Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 336 (2007) 542–555
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

On extremal problems related to integral transforms of a class of analytic functions

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Received 23 November 2006
Available online 3 March 2007
Submitted by K.A. Lurie

Abstract

For $\gamma \geqslant 0$ and $\beta < 1$ given, let $\mathcal{P}_\gamma(\beta)$ denote the class of all analytic functions f in the unit disk with the normalization $f(0) = f'(0) - 1 = 0$ and satisfying the condition

$$\mathrm{Re}\big\{e^{i\phi}\big(f'(z) + \gamma z f''(z) - \beta\big)\big\} > 0, \quad z \in \mathbb{D},$$

for some $\phi \in \mathbb{R}$. We shall investigate the integral transform

$$V_\lambda(f)(z) = \int_0^1 \lambda(t) \frac{f(tz)}{t}\, dt,$$

where λ is a nonnegative real valued function normalized by $\int_0^1 \lambda(t)\, dt = 1$. From our main results we get conditions on the number β and the function λ such that $V_\lambda(f)$ is starlike of order μ $(0 \leqslant \mu \leqslant 1/2)$ when $f \in \mathcal{P}_\gamma(\beta)$. As applications we study various choices of $\lambda(t)$, related to classical integral transforms.

Keywords: Hadamard product; Analytic, univalent, starlike and convex functions

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.02.076

1. Introduction and preliminaries

In the present work we study certain class of analytic functions on the unit disk $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$ of the complex plane and aim at formulating results to illustrate the powerful method of duality principle in function theory developed by Ruscheweyh [12]. As a result of it, we present several applications based on a general method of description dealing with certain integral transforms acting on a class of analytic functions.

Let $\mathcal{A}$ denote the set of all functions f analytic in the unit disk $\mathbb{D}$, normalized by $f(0)=f'(0)-1=0$. Let $\mathcal{A}_0=\{g\colon\ g(z)=f(z)/z,\ f\in\mathcal{A}\}$. Let $\mathcal{S}\subset\mathcal{A}$ be the class of functions univalent in $\mathbb{D}$, and denote by $\mathcal{S}^*(\mu)$, $0\leqslant\mu<1$, the set of all starlike functions of order μ. As is well known, $f\in\mathcal{S}^*(\mu)$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\mu,\quad z\in\mathbb{D}.$$

For $\gamma\geqslant 0$ and $\beta<1$ given, define

$$\mathcal{P}_\gamma(\beta)=\bigl\{f\in\mathcal{A}\colon \exists\phi\in\mathbb{R}\text{ such that }\operatorname{Re}\bigl\{e^{i\phi}\bigl(f'(z)+\gamma zf''(z)-\beta\bigr)\bigr\}>0,\ z\in\mathbb{D}\bigr\}.$$

For $\gamma=0$, we set $\mathcal{P}(\beta)\equiv\mathcal{P}_0(\beta)$, and $\mathcal{P}=\mathcal{P}(0)$.

Before getting into our main result, we introduce the following terminology: If f and g are analytic functions in $\mathbb{D}$ with $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $g(z)=\sum_{n=0}^{\infty}b_nz^n$, then the convolution (Hadamard product) of f and g, denoted by $f*g$, is an analytic function given by

$$(f*g)(z)=\sum_{n=0}^{\infty}a_nb_nz^n,\quad z\in\mathbb{D}.$$

For a nonnegative real valued integrable function $\lambda(t)$ satisfying the normalizing condition $\int_0^1\lambda(t)\,dt=1$, define

$$F(z)=V_\lambda(f)(z)=\int_0^1\lambda(t)\frac{f(tz)}{t}\,dt. \tag{1.1}$$

In the theory of univalent functions, a variety of linear and nonlinear integral operators have been considered. For a detailed discussion, especially for operators other than the form (1.1), we refer to the monograph by Miller and Mocanu [6], where a long list of references can be found. The usual methods in handling such problems have been from the theory of subordinations, but the paper by Fournier and Ruscheweyh [4] led the way to the application of duality for Hadamard product in connection with linear integral operators such as (1.1). Using the duality technique, this operator has been studied in detail in [1,2,4,5,8,9]. As the present investigation relies on the duality theory for convolutions, so we include here some basic concepts and results from this theory. For a subset $\mathcal{B}\subset\mathcal{A}_0$ we define

$$\mathcal{B}^*=\bigl\{g\in\mathcal{A}_0\colon\ (f*g)(z)\neq 0,\ z\in\mathbb{D},\ \text{for all } f\in\mathcal{B}\bigr\}.$$

The set $\mathcal{B}^*$ is called the dual of $\mathcal{B}$. Further, the second dual, or *dual hull*, of $\mathcal{B}$ is defined as $\mathcal{B}^{**}=(\mathcal{B}^*)^*$. The basic reference to this theory is the book by Ruscheweyh [12] (see also [10]). We may need the following fundamental result and for a general statement we refer to [10, Theorem 1] and [12].

