on the system. This is because the closed loop system with the control law* (13) *given by* (14d) *guarantees that this assumption is fulfilled.*

Lemma 3 *Consider the subsystems* (14c) *and* (14d). *Let Assumption 1 holds. Assume that $\tilde{\sigma}_{\mathbf{v}} = 0$. Then, $\sigma_{\mathbf{v}} = 0$ is finite-time stable if $\epsilon_1 > \epsilon_2 + \mu_2$ and $\epsilon_2 > \mu_2$.*

Proof. Omitted due to page limitation. ■

We now prove our main result for the closed loop system when the estimated state is being used in the controller. The idea is to design the differentiator gains κ_1 and κ_2 large enough to ensure $\tilde{\sigma}_{\mathbf{v}} = 0$ in a sufficiently small time such that $\sigma_{\mathbf{v}}(t)$ belongs to the region of interest for all time.

Define $\Omega_\rho = \{\sigma_{\mathbf{v}} \in \mathbb{R}^2 : V_1(\sigma_{\mathbf{v}}) \leq \rho\}$ for some $\rho > 0$ where $V_1(\sigma_{\mathbf{v}}) = \epsilon_1|\sigma_1| + \sigma_2^2/2$. Then, for any $\rho_1 > 0$, we define the set Ω_{ρ_1} in the similar manner. Note that $\Omega_{\rho_1} \subset \Omega_\rho$ for any $\rho_1 < \rho$. Also, define $\widetilde{\mathcal{X}}_\rho = \widetilde{\mathcal{X}}_1 \times \widetilde{\mathcal{X}}_2 \times \Omega_\rho$ for later use.

Theorem 1 *Consider the system* (14c)–(14f). *Suppose that $\sigma_{\mathbf{v}}(0) \in \Omega_{\rho_1}$ for any $\rho_1 < \rho$. Then, for any $s_{\mathbf{v}}(0) \in \mathbb{R}^2$ there exist the differentiator gains κ_1 and κ_2 and the controller gains $\epsilon_2 > \mu_2$ and $\epsilon_1 > \epsilon_2 + \mu_2$ such that $\sigma_{\mathbf{v}}(t)$ belongs to Ω_ρ for all $t \geq 0$, and moreover, it approaches to $\sigma_{\mathbf{v}} = 0$ in some finite-time.*

Proof. By noting $\sigma_{\mathbf{v}}(0) \in \Omega_{\rho_1}$, we can show that for any $(\sigma_{\mathbf{v}}, s_{\mathbf{v}}) \in \Omega_\rho \times \mathbb{R}^2$,

$$|\dot{V}_1(\sigma_{\mathbf{v}})| \leq \kappa_{\mathbf{v}}$$

for some $\kappa_{\mathbf{v}} > 0$. Let $t_1 = (\rho - \rho_1)/\kappa_{\mathbf{v}}$. Then, from the above relations, we conclude that $V_1(\sigma_{\mathbf{v}}(t)) \leq \rho$ for all $t \in [0, t_1]$. Now, choose the differentiator gains κ_1 and κ_2 sufficiently large such that $\tilde{\sigma}_{\mathbf{v}}(t) = 0$ for all $t \geq t_1$. This is always possible since Lemma 2 guarantees the finite-time estimation of $\sigma_{\mathbf{v}}$. Now, since $s_2(t) = \sigma_2(t)$ for all $t \geq t_1$, we have

$$\dot{V}_1(\sigma_{\mathbf{v}}(t)) < 0, \qquad \forall\, t \geq t_1$$

for $\sigma_2(t) \neq 0$. Thus, the finite-time stability of $\sigma_{\mathbf{v}} = 0$ follows by Lemma 3 since $\tilde{\sigma}_{\mathbf{v}}(t) = 0$ for all $t \geq t_1$. ■

Proposition 2 *Consider the system* (14a)–(14d). *Let the conditions in Theorem 1 hold. Then, there exists a subset of $\widetilde{\mathcal{X}}_\rho$ such that the system trajectories starting within this region remain within $\widetilde{\mathcal{X}}_\rho$ for all time and converge to zero asymptotically.*