Theorem A. *Let*

$$\mathcal{B} = \left\{\beta + (1-\beta)\left(\frac{1+xz}{1+yz}\right)\colon |x| = |y| = 1\right\}, \quad \beta \in \mathbb{R},\ \beta \neq 1.$$

We have

(1) $\mathcal{B}^{**} = \{g \in \mathcal{A}_0\colon \exists \phi \in \mathbb{R}$ *such that* $\operatorname{Re}\{e^{i\phi}(g(z)-\beta)\} > 0,\ z \in \mathbb{D}\}$.
(2) *if* Γ_1 *and* Γ_2 *are two continuous linear functionals on* $\mathcal{B}$ *with* $0 \notin \Gamma_2(\mathcal{B})$, *then for every* $g \in \mathcal{B}^{**}$ *we can find* $v \in \mathcal{B}$ *such that*

$$\frac{\Gamma_1(g)}{\Gamma_2(g)} = \frac{\Gamma_1(v)}{\Gamma_2(v)}.$$

Starlikeness of integral transforms $V_\lambda(f)$ of functions in $\mathcal{P}_0(\beta)$ has been investigated. For example, we have the following reformulated version from [4].

Corollary A. *Let* $\gamma \geqslant 1/3$ *and* $\beta(\gamma)$ *be given by*

$$\frac{\beta(\gamma)}{1-\beta(\gamma)} = -\frac{1}{\gamma}\int_0^1 t^{-1+1/\gamma}\frac{1-t}{1+t}\,dt. \tag{1.2}$$

If $f \in \mathcal{A}$ *satisfies the condition* $\operatorname{Re}\{e^{i\phi}(f'(z)+\gamma z f''(z)-\beta(\gamma))\} > 0$ *in* $\mathbb{D}$ *and for some* $\phi \in \mathbb{R}$, *then* f *is starlike and the value of* $\beta(\gamma)$ *is sharp.*

In particular, if $\gamma = 1$, then we have the following sharp inclusion:

$$\mathcal{P}_1(\beta) \subset \mathcal{S}^*,$$

whenever $\beta \geqslant \beta(1) = 1 - \frac{1}{2(1-\log 2)} \approx -0.6294$ and the value of $\beta(1)$ cannot be replaced by a smaller number. A general condition on $\lambda(t)$ for the starlikeness of integral transforms $V_\lambda(f)$ of functions in the full class $\mathcal{S}^*(\mu)$ is still not known. However, in view of Corollary A, our results (e.g. Theorem 3.1) motivate the study by producing such an inclusion result for a smaller class $\mathcal{P}_\gamma(\beta)$.

We recall that the class $\mathcal{P}_\gamma(\beta)$ is closely related to the class $\mathcal{R}_\gamma(\beta)$, where $\mathcal{R}_\gamma(\beta)$ denotes the class of all functions $f \in \mathcal{A}$ such that there exists $\phi \in \mathbb{R}$ satisfying the condition

$$\operatorname{Re} e^{i\phi}\left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - \beta\right) > 0, \quad z \in \mathbb{D}.$$

We make the following simple observations about the relationship between these classes:

(a) $f \in \mathcal{P}_\gamma(\beta) \Leftrightarrow zf' \in \mathcal{R}_\gamma(\beta)$,
(b) $f \in \mathcal{P}_\gamma(\beta) \Leftrightarrow (1-\gamma)f + \gamma z f' \in \mathcal{P}_0(\beta)$,
(c) $\mathcal{P}_0(\beta) = \mathcal{R}_1(\beta)$.

Note that, with $\mathcal{B}$ as in Theorem A, we can express $\mathcal{P}_0(\beta)$ as

$$\mathcal{P}_0(\beta) = \left\{f \in \mathcal{A}\colon f' \in \mathcal{B}^{**}\right\}$$

and thus we realize that we can obtain results about $\mathcal{P}_\gamma(\beta)$ by investigating $\mathcal{B}$.

Starlikeness of integral transforms $V_\lambda(f)$ of functions in $\mathcal{R}_\gamma(\beta)$ has been investigated by Kim and Rønning in [5]. In this paper, we want to find conditions on the function λ and the number β such that when $f \in \mathcal{P}_\gamma(\beta)$ the integral transform $V_\lambda(f)$ will be starlike of order μ. For $\gamma = 0$, this problem has been investigated recently by Ponnusamy and Rønning in [8]. For $f \in \mathcal{R}_\gamma(\beta)$, the corresponding problem has been investigated in [5] whereas the case $\gamma = 1$ and $\mu = 0$ was discussed in [4]. In [3] the authors state a connection between α and β such that for $\gamma \geqslant 1$ and $f \in \mathcal{R}_\gamma(\beta)$, $V_\lambda(f)$ will be in $\mathcal{R}_1(\alpha)$. Also, in the same paper [3], the authors consider certain special choices of $\lambda(t)$ and obtain conditions on β' so that

$$\operatorname{Re} e^{i\phi}\left(\frac{f(z)}{z} - \beta\right) > 0 \quad \Rightarrow \quad V_\lambda(f) \in \mathcal{P}_0(\beta').$$

Again, the general problem of finding conditions on β' and $\lambda(t)$ satisfying the last implication remains an open question.

2. Main results

For our investigation, we introduce the function $g(t) := g_\gamma^\mu(t)$ as the solution of the initial value problem

$$g(t) + \gamma g'(t) = \frac{1+\mu-(1-\mu)t}{(1-\mu)(1+t)} - \frac{2\mu}{(1-\mu)}\frac{\log(1+t)}{t}, \qquad g(0) = 1.$$

The solution $g(t)$ can be written as

$$g(t) = \frac{1}{\gamma}\int_0^1 s^{(1/\gamma)-1}\left[\frac{1+\mu-(1-\mu)st}{(1-\mu)(1+st)} - \frac{2\mu}{(1-\mu)}\frac{\log(1+st)}{st}\right] ds. \tag{2.1}$$

We also introduce

$$\Pi_\gamma(t) = \begin{cases} \int_t^1 \left(\int_s^1 \frac{\lambda(\sigma)}{\sigma^{1/\gamma}}\, d\sigma\right) s^{(1/\gamma)-2}\, ds & \text{if } \gamma > 0, \\ \int_t^1 \frac{\lambda(s)}{s}\, ds & \text{if } \gamma = 0. \end{cases}$$

In this setting we first formulate the following result which extend the recent work of Ponnusamy and Rønning [9].

Theorem 2.1. *Let $f \in \mathcal{P}_\gamma(\beta)$, $\gamma \geqslant 0$ and $\beta < 1$ with*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\, dt. \tag{2.2}$$

Then $F(z) = V_\lambda(f)(z) \in \mathcal{S}^*(\mu)$, $0 \leqslant \mu \leqslant 1/2$, if and only if

$$\operatorname{Re}\int_0^1 \Pi_\gamma(t)\left(\frac{h(tz)}{tz} - \frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right) dt \geqslant 0,$$

where, for $\gamma > 0$,

$$\Pi_\gamma(t) = \int_t^1 \Lambda_\gamma(s) s^{(1/\gamma)-2}\, ds, \quad \textit{with } \Lambda_\gamma(t) = \int_t^1 \frac{\lambda(\sigma)}{\sigma^{1/\gamma}}\, d\sigma,$$

and

$$h(z)=\frac{z[1+(\frac{\epsilon+2\mu-1}{2-2\mu})z]}{(1-z)^2},\quad |\epsilon|=1.$$

The value of β is sharp.

Proof. The case $\gamma=0$ corresponds to Corollary 2.2 in Ponnusamy and Rønning [8]. For the sake of completeness we have included it in the statement. So, we consider the case $\gamma>0$ in the following proof. As in [9], we define

$$\phi(z)=1+\sum_{n=1}^{\infty}(1+\gamma n)z^n \tag{2.3}$$

and

$$\psi(z)=\phi^{(-1)}(z)=1+\sum_{n=1}^{\infty}\frac{z^n}{1+\gamma n}=\int_0^1\frac{dt}{1-t^{\gamma}z}. \tag{2.4}$$

Using a change of variable, we can rewrite $\psi(z)$ as

$$\psi(z)=\begin{cases}\frac{1}{\gamma}\int_0^1\frac{s^{(1/\gamma)-1}}{1-sz}\,ds & \text{if } \gamma>0,\\ \frac{1}{1-z} & \text{if } \gamma=0.\end{cases}$$

In view of these representations, we can write

$$f'(z)+\gamma zf''(z)=f'(z)*\phi(z) \tag{2.5}$$

and so

$$\bigl(f'(z)+\gamma zf''(z)\bigr)*\psi(z)=f'(z). \tag{2.6}$$

Now, we let $f\in\mathcal{P}_{\gamma}(\beta)$. Then, in view of the Duality Principle [12], we may restrict our attention to functions $f\in\mathcal{P}_{\gamma}(\beta)$ for which

$$f'(z)+\gamma zf''(z)=\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right),\quad |x|=|y|=1.$$

Thus, by (2.6), this observation gives

$$f'(z)=\left[\beta+(1-\beta)\left(\frac{1+xz}{1+yz}\right)\right]*\psi(z).$$

Also, we recall a well-known result from the theory of convolutions [11] (see also [12]):

$$F\in\mathcal{S}^*(\mu)\quad\text{if and only if}\quad \frac{1}{z}\bigl(F(z)*h(z)\bigr)\neq 0,\quad z\in\mathbb{D},$$

where

$$h(z)=\frac{z[1+(\frac{\epsilon+2\mu-1}{2-2\mu})z]}{(1-z)^2},\quad |\epsilon|=1.$$

Let $F(z)=V_\lambda(f)$. Thus, proceeding exactly as in [9], it follows that $F\in\mathcal{S}^*(\mu)$ if and only if $0\neq\frac{F(z)}{z}*\frac{h(z)}{z}$. This holds if and only if

$$\operatorname{Re}\int_0^1\lambda(t)\frac{1}{z}\int_0^z\int_0^1\left\{s^{(1/\gamma)-1}\frac{1}{\gamma}\frac{h(stw)}{stw}-\frac{1+g(t)}{2}\right\}ds\,dw\,dt\geqslant 0,$$

where $g(t)$ is defined by (2.1). Also, from (2.1), it is easy to see that

$$\frac{1+g(t)}{2}=\frac{1}{\gamma}\int_0^1 s^{(1/\gamma)-1}\frac{1}{2}\left[1+\frac{1+\mu-(1-\mu)st}{(1-\mu)(1+st)}-\frac{2\mu}{(1-\mu)}\frac{\log(1+st)}{st}\right]ds.$$