Proof. Following same notations as in the proof of Proposition 1, construct the set $\Omega_{\rho_0} = \{\sigma_{\mathbf{v}} \in \mathbb{R}^2 : V_1(\sigma_{\mathbf{v}}) \leq \rho_0\}$ with any $\rho_0 < \rho$ where $\rho \leq \epsilon_1 \lambda_{\min}\{Q_2\}/(2\|P_2 A_{22}\|) \min_{\tilde{x}_2 \in \partial \widetilde{\mathcal{X}}_{20}} \|\tilde{x}_2\|$. Let $\widetilde{\mathcal{X}}_{\rho_0} = \widetilde{\mathcal{X}}_{10} \times \widetilde{\mathcal{X}}_{20} \times \Omega_{\rho_0}$. Then, with the help of Theorem 1 the proof follows similar lines that of Proposition 1. ■

V. NUMERICAL SIMULATION

The proposed design methods can be applied to practical systems such as underactuated slosh control in a container confined track length ([26]), TORA system. To demonstrate the effectiveness of proposed control design we consider the TORA system. A TORA system, which is also called as rotational/translational Actuator (RTAC), was proposed in [21] to study the control of excited nutation in the dual-spin spacecraft. TORA model also found to be used to study the problem of stabilizing the translational motion of multi-mode systems with a rotational actuator [5].

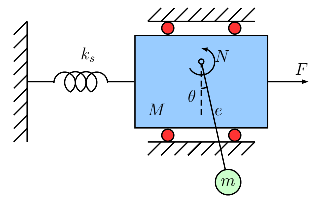

Fig. 2. Translational Oscillator with Rotating Actuator.

As posed in [7], TORA consists of a platform of mass M and eccentric rotating mass m that stabilizes the translational motion of the platform. The rotating mass is located at a distance e from the point about which it rotates with the moment of inertia I. The cart is connected to a fixed wall through a linear spring of stiffness k. Let $N(t)$ be the control torque applied to the mass m and $F(t)$ be the bounded disturbance force on the platform. Let $q(t)$ be the position of the platform and $\theta(t)$ be the angular position of the rotational mass m. A typical TORA system is shown in Fig. 2.

The equations of motion are given by

$$(M + m)\ddot{q} + k_s q = -me(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta) + F \tag{15}$$

$$(I + me^2)\ddot{\theta} = -me\ddot{q}\cos\theta + N. \tag{16}$$

Where, $k_s = m_0(I + me^2)$ and $m_0 = (M + m)/(I + me^2)$. Define $z := \sqrt{m_0}\, q$, $\epsilon := me\sqrt{m_0}/(M + m)$, $v := m_0 N/k_s$ and $w := \sqrt{m_0}\, F/k_s$. Then, the normalized equations of motion (15) and (16) for the TORA system can be expressed as

$$\ddot{z} + z = \epsilon\left(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta\right) + w$$
$$\ddot{\theta} = -\epsilon\ddot{z}\cos\theta + v$$

where z is the normalized cart position, v is dimensionless control torque, w is disturbance force, and ϵ is the coupling factor between the translational and rotational motions. Assume that $\epsilon < 1$ and there exists a $w_{\max} > 0$ such that $|w(t)| \leq w_{\max}$ for all $t \geq 0$.

Define state variables as $z_1 = z$, $z_2 = \dot{z}$, $z_3 = \theta$ and $z_4 = \dot{\theta}$. Then, the dynamics of the system can be described by following state equations

$$\dot{z}_1 = z_2 \tag{17a}$$

$$\dot{z}_2 = \frac{-z_1 + \epsilon z_4^2 \sin z_3}{1 - \epsilon^2 \cos^2 z_3} - \frac{\epsilon \cos z_3}{1 - \epsilon^2 \cos^2 z_3} v + \frac{1}{1 - \epsilon^2 \cos^2 z_3} w \tag{17b}$$

$$\dot{z}_3 = z_4 \tag{17c}$$

$$\dot{z}_4 = \frac{\epsilon \cos z_3 (z_1 - \epsilon z_4^2 \sin z_3)}{1 - \epsilon^2 \cos^2 z_3} + \frac{1}{1 - \epsilon^2 \cos^2 z_3} v - \frac{\epsilon \cos z_3}{1 - \epsilon^2 \cos^2 z_3} w. \tag{17d}$$