Substituting this in the left-hand side of the last inequality and then introducing $v=st$ in the resulting equation, it follows that

$$\operatorname{Re}\int_0^1\frac{\lambda(t)}{t^{1/\gamma}}\left[\int_0^t v^{(1/\gamma)-1}\left\{\frac{1}{z}\int_0^z\left(\frac{h(wv)}{wv}\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}\left[1+\frac{1+\mu-(1-\mu)v}{(1-\mu)(1+v)}-\frac{2\mu}{(1-\mu)}\frac{\log(1+v)}{v}\right]\right)dw\right\}dv\right]dt\geqslant 0.$$

An integration by parts here yields that

$$\operatorname{Re}\int_0^1\Lambda_\gamma(t)t^{(1/\gamma)-2}\left\{\frac{1}{z}\int_0^z\left(\frac{h(tw)}{w}\right.\right.$$
$$\left.\left.-\frac{t}{2}\left[1+\frac{1+\mu-(1-\mu)t}{(1-\mu)(1+t)}-\frac{2\mu}{(1-\mu)}\frac{\log(1+t)}{t}\right]\right)dw\right\}dt\geqslant 0,$$

where $\Lambda_\gamma(t)=\int_t^1\frac{\lambda(\sigma)}{\sigma^{1/\gamma}}\,d\sigma$. Another integration by parts gives the following compact form:

$$\operatorname{Re}\int_0^1\Pi_\gamma(t)\left(\frac{h(tz)}{tz}-\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right)dt\geqslant 0,$$

where $\Pi_\gamma(t)=\int_t^1\Lambda_\gamma(s)s^{(1/\gamma)-2}\,ds$. According to the result of Ponnusamy and Rønning [8, Theorems 2.1 and 2.3], it follows that $F\in\mathcal{S}^*(\mu)$.

Finally, to prove the sharpness, let $f\in\mathcal{P}_\gamma(\beta)$ be of the form for which

$$f'(z)+\gamma zf''(z)=\beta+(1-\beta)\frac{1+z}{1-z}.$$

Using a series expansion we obtain that

$$f(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{1}{n(n\gamma+1-\gamma)}z^n.$$

Then we can write

$$F(z)=V_\lambda(f)(z)=z+2(1-\beta)\sum_{n=2}^{\infty}\frac{\mu_n}{n(n\gamma+1-\gamma)}z^n,$$

where $\mu_n = \int_0^1 \lambda(t)t^{n-1}\,dt$. Further, it is a simple exercise to write $g(t)$ in (2.1) in a series expansion as

$$g(t) = 1 + \frac{2}{1-\mu}\sum_{n=1}^{\infty}(-1)^n \frac{n+1-\mu}{(1+\gamma n)(n+1)}t^n. \tag{2.7}$$

Now, by (2.2) and (2.7), we have

$$\begin{aligned}\frac{\beta}{1-\beta} &= -\int_0^1 \lambda(t)g(t)\,dt \\ &= -\int_0^1 \lambda(t)\left\{1 + \frac{2}{1-\mu}\sum_{n=1}^{\infty}(-1)^n \frac{n+1-\mu}{(n\gamma+1)(n+1)}t^n\right\}dt \\ &= -1 - \frac{2}{1-\mu}\sum_{n=1}^{\infty}(-1)^n \frac{n+1-\mu}{(n\gamma+1)(n+1)}\int_0^1 \lambda(t)t^n\,dt\end{aligned}$$

and so

$$\frac{1}{1-\beta} = -\frac{2}{1-\mu}\sum_{n=2}^{\infty}(-1)^{n-1}\frac{(n-\mu)\mu_n}{(n\gamma+1-\gamma)n}. \tag{2.8}$$

Finally, we see that

$$F'(z) = 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{\mu_n}{n\gamma+1-\gamma}z^{n-1}$$

which for $z=-1$, by (2.8), gives the value

$$\begin{aligned}F'(-1) &= 1 + 2(1-\beta)\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\mu_n}{n\gamma+1-\gamma} \\ &= \mu + 2(1-\beta)\mu\sum_{n=2}^{\infty}\frac{(-1)^{n-1}\mu_n}{(n\gamma+1-\gamma)n} \\ &= -\mu F(-1)\end{aligned}$$

so that $zF'(z)/F(z)$ at $z=-1$ equals μ. This implies that the result is sharp for the order of starlikeness. □

In our next result we provide a necessary condition so that $V_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}^*(\mu)$, $0 \leqslant \mu \leqslant 1/2$.

Theorem 2.2. *Assume that both Λ_γ and Π_γ are integrable on $[0,1]$ and positive on $(0,1)$. Assume further that*

$$\frac{\Pi_\gamma(t)}{(1+t)(1-t)^{1+2\mu}} \tag{2.9}$$

is decreasing on $(0,1)$. If $\beta = \beta(\lambda,\mu)$, $\Lambda_\gamma(t)$, $\Pi_\gamma(t)$ are defined as in Theorem 2.1, then we have $V_\lambda(\mathcal{P}_\gamma(\beta)) \subset \mathcal{S}^*(\mu)$, where $V_\lambda(f)$ is defined by (1.1).