Note that here $0 < \epsilon < 1$ which implies that $|\epsilon \cos(\cdot)| < 1$. Define the transformation $x = \Phi_r(z)$ where $\Phi_r(z) = \begin{bmatrix} z_1 + \epsilon \sin z_3 & z_2 + \epsilon z_4 \cos z_3 & z_3 & z_4 \end{bmatrix}^\top$. Therefore, the system (17) in new coordinate space $x = \Phi_r(z)$ can be given as

$$\dot{x}_1 = x_2 \tag{18a}$$

$$\dot{x}_2 = -x_1 + \epsilon \sin x_3 + w \tag{18b}$$

$$\dot{x}_3 = x_4 \tag{18c}$$

$$\dot{x}_4 = u + \mathbf{w} \tag{18d}$$

where $v = -\left(\alpha(x) - u\right)/\beta(x)$ and $\mathbf{w} = -\epsilon w \beta(x) \cos x_3$ with

$$\beta(x) = 1/(1 - \epsilon^2 \cos^2 x_3)$$
$$\alpha(x) = \epsilon \cos x_3 [x_1 - \epsilon(1 + x_4^2) \sin x_3] \beta(x).$$

The system (18) is already in the desired form. Now, we have to look for a function $x_3 = \xi(\mathbf{x}_1)$ such that $\mathbf{x}_1 = 0$ of (18a) and (18b) with $w = 0$ is asymptotically stable where $\mathbf{x}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$. Let $V(\mathbf{x}_1) = \mathbf{x}_1^\top \mathbf{x}_1/2$ be the Lyapunov function for the subsystem (18a) and (18b). Then, time derivative of $V(\mathbf{x}_1)$ along the solutions of (18a) and (18b) with $w = 0$ yields

$$\dot{V}(\mathbf{x}_1) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = \epsilon x_2 \sin x_3.$$

Let $x_3 = \xi(\mathbf{x}_1) = -\arctan \kappa x_2$ for any $\kappa > 0$. Then, the above relation becomes

$$\dot{V}(\mathbf{x}_1) = \epsilon x_2 \sin(-\arctan \kappa x_2) = -\left(\frac{\epsilon \kappa x_2^2}{\sqrt{1 + \kappa^2 x_2^2}}\right). \tag{19}$$

The convergence of system trajectory to the origin cannot be guaranteed by (19). So, we apply LaSalle's invariance principle to show the asymptotic convergence of the system trajectories to $\mathbf{x}_1 = 0$. Let $\Omega \subset \mathbb{R}^2$ be a positively invariant compact set containing the equilibrium. Let $E_0 = \{\mathbf{x}_1 \in \mathbb{R}^2 : \dot{V}(\mathbf{x}_1) = 0\} = \{\mathbf{x}_1 \in \mathbb{R}^2 : x_2 = 0\}$ such that $E_0 \subset \Omega$. Now, we find a largest invariant set E_1 within E_0. Since $\xi(\mathbf{x}_1) = 0$ for $x_2 = 0$, we see that $\dot{x}_2(t) = 0$ if $x_1(t) = x_2(t) = 0$ for all $t \geq 0$ provided $w = 0$. Thus, $E_1 = \{\mathbf{x}_1 \in \mathbb{R}^2 : \mathbf{x}_1 = 0\}$. Therefore, by LaSalle's invariance principle we conclude that for $w = 0$ all the trajectories $\mathbf{x}_1(t)$ will converge to E_1 as $t \to \infty$.

Let's define the transformation $\tilde{x} = T(x)$ as in (6)

$$\tilde{x} := \begin{bmatrix} \tilde{\mathbf{x}}_1 \\ \tilde{\mathbf{x}}_2 \end{bmatrix} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 + \arctan \kappa x_2 \\ x_4 + \frac{\kappa(-x_1 + \epsilon \sin x_3)}{1 + \kappa^2 x_2^2} \end{bmatrix}. \tag{20}$$

Therefore, the system after transformation (20) is mapped to

$$\dot{\tilde{x}}_1 = \tilde{x}_2 \tag{21a}$$

$$\dot{\tilde{x}}_2 = -\tilde{x}_1 + \epsilon \sin(\tilde{x}_3 - \arctan \kappa \tilde{x}_2) + w \tag{21b}$$

$$\dot{\tilde{x}}_3 = \tilde{x}_4 \tag{21c}$$

$$\dot{\tilde{x}}_4 = \zeta_4(\tilde{x}) + u + \tilde{\mathbf{w}} \tag{21d}$$

where $u = (\beta(x)v + \alpha(x))|_{x=T^{-1}(\tilde{x})}$ and

$$\zeta_4(\tilde{x}) = \left[\frac{\kappa(-x_2 + \epsilon x_4 \cos x_3)}{1 + \kappa^2 x_2^2} \cdots - \frac{\kappa(-x_1 + \epsilon \sin x_3)^2 * 2\kappa^2 x_2}{(1 + \kappa^2 x_2^2)^2}\right]\Bigg|_{x=T^{-1}(\tilde{x})}$$

$$\tilde{\mathbf{w}} = -\frac{\kappa(-x_1 + \epsilon \sin x_3) * 2\kappa^2 x_2}{(1 + \kappa^2 x_2^2)^2} w\Bigg|_{x=T^{-1}(\tilde{x})}.$$

The TORA system in the similar form has been utilized in design of passivity based controller in [22]. For our numerical simulation, we consider $\epsilon = 0.2$ and normalized disturbing force $w = 0.2 \sin(10t)$.

5.1. Switching function

A reduced order sliding surface for $\tilde{\mathbf{x}}_2$-subsystem is designed with $c_1 = 4$ as

$$\sigma = 4\tilde{x}_3 + \tilde{x}_4 \tag{22}$$

so that the dynamics $\dot{\tilde{x}}_3 = -4\tilde{x}_3$ is asymptotically stable to $\tilde{x}_3 = 0$ when $\sigma \equiv 0$.

5.2. First order SMC

For the finite time reachability of $\tilde{\mathbf{x}}_2$ to the surface $\sigma = \sigma(\tilde{x}_2) = 0$, the reaching law based control law as in (9) is given by

$$u = -4\tilde{x}_4 - \zeta_4 - k\sigma - Q\text{sgn}\,(\sigma) \tag{23}$$

where $k > 0$ and $Q > \mu_1$. As the magnitude of matched disturbance, $|\tilde{\mathbf{w}}_3| \leq \mu_1 = 0.0417$, we choose $Q = 1$ and for faster convergence towards the surface we select $k = 2$. Fig. 3 shows the state trajectory of the system (21) and Fig. 4 shows the evolution of switching function and the conventional SMC input.

5.3. Twisting control

As in the control law (13), the twisting control for TORA can be given by

$$u(t) = -\int_0^t \hat{v}(\tau)d\tau \tag{24}$$

where $\hat{v} = -M(\tilde{x}) - \epsilon_1\text{sgn}(\sigma_1) - \epsilon_2\text{sgn}(s_2)$, $M(\tilde{x}) = 64\tilde{x}_2 - 16\sigma_1 + 4s_2 + \dot{\zeta}_4(\tilde{x})$ and $\nu = -16\tilde{x}_2 + 4\sigma_1 + \zeta_4(\tilde{x}) + u + \tilde{\mathbf{w}}$. Fig. 5 shows the state trajectories and the twisting algorithm based control signal with $\epsilon_1 = 5$ and $\epsilon_2 = 2$.

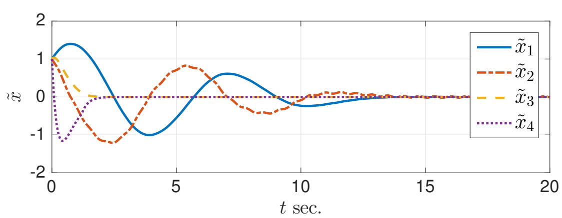

Fig. 3. State trajectory for the TORA using first order SMCl.