Proof. By (2.9) and the result of Ponnusamy and Rønning [4, Theorem 2.3], we have

$$\mathrm{Re}\int_0^1 \Pi_\gamma(t)\left(\frac{h(tz)}{tz}-\frac{1-\mu(1+t)}{(1-\mu)(1+t)^2}\right)dt \geqslant 0.$$

The desired conclusion now follows from Theorem 2.1. □

3. Main consequence of Theorem 2.2

In order to apply Theorem 2.2 in specific situation, it suffices to show that

$$p(t)=\frac{\Pi_\gamma(t)}{(1+t)(1-t)^{1+2\mu}}$$

is decreasing on the interval (0, 1), where $\gamma>0$,

$$\Lambda_\gamma(t)=\int_t^1 \frac{\lambda(\sigma)}{\sigma^{1/\gamma}}\,d\sigma \quad \text{and} \quad \Pi_\gamma(t)=\int_t^1 \Lambda_\gamma(s)s^{(1/\gamma)-2}\,ds.$$

Again we omit the limiting case $\gamma=0$ as this has been dealt in [8]. Taking the logarithmic derivative of $p(t)$ and using the fact that $\Pi_\gamma'(t)=-\Lambda_\gamma(t)t^{(1/\gamma)-2}$, we have

$$\frac{p'(t)}{p(t)}=-\frac{\Lambda_\gamma(t)}{t^{2-(1/\gamma)}\Pi_\gamma(t)}+\frac{2(t+\mu(1+t))}{1-t^2}.$$

Note that $p(t)>0$. Therefore, we observe that $p'(t)\leqslant 0$, for $t\in(0,1)$, is equivalent to the inequality

$$q(t)=\Pi_\gamma(t)-\frac{(1-t^2)\Lambda_\gamma(t)t^{(1/\gamma)-2}}{2(t+\mu(1+t))}\leqslant 0, \quad \text{for } t\in(0,1). \tag{3.1}$$

Clearly $q(1)=0$ and we note that if $q(t)$ is increasing on (0, 1), then $p(t)$ is decreasing on (0, 1) and the proof will be completed. In view of this observation, it suffices to prove that $q(t)$ is increasing on (0, 1). We compute $q'(t)$ explicitly and after a simple calculation we obtain that

$$\begin{aligned} q'(t)=-\frac{t^{(1/\gamma)-2}(1+t)}{2(t+\mu(1+t))^2}\Big[&-\lambda(t)t^{-1/\gamma}(1-t)\big(t+\mu(1+t)\big)\\ &+\Lambda_\gamma(t)\left\{\frac{(1-t)}{t}\big(t+\mu(1+t)\big)\left(\frac{1}{\gamma}-2\right)-\big(1-t-\mu(1+t)\big)(1+2\mu)\right\}\Big]\end{aligned}$$

and therefore, $q'(t)\geqslant 0$, for $t\in(0,1)$, is equivalent to the inequality

$$\begin{aligned}\Delta(t)&=-t^{-1/\gamma}\lambda(t)(1-t)\big(t+\mu(1+t)\big)\\ &\quad+\Lambda_\gamma(t)\left\{\frac{(1-t)}{t}\big(t+\mu(1+t)\big)\left(\frac{1}{\gamma}-2\right)-\big(1-t-\mu(1+t)\big)(1+2\mu)\right\}\\ &\leqslant 0\end{aligned} \tag{3.2}$$

which may be rewritten as

$$\Delta(t)=-A(t)X(t)+\frac{Y(t)}{t}\int_t^1 A(s)\,ds,$$

where

$$
\begin{aligned}
A(t) &= t^{-1/\gamma}\lambda(t),\\
X(t) &= (1-t)\bigl(t+\mu(1+t)\bigr),\\
Y(t) &= X(t)\left(\frac{1}{\gamma}-2\right)+Z(t),\\
Z(t) &= -t\bigl(1-t-\mu(1+t)\bigr)(1+2\mu).
\end{aligned}
$$

Case (1). If $Y(t) \leqslant 0$ on $(0,1)$, then $\Delta(t) \leqslant 0$ on $(0,1)$ is obvious and the result follows.

Case (2). It is possible that either $Y(t) > 0$ on $(0,t_0)$ and $Y(t) \leqslant 0$ on $[t_0,1)$ or $Y(t) > 0$ throughout the interval $(0,1)$. Equally, it is also possible that $Y(t) \leqslant 0$ on $(0,t_0]$ and $Y(t) > 0$ on $(t_0,1)$. We remark that there is nothing to prove on the interval, where $Y(t) \leqslant 0$, since $\Delta(t) \leqslant 0$ on that interval. Thus, it suffices deal with the case where $Y(t) > 0$ on the subinterval, say I, of $(0,1)$, for instance on $(0,t_0)$ in the first instance. In this case, we may set

$$
\Delta(t) = \frac{Y(t)}{t}B(t), \quad B(t) = -A(t)\frac{tX(t)}{Y(t)} + \int_t^1 A(s)\,ds.
$$