Remark 4 *It can be observed that in both conventional SMC* (23) *and twisting control* (24) *based on reduced order switching function* (22)*, the trajectory of the TORA system converges to the origin as shown in Fig. 3 and Fig. 5(a), respectively. Note that the twisting control reduces the chattering significantly as it is evident from Fig. 5(b). However, the conventional first order SMC exhibits chattering which may be unacceptable in case of large disturbances (see Fig. 4(b)).*

VI. Conclusion

For the asymptotic stability of the nonlinear system, SMC based on reduced order sliding function

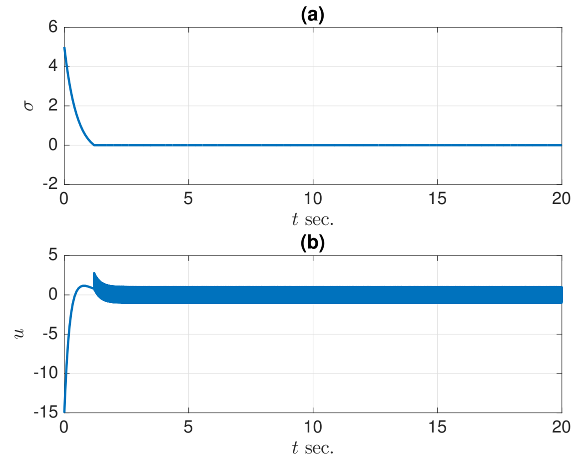

Fig. 4. (a) Evolution of switching function σ; (b) First order SMC for TORA.

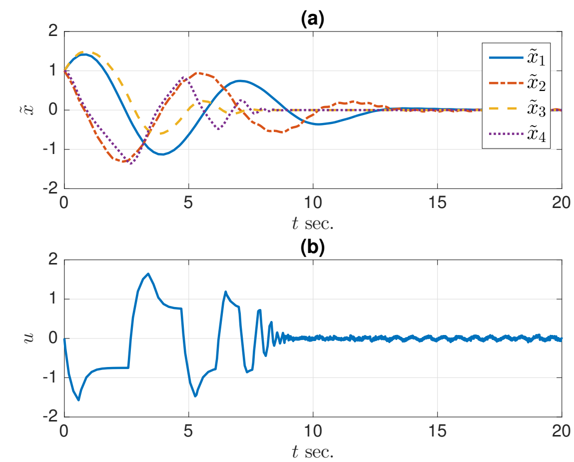

Fig. 5. (a) State trajectory for the TORA using twisting control; (b) Twisting control.

are designed for the system represented by the cascade of two subsystems which consitss of asymptotically stabilizable subsystem and controlled subsystem in phase variable form. Therefore, by designing SMC for the controlled subsystem, the trajectory of the whole system can be made asymptotically stable to the origin. The twisting control based on reduced order sliding function is also designed for stability of the phase variable subsystem that ensures the asymptotic stability of the entire system.

REFERENCES

1. B. Bandyopadhyay, Alemayehu G/Egziabher Abera, and S. Janardhanan. Sliding mode control

design via reduced order model approach. In *Proc. of IEEE ICIT*, pages 1538–1541, Mumbai, India, Dec. 2006.