We note that $B(1)=0$. Thus, to prove that $\Delta(t) \leqslant 0$, it suffices to show that $B(t)$ is an increasing function of t. Indeed

$$
\begin{aligned}
B'(t) &= -A(t)\left[\frac{A'(t)}{A(t)}\frac{tX(t)}{Y(t)} + \left(\frac{tX}{Y}\right)'(t)+1\right]\\
&= -t^{-1/\gamma}\lambda(t)\left[\left(\frac{t\lambda'(t)}{\lambda(t)}-\frac{1}{\gamma}\right)\frac{X(t)}{Y(t)} + \left(\frac{tX}{Y}\right)'(t)+1\right]
\end{aligned}
$$

which implies that we need to find conditions so that $B'(t) \geqslant 0$, which is same as

$$
\frac{t\lambda'(t)}{\lambda(t)} \leqslant \frac{1}{\gamma} - \left[1+\left(\frac{tX}{Y}\right)'(t)\right]\frac{Y(t)}{X(t)}, \tag{3.3}
$$

for $t \in I$.

Subcase (i). The case $\mu = 0$. Then X and Y reduce to the simple form

$$
X(t) = t(1-t) \quad \text{and} \quad Y(t) = \left(\frac{1}{\gamma}-3\right)t(1-t)
$$

so that $Y(t) \leqslant 0$ on $(0,1)$ if $\gamma \geqslant 1/3$ and so, the inequality (3.2) clearly holds in this case. If $0<\gamma<1/3$, then $Y(t) > 0$ on $(0,1)$ and we find that (3.3) becomes

$$
t\frac{\lambda'(t)}{\lambda(t)} \leqslant 2 \quad \text{on } (0,1),
$$

which leads to the following result of Ponnusamy and Rønning [9, Theorem 3.1].

Corollary 3.1. *Let $\lambda(t)$ be a normalized nonnegative real valued integrable function on $[0,1]$ satisfying the condition*

$$
t\frac{\lambda'(t)}{\lambda(t)} \leqslant 2 \quad \textit{on } (0,1).
$$

Assume further that both Λ_γ and Π_γ are integrable on $[0,1]$ *and positive on* $(0,1)$. *Let* $f \in \mathcal{P}_\gamma(\beta)$, $\gamma > 0$, *and* $\beta < 1$ *with*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\, dt, \tag{3.4}$$

where $g(t)$ *is defined by* (2.1) (*with* $\mu = 0$). *Then* $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{S}^*$. *The conclusion does not hold for smaller values of* β.

Subcase (ii). The case $0 < \mu \leqslant 1/2$. A computation shows that (3.3) is equivalent to

$$\left(\frac{1}{\gamma} - \frac{t\lambda'(t)}{\lambda(t)}\right) X(t)Y(t) \geqslant Y^2(t) + Y(t)\big(tX'(t) + X(t)\big) - Y'(t)tX(t).$$

Because $Y(t) = X(t)((1/\gamma) - 2) + Z(t)$, the above equation may be written as

$$\begin{aligned}&\left(\frac{1}{\gamma} - 2\right)\big[X(t) + Z(t)\big]X(t) - \left(2 - \frac{t\lambda'(t)}{\lambda(t)}\right)\left[\left(\frac{1}{\gamma} - 2\right)X(t) + Z(t)\right]X(t) \\ &\quad \leqslant Z'(t)\big(tX(t)\big) - Z(t)(tX)'(t) - Z^2(t).\end{aligned} \tag{3.5}$$

Set $D(t) = t(1+\mu) - (1-\mu)$. Then, $D^2(t) \leqslant 1$, for $t \in [0,1]$, and using the expressions for $Z(t)$ and $X(t)$ we see that

$$Z(t) = (1+2\mu)tD(t), \qquad X(t) = (1-t)\big(D(t) + 1\big)$$

and so, an algebraic calculation gives that

$$Z'(t)\big(tX(t)\big) - Z(t)(tX)'(t) - Z^2(t) = 2\mu(1+2\mu)t^2\big(1 - D^2(t)\big)$$

which is indeed nonnegative on $(0,1)$. Note also that $X(t) + Z(t)$ and $X(t)$ are nonnegative on $(0,1)$. In view of these observations if $\gamma \geqslant 1/2$, then the inequality (3.5) holds on the interval where $Y(t) > 0$ and hence, (3.2) holds throughout the unit interval $(0,1)$.

Thus, we see that the above corollary continues to hold for $\mu \in (0, 1/2]$ but with the restriction $\gamma \geqslant 1/2$. More precisely, we have

Theorem 3.1. *Let* $\lambda(t)$ *be a normalized nonnegative real valued integrable function on* $[0,1]$ *satisfying the condition*

$$t\frac{\lambda'(t)}{\lambda(t)} \leqslant 2 \quad \text{on } (0,1). \tag{3.6}$$

Let $f \in \mathcal{P}_\gamma(\beta)$ *with* $\gamma \geqslant 1/2$, *and* $\beta < 1$ *with*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\, dt, \tag{3.7}$$

where $g(t)$ *is defined by* (2.1) *with* $\mu \in (0, 1/2]$. *Assume further that both* Λ_γ *and* Π_γ *are integrable on* $[0,1]$ *and positive on* $(0,1)$. *Then* $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{S}^*(\mu)$. *The conclusion does not hold for smaller values of* β.