2. B. Bandyopadhyay and Machhindranath Patil. Sliding mode control with reduced order switching function: An scb approach. *International Journal of Control*, 88(5):1089–1101, 2015.
3. Giorgio Bartolini, Antonella Ferrara, and Elio Usai. Chattering avoidance by second-order sliding mode control. *IEEE Transactions on automatic control*, 43(2):241–246, 1998.
4. Abhisek K Behera, Asif Chalanga, and Bijnan Bandyopadhyay. A new geometric proof of super-twisting control with actuator saturation. *Automatica*, 87:437–441, 2018.
5. Robert T Bupp, Dennis S Bernstein, and Vincent T Coppola. Vibration suppression of multi-modal translational motion using a rotational actuator. In *Decision and Control, 1994., Proceedings of the 33rd IEEE Conference on*, volume 4, pages 4030–4034. IEEE, 1994.
6. Robert T Bupp, Dennis S Bernstein, and Vincent T Coppola. A benchmark problem for nonlinear control design: problem statement, experimental testbed, and passive nonlinear compensation. In *American Control Conference, Proceedings of the 1995*, volume 6, pages 4363–4367. IEEE, 1995.
7. Robert T Bupp, Dennis S Bernstein, and Vincent T Coppola. A benchmark problem for nonlinear control design. *Int. J. Robust Nonlinear Control*, 8:307–310, 1998.
8. R. A. Decarlo, S. H. Zak, and G. P. Matthews. Variable structure control of nonlinear multivariable systems: A tutorial. In *Proc. of the IEEE*, volume 76, no. 3, pages 212–232, 1988.
9. C. Edwards and S. K. Spurgeon. *Sliding Mode Control: Theory and Applications*. Taylor & Francis Ltd., 1998.
10. Bingtuan Gao, Yuqing Bao, Jiehua Xie, and Lijuan Jia. Passivity-based control of two-dimensional translational oscillator with rotational actuator. *Transactions of the Institute of Measurement and Control*, 36(1):111–118, 2014.
11. J. Y. Hung, W. Gao, and J. C. Hung. Variable structure control: A survey. *IEEE Trans. on Ind. Elect.*, 40(1):2–22, 1993.
12. Hassan K Khalil. *Noninear systems*. Prentice-Hall, New Jersey, 2nd edition, 1996.
13. P. R. Kumar, Abhisek K. Behera, and Bijnan Bandyopadhyay. Robust finite-time tracking of stewart platform: A super-twisting like observer-based forward kinematics solution. *IEEE Transactions on Industrial Electronics*, 64(5):3776–3785, 2017.
14. Arie Levant. Sliding order and sliding accuracy in sliding mode control. *International journal of control*, 58(6):1247–1263, 1993.
15. Arie Levant. Robust exact differentiation via sliding mode technique. *Automatica*, 34(3):379–384, 1998.
16. A. G. Lukyanov and V. I. Utkin. Methods of reducing equations for dynamic systems to a regular form. *Automation and Remote Control*, 42(4):413–420, 1981.
17. Maryam Malekzadeh, Maryam Rezayati, and Mobin Saboohi. Hardware-in-the-loop attitude control via a high-order sliding mode controller/observer. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, page 0954410017706992, 2017.
18. Yury Orlov. Finite time stability and robust control synthesis of uncertain switched systems. *SIAM Journal on Control and Optimization*, 43(4):1253–1271, 2005.
19. Machhindranath Patil and B. Bandyopadhyay. Discrete-time sliding mode tracking control for nmp systems using reduced order switching function. In *10th IFAC International Symposium on Dynamics and Control of Process Systems*, pages 469–474, Mumbai, India, Dec. 2013.
20. Arun K. Paul, J. K. Mishra, and M. G. Radke. Reduced order sliding mode control for pneumatic actuator. *IEEE Trans. on Control Sys. Tech.*, 2(3):271–276, 1994.
21. Richard H Rand, Kinsey RJ, and D Lewis Mingori. Dynamics of spinup through resonance. *International journal of non-linear mechanics*, 27(3):489–502, 1992.
22. R. Sepulchre, M. Janković, and P.V. Kokotović. *Constructive Nonlinear Control*. Communications and control engineering. W.H. Freeman., 1997.
23. Yuri Shtessel, Christopher Edwards, Leonid Fridman, and Arie Levant. *Sliding mode control and observation*. Springer, 2014.
24. Eduardo D Sontag. Smooth stabilization implies coprime factorization. *IEEE transactions on automatic control*, 34(4):435–443, 1989.
25. S K Spurgeon, L Yao, and X Y Lu. Robust tracking via sliding mode control for elastic joint manipulators. *Proceedings of the Institution of*

Mechanical Engineers, Part I: Journal of Systems and Control Engineering, 215(4):405–417, 2001.
26. Parth S Thakar, Bijnan Bandyopadhyay, and Prasanna Gandhi. Improved output-feedback second order sliding mode control design with implementation for underactuated slosh-container system having confined track length. *IET Control Theory & Applications*, 11(8):1316–1323, 2016.
27. V. I. Utkin. Variable structure systems with sliding modes. *IEEE Trans. on Automatic Control*, Ac-22(2):212–222, 1977.
28. Vadim Utkin, Jürgen Guldner, and Jingxin Shi. *Sliding mode control in electro-mechanical systems*, volume 34. CRC press, 2009.
29. B. A. White. Reduced-order switching functions in variable-structure control systems. In *IEE Proc. D, Control Theory and Applications*, pages 1199–1206, Mar. 1983.

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