Remark. The calculation to cover the case $\gamma \in (0, 1/2)$ leads to a delegate inequality which we wish to avoid in our presentation here although some of the special cases handled below may be considered by a direct approach by checking the inequality (3.2).

4. Applications

Now we are in a position to present a number of applications of Theorem 3.1. Define

$$\lambda(t) = \begin{cases} (a+1)(b+1)(\frac{t^a(1-t^{b-a})}{b-a}), & \text{for } b \neq a,\ a > -1,\ b > -1, \\ (a+1)^2 t^a \log(1/t), & \text{for } b = a,\ a > -1. \end{cases} \tag{4.1}$$

In this case $V_\lambda(f)(z)$ becomes the convolution operator $G_f(a, b; z)$, where

$$G_f(a, b; z) := \left(\sum_{n=1}^{\infty} \frac{(1+a)(1+b)}{(n+a)(n+b)} z^n \right) * f(z).$$

This operator was introduced by [7] and was studied later in [1,2,8]. In view of the symmetry, without loss of generality, we may assume that $b > a$ in the case $b \neq a$. With this observation, we may rewrite $G_f(a, b; z)$ as

$$G_f(a, b; z) = \begin{cases} \frac{(a+1)(b+1)}{b-a} \int_0^1 t^{a-1}(1 - t^{b-a}) f(tz)\, dt & \text{if } b > a,\ a > -1, \\ (1+a)^2 \int_0^1 t^{a-1} \log(1/t) f(tz)\, dt & \text{if } b = a,\ a > -1. \end{cases}$$

Now formulate our first application.

Theorem 4.1. *Let $a > -1$, $b > -1$ be such that $a \in (-1, 2]$ and $b \geqslant a$. Suppose that $g(t)$ is defined by* (2.1) *and $\lambda(t)$ is defined by* (4.1). *If β is given by*

$$\frac{\beta}{1-\beta} = -\int_0^1 \lambda(t) g(t)\, dt,$$

then, for $f \in \mathcal{P}_\gamma(\beta)$, the function $G_f(a, b; z) \in \mathcal{S}^*(\mu)$. The value of β is sharp.

Proof. If $\lambda(t)$ is defined by (4.1), then it is a simple exercise to see that the inequality (3.6) is equivalent to

$$\begin{cases} (2-a)t^a - (b-2)t^b \geqslant 0, & \text{for } b > a, a > -1, b > -1, \\ (2-a)\log(1/t) + 1 \geqslant 0, & \text{for } b = a, a > -1 \end{cases}$$

which is clearly true by hypothesis. The conclusion now follows from Theorem 3.1. □

Theorem 4.2. *Let $-1 < a \leqslant 2$, $p \geqslant 1$, $\mu \in (0, 1/2]$, $f \in \mathcal{P}_\gamma(\beta)$ with $\gamma \geqslant 1/2$, and $\beta < 1$ with*

$$\frac{\beta}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \big(\log(1/t)\big)^{p-1} g(t)\, dt, \tag{4.2}$$

where $g(t)$ is defined by (2.1). *Then the Hadamard product function $\Phi_p(a; z) * f(z)$ defined by*

$$\Phi_p(a; z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 (\log 1/t)^{p-1} t^{a-1} f(tz)\, dt$$

belongs to $\mathcal{S}^*(\mu)$. The value of β is sharp.

Proof. Consider

$$\lambda(t)=\frac{(1+a)^p}{\Gamma(p)}t^a\bigl(\log(1/t)\bigr)^{p-1},\quad a>-1,\ p\geqslant 0.$$

Then, in this case $V_\lambda(f)(z)$ becomes the convolution operator

$$V_\lambda(f)(z):=\left(\sum_{n=1}^{\infty}\frac{(1+a)^p}{(n+a)^p}z^n\right)*f(z)$$

which is same as $\Phi_p(a;z)*f(z)$. Clearly, to complete the proof, it suffices to verify inequality (3.6). Substituting the λ-value, we see that the inequality (3.6) is equivalent to

$$(2-a)\log(1/t)+p-1\geqslant 0$$

which is clearly true for all $t\in(0,1)$ by the hypothesis. □

Let $(a)_n$ denote symbol for the generalized factorial:

$$(a)_0=1,\quad \text{for } a\neq 0,\qquad (a)_n:=a(a+1)\cdots(a+n-1),\quad \text{for } n\in\mathbb{N},$$

and define

$$_2F_1(a,b;c;z):=F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n\quad \bigl(|z|<1\bigr),$$

the Gaussian hypergeometric function. This function is analytic in the unit disk $\mathbb{D}$. Also, we let ϕ by $\phi(1-t)=1+\sum_{n=1}^{\infty}b_n(1-t)^n$, where $b_n\geqslant 0$ for each $n\geqslant 1$, and consider

$$\lambda(t)=Kt^{b-1}(1-t)^{c-a-b}\phi(1-t),\tag{4.3}$$

where K is a constant chosen such that $\int_0^1\lambda(t)\,dt=1$. Now, we include our final result.

Theorem 4.3. *Let $a,b,c>0$, $\gamma\geqslant 1/2$, $\mu\in(0,1/2]$, and $\beta=\beta_{a,b,c,\gamma,\mu}$ with*

$$\frac{\beta}{1-\beta}=-K\int_0^1 t^{b-1}(1-t)^{c-a-b}\phi(1-t)g(t)\,dt,$$

where K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\phi(1-t)\,dt=1$ and $g(t)$ is defined by (2.1)*. Then for $f\in\mathcal{P}_\gamma(\beta)$, the function*

$$V_\lambda(f)(z)=K\int_0^1 t^{b-1}(1-t)^{c-a-b}\phi(1-t)\frac{f(tz)}{t}\,dt$$

belongs to $\mathcal{S}^*(\mu)$ whenever a,b,c are related by the condition $0<b\leqslant 1$ and $c\leqslant a+b$. The value of β is sharp.

Proof. To establish the theorem, it is enough to show that the inequality (3.6) holds under the stated conditions. Since $\phi(1-t)=1+\sum_{n=1}^{\infty}b_n(1-t)^n>0$ with $b_n\geqslant 0$ for all $n\geqslant 1$, $\phi'(1-t)$ in the expression of $\Psi(t)$ is nonnegative for $t\in(0,1)$ and so the inequality (3.6) is equivalent to

$$2t(1-t)+t(1-t)\frac{\phi'(1-t)}{\phi(1-t)}\geqslant(b-1)(1-t)+(c-a-b)t\tag{4.4}$$

which is clearly true under the hypothesis. □

An interesting case is obtained by choosing

$$\phi(1-t) = F\left(\begin{matrix} c-a,\ 1-a \\ c-a-b+1 \end{matrix}; 1-t\right),$$

where $c+1-a-b>0$. In this case $V_\lambda(f)(z)$ reduces to the convolution operator discussed in [1,2]

$$V_\lambda(f)(z) = zF(a,b;c;z) * f(z)$$

so that

$$V_\lambda(f)(z) = K\int_0^1 t^{b-1}(1-t)^{c-a-b} F\left(\begin{matrix} c-a,\ 1-a \\ c-a-b+1 \end{matrix}; 1-t\right)\frac{f(tz)}{t}\,dt,$$

where $a>0$, $b>0$, $c+1-a-b>0$ and

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.$$

Remark. We remark that in the case $\mu=0$, Theorems 4.1, 4.2 and 4.3 continue to hold with $\gamma>0$ instead of $\gamma \geqslant 1/2$.

The following special case which corresponds to Bernardi operator is easy to obtain from Theorem 4.3.

Corollary 4.1. *Let $\gamma \geqslant 1/2$, $\mu \in (0,1/2]$, $-1<c\leqslant 2$, and $\beta<1$ with*

$$\frac{\beta}{1-\beta} = -(1+c)\int_0^1 t^c g(t)\,dt,$$

where $g(t)$ is defined by (2.1). *Then, for $f \in \mathcal{P}_\gamma(\beta)$, the Bernardi transform $F(z)=(1+c)\times \int_0^1 t^{c-1} f(tz)\,dt$ is in $\mathcal{S}^*(\mu)$. The conclusion does not hold for smaller values of β.*

References

[1] R. Balasubramanian, S. Ponnusamy, M. Vuorinen, On hypergeometric functions and function spaces, J. Comput. Appl. Math. 139 (2002) 299–322.

[2] R. Balasubramanian, S. Ponnusamy, D.J. Prabhakaran, Duality techniques for certain integral transforms to be starlike, J. Math. Anal. Appl. 293 (2004) 355–373.

[3] R.W. Barnard, S. Naik, S. Ponnusamy, Univalency of a weighted integral transform of certain functions, J. Comput. Appl. Math. 193 (2006) 638–651.

[4] R. Fournier, St. Ruscheweyh, On two extremal problems related to univalent functions, Rocky Mountain J. Math. 24 (2) (1994) 529–538.

[5] Y.C. Kim, F. Rønning, Integral transforms of certain subclasses of analytic functions, J. Math. Anal. Appl. 258 (2001) 466–486.

[6] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, Monogr. Textbooks Pure Appl. Math., vol. 225, Marcel Dekker, Inc., New York, 2000, xii+459 pp.

[7] S. Ponnusamy, Inclusion theorems for convolution product of second order polylogarithms and functions with the derivative in a halfplane, Rocky Mountain J. Math. 28 (2) (1998) 695–733.

[8] S. Ponnusamy, F. Rønning, Duality for Hadamard products applied to certain integral transforms, Complex Var. Theory Appl. 32 (1997) 263–287.

[9] S. Ponnusamy, F. Rønning, Integral transforms of a class of analytic functions, Complex Var. Theory Appl., 14 pp., in press.

[10] St. Ruscheweyh, Duality for Hadamard products with applications to extremal problems for functions regular in the unit disk, Trans. Amer. Math. Soc. 210 (1975) 63–74.

[11] H. Silverman, E.M. Silvia, D. Telage, Convolution conditions for convexity starlikeness and spiral-likeness, Math. Z. 162 (2) (1978) 125–130.

[12] St. Ruscheweyh, Convolutions in Geometric Function Theory, Les Presses de l'Université de Montréal, Montréal, 1982.